ELECTROMAGNETIC RESEARCH CORPORATION

REC

A NONLINEAR TECHNIQUE FOR PLASMA DIAGNOSTICS

by

W. W. Zachary

Scientific Report No. 3
on
Contract NASw-835

Prepared for

National Aeronautics and Space Administration
Goddard Space Flight Center
Greenbelt, Maryland.

Report No. NAS 835-3
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ABSTRACT

A new technique for plasma diagnostics is presented which employs the nonlinear properties of the plasma medium.

The situation is considered in which two electromagnetic fields with different frequencies, ω_1 and ω_2, are externally applied to a plasma. Interest is then concentrated on the current densities associated with the beat frequencies, $\omega_1 \pm \omega_2$. One expects, on the basis of physical reasoning, that a resonance effect will be established if one chooses one of the beat frequencies equal to a characteristic frequency of the plasma.

Detailed calculations are carried out for the cases of wave propagation parallel and perpendicular to a static magnetic field. The most promising results, for the purposes of plasma diagnostics, are found in the case of perpendicular propagation, where one chooses either

$$\omega_1 - \omega_2 = \omega_c$$

or

$$\omega_1 - \omega_2 = 2\,\omega_c$$

where ω_c is the electron cyclotron frequency, and measures currents flowing perpendicular to the static magnetic field. In these cases it is found that the currents associated with the difference frequency, $\omega_1 - \omega_2$, have measurable values and are strongly dependent on the values of the electron temperature perpendicular to the static magnetic field.

Measured values of these currents should, therefore, yield values for this electron temperature. These currents are independent of the electron temperature parallel to the static magnetic field. If the work presented herein is extended to include wave propagation directions at an arbitrary angle with respect to the static magnetic field, the corresponding currents will depend on the longitudinal as well as the transverse temperature. It is hoped that the longitudinal

(over)

temperature can also be measured in this way. This extension is straightforward in principle, involving only more tedious calculations.

The diagnostic method proposed in this work should have some advantages over present techniques. This method should enable one to measure anisotropic temperature effects, which is beyond the capability of present probe techniques.

Since there are two resonance conditions in the case discussed above, a frequency sweep technique is suggested for measurements of electron temperatures in the ionosphere.

author

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1. INTRODUCTION

In this report we discuss a new method for determining plasma properties, with particular emphasis on the electron density and electron temperature. Before proceeding with the development of the theoretical basis of the new method, however, we first give a brief discussion of present diagnostic techniques revealing in the process the motivation for the present investigation.

There are, at present, several methods in use for making measurements on low density plasmas such as occur in the ionosphere. The most common method is the use of the Langmuir probe. There are, however, certain ambiguities associated with the operation of these probes [1, 2]*. Certain types of double probes, modifications of the Langmuir probe, are also in use [3, 4]. A comparison of measurements obtained on rockets and satellites by use of these probes has recently been given by Evans [5], along with a comparison with the radar backscatter technique. This latter method will not interest us in the present discussion, however, because we are interested in methods of obtaining in situ measurements. There has also been some work done on a triple probe [6], although this device has not yet been used for ionosphere studies. The double and triple probes have essentially the same ambiguities as the Langmuir probes.

It was principally because of the difficulties with the probes listed above that the so-called "resonance probe" was developed [2, 7, 8]. This method takes advantage of the resonance at a certain frequency. One advantage of this type of probe over the Langmuir type is that the results are relatively insensitive to the shape or size of the probe and sheath.

The analysis for the resonance probe has recently been extended to include the effect of a static magnetic field, under certain conditions [9]. There is, however, no provision for the possibility that the electron temperature may be anisotropic.

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* Numbers in brackets refer to the correspondingly numbered references on p. 90.

With this short discussion of the probes now being used for the study of the low density ionosphere plasma, we will now discuss the basic ideas behind the new method being presented here. More exhaustive discussions of probes, and also other methods used mostly for the study of laboratory plasmas, can be found in the books by Heald and Wharton [10] and by Huddlestone and Leonard [11].

Suppose that two electromagnetic fields with angular frequencies ω_1 and ω_2 are externally applied to a plasma. It will be supposed in the following discussion that these frequencies are completely arbitrary in nature, i.e., they are not harmonically related to each other or to the characteristic frequencies of the plasma — the electron and ion plasma and cyclotron frequencies. It is well known that fields are generated in the plasma characterized by harmonics of ω_1 and ω_2 and the beat frequencies $\omega_1 + \omega_2$ and $\omega_1 - \omega_2$. As a result of the application of these two external fields, a current density will be produced. This current density will consist of a sum of terms involving the single frequencies, ω_1 and ω_2, (which can be investigated by means of a linearized theory) and, in addition, terms involving higher harmonics of ω_1 and ω_2 and also terms involving sum and differences of the various frequencies. These terms are all consequences of the nonlinear interactions occurring within the plasma.

We shall single out for study the contributions to the current density associated with the beat frequencies, $\omega_1 \pm \omega_2$. The main objective here is to provide a better means for measuring the electron temperature in a plasma (in particular, a low density, low temperature ionosphere plasma) than is currently available in the probe methods discussed above.

One distinctive feature of this method, which can be stated at the outset, is that it involves measurements of only AC currents, whereas all the conventional probe methods involve measurements of DC currents.

The theoretical problem is mathematically formulated in Section 2. Section 3 is then devoted to a discussion of the linearized theory of plasma waves, with emphasis on the problem under consideration. Section 4 contains the solution of the nonlinear problem, i.e., the calculation of the current densities associated with the sum and difference frequencies, $\omega_1 \pm \omega_2$. In Section 5 we give some numerical examples of the results obtained in the previous section, and finally, Section 6 consists of concluding remarks.

2. FORMULATION OF THE PROBLEM

The plasma will be described by considering Maxwell's equations in conjunction with the Boltzmann equations for the electrons, ions, and neutral particles. For simplicity, the ions will be considered to be immobile, the positive ions being replaced by a smeared out positive charge background. The frequencies to be considered will always be sufficiently high so that this approximation is justified.

The Boltzmann equation for the electrons is given by (in MKS units)

$$\left[\frac{\partial}{\partial t} + \underset{\sim}{v} \cdot \underset{\sim}{\nabla} - \frac{e}{m}\left(\underset{\sim}{E} + \underset{\sim}{v} \times \underset{\sim}{B}\right) \cdot \frac{\partial}{\partial \underset{\sim}{v}}\right] f(\underset{\sim}{x}, \underset{\sim}{v}, t) = \left(\frac{\partial f}{\partial t}\right)_{coll} \tag{1}$$

where the right side represents the change of the electron distribution, f, due to collisions between electrons and other particles. The vector, $\underset{\sim}{B}$, will be decomposed into

$$\underset{\sim}{B} = \underset{\sim}{B}_o + \underset{\sim}{B}_i \tag{2}$$

where $\underset{\sim}{B}_o$ represents a static field and $\underset{\sim}{B}_i$ the internal field in the plasma due to motion of the electrons.

The collision term, in general, is quite complicated. We will, however, restrict consideration to the case of the F-layer of the ionosphere or, more exactly, to regions of the ionosphere for which the following inequality is satisfied,

$$\nu \ll \omega \tag{3}$$

where ν is the collision frequency for momentum transfer and ω is the frequency of one of the external fields. In this situation*, the collision term can be written as

$$\left(\frac{\partial f}{\partial t}\right)_{coll} = \nu \left(f_0 - f \right) \tag{4}$$

where f_0 is the zero-order distribution function and the collision frequency, ν, is independent of velocity.

We now decompose the distribution function into the form,

$$f = f_0 + f_1 \tag{5}$$

where f_0 is the zero-order distribution function which satisfies

$$\left(\underset{\sim}{V} \times \underset{\sim}{\omega_c} \right) \cdot \frac{\partial f_0}{\partial \underset{\sim}{V}} = 0 \tag{6}$$

where

$$\underset{\sim}{\omega_c} \equiv \frac{e}{m} \underset{\sim}{B_0} .$$

The scalar $|\underset{\sim}{\omega_c}|$ is the electron cyclotron (angular) frequency.

The most general solution of (6) is

$$f_0 = f_0 \left(V_\perp^2 , V_\parallel \right) \tag{7}$$

where

$$\underset{\sim}{V_\parallel} = \underset{\sim}{B_0} \frac{\left(\underset{\sim}{B_0} \cdot \underset{\sim}{V} \right)}{B_0^2}$$

$$\underset{\sim}{V_\perp} = \underset{\sim}{V} - \underset{\sim}{V_\parallel}$$

* There is, in addition, another condition in order that (4) be valid [12]. Namely, the Larmor radius of the electrons in the static magnetic field must be much less than the wavelengths of the waves propagating in the plasma. This condition is always satisfied in the cases considered here.

and f_0 is an arbitrary function of its arguments.

We will assume that the plasma (ionosphere) is in thermal equilibrium before the external fields are applied, and take f_0 to be:

$$f_0 = n_0 \left(\frac{m}{2\pi}\right)^{3/2} \theta_\perp^{-1} \theta_\parallel^{-\frac{1}{2}} \exp\left[-\frac{m}{2}\left(\frac{v_\perp^2}{\theta_\perp} + \frac{v_\parallel^2}{\theta_\parallel}\right)\right] \tag{8}$$

where $\theta_\perp$ and $\theta_\parallel$ are electron temperatures expressed in energy units. We have allowed for the fact that the mathematics of the problem allows the possibility that the electron temperatures parallel and perpendicular to the static magnetic field may be different. The plasma probes now in use cannot measure this anisotropy of the temperature.

There is some controversy at present concerning the question of whether or not the ionosphere is in thermal equilibrium [3, 13-15]. The assumption (8) is not necessary for the development of the present method, and is made only for definiteness. We could, for example, use, instead of (8), a distribution function which included the effects of high velocity particles (i.e., velocity $> \left(\frac{\theta_{\perp,\parallel}}{m}\right)^{\frac{1}{2}}$).

Substituting (5) into (1) and making use of (2), (4), and (6) we obtain

$$\left[\nu + \frac{\partial}{\partial t} + \underset{\sim}{v} \cdot \underset{\sim}{\nabla} - (\underset{\sim}{v} \times \underline{\omega_c}) \cdot \frac{\partial}{\partial \underset{\sim}{v}}\right] f_1 - \frac{e}{m}\left(\underset{\sim}{E} + \underset{\sim}{v} \times \underset{\sim}{B_i}\right) \cdot \frac{\partial f_0}{\partial \underset{\sim}{v}}$$
$$= \frac{e}{m}\left(\underset{\sim}{E} + \underset{\sim}{v} \times \underset{\sim}{B_i}\right) \cdot \frac{\partial f_1}{\partial \underset{\sim}{v}}, \tag{9}$$

(9) has been written in such a way that all nonlinear terms are on the right side. It is these terms that give rise to the beat frequencies. Note also that we are not neglecting the terms involving the internal magnetic field, as is quite frequently done. The reason is that we are looking for a small effect so that the magnetic terms are important. Indeed, there are cases even in the linear theory for which the magnetic terms are important, as pointed out by Scarf [16].

The problem before us is to solve (9) in conjunction with

$$\left(\underset{\sim}{\nabla} \times \underset{\sim}{\nabla} \times + \frac{1}{c^2} \frac{\partial^2}{\partial t^2}\right) \underset{\sim}{E} = \frac{e}{\epsilon_0 c^2} \frac{\partial}{\partial t} \int \underset{\sim}{v} f_1 \, d^3 v \tag{10}$$

which is easily derived from Maxwell's equations if we note that the current density is defined in terms of the distribution function by

$$\underset{\sim}{j} \equiv -e \int \underset{\sim}{v} f \, d^3 v = -e \int \underset{\sim}{v} f_1 \, d^3 v \tag{11}$$

The procedure will be to solve (9) for f_1 by an iteration procedure, and then obtain $\underset{\sim}{E}$ from (10). The internal magnetic field, $\underset{\sim}{B}_i$, is found from the differential formulation of Faraday's law,

$$\underset{\sim}{\nabla} \times \underset{\sim}{E} = -\frac{\partial}{\partial t} \underset{\sim}{B}_i . \tag{12}$$

In using an iteration method to solve (9), the assumption is made that the nonlinearities are "small". Similar procedures have been used previously [17-21] and are convenient because all that is required is a straightforward generalization of the well-known linearized theory. The manner in which the iteration process is formulated in the present work is different, however, from that of the papers just cited.

It is clear from physical considerations that such an approach must work in the present problem, since nonlinear propagation effects have been observed in the ionosphere at quite low field strengths [17]. It is expected, therefore, that the iteration procedure will produce meaningful results. In this sense, the ionosphere can be characterized as a weakly nonlinear medium, in contrast with some of the strongly nonlinear media currently of interest in nonlinear optics [22]. This idea is essentially contained in Danilkin's paper [19].

In order to carry out the iteration process, we need the solution to the linearized problem, i.e., when the right side of (9) is set equal to zero. This topic is discussed in the next section.

3. LINEARIZED THEORY

The linearized form of (9) is:

$$\left[\nu + \frac{\partial}{\partial t} + \underset{\sim}{v} \cdot \underset{\sim}{\nabla} - \left(\underset{\sim}{v} \times \underset{\sim}{\omega}_c\right) \cdot \frac{\partial}{\partial \underset{\sim}{v}}\right] f_1 = \frac{e}{m}\left(\underset{\sim}{E} + \underset{\sim}{v} \times \underset{\sim}{B}_i\right) \cdot \frac{\partial f_0}{\partial \underset{\sim}{v}} . \tag{13}$$

To examine the steady-state plane wave solutions of this equation along with (10), we make the ansatz:

$$\left.\begin{array}{c} f_1 \\ \underset{\sim}{E} \\ \underset{\sim}{B}_i \end{array}\right\} \sim \exp\left[i\left(\underset{\sim}{k} \cdot \underset{\sim}{x} - \omega t\right)\right], \tag{14}$$

The frequency, ω, is taken to be real (it will later be taken to be either ω_1, ω_2, or a linear combination of the two) but the propagation vector, $\underset{\sim}{k}$, may be complex. Substitution of (14) into (13) gives:

$$\left[\nu + i\left(\underset{\sim}{k} \cdot \underset{\sim}{v} - \omega\right) - \left(\underset{\sim}{v} \times \underset{\sim}{\omega}_c\right) \cdot \frac{\partial}{\partial \underset{\sim}{v}}\right] f_1 = \frac{e}{m}\left(\underset{\sim}{E} + \underset{\sim}{v} \times \underset{\sim}{B}_i\right) \cdot \frac{\partial f_0}{\partial \underset{\sim}{v}} . \tag{15}$$

It is now convenient to choose the coordinate system so that the static magnetic field lies along the z-axis. Introducing cylindrical coordinates

$$\begin{aligned} v_x &= v_\perp \cos\varphi \\ v_y &= v_\perp \sin\varphi \\ v_z &= v_\parallel \end{aligned} \tag{16}$$

we find [23,24]

$$f_1(v_\perp, \varphi, v_\parallel) = \frac{e}{m\omega_c} \int_{-\infty}^{\varphi} \left(\underset{\sim}{E} + \underset{\sim}{v}' \times \underset{\sim}{B}_i\right) \cdot \frac{\partial f_0}{\partial \underset{\sim}{v}'}\, G(\underset{\sim}{k}, \omega; \varphi, \varphi')\, d\varphi' \tag{17}$$

where

$$G(\underset{\sim}{K},\omega;\varphi,\varphi') \equiv \exp\left\{-\frac{1}{\omega_c}\int_{\varphi'}^{\varphi}\left[\nu+i(\underset{\sim}{K}\cdot\underset{\sim}{V}''-\omega)\right]d\varphi''\right\} \tag{18}$$

$$= \exp\left\{-\frac{i}{\omega_c}\left[K_x V_\perp(\sin\varphi-\sin\varphi')-K_y V_\perp(\cos\varphi-\cos\varphi')\right] - \frac{1}{\omega_c}\left[\nu+i(K_z V_\parallel-\omega)\right](\varphi-\varphi')\right\}.$$

The primed velocity vector in (17) is a shorthand notation and is defined, by means of (16), in terms of unprimed radial and axial variables and a primed angular variable. The lower limit in (17) has been chosen in such a way that f_1 is bounded and is periodic in φ with period 2π.

The equation for the electric field, $\underset{\sim}{E}$, is determined by substituting (17) into (10), making use of (14). After some manipulations, we find

$$\mathcal{D}_{\alpha\beta} E_\beta = 0 \tag{19}$$

where

$$\mathcal{D}_{\alpha\beta} \equiv \left(\frac{\omega^2}{c^2}-K^2\right)\delta_{\alpha\beta} + K_\alpha K_\beta + \frac{i\omega}{\epsilon_0 c^2}\sigma_{\alpha\beta}(\underset{\sim}{K},\omega). \tag{20}$$

In (19) we have introduced the summation convention in which a repeated subscript is summed from 1 to 3. The quantity $\sigma_{\alpha\beta}$ is the conductivity tensor defined by

$$j_\alpha = -e\int V_\alpha f_1 d^3V \equiv \sigma_{\alpha\beta}E_\beta \tag{21}$$

and is equal to

$$\sigma_{\alpha\beta} \equiv -\frac{e^2}{m\omega_c}\int d^3V\, V_\alpha \int_{-\infty}^{\varphi}\left[\left(1-\frac{K_\gamma V_\gamma'}{\omega}\right)\frac{\partial f_0}{\partial V_\beta'}+\frac{K_\gamma}{\omega}\frac{\partial f_0}{\partial V_\gamma'}V_\beta'\right]G(\underset{\sim}{K},\omega;\varphi,\varphi')\,d\varphi', \tag{22}$$

In order that non-trivial solutions of (19) exist we must have

$$\det(D_{\alpha\beta}) = 0. \tag{23}$$

The solutions of this equation determine the dispersion relations of the waves which can propagate in the plasma.

In order to solve (23), or even to write it out explicitly, the conductivity tensor must be determined; i.e., the velocity space integrations indicated in (22) must be evaluated. These integrations, in fact, can be done exactly [25], the results being expressed in terms of Bessel functions and error functions. (23) then becomes a transcendental equation which must be solved to obtain the dispersion relations of the various waves in the plasma. This procedure is, of course, very difficult to carry out. There is, however, a simplification which can be made here which is standard procedure in the linearized theory of plasma waves. The idea is to consider the case for which the phase velocity is much greater than the thermal velocity of the electrons. This is, in fact, the actual physical situation except near resonances [12]. For the cases in which this approximation is not satisfied the waves are strongly damped, so that this situation is not interesting. This damping is the analog of the damping of longitudinal waves in the absence of an external magnetic field, first found by Landau [26]. It is now known [23-25,27] that analogous damping phenomena exist for other types of waves.

The procedure, then, is to introduce the small-argument expansions for the Bessel functions and the asymptotic expansions for the error functions, so that the dispersion relations are determined to lowest order in k^2. This procedure is sufficient for the waves being considered here, since the wavelengths are very long, i.e., $\lambda \gg \lambda_D$ = Debye length.

Using the above procedure (23) becomes a cubic equation in k^2. The situation simplifies even further if consideration is restricted to the cases of propagation either parallel or perpendicular to the static magnetic field. In these cases the cubic equation degenerates into a product of two factors, one linear and one quadratic in k^3.

This last step is done in order to simplify the analysis, but is not a necessary restriction of the present theory. The consideration of these two limiting cases should be sufficient to determine whether or not the diagnostic method being proposed in this report is feasible.

3.1 Propagation Parallel to the Static Magnetic Field

Following the procedure discussed above, (23) becomes, in this case:

$$\begin{vmatrix} \mathcal{D}_{11} & \mathcal{D}_{12} & 0 \\ \mathcal{D}_{21} & \mathcal{D}_{22} & 0 \\ 0 & 0 & \mathcal{D}_{33} \end{vmatrix} = 0 \tag{24}$$

where:

$$\begin{aligned} \mathcal{D}_{11} &= \mathcal{D}_{22} = K_0^2 \left\{1 - n^2 + (\alpha_1 - 1)\left[1 + n^2\beta_\perp \mu_{1\perp} + n^2 \beta_\parallel \mu_{1\parallel}\right]\right\} \\ \mathcal{D}_{12} &= i K_0^2 \alpha_2 \left\{1 + n^2 \beta_\perp \mu_{2\perp} + n^2 \beta_\parallel \mu_{2\parallel}\right\} \\ \mathcal{D}_{33} &= K_0^2 \left\{1 + (\alpha_3 - 1)\left(1 + n^2 \beta_\parallel \mu_{3\parallel}\right)\right\}. \end{aligned} \tag{25}$$

In (25), $k_0 \equiv \frac{\omega}{c}$ is the vacuum wave number and $n \equiv \frac{ck}{\omega}$ is the refractive index. The temperature effects are contained in the dimensionless quantities,

$$\beta_{\perp,\parallel} \equiv \frac{\Theta_{\perp,\parallel}}{mc^2}. \tag{26}$$

The following normalized quantities are commonly used in the magneto-ionic theory and are convenient for the present problem,

$$\begin{aligned} \alpha_1 &\equiv 1 - \frac{XU}{U^2 - Y^2} \\ \alpha_2 &\equiv \frac{XY}{U^2 - Y^2} \\ \alpha_3 &\equiv 1 - \frac{X}{U} \end{aligned} \tag{27}$$

where X, Y, and U are the following normalized frequencies:

$$X \equiv \left(\frac{\omega_p}{\omega}\right)^2 = \frac{n_0 e^2}{m \epsilon_0 \omega^2}$$

$$Y \equiv \frac{\omega_c}{\omega} \tag{28}$$

$$U \equiv 1 + i Z , \quad Z \equiv \frac{\nu}{\omega}$$

ω_p being the familiar electron plasma frequency. It will be noted that the definition of U used here corresponds to the choice of sign for the exponential in (14).

The significance of the quantities defined by (27) is that the elements of the dielectric tensor of a zero temperature plasma can be completely expressed in terms of them.

The remaining quantities in (25) are functions only of the normalized parameters Y and U:

$$\mu_{1\perp} \equiv \frac{U^2 + Y^2}{U(U^2 - Y^2)}$$

$$\mu_{1\parallel} \equiv \frac{U^3(1-U) + Y^2(3U + Y^2)}{U(U^2 - Y^2)^2}$$

$$\mu_{2\perp} \equiv \frac{2U}{U^2 - Y^2}$$

$$\mu_{2\parallel} \equiv \frac{U^2(1-2U) + Y^2(1+2U)}{(U^2 - Y^2)^2}$$

$$\mu_{3\parallel} \equiv 3U^{-1} ,$$

One solution of (24) occurs when

$$\mathcal{D}_{33} = 0$$

or

$$\left(n^{(1)}\right)^2 = \frac{\alpha_3}{3\mu_{3\parallel}\,\beta_\parallel\,(1-\alpha_3)} , \tag{29}$$

The superscript indicates that this solution will henceforth be called "mode no. 1".

The remaining solutions of (24) are obtained from the condition

$$D_{11} = \pm i\, D_{12}$$

or,

$$n^2 = \frac{\alpha_1 \pm \alpha_2}{1+(1-\alpha_1)(\beta_\perp \mu_{1\perp} + \beta_\parallel \mu_{1\parallel}) \mp \alpha_2(\beta_\perp \mu_{2\perp} + \beta_\parallel \mu_{2\parallel})} , \tag{30}$$

Henceforth the solution for the upper sign in (30) will be denoted by "mode no. 2" and that for the lower sign by "mode no. 3". These solutions correspond to the ordinary and extraordinary waves of the magneto-ionic theory [28]. The corrections for finite temperature or compressibility are given in the second and third terms of the denominator. It will be shown in Section 5 that these corrections are completely negligible for the cases of interest in the present investigation, so that (30) will reduce to the results obtained from the zero-temperature magneto-ionic theory.

The wave described by (29) is longitudinal and linearly polarized along its direction of propagation, which is also the direction of the static magnetic field. It will be noted that the refractive index is independent of the static magnetic field. This is the familiar longitudinal plasma wave. It will be noted that $\beta_\parallel$ appears in the denominator of (29), so that $|n^{(1)}|$ will be very large and the phase velocity very small (of the order of the thermal velocity).

The waves described by (30) are circularly polarized in the plane perpendicular to the static magnetic field. For very high frequency, $\omega \to \infty$, (30) reduces to the dispersion relation for electromagnetic waves in vacuum, $n^2 = 1$,

3.2 Propagation Perpendicular to the Static Magnetic Field

We will next consider propagation along the x-axis.

Following the same procedure, (23) can again be written in the form (24) with the elements of the determinant now defined by:

$$\begin{aligned}
\mathcal{D}_{11} &= K_0^2 \{1 + (\alpha_1 - 1)(1 + n^2 \beta_\perp \lambda_{1\perp})\} \\
\mathcal{D}_{22} &= K_0^2 \{1 - n^2 + (\alpha_1 - 1)(1 + n^2 \beta_\perp \lambda_{2\perp})\} \\
\mathcal{D}_{33} &= K_0^2 \{1 - n^2 + (\alpha_3 - 1)[1 + n^2 \beta_\parallel \lambda_{3\parallel} + n^2 \beta_\perp \lambda_{3\perp}]\} \\
\mathcal{D}_{12} &= i K_0^2 \alpha_2 \, (1 + n^2 \beta_\perp \lambda_{0\perp})
\end{aligned} \tag{31}$$

where:

$$\lambda_{1\perp} \equiv \frac{3}{U^2 - 4Y^2} \quad ; \quad \lambda_{2\perp} \equiv \frac{U^2 + 8Y^2}{U^2(U^2 - 4Y^2)}$$

$$\lambda_{3\parallel} \equiv \frac{U}{U^2 - Y^2} \quad ; \quad \lambda_{3\perp} \equiv \frac{1 - U}{U^2 - Y^2}$$

$$\lambda_{0\perp} \equiv \frac{6}{U^2 - 4Y^2} \, .$$

One solution is again given by

$$\mathcal{D}_{33} = 0$$

or,

$$\left(n^{(1)}\right)^2 = \frac{\alpha_3}{1 + (1 - \alpha_3)(\beta_\parallel \lambda_{3\parallel} + \beta_\perp \lambda_{3\perp})} \, , \tag{32}$$

It will be noted that the dependence of $n^{(1)}$ on the transverse electron temperature vanishes if collisions are neglected. As already noted in connection with (30), the temperature dependent terms in (32) are very small for the cases of interest in the present work. Numerical examples will be considered later. Thus, (32) can be simplified to read

$$\left(n^{(1)}\right)^2 = \alpha_3 , \tag{32a}$$

the same result obtained from the magneto-ionic theory.

The other solutions are determined by the condition

$$\mathcal{D}_{11}\,\mathcal{D}_{22} + \mathcal{D}_{12}^2 = 0$$

which can be written as the following algebraic equation

$$\begin{aligned} & n^4\left[(1-\alpha_1)\beta_\perp\,\lambda_{1\perp}\left(1+(1-\alpha_1)\beta_\perp\,\lambda_{2\perp}\right)-\beta_\perp^2\,\lambda_{0\perp}^2\,\alpha_2^2\right] \\ & -n^2\left[\alpha_1\left(1+(1-\alpha_1)\beta_\perp(\lambda_{1\perp}+\lambda_{2\perp})\right)+2\,\beta_\perp\,\lambda_{0\perp}\,\alpha_2^2\right] \\ & +\alpha_1^2-\alpha_2^2 = 0 . \end{aligned} \tag{33}$$

By using the fact that the temperature effects are small, we obtain the simpler equation

$$(1-\alpha_1)\,\beta_\perp\,\lambda_{1\perp}\,n^4 - \alpha_1 n^2 + \alpha_1^2 - \alpha_2^2 = 0 . \tag{33a}$$

The solutions to this equation (taking into account the small magnitude of the temperature terms) are:

$$\left(n^{(2)}\right)^2 = \frac{\alpha_1}{(1-\alpha_1)\,\beta_\perp\,\lambda_{1\perp}} \tag{34}$$

$$\left(n^{(3)}\right)^2 = \frac{\alpha_1^2-\alpha_2^2}{\alpha_1} . \tag{35}$$

The wave described by (32) is linearly polarized parallel to the static magnetic field and is, therefore, a transverse wave. In the limit $\omega \to \infty$,

(32) reduces to the dispersion relation for electromagnetic waves in vacuum. At lower frequencies there is a correction to the vacuum dispersion relation due to presence of the plasma. The static magnetic field has no appreciable effect on this wave because the temperature corrections are negligible, as is readily seen from (32).

The waves described by (34) and (35) are elliptically polarized in the xy-plane. Thus, the polarization is mixed, being neither purely longitudinal nor purely transverse.

It will be noted that the temperature corrections for modes no. 1 and no. 3 are negligible, whereas this is not true for mode no. 2. Note also that the dispersion relations given by (32) and (35) reduce to the dispersion relation for vacuum electromagnetic waves in the limit $\omega \to \infty$, whereas (34) $\to \infty$ (mode no. 2).

The solutions for the refractive indices derived in this section, for both parallel and perpendicular propagation, will be discussed in more detail in later sections, after expressions for the current densities are derived.

We have thus obtained the plane wave solutions of the equations of motion. We can now use the fact that the equations are linear, and therefore that the superposition principle is valid, to construct the general solutions (in the linear approximation) from the plane wave solutions obtained above.

The electric field will be written as

$$\underset{\sim}{E}^{(0)} = \sum_{\alpha=1}^{3} \sum_{\gamma=1}^{2} \left[\underset{\sim}{a}_{\gamma}^{\alpha} \exp\left(i \left(\underset{\sim}{k}_{\gamma}^{\alpha} \cdot \underset{\sim}{x} - \omega_{\gamma} t \right) \right) + c.c. \right] \tag{36}$$

where "c.c." denote the complex conjugate of the preceding term. The summation over α indicates that we add the contributions of all the propagating plasma wave modes, which in general are three. The summation over γ is a consequence of the fact that two external fields are applied to the plasma. The amplitudes of the electric fields, $\underset{\sim}{a}_\gamma^\alpha$, are solutions of (19). The zero superscript is used above, and henceforth, to denote quantities obtained in the linear approximation.

We use the form (36) instead of the usual prescription in which a single plane wave is used, with the understanding that only the real part is to be taken after the calculation. The form (36) is used by Bloembergen [22] and others in nonlinear optics. It produces an additional factor of 2 over that of the usual method, but this is easily remedied at the end of the calculation.

By writing expressions in the form (36), the results will be real quantities, as they should be.

In a similar manner the distribution function, $f_1 \equiv f - f_0$, will be written in the following way:

$$f_1^{(0)} = \sum_{\alpha=1}^{3} \sum_{\gamma=1}^{2} \left[F_\gamma^\alpha \exp\left(i(\underset{\sim}{k}_\gamma^\alpha \cdot \underset{\sim}{x} - \omega_\gamma t)\right) + c.c. \right] \tag{37}$$

where:

$$F_\gamma^\alpha \equiv \frac{e}{m\omega_c} \int_{-\infty}^{\varphi} \left(\underset{\sim}{a}_\gamma^\alpha + \underset{\sim}{v}' \times \underset{\sim}{b}_\gamma^\alpha \right) \cdot \frac{\partial f_0}{\partial \underset{\sim}{v}'} G(\underset{\sim}{k}_\gamma^\alpha, \omega_\gamma; \varphi, \varphi')\, d\varphi'. \tag{38}$$

The amplitude of the internal magnetic field, $\underset{\sim}{b}_\gamma^\alpha$, is related to $\underset{\sim}{a}_\gamma^\alpha$ by the relation

$$\underset{\sim}{b}_\gamma^\alpha = \frac{1}{\omega_\gamma} \underset{\sim}{k}_\gamma^\alpha \times \underset{\sim}{a}_\gamma^\alpha \tag{39}$$

which is a consequence of (12).

We are now ready to obtain the solution of the nonlinear system (9) and (10). This will be discussed in the next section.

Before proceeding, however, we should discuss in somewhat more detail the consequences of writing (36) and (37) as a sum of plane waves, since there is an approximation involved.

There are known examples of cases in which plane wave solutions are not valid [26,29,30]. Landau [26] considered the problem of penetration of an external field into a plasma in the absence of a static magnetic field, considering only the longitudinal field. This analysis was extended by Shafranov [30] and by Platzman and Bucksbaum [29] to the case of propagation parallel to a static magnetic field. The results of these investigations are that the asymptotic results for the fields (i.e., far from the boundary inside the plasma) are (z = distance from the boundary)

$$\sim \exp\left(-z^{2/3}\right)$$

for a Maxwellian distribution [26,30] and

$$\sim z^{-2}$$

for a resonance distribution [29]. Platzman and Bucksbaum have pointed out that these effects can be ignored if one is not near a resonance and not too far from the boundary. We must point out, however, that these results were obtained for plasmas with thermal energies of the order of 30 ev [29]. On the contrary, the present work is concerned with plasmas for which the maximum thermal energies are of the order 0.1 ev. Therefore, the effects discussed above should not be significant for the present work even far from the plasma boundary. We will not consider cases for which the plasma is near resonance.

4. LOWEST ORDER NONLINEAR SOLUTION

We now evaluate the right side of (9) by using (36) and (37) for the zero-order quantities. The resulting linear equation for the next iterative solution (lowest order nonlinear solution) for f_1, hereafter called $f_1^{(1)}$ is:

$$\left[\nu+\frac{\partial}{\partial \tau}+\underset{\sim}{v}\cdot\underset{\sim}{\nabla}+\omega_c\frac{\partial}{\partial \varphi}\right]f_1^{(1)}-\frac{e}{m}\left(\underset{\sim}{E}^{(1)}+\underset{\sim}{v}\times\underset{\sim}{B}^{(1)}\right)\cdot\frac{\partial f_0}{\partial \underset{\sim}{v}}=R \tag{40}$$

where:

$$R \equiv \frac{e}{m} \sum_{\alpha=1}^{3} \sum_{\gamma=1}^{2} \sum_{\alpha'=1}^{3} \sum_{\gamma'=1}^{2}$$

$$\Big\{ (\underset{\sim}{a}_\gamma^\alpha + \underset{\sim}{v} \times \underset{\sim}{b}_\gamma^\alpha) \cdot \frac{\partial}{\partial \underset{\sim}{v}} F_{\gamma'}^{\alpha'} \exp\left[i\left((\underset{\sim}{K}_\gamma^\alpha + \underset{\sim}{K}_{\gamma'}^{\alpha'}) \cdot \underset{\sim}{x} - (\omega_\gamma + \omega_{\gamma'})\tau\right)\right]$$

$$+ (\underset{\sim}{a}_\gamma^\alpha + \underset{\sim}{v} \times \underset{\sim}{b}_\gamma^\alpha) \cdot \frac{\partial}{\partial \underset{\sim}{v}} F_{\gamma'}^{\alpha' *} \exp\left[i\left((\underset{\sim}{K}_\gamma^\alpha - \underset{\sim}{K}_{\gamma'}^{\alpha'}) \cdot \underset{\sim}{x} - (\omega_\gamma - \omega_{\gamma'})\tau\right)\right]$$

$$+ \text{c. c.} \quad \Big\} . \tag{41}$$

The quantities $\underset{\sim}{E}^{(1)}$ and $\underset{\sim}{B}_i^{(1)}$ denote the electric field and internal magnetic field, respectively, in the lowest order nonlinear approximation, and the * denotes the complex conjugate.

The current density is defined by

$$\underset{\sim}{j} = -e \int \underset{\sim}{v}\, f\, d^3v = -e \int \underset{\sim}{v}\, (f_0 + f_1^{(0)} + f_1^{(1)})\, d^3v$$
$$= -e \int \underset{\sim}{v}\, (f_1^{(0)} + f_1^{(1)})\, d^3v. \tag{42}$$

We will concentrate, in the following work, on the contribution to (42) from the beat frequencies; i.e., the sum and difference frequencies $\omega_1 \pm \omega_2$. Thus, we will not be concerned with the contributions to (42) from single frequencies, contained in $f_1^{(0)}$, nor from the terms produced by harmonic generation, contained in $f_1^{(1)}$. The contributions to (42) associated with the frequencies $\omega_1 + \omega_2$ and $\omega_1 - \omega_2$ will be denoted by $\underset{\sim}{j}^{(+)}$ and $\underset{\sim}{j}^{(-)}$, respectively.

In view of the above remarks, it will not be necessary to consider all the terms in (41). The terms of R which will be of interest to us are the following, denoted by $\bar{R}$, with corresponding notation for $\bar{f}_1^{(1)}$:

$$
\begin{aligned}
\overline{R} = \frac{e}{m}\sum_{\alpha=1}^{3}\sum_{\alpha'=1}^{3} \Big\{ & \left(\underset{\sim}{a}_1^{\alpha} + \underset{\sim}{v}\times\underset{\sim}{b}_1^{\alpha}\right)\cdot\frac{\partial}{\partial \underset{\sim}{v}} F_2^{\alpha'} \exp\left[i\left((\underset{\sim}{K}_1^{\alpha}+\underset{\sim}{K}_2^{\alpha'})\cdot\underset{\sim}{x} - (\omega_1+\omega_2)\tau\right)\right] \\
& + \left(\underset{\sim}{a}_1^{\alpha} + \underset{\sim}{v}\times\underset{\sim}{b}_1^{\alpha}\right)\cdot\frac{\partial}{\partial \underset{\sim}{v}} F_2^{\alpha' *} \exp\left[i\left((\underset{\sim}{K}_1^{\alpha}-\underset{\sim}{K}_2^{\alpha'})\cdot\underset{\sim}{x} - (\omega_1-\omega_2)\tau\right)\right] \\
& + \left(\underset{\sim}{a}_2^{\alpha} + \underset{\sim}{v}\times\underset{\sim}{b}_2^{\alpha}\right)\cdot\frac{\partial}{\partial \underset{\sim}{v}} F_1^{\alpha'} \exp\left[i\left((\underset{\sim}{K}_2^{\alpha}+\underset{\sim}{K}_1^{\alpha'})\cdot\underset{\sim}{x} - (\omega_2+\omega_1)\tau\right)\right] \\
& + \left(\underset{\sim}{a}_2^{\alpha} + \underset{\sim}{v}\times\underset{\sim}{b}_2^{\alpha}\right)\cdot\frac{\partial}{\partial \underset{\sim}{v}} F_1^{\alpha' *} \exp\left[i\left((\underset{\sim}{K}_2^{\alpha}-\underset{\sim}{K}_1^{\alpha'})\cdot\underset{\sim}{x} - (\omega_2-\omega_1)\tau\right)\right] \\
& + c.\,c. \Big\}
\end{aligned}
\tag{43}
$$

We now consider (40) with R replaced by $\overline{R}$ and $f_1^{(1)}$ by $\overline{f}_1^{(1)}$. Since $\overline{R}$ is expressed as a sum of plane waves, we can decompose the variables in (40) in a similar manner:

$$
\begin{aligned}
\overline{f}_1^{(1)} = \sum_{\alpha=1}^{3}\sum_{\alpha'=1}^{3} \Big\{ & A_1^{\alpha\alpha'} \exp\left[i\left((\underset{\sim}{K}_1^{\alpha}+\underset{\sim}{K}_2^{\alpha'})\cdot\underset{\sim}{x} - (\omega_1+\omega_2)\tau\right)\right] \\
& + A_2^{\alpha\alpha'} \exp\left[i\left((\underset{\sim}{K}_2^{\alpha}+\underset{\sim}{K}_1^{\alpha'})\cdot\underset{\sim}{x} - (\omega_2+\omega_1)\tau\right)\right] \\
& + A_3^{\alpha\alpha'} \exp\left[i\left((\underset{\sim}{K}_1^{\alpha}-\underset{\sim}{K}_2^{\alpha'})\cdot\underset{\sim}{x} - (\omega_1-\omega_2)\tau\right)\right] \\
& + A_4^{\alpha\alpha'} \exp\left[i\left((\underset{\sim}{K}_2^{\alpha}-\underset{\sim}{K}_1^{\alpha'})\cdot\underset{\sim}{x} - (\omega_2-\omega_1)\tau\right)\right] \\
& + c.\,c. \Big\}
\end{aligned}
\tag{44}
$$

$$
\begin{aligned}
\underset{\sim}{E}^{(1)} = \sum_{\alpha=1}^{3} \sum_{\alpha'=1}^{3} \Big\{ & \underset{\sim}{M}_1^{\alpha\alpha'} \exp\left[i \left((\underset{\sim}{K}_1^{\alpha} + \underset{\sim}{K}_2^{\alpha'}) \cdot \underset{\sim}{X} - (\omega_1 + \omega_2) \tau \right) \right] \\
& + M_2^{\alpha\alpha'} \exp\left[i \left((\underset{\sim}{K}_2^{\alpha} + \underset{\sim}{K}_1^{\alpha'}) \cdot \underset{\sim}{X} - (\omega_2 + \omega_1) \tau \right) \right] \\
& + M_3^{\alpha\alpha'} \exp\left[i \left((\underset{\sim}{K}_1^{\alpha} - \underset{\sim}{K}_2^{\alpha'}) \cdot \underset{\sim}{X} - (\omega_1 - \omega_2) \tau \right) \right] \\
& + M_4^{\alpha\alpha'} \exp\left[i \left((\underset{\sim}{K}_2^{\alpha} - \underset{\sim}{K}_1^{\alpha'}) \cdot \underset{\sim}{X} - (\omega_2 - \omega_1) \tau \right) \right] \\
& + c.c. \Big\}
\end{aligned}
\tag{45}
$$

$$
\begin{aligned}
\underset{\sim}{B}_i^{(1)} = \sum_{\alpha=1}^{3} \sum_{\alpha'=1}^{3} \Big\{ & \underset{\sim}{N}_1^{\alpha\alpha'} \exp\left[i \left((\underset{\sim}{K}_1^{\alpha} + \underset{\sim}{K}_2^{\alpha'}) \cdot \underset{\sim}{X} - (\omega_1 + \omega_2) \tau \right) \right] \\
& + N_2^{\alpha\alpha'} \exp\left[i \left((\underset{\sim}{K}_2^{\alpha} + \underset{\sim}{K}_1^{\alpha'}) \cdot \underset{\sim}{X} - (\omega_2 + \omega_1) \tau \right) \right] \\
& + N_3^{\alpha\alpha'} \exp\left[i \left((\underset{\sim}{K}_1^{\alpha} - \underset{\sim}{K}_2^{\alpha'}) \cdot \underset{\sim}{X} - (\omega_1 - \omega_2) \tau \right) \right] \\
& + N_4^{\alpha\alpha'} \exp\left[i \left((\underset{\sim}{K}_2^{\alpha} - \underset{\sim}{K}_1^{\alpha'}) \cdot \underset{\sim}{X} - (\omega_2 - \omega_1) \tau \right) \right] \\
& + c.c. \Big\} .
\end{aligned}
\tag{46}
$$

The amplitudes in (46) are related to those in (45) as a consequence of (12). For example,

$$
N_1^{\alpha\alpha'} = \frac{1}{\omega_1 + \omega_2} \left(\underset{\sim}{K}_1^{\alpha} + \underset{\sim}{K}_2^{\alpha'} \right) \times \underset{\sim}{M}_1^{\alpha\alpha'} . \tag{47}
$$

The relations between the other amplitudes are obtained in an analogous manner.

Comparison of (40) with (43)-(46) indicates that the solutions for each plane wave can be obtained separately. In the following development we will consider only the derivation of results for the first term listed in (43)-(46). This is actually all that is required in order to determine $\underset{\sim}{j}^{(+)}$ and $\underset{\sim}{j}^{(-)}$, because the contributions of the remaining terms can be obtained simply by changes in notation. Thus, the contributions of the second terms can be obtained from those of the first terms by the transformations $1 \rightleftarrows 2$ on the subscipts of the propagation vectors and frequencies. The contributions of the third terms are found from those of the first terms by means of the transformations,

$$\begin{aligned} \underset{\sim}{K}_1^{\alpha} + \underset{\sim}{K}_2^{\alpha'} &\longrightarrow \underset{\sim}{K}_1^{\alpha} - \underset{\sim}{K}_2^{\alpha'} \\ \omega_1 + \omega_2 &\longrightarrow \omega_1 - \omega_2 \, . \end{aligned} \tag{48}$$

Finally, the contributions of the fourth terms are obtained from those of the third terms by the transformations $1 \rightleftarrows 2$ on the subscripts. Of course, another contribution is obtained from each of the four terms discussed above by taking the complex conjugate.

Following the procedure of Section 3, we obtain:

$$\begin{aligned} A_1^{\alpha\alpha'}(v_\perp, \varphi, v_\parallel) = \frac{1}{\omega_c} \int_{-\delta}^{\varphi} \Big[\frac{e}{m} \big(\underset{\sim}{M}_1^{\alpha\alpha'} + \underset{\sim}{v}' \times \underset{\sim}{N}_1^{\alpha\alpha'} \big) \cdot \frac{\partial f_0}{\partial \underset{\sim}{v}'} + \bar{R}^{\alpha\alpha'}(\underset{\sim}{v}') \Big] \\ \times G(\underset{\sim}{K}_+, \omega_+ ; \varphi, \varphi')\, d\varphi' \end{aligned} \tag{49}$$

where

$$\bar{R}^{\alpha\alpha'}(\underset{\sim}{v}') \equiv \frac{e}{m} \big(\underset{\sim}{a}_1^{\alpha} + \underset{\sim}{v}' \times \underset{\sim}{b}_1^{\alpha} \big) \cdot \frac{\partial}{\partial \underset{\sim}{v}'} F_2^{\alpha'}$$

and we have introduced the shorthand notation,

$$\underset{\sim}{K}_+ \equiv \underset{\sim}{K}_1^{\alpha} + \underset{\sim}{K}_2^{\alpha'}$$
$$\omega_+ \equiv \omega_1 + \omega_2 \, . \tag{50}$$

We now eliminate $A_1^{\alpha\alpha'}$ by substituting from (49) into the plane wave version of (10), making use of (47):

$$\mathcal{D}^{(1)}_{\alpha\gamma} M_{1\gamma}^{\alpha\alpha'} = \frac{ie}{\epsilon_0 c^2} \frac{\omega_+}{\omega_c} \int \underset{\sim}{v} \, d^3v \int_{-\infty}^{\varphi} d\varphi' \, \bar{R}^{\alpha\alpha'}(\underset{\sim}{x}') G(\underset{\sim}{K}_+, \omega_+ ; \varphi, \varphi') \tag{51}$$

where $\mathcal{D}^{(1)}_{\alpha\beta}$ is defined in analogy with (20),

$$\mathcal{D}^{(1)}_{\alpha\gamma} = \left(K_{0+}^2 - K_+^2\right)\delta_{\alpha\gamma} + K_{+\alpha} K_{+\beta} + \frac{i\omega_+}{\epsilon_0 c^2} \sigma_{\alpha\gamma}(\underset{\sim}{K}_+, \omega_+), \tag{52}$$

It follows from (23) that

$$\det\left(\mathcal{D}^{(1)}_{\alpha\gamma}\right) = 0$$

so that (51) can be solved for $\underset{\sim}{M}_1^{\alpha\alpha'}$. The remaining task then is to perform the velocity-space integrations.

The contribution to the current density is:

$$J_\gamma^{\alpha\alpha'} = -e \int v_\gamma A_1^{\alpha\alpha'} d^3v \, \exp\left[i(\underset{\sim}{K}_+ \cdot \underset{\sim}{X} - \omega_+ T)\right]$$

$$= \left[\frac{-i\omega_+}{\epsilon_0 c^2} \mathcal{D}^{(1)^{-1}}_{\delta\chi} \Gamma_\chi^{\alpha\alpha'} \sigma_{\gamma\delta}(\underset{\sim}{K}_+, \omega_+) + \Gamma_\gamma^{\alpha\alpha'}\right] \exp\left[i(\underset{\sim}{K}_+ \cdot \underset{\sim}{X} - \omega_+ T)\right] \tag{53}$$

which follows from (22), (47), (49), and (51), and in which the quantity, $\underset{\sim}{\Gamma}$, is defined by:

$$\underset{\sim}{\Gamma}^{\alpha\alpha'} \equiv -\frac{e}{\omega_c} \int \underset{\sim}{v} \, d^3v \int_{-\infty}^{\varphi} d\varphi' \, \bar{R}^{\alpha\alpha'}(\underset{\sim}{x}') \, G(\underset{\sim}{K}_+, \omega_+ ; \varphi, \varphi'). \tag{54}$$

Thus, the velocity-space integrations which remain to be done are embodied in the single function $\underset{\sim}{\Gamma}$.

The types of integrals encountered in the evaluation of (54) are similar to those discussed in Section 3, but are much more complicated. Thus, the procedure used before, i.e., evaluating the integrals exactly and then using small-argument and asymptotic expansions for the various functions, turns out to be very involved, although this procedure could conceivably be carried out.

It is fortunate, therefore, that an alternative and equivalent procedure to that described above exists. The use of the expansions described above involves the assumption, as was discussed in Section 3, that the ratio of the thermal velocity to the phase velocity of the various waves involved in the problem can be treated as a small parameter. This is also the condition that Landau-type damping can be neglected [12]. In the integrals contained in (54), we have not only the phase velocities $\frac{\omega_1}{k}$, $\frac{\omega_2}{k}$, but also the quantities $\frac{\omega_p}{k}$ and $\frac{\omega_c}{k}$, which must be considered large (i.e., large compared with the thermal velocities). It will become evident from the numerical examples considered later that these conditions are satisfied for the data to be chosen.

The procedure used in the evaluation of the integrals in (54) is to expand the integrand in powers of $\underset{\sim}{k}$, and then perform the integrations. A typical combination involved in the expansion is

$$\frac{K_+ V}{\omega_c} \tag{55}$$

where v is either $v_\perp$ or $v_\parallel$ and k_+ denotes a cartesian component of the vector $\underset{\sim}{k}_+$. Now, since v is integrated over an infinite range, it is not clear a priori that this procedure is valid. However, because of the presence of the Maxwellian distribution, most of the contribution to the integrals comes from the small velocity region. Therefore, when we expand in the parameter given by (55) we are, in effect, expanding in a parameter equal to the thermal velocity divided by the "phase velocity" $\frac{\omega_c}{k_+}$. Moreover, it has been verified by direct computation that this procedure gives the same results as the first method, viz., evaluating the integrals exactly and then introducing the small-argument and asymptotic expansions.

In the event that a different zero-order function than Maxwellian were chosen, the procedure just described would have to be reexamined. In any case, it would always be possible to use the method employed in the linearized case. The calculations would just become more complicated than those for the present (Maxwellian) case.

By expanding the integrand of (54) in powers of $\underset{\sim}{k}_+$ through the second power, it is a straightforward but tedious process to show that only the linear terms give non-vanishing results. Because of the low temperature of the electrons, there is no advantage in carrying the expansion farther.

Even with these simplifications it is exceedingly tedious to evaluate (54) in the general case, i.e., in the case of propagation at an arbitrary angle with respect to the static magnetic field. The reason is that, although the integrations are trivial, there are so many of them that the bookkeeping becomes horrendous. For this reason we will confine our consideration in the present treatment to the special cases of propagation parallel and perpendicular to the static magnetic field.

As we stated in Section 3, this restriction is for purposes of simplicity and is not a necessary restriction of the theory. The consideration of these two limiting cases should be sufficient to determine whether or not the diagnostic method being proposed in this report is feasible.

4.1 Propagation Parallel to the Static Magnetic Field

We see from the analysis in Section 3.1, in particular (24), that the amplitudes of the waves are still not determined. Therefore, we turn our attention to this problem before discussing the results of carrying out the integrations in (54).

In order to determine the wave amplitudes, we consider a boundary value problem in which the plasma is considered to exist in the half-space $z > 0$, whereas the half-space $z < 0$ is a vacuum. Now suppose that a transverse electromagnetic wave is normally incident on the plasma-vacuum boundary from the vacuum side. We will apply boundary conditions at $z = 0$ in order to determine the wave amplitudes in the plasma.

There has been a great deal of work done on problems of the type described in the above paragraph [31-37]. The idea behind most of this work was to investigate the question of radiation by plasma oscillations, which is thought to be an important mechanism in solar flares [38]. Our motivation for considering this problem is, of course, different.

It follows from (24) and the relations above (30) that

$$a_y^{(2)} = -i\, a_x^{(2)} \tag{56}$$

and

$$a_y^{(3)} = i\, a_x^{(3)} \;. \tag{57}$$

In the discussion of boundary conditions we will not indicate the frequency dependence. The same analysis applies to either ω_1 or ω_2.

Using the usual boundary conditions (at $z = 0$) of the continuity of the tangential electric and magnetic fields, taking note of the fact that mode 1 is longitudinal and modes 2 and 3 are transverse, and eliminating the amplitudes of the reflected waves, we obtain:

$$\left(K^{(2)} + K_0\right) a_{x,y}^{(2)} + \left(K^{(3)} + K_0\right) a_{x,y}^{(3)} = 2 K_0 A_{x,y} \tag{58}$$

where $\underset{\sim}{A} = (A_x, A_y, 0)$ is the vector amplitude of the incident wave. Eliminating the y-components among (56)-(58) we obtain

$$a_x^{(2)} = \frac{K_0}{K^{(2)}+K_0}\left(A_x + i A_y\right) \tag{59}$$

$$a_x^{(3)} = \frac{K_0}{K^{(3)}+K_0}\left(A_x - i A_y\right). \tag{60}$$

Thus, the amplitudes of the ordinary and extraordinary waves in the plasma are given in terms of the amplitudes of the external field by (56), (57), (59), and (60).

It will be noted that the above considerations do not determine the amplitude of the longitudinal wave. In order to obtain this quantity another boundary condition must be imposed on the system. The condition usually imposed in the fluid formulations of the problem [31,34,35,37] is that the normal component of the mean velocity of the electrons vanishes at the boundary. An argument given by Field [31] shows that this condition is equivalent to the condition of specular reflection used in formulations by means of the Boltzmann theory [26,29,30,39].

The normal component (i.e., in the z direction) of the mean velocity of the electrons is defined by

$$V_z \equiv \frac{\int v_{\parallel} f_1 \, d^3v}{\int f \, d^3v} \tag{61}$$

since the contribution of f_0 to the numerator vanishes.

Substituting the expression (obtained from (17))

$$f_1 = \frac{e}{m\omega_c} \sum_{\alpha=1}^{3} \int_{-\infty}^{\varphi} d\varphi' \left(\underset{\sim}{a}^{(\alpha)} + \underset{\sim}{v}' \times \underset{\sim}{b}^{(\alpha)}\right) \cdot \frac{\partial f_0}{\partial \underset{\sim}{v}'} \exp\left[-\frac{i}{\omega_c}\left(K^{(\alpha)} v_{\parallel} - \omega - i\nu\right)(\varphi - \varphi')\right]$$

$$\times \exp\left[i\left(K^{(\alpha)} z - \omega t\right)\right]$$

into (61), performing the integrations, and requiring that $V_z\big|_{z=0} = 0$ leads to the result,

$$a_z^{(1)} = 0 \ , \tag{62}$$

Thus, when propagation is parallel to the static magnetic field no longitudinal waves are generated at the density discontinuity. This result is in agreement with previous investigations [29,30].

We are now ready to derive the expressions for $\underset{\sim}{j}^{(\pm)}$. We will first derive the expression for the component along the static magnetic field (which is also the direction of propagation).

The elements of the inverse matrices in (53) are easily found from (24) and (25). Putting $\gamma = 3$ in (53) we find

$$J_3^{\alpha\alpha'} = \Gamma_3^{\alpha\alpha'}\left[1 + (\alpha_{3+} - 1)(1 + n_+^2 \beta_{\parallel} \mu_{3\parallel+})\right]^{-1} \exp\left[i(K_+ z - \omega_+ \tau)\right] \tag{63}$$

where the quantities with the "+" subscript are the same as previously given in Section 3, except that they now refer to the frequency ω_+, as defined in (50).

We see that for this component of current density only one component of $\underset{\sim}{\Gamma}^{\alpha\alpha'}$ is required. As we have mentioned previously, the evaluation of $\underset{\sim}{\Gamma}^{\alpha\alpha'}$ from (54) is a straightforward but tedious process. For this reason we will only quote the result at this point. The procedure will be given in Appendix I. We find:

$$\Gamma_3^{\alpha\alpha'} = -\epsilon_0 \omega_p^2 \Big\{ \left(a_{11}^{\alpha} a_{22}^{\alpha'} - a_{12}^{\alpha} a_{21}^{\alpha'}\right)$$

$$\times \left[\frac{i K_1^{\alpha} Y_2}{\omega_1 \omega_2 \omega_+ U_+ (U_2^2 - Y_2^2)}\left(1 + \frac{1}{2}\frac{\beta_{\parallel}}{\beta_{\perp}} \frac{U_2(1 - U_2)}{Y_2^2}\right)\right.$$

$$+ \cdots$$

$$+\frac{iK_2^{\alpha'}}{2\omega_2^2\omega_+U_+(U_2^2-Y_2^2)^2Y_2}\Big(U_2(U_2-1)(U_2^2-Y_2^2)$$
$$+\frac{\beta_{\parallel}}{\beta_{\perp}}\big(U_2^2(2U_2-U_2^2-1)+Y_2^2(U_2^2-1)\big)\Big)$$
$$+\frac{iK_+}{2}\frac{U_2(U_2-1)}{\omega_+^3U_+^2(U_2^2-Y_2^2)Y_+}\frac{\beta_{\parallel}}{\beta_{\perp}}\Big]$$
$$+(a_{11}^{\alpha}a_{21}^{\alpha'}+a_{12}^{\alpha}a_{22}^{\alpha'})\Big[\frac{-K_1^{\alpha}\left(U_2+\frac{1}{2}\frac{\beta_{\parallel}}{\beta_{\perp}}(U_2-1)\right)}{\omega_1\omega_2\omega_+U_+(U_2^2-Y_2^2)}$$
$$+\frac{K_2^{\alpha'}(1-U_2)}{\omega_+\omega_2^2U_+(U_2^2-Y_2^2)^2}\left(\frac{1}{2}(U_2^2-Y_2^2)+\frac{\beta_{\parallel}}{\beta_{\perp}}U_2\right)$$
$$-\frac{K_+(U_2-1)}{2\omega_+^2\omega_2U_+^2(U_2^2-Y_2^2)}\frac{\beta_{\parallel}}{\beta_{\perp}}\Big\}\exp[i(K_+z-\omega_+\tau)]\Big] \tag{64}$$

where use has been made of (62). In the double subscripts used above, the first applies to the frequency dependence (ω_1 or ω_2) and the second denotes the cartesian component of the quantity, $(1,2,3) \equiv (x,y,z)$.

By substituting (64) into (63), summing over α and α' from 2 to 3, and using (50), the contribution to the beat-frequency current densities stemming from the first term in (44) is obtained. The remaining contributions are obtained by the transformations discussed on p. 21. The total result for these current densities is:

$$j_z^{(\pm)}=\frac{e}{m}\epsilon_0\omega_p^2\frac{K_{01}K_{02}}{\omega_\pm}\Big\{\left(P^{(2,3)}\right)^{-1}e^{-|K_{\pm i}^{(2,3)}|z}$$
$$\times\Big[\left(M_+^{(2,3)}\right)^{-1}\left[A_+^{(2,3)}\left(C^{(2,3)}G-D^{(2,3)}H\right)+B_+^{(2,3)}\left(D^{(2,3)}G+C^{(2,3)}H\right)\right]$$
$$\mp\left(M_\pm^{(2,3)}\right)^{-1}\left[A_\pm^{(2,3)}\left(C^{(2,3)}G'-D^{(2,3)}H'\right)+B_\pm^{(2,3)}\left(D^{(2,3)}G'+C^{(2,3)}H'\right)\right]\Big]$$
$$+\cdots \tag{65}$$

$$+ (P^{(3,2)})^{-1} e^{-|\kappa_{\pm i}|z}$$

$$\times \Big[(M_+^{(3,2)})^{-1} \big[A_+^{(3,2)} (C^{(3,2)} G'' - D^{(3,2)} H'') + B_+^{(3,2)} (D^{(3,2)} G'' + C^{(3,2)} H'') \big]$$

$$\pm (M_\pm^{(3,2)})^{-1} \big[A_\pm^{(3,2)} (C^{(3,2)} G''' - D^{(3,2)} H''') + B_\pm^{(3,2)} (D^{(3,2)} G''' + C^{(3,2)} H''') \big] \Big] .$$

In order not to interrupt the continuity of the discussion we list the definitions of the quantities appearing in (65) in Appendix II.

A factor of 4 has been deleted from (65), in accordance with the discussion on p. 16.

It will be noted that, if one takes the limit $\omega_c \to 0$ in (64) in a straightforward way, divergences appear in some of the terms. This situation is unsatisfactory because this limit must exist. The resolution of the difficulty is that the limit must be taken as

$$\frac{\nu}{\omega_c} \longrightarrow 0 . \tag{66}$$

When this is done the correct limit is obtained. The need for this subtlety does not arise in the linearized theory.

We also note that (64) only has a temperature dependence if the temperatures are unequal <u>and</u> if collisions are present. The temperature-dependent terms vanish in the collisionless case.

In order to derive corresponding results for the other two coordinate directions, we note that it follows from (53) that $J_1^{\alpha\alpha'}$ and $J_2^{\alpha\alpha'}$ depend on $\Gamma_1^{\alpha\alpha'}$ and $\Gamma_2^{\alpha\alpha'}$, but are independent of $\Gamma_3^{\alpha\alpha'}$. These conclusions are easily derived by using the same procedure which led to (63) for $J_3^{\alpha\alpha'}$. It now can be shown by applying the procedure given in Appendix I to (54) that

$$\Gamma_1^{\alpha\alpha'} = \Gamma_2^{\alpha\alpha'} = 0$$

because of (62). Thus we see that

$$j_x^{(\pm)} = j_y^{(\pm)} = 0. \tag{67}$$

Thus, the current densities associated with the beat frequencies have only one non-vanishing component, the one parallel to the direction of propagation and the static magnetic field. This result is reasonable from the physical point of view, because when propagation occurs parallel to the static magnetic field the plasma acts essentially like an isotropic medium in that the current flows parallel to the field. The static magnetic field still has an important effect, however, in that it makes the plasma birefringent.

The numerical aspects of (65) will be discussed later in Section 5.

4.2 Propagation Perpendicular to the Static Magnetic Field

The analysis for this case proceeds in an exactly analogous manner to that for parallel propagation in Sec. 4.1. The first step is the determination of the wave amplitudes.

The boundary value problem is formulated in essentially the same way, except that now we consider the case of propagation along the x-axis, so that the plasma occupies the half-space x > 0, and vacuum the half-space x < 0. The static magnetic field is again directed along the z-axis.

From the form of (24) and the expressions listed in (31) we obtain:

$$a_x^{(2,3)} = \frac{-i\,\alpha_2\left(1+n^2\beta_\perp\lambda_{0\perp}\right)}{1+(\alpha_1-1)\left(1+n^2\beta_\perp\lambda_{1\perp}\right)}\;a_y^{(2,3)}. \tag{68}$$

As in the treatment of the case of parallel propagation, we will not indicate the frequency dependence in discussing the boundary conditions. The same analysis will apply to either ω_1 or ω_2.

We have previously noted that the temperature-dependent terms in expressions involving single frequencies are negligible (see Section 5) so that (68) becomes:

$$a_x^{(2,3)} = -\frac{i\alpha_2}{\alpha_1} a_y^{(2,3)} . \tag{69}$$

Using the usual boundary conditions (at x = 0) of the continuity of the tangential electric and magnetic fields, taking note of the fact that mode 1 is polarized along the z-axis and modes 2 and 3 are polarized in the xy-plane, and eliminating the amplitudes of the reflected waves, we obtain:

$$a_z^{(1)} = \frac{2K_0}{K_0 + K^{(1)}} A_z \tag{70}$$

$$(K_0 + K^{(2)}) a_y^{(2)} + (K_0 + K^{(3)}) a_y^{(3)} = 2K_0 A_y . \tag{71}$$

It is obvious that the system of three equations (69) and (71) is not sufficient to determine the four unknowns $a_x^{(2,3)}$, $a_y^{(2,3)}$.

In a proper formulation of the problem another boundary condition must be imposed in order to provide the required additional equation. The boundary condition is the same as that used in the case of parallel propagation, i.e., the vanishing of the normal component of the mean electron velocity at the plasma boundary.

For our present purposes, however, it will not be necessary to carry out these calculations. It was pointed out in Section 3 that the propagation vectors of modes 1 and 3 are temperature-independent, whereas the propagation vector of mode 2 is strongly temperature-dependent. This means that the phase velocity of mode 2 is approximately equal to the thermal electron velocity. As was also discussed in Section 3, this is the condition for the wave to be strongly damped. Thus, mode 2 will be strongly damped in a very short distance and, therefore, will not give a significant contribution to our results. We can, therefore, put

$$a_x^{(2)} = a_y^{(2)} = 0. \tag{72}$$

(69) and (71) can now be solved for the amplitudes of mode 3:

$$a_y^{(3)} = \frac{2 K_0}{K_0 + K^{(3)}} A_y \tag{73}$$

$$a_x^{(3)} = - i \frac{\alpha_2}{\alpha_1} a_y^{(3)}. \tag{74}$$

We are now ready to derive the expressions for $\underset{\sim}{j}^{(2)}$. The first such expression to be considered will be the component of current density parallel to the direction of the static magnetic field.

Following the same procedure that led to (63), we obtain:

$$J_3^{\alpha\alpha'} = \Gamma_3^{\alpha\alpha'}(1-n_+^2)\left[1-n_+^2+(\lambda_{3+}-1)\left(1+n_+^2\{\beta_\parallel\lambda_{3\parallel+}+\beta_\perp\lambda_{3\perp+}\}\right)\right]^{-1} \cdot \exp\left[i(K_+x-\omega_+t)\right] \tag{75}$$

and we are again using the shorthand notation (50). As was the case with (63), we see that the above result depends on only one component of $\underset{\sim}{\Gamma}^{\alpha\alpha'}$.

Following the procedure given in Appendix I, we obtain for the contribution from the first term in (44):

$$\Gamma_3^{\alpha\alpha'} = -\epsilon_0\,\omega_p^2\left\{\frac{K_2^{\alpha'} a_{11}^{\alpha} a_{23}^{\alpha'}(U_2-1)}{2\omega_+\omega_2^2 U_+(U_2^2-Y_2^2)}\left(U_2-1-\frac{\beta_\parallel}{\beta_\perp}\right)\right.$$

$$\left. - \frac{i K_2^{\alpha'} a_{12}^{\alpha} a_{23}^{\alpha'}(U_2-1)}{2\omega_+\omega_2^2 Y_2 U_+(U_2^2-Y_2^2)}\left(U_2-1+\frac{\beta_\parallel}{\beta_\perp}U_2\right) - \frac{K_2^{\alpha'} a_{13}^{\alpha} a_{21}^{\alpha'}}{\omega_+\omega_2^2 U_+(U_2^2-Y_2^2)} + \cdots\right.$$

$$+\frac{K_2^{\alpha'} a_{13}^{\alpha} a_{22}^{\alpha'} i Y_2}{\omega_+ \omega_2^2 U_+ U_2 (U_2^2 - Y_2^2)} + \frac{K_1^{\alpha} a_{13}^{\alpha} a_{21}^{\alpha'} \left(U_2 - \frac{\beta_\parallel}{2\beta_\perp}(U_2 - 1)\right)}{\omega_+ \omega_1 \omega_2 U_+ (U_2^2 - Y_2^2)}$$

$$-\frac{i K_1^{\alpha} a_{13}^{\alpha} a_{22}^{\alpha'}}{\omega_+ \omega_1 \omega_2 U_+ (U_2^2 - Y_2^2)} \left(Y_2 - \frac{\beta_\parallel}{2\beta_\perp} \frac{U_2 (U_2 - 1)}{Y_2}\right) \tag{76}$$

$$+\frac{i K_1^{\alpha} a_{12}^{\alpha} a_{23}^{\alpha'} (U_2 - 1)}{\omega_+ \omega_1 \omega_2 Y_2 U_+ U_2} + \frac{K_+ a_{11}^{\alpha} a_{23}^{\alpha'} \left((U_2 - 1)\omega_2 - 2\omega_+ U_+\right)}{2\omega_+^3 \omega_2 U_2 U_+ (U_+^2 - Y_+^2)}$$

$$+\frac{i K_+ a_{12}^{\alpha} a_{23}^{\alpha'} \left(Y_+ Y_2 - U_+ (U_2 - 1)\right)}{\omega_+^2 \omega_2 Y_2 U_+ U_2 (U_+^2 - Y_+^2)}$$

$$-\frac{K_+ a_{13}^{\alpha} a_{21}^{\alpha'} (\omega_2 U_+ U_2 + \omega_+ Y_+^2)}{\omega_2^2 \omega_+^2 U_+ (U_2^2 - Y_2^2)(U_+^2 - Y_+^2)}$$

$$\left. + \frac{i K_+ a_{13}^{\alpha} a_{22}^{\alpha'} \left(U_+ (U_2 - U_+) + Y_+ (Y_2 + Y_+)\right)}{\omega_+ \omega_2^2 Y_2 (U_2^2 - Y_2^2)(U_+^2 - Y_+^2)} \right\}$$

where use has been made of the same double subscript notation as in (64).

We note that (76) is a function of the ratio of the electron temperatures, but not of the temperatures individually, and that this temperature dependence vanishes in the collisionless case. An analogous situation was encountered in the case of (64).

Substitution of (76) into (75), letting α and α' equal 1 and 3 successively, and using (50), gives one contribution to the current densities associated with the beat frequencies. The remaining contributions are obtained by the transformations discussed on p. 21. The total result for these current densities is:

$$
\begin{aligned}
j_z^{(\pm)} = {} & 2 \frac{e}{m} \epsilon_0 \omega_p^2 K_{01} K_{02} \\
& \times \Bigg\{ \frac{e^{-|K_{+i}^{1,3}|x} A_{1z} A_{2y}}{\left[(C^{1,3})^2 + (D^{1,3})^2\right]\left[(E^{1,3})^2 + (F^{1,3})^2\right]} \Big[(B^{1,3} C^{1,3} + A^{1,3} D^{1,3})(E^{1,3} G + F^{1,3} H) \\
& \qquad + (A^{1,3} C^{1,3} - B^{1,3} D^{1,3})(E^{1,3} H - F^{1,3} G) \Big] \\
& + \frac{e^{-|K_{+i}^{3,1}|x} A_{2z} A_{1y}}{\left[(C^{3,1})^2 + (D^{3,1})^2\right]\left[(E^{3,1})^2 + (F^{3,1})^2\right]} \Big[(B^{3,1} C^{3,1} + A^{3,1} D^{3,1})(E^{3,1} G' + F^{3,1} H') \\
& \qquad + (A^{3,1} C^{3,1} - B^{3,1} D^{3,1})(E^{3,1} H' - F^{3,1} G') \Big] \qquad (77) \\
& + T \qquad \Bigg\}
\end{aligned}
$$

where "T" stands for all the preceding terms with the transformations $1 \rightleftarrows 2$ on the subscripts applied to them. The definitions of the various quantities appearing in (77) are given in Appendix III.

A factor of 4 has been deleted in the derivation of (77) for the same reason as was done previously in the derivation of (65).

We now consider the derivation of results for the current densities parallel to the direction of wave propagation, i.e., along the x-axis. One contribution to the current density is:

$$
J_1^{\alpha\alpha'} = \frac{K_{o+}^2 \left[\Gamma_1^{\alpha\alpha'} \mathcal{D}_{22} - \mathcal{D}_{12} \Gamma_2^{\alpha\alpha'} \right]}{\mathcal{D}_{11} \mathcal{D}_{22} + \mathcal{D}_{12}^2} \exp\left[i \left(K_+ x - \omega_+ t \right) \right] \qquad (78)
$$

and we again use the shorthand notation defined in (50). The quantities $\mathcal{D}_{11}$, $\mathcal{D}_{22}$, $\mathcal{D}_{12}$, and σ_{12} are given by (20) and (31). They are functions of the variables k_+ and ω_+, as given by (50). Thus, we need expressions for both $\Gamma_1^{\alpha\alpha'}$ and $\Gamma_2^{\alpha\alpha'}$.

Following the procedure given in Appendix I, we obtain:

$$\Gamma_1^{\alpha\alpha'} = -\epsilon_0 \omega_p^2 \Big[K_1^{\alpha} \gamma_1 a_{12}^{\alpha} a_{21}^{\alpha'} + K_1^{\alpha} \gamma_2 a_{12}^{\alpha} a_{22}^{\alpha'} + K_1^{\alpha} \gamma_3 a_{13}^{\alpha} a_{23}^{\alpha'} + K_+ \gamma_4 a_{11}^{\alpha} a_{22}^{\alpha'} + K_+ \gamma_5 a_{12}^{\alpha} a_{22}^{\alpha'} + K_+ \gamma_6 a_{11}^{\alpha} a_{21}^{\alpha'} + K_+ \gamma_7 a_{12}^{\alpha} a_{21}^{\alpha'} + K_2^{\alpha'} \gamma_8 a_{11}^{\alpha} a_{21}^{\alpha'} + K_2^{\alpha'} \gamma_9 a_{12}^{\alpha} a_{22}^{\alpha'} + K_2^{\alpha'} \gamma_{10} a_{12}^{\alpha} a_{21}^{\alpha'} + K_2^{\alpha'} \gamma_{11} a_{11}^{\alpha} a_{22}^{\alpha'} \Big] \tag{79}$$

and

$$\Gamma_2^{\alpha\alpha'} = -\epsilon_0 \omega_p^2 \Big[K_1^{\alpha} \gamma_{12} a_{12}^{\alpha} a_{21}^{\alpha'} + K_1^{\alpha} \gamma_{13} a_{12}^{\alpha} a_{22}^{\alpha'} + K_1^{\alpha} \gamma_{14} a_{13}^{\alpha} a_{23}^{\alpha'} + K_+ \gamma_{15} a_{11}^{\alpha} a_{22}^{\alpha'} + K_+ \gamma_{16} a_{12}^{\alpha} a_{21}^{\alpha'} + K_+ \gamma_{17} a_{11}^{\alpha} a_{21}^{\alpha'} + K_+ \gamma_{18} a_{12}^{\alpha} a_{22}^{\alpha'} + K_2^{\alpha'} \gamma_{19} a_{11}^{\alpha} a_{21}^{\alpha'} + K_2^{\alpha'} \gamma_{20} a_{12}^{\alpha} a_{22}^{\alpha'} + K_2^{\alpha'} \gamma_{21} a_{11}^{\alpha} a_{22}^{\alpha'} + K_2^{\alpha'} \gamma_{22} a_{12}^{\alpha} a_{21}^{\alpha'} \Big] \tag{80}$$

where we again use the shorthand notation given in (50). The coefficients γ_i in (79) and (80) are given in Appendix IV.

Substitution of (79) and (80) into (78) gives one contribution to the current densities. Then by using the transformations discussed previously we obtain, after some algebra:

$$
j_x^{(\pm)} = \frac{e}{m}\epsilon_0 \frac{\omega_p^2 K_{01} K_{02} A_{1z} A_{2z}}{\omega_1\omega_2 (C_1^2 + D_1^2)\, P_1} e^{-|K''_{\pm i}| x}
$$

$$
\times \Big\{ (A_1 C_1 + B_1 D_1) \Big[K^{(1)}_{1r} (M_1 G - N_1 H - I_1 \underline{K} + J_1 L) - K^{(1)}_{1i} (N_1 G + M_1 H - J_1 \underline{K} - I_1 L) \Big]
$$

$$
- (B_1 C_1 - A_1 D_1) \Big[K^{(1)}_{1i} (M_1 G - N_1 H - I_1 \underline{K} + J_1 L) + K^{(1)}_{1r} (N_1 G + M_1 H - J_1 \underline{K} - I_1 L) \Big] \Big\}
$$

$$
+ \frac{e}{m}\epsilon_0 \omega_p^2 \frac{K_{01} K_{02} A_{1y} A_{2y}}{(C_3^2 + D_3^2)\, Q_3 \omega_1 \omega_2} e^{-|K^{33}_{\pm i}| x} \sum_{\mathcal{H}} \tag{81}
$$

$$
\times \Big\{ (A_3 C_3 + B_3 D_3) \Big[M_3 (K_r^{\mathcal{H}} G_{\mathcal{H}} - K_i^{\mathcal{H}} H_{\mathcal{H}}) - N_3 (K_i^{\mathcal{H}} G_{\mathcal{H}} + K_r^{\mathcal{H}} H_{\mathcal{H}}) - I_3 (K_r^{\mathcal{H}} B_{\mathcal{H}} - K_i^{\mathcal{H}} L_{\mathcal{H}}) + J_3 (K_i^{\mathcal{H}} B_{\mathcal{H}} + K_r^{\mathcal{H}} L_{\mathcal{H}}) \Big]
$$

$$
- (B_3 C_3 - A_3 D_3) \Big[N_3 (K_r^{\mathcal{H}} G_{\mathcal{H}} - K_i^{\mathcal{H}} H_{\mathcal{H}}) + M_3 (K_i^{\mathcal{H}} G_{\mathcal{H}} + K_r^{\mathcal{H}} H_{\mathcal{H}}) - J_3 (K_r^{\mathcal{H}} B_{\mathcal{H}} - K_i^{\mathcal{H}} L_{\mathcal{H}}) - I_3 (K_i^{\mathcal{H}} B_{\mathcal{H}} + K_r^{\mathcal{H}} L_{\mathcal{H}}) \Big] \Big\} + T .
$$

In the process of deriving (81), we have split the propagation vectors into their real and imaginary parts,

$$K_\alpha^{(\beta)} = K_{\alpha r}^{(\beta)} + i\, K_{\alpha i}^{(\beta)} \quad ; \quad K_{\alpha r}^{(\beta)} \, , \; K_{\alpha i}^{(\beta)} \text{ real.}$$

The letter "T" denotes, as in (77), the operation of writing all the preceding terms with the transformation $1 \rightleftharpoons 2$ on the subscripts. The remaining quantities in (81) are defined in Appendix V.

The last component of current density is that parallel to the y-axis. Using the same procedure as in the derivation of (78), we obtain:

$$J_2^{\alpha\alpha'} = \frac{(K_{o+}^2 - K_+^2)\left[\Gamma_2^{\alpha\alpha'} \mathcal{D}_{11} + \Gamma_1^{\alpha\alpha'} \mathcal{D}_{12}\right]}{\mathcal{D}_{11}\mathcal{D}_{22} + \mathcal{D}_{12}^2} \exp\left[i(K_+ x - \omega_+ t)\right]. \tag{82}$$

Comparing the quantities in brackets in (78) and (82), we see that they are composed of the same quantities in different orders. This observation enables us to derive $j_y^{(\pm)}$ by means of trivial manipulations on (81). We obtain

$$\begin{aligned} j_y^{(\pm)} = {} & \frac{\frac{e}{m}\epsilon_0 \omega_p^2 K_{01} K_{02} A_{1z} A_{2z}}{\omega_1 \omega_2 (C_1^2 + D_1^2)\, Q_1} e^{-|K''_{\pm i}|x} \\ & \times \Big\{ (A_1 C_1 + B_1 D_1)\Big[K_{1r}^{(1)}(E_1 K - F_1 L + I_1 G - J_1 H) \\ & \qquad - K_{1i}^{(1)}(F_1 K + E_1 L + J_1 G + I_1 H)\Big] \\ & - (B_1 C_1 - A_1 D_1)\Big[K_{1i}^{(1)}(E_1 K - F_1 L + I_1 G - J_1 H) \\ & \qquad + K_{1r}^{(1)}(F_1 K + E_1 L + J_1 G + I_1 H)\Big]\Big\} + \cdots \end{aligned}$$

$$+\frac{\frac{e}{m}\epsilon_0\omega_p^2 K_{01}K_{02}A_{1y}A_{2y}\,e^{-|K_{1i}^{33}|x}}{\omega_1\omega_2(C_3^2+D_3^2)\,Q_3}\sum_{\varkappa} \tag{83}$$

$$\times\Big\{(A_3C_3+B_3D_3)\Big[E_3\left(K_r^{\varkappa}B_{\varkappa}-K_i^{\varkappa}L_{\varkappa}\right)$$

$$-F_3\left(K_i^{\varkappa}B_{\varkappa}+K_r^{\varkappa}L_{\varkappa}\right)+I_3\left(K_r^{\varkappa}G_{\varkappa}-K_i^{\varkappa}H_{\varkappa}\right)$$

$$-J_3\left(K_i^{\varkappa}G_{\varkappa}+K_r^{\varkappa}H_{\varkappa}\right)\Big]$$

$$-(B_3C_3-A_3D_3)\Big[F_3\left(K_r^{\varkappa}B_{\varkappa}-K_i^{\varkappa}L_{\varkappa}\right)$$

$$+E_3\left(K_i^{\varkappa}B_{\varkappa}+K_r^{\varkappa}L_{\varkappa}\right)+J_3\left(K_r^{\varkappa}G_{\varkappa}-K_i^{\varkappa}H_{\varkappa}\right)$$

$$+I_3\left(K_i^{\varkappa}G_{\varkappa}+K_r^{\varkappa}H_{\varkappa}\right)\Big]\Big\}+T.$$

Factors of 4 have been deleted from (81) and (83) for the same reason as previously discussed in connection with (65) and (77); i.e., because we are writing the fields in the form (36).

The summations over $\varkappa$ in (81) and (83) are to be taken over the values $\varkappa = 1,+,2$ (see the definitions of the quantities given in Appendix V).

We have now completed the derivations of the contributions to the current densities associated with the beat frequencies, $\omega_1 \pm \omega_2$. Some numerical consequences of these results will be discussed in the next section. We first make a few observations.

It will be noted that the static magnetic field plays a much more important role in the results for propagation perpendicular to the magnetic field, as given by (77), (81), and (83) than in the results for propagation parallel to the magnetic field as given by (65). Indeed, this is exactly what one expects on intuitive grounds.

It will be shown in the next section that the results for the current densities associated with the sum frequency, $\omega_1 + \omega_2$, are independent of electron temperature. The current densities can be made temperature-dependent, however, if the frequencies are judiciously chosen.

One intuitively expects that some enhancement may be obtained if one chooses the difference frequency equal to a characteristic frequency of the plasma. In the case of perpendicular propagation it is obvious from the appearance of (77), (81), and (83) (and the definitions given in Appendices III and V) that enhancement can occur if we choose,

$$\omega_1 - \omega_2 = 2\,\omega_c \quad \left(\text{for } \dot{J}_x^{(-)}, \dot{J}_y^{(-)}\right)$$

or $$\tag{84}$$

$$\omega_1 - \omega_2 = \omega_c \left(\text{for } \dot{J}_x^{(-)}, \dot{J}_y^{(-)}, \dot{J}_z^{(-)}\right).$$

In the case of parallel propagation, it is not possible to make the choices (84). From the appearance of (65) (and the definitions listed in Appendix II) it appears a priori that we may make the choice

$$\omega_1 - \omega_2 = \nu \tag{85}$$

although the resonance structure of the equations is not so obvious in this case. The numerical consequences of the choices (84) and (85) will be explored in the next section.

The term "enhancement", as used in the above discussion, has a double meaning. In the first place, we mean that the magnitudes of the current densities are increased, and, secondly, that the magnitudes of the temperature-dependent terms are increased relative to the temperature-independent terms.

5. NUMERICAL CONSIDERATIONS

We have made certain remarks, at various places in this report, concerning the temperature-dependent terms in comparison with the temperature-independent terms. These statements remain to be verified. In addition, we should demonstrate that the current densities correspond to measurable currents. These topics will be discussed in the present section.

Suppose, for illustrative purposes, that we consider a value of electron density of $n_0 = 10^5\,\mathrm{cm}^{-3} = 10^{11}\,\mathrm{m}^{-3}$. Then the plasma frequency is

$$\omega_p = 1.78 \times 10^7\ \mathrm{sec}^{-1}.$$

We will take the earth's magnetic field to be 0.5 gauss, so that

$$\omega_c = 8.82 \times 10^6\ \mathrm{sec}^{-1}.$$

The first case to be considered will be that of parallel propagation. For purposes of computing $j_z^{(+)}$, as given by (65), we take

$$\omega_1 = \omega_2 = 3 \times 10^7\ \mathrm{sec}^{-1}. \tag{86}$$

The condition (86) is made for purposes of simplicity in calculations of $\underset{\sim}{j}^{(+)}$, but will not be used in calculations of $\underset{\sim}{j}^{(-)}$. It will be seen below that measured values of $\underset{\sim}{j}^{(+)}$ cannot be used for determination of electron temperatures. These calculations are actually only made for purposes of orientation, so that the use of (86) is not critical insofar as the results are concerned.

For the calculations in this section we will use values for electron temperatures and collision frequency appropriate for the F-layer [15],

$$T_\parallel = T_\perp = 2000\,^\circ K$$

which corresponds to

$$\beta_\parallel = \beta_\perp = 3.37 \times 10^{-7} \tag{87}$$

and $$\nu = 10^{3}$$

which corresponds to,

$$z_1 = z_2 = \frac{1}{3} \times 10^{-4}. \tag{88}$$

These choices of data insure that the plasma waves are only weakly damped. We find, for the propagation vectors,

$$\begin{aligned} K_r^{(2)} &= 8.55 \times 10^{-2}\ m^{-1} \\ K_i^{(2)} &= 4.15 \times 10^{-7}\ m^{-1} \\ K_r^{(3)} &= 7.1 \times 10^{-2}\ m^{-1} \\ K_i^{(3)} &= 1.65 \times 10^{-6}\ m^{-1}. \end{aligned} \tag{89}$$

It follows from (86) that,

$$K_{+i}^{(2,3)} = K_{+i}^{(3,2)} = 2.07 \times 10^{-6}\ m^{-1}. \tag{90}$$

We see that, for reasonable values of z in (65), the spatial damping of the current density can be neglected.

It will be noted that we are assuming that the two electron temperatures are equal. This is for purposes of simplicity in the present calculation. The expressions for the current densities are such that the temperature difference will be measurable if we can show that the temperature is itself measurable.

With the above data, the temperature-dependent terms in (30) are of order 10^{-6} and are therefore completely negligible in comparison with unity. This justifies the procedure of neglecting the temperature dependence of the propagation vectors in the previous sections.

In order to calculate the current densities, it is necessary to specify the amplitudes of the incident fields. These amplitudes cannot be specified arbitrarily, however. There are two points to be considered. Firstly, a perturbation method has been used to calculate the current densities. Obviously, then, the amplitudes cannot be chosen too large. It is not necessary, however, to restrict the amplitudes to be sufficiently small so that the condition for linearization, as described by Platzman and Buchsbaum [29], is satisfied. The theory developed herein is similar in spirit to the "fairly small amplitude" theory of Sturrock [20,21], but the approach is different. Secondly, we want the measured current densities to be representative of the ambient electron density and electron temperature. This is only true if the amplitudes of the incident fields are bounded in magnitude. From the work of Ginzburg and Gurevich [15,17] it is known that the temperature is essentially unchanged if the incident fields are small in comparison with a quantity, E_p, called the plasma field. For the F-layer, it has the approximate value [15,17],

$$E_p \cong 2 \ V/m.$$

It will be noted from (65) that there are four types of polarizations which contribute to the current densities: $A_{1x}A_{2x}$, $A_{1y}A_{2y}$, $A_{1y}A_{2x}$, and $A_{1x}A_{2y}$. For definiteness, we will choose,

$$\begin{aligned} A_{1x} &= A_{2y} = 10^{-1} \ V/m \\ A_{1y} &= A_{2x} = 0. \end{aligned} \tag{91}$$

Inserting the above data into (65), we find

$$j_z^{(+)} \cong 10^{-11} \ amp/m^2 \tag{92}$$

which is independent of temperature. If a current detector of fairly large size (equivalent cross-sectional area $\approx 1 \ m^2$) is used, then (92) may be measurable if the proper bandwidth is chosen.

The order of magnitude of the result (92) does not change if different polarizations than (91) are chosen (for given field magnitudes). The situation can be somewhat improved if the field strengths are increased. We cannot proceed too far in this direction, however, because of the limitation discussed above. Perhaps an order of magnitude increase over (92) can be realized.

In order to calculate $j_z^{(-)}$ we must make a choice for the difference frequency, $\omega_1 - \omega_2$. One of the basic ideas behind the investigation being described in this report is the possibility that the current density may be enhanced if the difference frequency is chosen equal to one of the characteristic frequencies of the plasma. It is evident from (65) and Appendix II that the only possibility of obtaining a resonance effect in the case of parallel propagation is to choose,

$$Z_- = 1 \tag{93}$$

i.e.,

$$\omega_1 - \omega_2 = \nu.$$

In the following calculation we will suppose that ω_2 again has the value (86), but that ω_1 is determined from (93). Following the same procedure as in the calculation of $j_z^{(+)}$, we obtain from (65),

$$j_z^{\prime(-)} \cong 10^{-16} \text{ amp/m}^2. \tag{94}$$

For current detectors of reasonable size the current obtained from (94) is too small to be measured.

By comparing (92) and (94) we see that

$$j_z^{\prime(-)} \ll j_z^{\prime(+)}. \tag{95}$$

Thus, the choice (93) for the difference frequency does not reveal a resonance effect as one might suspect a priori from the appearance of the quantities defined in Appendix II. The physical reason for this result is that the collision frequency is not a true resonance frequency in the same sense as the cyclotron and plasma frequencies. It will be shown below that resonance effects do occur for the case of perpendicular propagation.

Using the same data as in the derivation of (89), we obtain for the real parts of the propagation vectors for the case of perpendicular propagation,

$$K_r^{(1)} = 8.07 \times 10^{-2}\ m^{-1} \tag{96}$$

$$K_r^{(2)} = 1.02 \times 10^{2}\ m^{-1} \tag{97}$$

$$K_r^{(3)} = 7.8 \times 10^{-2}\ m^{-1} \tag{98}$$

where we have set, for definiteness, $T_\perp = T_\parallel$. We note that $K_r^{(2)}$ is much larger than $K_r^{(1)}$ and $K_r^{(3)}$. This is a direct consequence of the fact, previously noted in Section 3, that $K_r^{(2)}$ is strongly dependent upon temperature, whereas $K_r^{(1)}$ and $K_r^{(3)}$ are not (see below).

The phase velocity corresponding to (97) is

$$\frac{\omega_{1,2}}{K^{(2)}} = 2.94 \times 10^{5}\ m\ sec^{-1}.$$

This result is of the same order of magnitude as the thermal velocity and, therefore, mode 2 will be strongly affected by Landau-type damping. This is the justification for neglecting this mode in calculating the current densities. It is easily verified that the phase velocities corresponding to (96) and (98) are greater than the velocity of light in vacuum. Therefore, these modes are not damped by this mechanism.

It is easily verified, using the same data as before, that $K^{(1)}$ and $K^{(3)}$ are temperature-independent, the temperature terms being of order 10^{-6}.

Use of the same data leads to the result,

$$K_{+i}^{(1,3)} \equiv K_{1i}^{(1)} + K_{2i}^{(3)} = K_{+i}^{(3,1)} \equiv K_{1i}^{(3)} + K_{2i}^{(1)}$$

$$= 4.25 \times 10^{-6}\ m^{-1}.$$

It follows from (77) that the spatial damping is small for the z-component of current density, as in the case of parallel propagation. The remaining components will be discussed later.

Taking, for definiteness, the polarizations of the external fields to be

$$A_{1z} = A_{2y} = 10^{-1}\ v/m$$
$$A_{1y} = A_{2z} = 0 \tag{99}$$

and using the same data as before, we find from (77):

$$j_z^{(+)} \cong 10^{-12}\ amp/m^2 \tag{100}$$

which is independent of temperature. As in the case of (92), this result corresponds to currents which may be measurable, if the size of the current detector and the bandwidth are properly chosen.

In order to calculate $j_z^{(-)}$, we again must make a choice for the difference frequency. In the present case it is advantageous, as can be seen from (77) and Appendix III, to pick

$$Y_- = 1 \tag{101}$$

i.e.,

$$\omega_1 - \omega_2 = \omega_c$$

instead of (93), which was used in the case of parallel propagation. Using (101) and the previous data, we find,

$$j_z^{(-)} \cong 10^{-11}\ amp/m^2. \tag{102}$$

It will be noted that this result is an order of magnitude greater than (100), and of the same order of magnitude as (92). By using the same argument as before, we conclude that (102) should be measurable under proper experimental conditions. This result is, however, temperature-dependent, in contradistinction to the cases considered previously.

It will be observed that (102) does not represent much of an enhancement over (100). This situation is not what one usually encounters in a resonance effect. The reason that the enhancement is so low is that, although the resonance denominator does become small in the case of (102) (it is down by a factor 10^8 from the calculation of (100)), the numerator also becomes small, so that the resonant character of the phenomenon is masked. Actually, the resonant terms give the major contribution to the quantities defined in Appendix III, so that the increase in magnitude in the current density between (100) and (102) is indeed the result of a resonance effect.

The current densities $j_x^{(\pm)}$ and $j_y^{(\pm)}$ can be calculated in a similar way by use of the same data from (81) and (83), respectively. It is found that these current densities are enhanced by several orders of magnitude if the choices (84) are made.

For example, if we put

$$Y_- = 1 \tag{103}$$

and use the previous data in (81) and Appendix V, we obtain,

$$j_x^{(-)} \cong 10^{-8}\ \mathrm{amp}/m^2 \tag{104}$$

which is seen to be three-to-four orders of magnitude greater than the previous results. The current corresponding to (104) should easily be measurable under proper experimental conditions.

Similar results are obtained if one puts (103) into (83) or

$$Y_-^2 = \frac{1}{4} \tag{105}$$

into either (81) and (83).

The result (104), and its analogues just referred to, is strongly temperature-dependent. This is in contradistinction to the results obtained previously, for which there was only a slight (i.e., barely measurable) temperature dependence.

The results (104) and its analogs are a measure of $T_\perp$, but not of $T_\parallel$.

6. CONCLUDING REMARKS

We have calculated the contributions to the current densities associated with the sum and difference frequencies of the external fields. The technique used in order to perform the calculations was a perturbation technique. Arguments were given to indicate that this method should be valid for the present problem. The results verify a posteriori that this argument is correct, since the current densities obtained correspond to very small currents.

It has been shown that $\underset{\sim}{j}^{(+)}$, the contribution to the current density associated with the sum frequency, $\omega_1 + \omega_2$, is independent of temperature and, therefore, provides a measure of the electron density. On the other hand, $\underset{\sim}{j}^{(-)}$, the contribution associated with the difference frequency, $\omega_1 - \omega_2$, can be made temperature-dependent if the difference frequency is chosen properly.

Calculations were carried out for two situations, propagation parallel and perpendicular to the static magnetic field. The general case of propagation at an arbitrary angle with respect to the magnetic field can be done; the calculations just become much more tedious. This extension of the method is reserved for future work. It was felt that the two limiting cases considered herein would be sufficient to determine whether or not the diagnostic method being proposed is feasible.

In the case of parallel propagation, the only non-vanishing component of current density is that parallel to the magnetic field and the direction of propagation. It was shown in Section 5 that the current density corresponding to the sum frequency, $\omega_1 + \omega_2$, has numerical values near the borderline of measurability, for proper experimental conditions, and is independent of electron temperature. A calculation of the current density corresponding to the difference frequency, $\omega_1 - \omega_2$, was carried out by setting

$$\omega_1 - \omega_2 = \nu. \qquad (106)$$

It was found that this current density is too small, by several orders of magnitude, to be measurable. It was pointed out that this situation occurs (i.e., $j_z^{(-)} \ll j_z^{(+)}$) because the collision frequency is not a resonant frequency of the plasma in the same sense as the cyclotron and plasma frequencies, so that a resonant enhancement is not obtained.

In the case of perpendicular propagation, there are no vanishing components of current density. Calculations carried out show that $j_z^{(\pm)}$ are both on the borderline of measurability, for proper experimental conditions. The current density, $j_z^{(-)}$, was calculated by choosing the difference frequency as

$$\omega_1 - \omega_2 = \omega_c , \tag{107}$$

It was found that $j_z^{(-)}$ is an order of magnitude greater than $j_z^{(+)}$. Thus, although we take advantage of the condition of cyclotron resonance, the enhancement is much less than one would expect. It was pointed out that this situation arises because the numerator of the expression for $j_z^{(-)}$ becomes small along with the denominator so that the enhancement is reduced.

A calculation was carried out for $j_x^{(-)}$ by again choosing the difference frequency from (107). It was found that the resulting current density corresponds to currents, under suitable conditions, which are easily measurable. These currents are strongly temperature-dependent, in contradistinction to the previous cases considered in which the temperature dependence was very slight. It was also pointed out that similar results are obtained for $j_x^{(-)}$ by choosing,

$$\omega_1 - \omega_2 = 2\,\omega_c \tag{108}$$

and for $j_y^{(-)}$ by choosing either (107) or (108).

The conclusions reached from the above considerations are that, if the case of perpendicular propagation is considered, and the currents perpendicular to the direction of the static magnetic field are measured, then the electron temperature can be determined. The currents along the magnetic field in both the cases of perpendicular and parallel propagation were smaller by several orders of magnitude and were only slightly temperature-dependent, even if they could be measured.

The electron temperature determined by measuring the currents corresponding to the current densities, $j_x^{(-)}$, $j_y^{(-)}$ is $T_\perp$. The temperature $T_\parallel$ cannot be determined by this procedure. The reason for this is that we have considered a special case, i.e., the propagation was perpendicular to the static magnetic field. If the treatment is generalized to include other directions of propagation, then $T_\parallel$ should also be capable of measurement. This aspect of the problem is reserved for future work. We have only applied the results derived in this report to the F-region of the ionosphere. Generalization of these results to different ionosphere regions will be considered in future work. It is expected that the method advocated herein will also prove to be a useful diagnostic tool in these situations.

The diagnostic method suggested herein should have some advantages over the methods currently being employed.

The present technique allows the measurement of electron temperatures both parallel and perpendicular to the static magnetic field, whereas the other diagnostic (probe) methods do not consider this possibility.

We have not included sheath effects in the present analysis. The philosophy has been to consider the present work as a preliminary investigation of a new diagnostic technique rather than a final analysis for practical experimental conditions. Sheath effects should not change the order of magnitude of the results.

If the experiments are performed on a rocket the value of ω_c will change as the rocket passes through the ionosphere. The conditions (107) and (108) suggest that a frequency sweep technique be employed. This procedure is feasible, since the variation of ω_c with altitude is small.

APPENDIX I

In this appendix we discuss the evaluation of $\underset{\sim}{\Gamma}^{\alpha\alpha'}$, as given by (54). The procedure of expanding the integrand of (54) in powers of $\underset{\sim}{k}$ ($\equiv \underset{\sim}{k}_1^{\alpha} + \underset{\sim}{k}_3^{\alpha'}$, $\underset{\sim}{k}_1^{\alpha} - \underset{\sim}{k}_3^{\alpha'}$, $\underset{\sim}{k}_3^{\alpha} + \underset{\sim}{k}_1^{\alpha'}$, or $\underset{\sim}{k}_3^{\alpha} - \underset{\sim}{k}_1^{\alpha'}$) before integrating has been justified in Section 4. We will now indicate the steps in the evaluation of (54) using this procedure.

Writing out Eq. (54) we have:

$$\underset{\sim}{\Gamma}^{\alpha\alpha'} = \frac{e}{\theta_\perp \omega_c^2} \int \underset{\sim}{v}\, d^3v \int_{\infty}^{0} d\eta \; G(\underset{\sim}{k}, \omega; \varphi, \varphi-\eta)\; f_0(v_\perp, v_\parallel)$$

$$\times \Big\{ \frac{1}{v_\perp} \Big[-(a_{11}^{\alpha} - v_\parallel b_{12}^{\alpha}) \sin(\varphi-\eta) + (a_{12}^{\alpha} + v_\parallel b_{11}^{\alpha}) \cos(\varphi-\eta) - v_\perp h_{13}^{\alpha} \Big]$$

$$\times \Big[v_\perp \big\{ (a_{21}^{\alpha'} - v_\parallel b_{22}^{\alpha'}) \cos(\varphi-\eta) + (a_{22}^{\alpha'} + v_\parallel b_{21}^{\alpha'}) \sin(\varphi-\eta) \big\}$$

$$+ \frac{\theta_\perp}{\theta_\parallel} v_\parallel \big(a_{23}^{\alpha'} + v_\perp \{ b_{22}^{\alpha'} \cos(\varphi-\eta) - b_{21}^{\alpha'} \sin(\varphi-\eta) \} \big) \Big]$$

$$+ \int_{\infty}^{0} d\xi \; G(\underset{\sim}{k}_2^{\alpha'}, \omega_2; \varphi-\eta, \varphi-\eta-\xi)$$

$$\times \Big[\big\{ (a_{11}^{\alpha} - v_\parallel b_{12}^{\alpha}) \cos(\varphi-\eta) + (a_{12}^{\alpha} + v_\parallel b_{11}^{\alpha}) \sin(\varphi-\eta) \big\}$$

$$\times \Big[-\frac{i}{\omega_c} \big\{ K_{21}^{\alpha'} (\sin(\varphi-\eta) - \sin(\varphi-\eta-\xi)) - K_{22}^{\alpha'} (\cos(\varphi-\eta) - \cos(\varphi-\eta-\xi)) \big\}$$

$$\times \big\{ v_\perp \big[(a_{21}^{\alpha'} - v_\parallel b_{22}^{\alpha'}) \cos(\varphi-\eta-\xi) + (a_{22}^{\alpha'} + v_\parallel b_{21}^{\alpha'}) \sin(\varphi-\eta-\xi) \big]$$

$$+ \frac{\theta_\perp}{\theta_\parallel} v_\parallel \big(a_{23}^{\alpha'} + v_\perp \{ b_{22}^{\alpha'} \cos(\varphi-\eta-\xi) - b_{21}^{\alpha'} \sin(\varphi-\eta-\xi) \} \big) \big\}$$

$$+ \cdots$$

$$
\begin{aligned}
&+\left(1-\frac{mV_\perp^2}{\Theta_\perp}\right)\left\{\left(a_{21}^{\alpha'}-V_\parallel b_{22}^{\alpha'}\right)\cos(\varphi-\eta-\zeta)+\left(a_{22}^{\alpha'}+V_\parallel b_{21}^{\alpha'}\right)\sin(\varphi-\eta-\zeta)\right\} \\
&\quad+\frac{\Theta_\perp}{\Theta_\parallel}V_\parallel\left(b_{22}^{\alpha'}\cos(\varphi-\eta-\zeta)-b_{21}^{\alpha'}\sin(\varphi-\eta-\zeta)\right) \\
&-\frac{m}{\Theta_\parallel}V_\perp V_\parallel\left(a_{23}^{\alpha'}+V_\perp\left\{b_{22}^{\alpha'}\cos(\varphi-\eta-\zeta)-b_{21}^{\alpha'}\sin(\varphi-\eta-\zeta)\right\}\right)\Big] \\
&-\frac{i}{\omega_c V_\perp}\left\{-\left(a_{11}^{\alpha}-V_\parallel b_{12}^{\alpha}\right)\sin(\varphi-\eta)+\left(a_{12}^{\alpha}+V_\parallel b_{11}^{\alpha}\right)\cos(\varphi-\eta)-V_\perp b_{13}^{\alpha}\right\} \\
&\quad\cdot\left\{K_{21}^{\alpha'}V_\perp\cos(\varphi-\eta)+K_{22}^{\alpha'}V_\perp\sin(\varphi-\eta)+K_{23}^{\alpha'}V_\parallel-\omega_2\right\} \\
&\times\Big\{V_\perp\left[\left(a_{21}^{\alpha'}-V_\parallel b_{22}^{\alpha'}\right)\cos(\varphi-\eta-\zeta)+\left(a_{22}^{\alpha'}+V_\parallel b_{21}^{\alpha'}\right)\sin(\varphi-\eta-\zeta)\right] \\
&\quad+\frac{\Theta_\perp}{\Theta_\parallel}V_\parallel\left(a_{23}^{\alpha'}+V_\perp\left\{b_{22}^{\alpha'}\cos(\varphi-\eta-\zeta)-b_{21}^{\alpha'}\sin(\varphi-\eta-\zeta)\right\}\right)\Big\} \\
&+\left\{a_{13}^{\alpha}+V_\perp\left(b_{12}^{\alpha}\cos(\varphi-\eta)-b_{11}^{\alpha}\sin(\varphi-\eta)\right)\right\} \\
&\times\Big[-\frac{i}{\omega_c}K_{23}^{\alpha'}\zeta\,\Big\{V_\perp\Big[\left(a_{21}^{\alpha'}-V_\parallel b_{22}^{\alpha'}\right)\cos(\varphi-\eta-\zeta) \\
&\qquad\qquad+\left(a_{22}^{\alpha'}+V_\parallel b_{21}^{\alpha'}\right)\sin(\varphi-\eta-\zeta)\Big] \\
&\qquad\qquad+\frac{\Theta_\perp}{\Theta_\parallel}V_\parallel\left(a_{23}^{\alpha'}+V_\perp\left\{b_{22}^{\alpha'}\cos(\varphi-\eta-\zeta)-b_{21}^{\alpha'}\sin(\varphi-\eta-\zeta)\right\}\right)\Big\} \\
&+V_\perp\left\{-b_{22}^{\alpha'}\cos(\varphi-\eta-\zeta)+b_{21}^{\alpha'}\sin(\varphi-\eta-\zeta)\right\} \\
&-\frac{m}{\Theta_\parallel}V_\perp V_\parallel\left\{\left(a_{21}^{\alpha'}-V_\parallel b_{22}^{\alpha'}\right)\cos(\varphi-\eta-\zeta)+\left(a_{22}^{\alpha'}+V_\parallel b_{21}^{\alpha'}\right)\sin(\varphi-\eta-\zeta)\right\} \\
&+\frac{\Theta_\perp}{\Theta_\parallel}\left(1-\frac{mV_\parallel^2}{\Theta_\parallel}\right)\left(a_{23}^{\alpha'}+V_\perp\left\{b_{22}^{\alpha'}\cos(\varphi-\eta-\zeta)-b_{21}^{\alpha'}\sin(\varphi-\eta-\zeta)\right\}\right)\Big]\Big]\Big\}
\end{aligned}
\tag{I-1}
$$

where we have used the definition of $\bar{R}$,

$$\bar{R}^{\alpha\alpha'}(\underset{\sim}{v}') \equiv \frac{e}{m}\left(\underset{\sim}{a}_1^{\alpha} + \underset{\sim}{v}' \times \underset{\sim}{b}_1^{\alpha}\right) \cdot \frac{\partial}{\partial \underset{\sim}{v}'} F_2^{\alpha'}$$

and the definition of $F_2^{\alpha'}$ given by (38). In addition the following changes of variable have been made

$$\eta = \varphi - \varphi'$$
$$\varsigma = \varphi' - \varphi''.$$

The procedure for performing the integrations in (I-1) is the same as that discussed by, for example, Montgomery and Tidman [27] in the case of the linearized theory. That is, the φ-integration is done first, then the η- and ς-integrations, and finally the integrations over $v_\perp$ and $v_\parallel$.

We first consider the terms in (I-1) which are independent of $\underset{\sim}{k}$:

$$\underset{\sim}{\Gamma}^{(0)} = \frac{e}{\theta_\perp \omega_c^2} \int \underset{\sim}{v}\, d^3 v \int_\infty^0 d\eta\, e^{i\frac{(\omega + i\nu)}{\omega_c}\eta} f_0(v_\perp, v_\parallel)$$
$$\times \left\{ \frac{1}{v_\perp} \left[-a_{11}^{\alpha} \sin(\varphi - \eta) + a_{12}^{\alpha} \cos(\varphi - \eta) \right] \right.$$
$$\cdot \left[v_\perp \left(a_{21}^{\alpha'} \cos(\varphi - \eta) + a_{22}^{\alpha'} \sin(\varphi - \eta) \right) + \frac{\theta_\perp}{\theta_\parallel} v_\parallel a_{23}^{\alpha'} \right]$$
$$- \int_\infty^0 d\varsigma\, e^{i\frac{(\omega_2 + i\nu)}{\omega_c}\varsigma} \left[\left\{ a_{11}^{\alpha} \cos(\varphi - \eta) + a_{12}^{\alpha} \sin(\varphi - \eta) \right\} \right.$$
$$\cdot \left[\left(1 - \frac{m v_\perp^2}{\theta_\perp}\right) \left\{ a_{21}^{\alpha'} \cos(\varphi - \eta - \varsigma) + a_{22}^{\alpha'} \sin(\varphi - \eta - \varsigma) \right\} \right.$$
$$\left. - \frac{m}{\theta_\parallel} v_\perp v_\parallel a_{23}^{\alpha'} \right] \quad + \cdots$$

$$+\frac{1}{v_\perp}\frac{i\omega_2}{\omega_c}\left\{-a_{11}^{\alpha}\sin(\varphi-\eta)+a_{12}^{\alpha}\cos(\varphi-\eta)\right\}$$

$$\cdot\left\{v_\perp\left(a_{21}^{\alpha'}\cos(\varphi-\eta-\xi)+a_{22}^{\alpha'}\sin(\varphi-\eta-\xi)\right)\right.$$

$$\left.+\frac{\theta_\perp}{\theta_\parallel}v_\parallel a_{23}^{\alpha'}\right\}$$

$$+a_{23}^{\alpha'}\left[-\frac{m}{\theta_\parallel}v_\perp v_\parallel\left(a_{21}^{\alpha'}\cos(\varphi-\eta-\xi)+a_{22}^{\alpha'}\sin(\varphi-\eta-\xi)\right)\right.$$

$$\left.\left.\left.+\frac{\theta_\perp}{\theta_\parallel}\left(1-\frac{m v_\parallel^2}{\theta_\parallel}\right)a_{23}^{\alpha'}\right]\right]\right\}. \qquad \text{(I-2)}$$

It is now easily shown that (I-2) vanishes, because (for a given term) either the integration over $v_\parallel$ or φ vanishes. The integrations are all trivial.

For the first order terms, we consider the special case in which propagation is parallel to the static magnetic field, i.e., along the z-axis. This is the simplest case to consider, the remaining cases being evaluated in an analogous manner, but more tediously.

For this case we find:

$$\underset{\sim}{\Gamma}^{(1)}=\frac{e}{\theta_\perp\omega_c^2}\int\underset{\sim}{v}\,d^3v\int_\infty^0 d\eta\; f_0(v_\perp,v_\parallel)\;e^{i\frac{(\omega+i\nu)}{\omega_c}\eta}$$

$$\times \left\{ \frac{1}{V_\perp} \left(a_{11}^\alpha \sin(\varphi-\eta) - a_{12}^\alpha \cos(\varphi-\eta) \right) \right.$$

$$\times \left[\frac{V_\perp V_\parallel}{\omega_2} K_2^{\alpha'} \left(1 - \frac{\theta_\perp}{\theta_\parallel}\right)\left(a_{21}^{\alpha'} \cos(\varphi-\eta) + a_{22}^{\alpha'} \sin(\varphi-\eta) \right) \right.$$

$$\left. + \frac{V_\parallel K_1^\alpha}{\omega_1} \left(V_\perp \left\{ a_{21}^{\alpha'} \cos(\varphi-\eta) + a_{22}^{\alpha'} \sin(\varphi-\eta) \right\} + \frac{\theta_\perp}{\theta_\parallel} V_\parallel a_{23}^{\alpha'} \right) \right]$$

$$+ \frac{i}{\omega_c} \frac{K V_\parallel \eta}{V_\perp} \left\{ a_{11}^\alpha \sin(\varphi-\eta) - a_{12}^\alpha \cos(\varphi-\eta) \right\}$$

$$\times \left\{ V_\perp \left(a_{21}^{\alpha'} \cos(\varphi-\eta) + a_{22}^{\alpha'} \sin(\varphi-\eta) \right) + \frac{\theta_\perp}{\theta_\parallel} V_\parallel a_{23}^{\alpha'} \right\}$$

$$- \int_\infty^0 d\varsigma \quad e^{i \frac{(\omega_2 + i\nu)}{\omega_c} \varsigma}$$

$$\times \left[\left\{ a_{11}^\alpha \cos(\varphi-\eta) + a_{12}^\alpha \sin(\varphi-\eta) \right\} \frac{V_\parallel K_2^{\alpha'}}{\omega_2} \right.$$

$$\times \left\{ m V_\perp^2 \left(\frac{1}{\theta_\perp} - \frac{1}{\theta_\parallel} \right) + \frac{\theta_\perp}{\theta_\parallel} - 1 \right\}$$

$$\times \left\{ a_{21}^{\alpha'} \cos(\varphi-\eta-\varsigma) + a_{22}^{\alpha'} \sin(\varphi-\eta-\varsigma) \right\}$$

$$- \frac{V_\parallel K_1^\alpha}{\omega_1} \left(a_{11}^\alpha \cos(\varphi-\eta) + a_{12}^\alpha \sin(\varphi-\eta) \right)$$

$$\times \left\{ \left(1 - \frac{m V_\perp^2}{\theta_\perp}\right) \left(a_{21}^{\alpha'} \cos(\varphi-\eta-\varsigma) + a_{22}^{\alpha'} \sin(\varphi-\eta-\varsigma) \right) \right.$$

$$\left. - \frac{m}{\theta_\parallel} V_\parallel V_\perp a_{23}^{\alpha'} \right\} \qquad + \cdots$$

$$+\frac{1}{V_\perp}\frac{iV_\parallel}{\omega_c}\left(a_{11}^{\alpha}\sin(\varphi-\eta)-a_{12}^{\alpha}\cos(\varphi-\eta)\right)\left(K_2^{\alpha'}+K_1^{\alpha}\frac{\omega_2}{\omega_1}\right)$$

$$\times\left\{V_\perp\left(a_{21}^{\alpha'}\cos(\varphi-\eta-\varsigma)+a_{22}^{\alpha'}\sin(\varphi-\eta-\varsigma)\right)+\frac{\theta_\perp}{\theta_\parallel}V_\parallel a_{23}^{\alpha'}\right\}$$

$$+a_{13}^{\alpha}\frac{V_\perp K_2^{\alpha'}}{\omega_2}\left(1-\frac{mV_\parallel^2}{\theta_\parallel}\right)\left(\frac{\theta_\perp}{\theta_\parallel}-1\right)$$

$$\times\left(a_{21}^{\alpha'}\cos(\varphi-\eta-\varsigma)+a_{22}^{\alpha'}\sin(\varphi-\eta-\varsigma)\right)$$

$$-\frac{i}{\omega_c}a_{13}^{\alpha}K_2^{\alpha'}\varsigma\left\{V_\perp\left(a_{21}^{\alpha'}\cos(\varphi-\eta-\varsigma)+a_{22}^{\alpha'}\sin(\varphi-\eta-\varsigma)\right)\right.$$

$$\left.+\frac{\theta_\perp}{\theta_\parallel}V_\parallel a_{23}^{\alpha'}\right\}$$

$$+\frac{V_\perp K_1^{\alpha}}{\omega_1}\left(a_{11}^{\alpha}\cos(\varphi-\eta)+a_{12}^{\alpha}\sin(\varphi-\eta)\right)$$

$$\times\left\{-\frac{m}{\theta_\parallel}V_\perp V_\parallel\left(a_{21}^{\alpha'}\cos(\varphi-\eta-\varsigma)+a_{22}^{\alpha'}\sin(\varphi-\eta-\varsigma)\right)\right.$$

$$\left.+\frac{\theta_\perp}{\theta_\parallel}\left(1-\frac{mV_\parallel^2}{\theta_\parallel}\right)a_{23}^{\alpha'}\right\}$$

$$-\frac{iV_\parallel}{\omega_c}K_2^{\alpha'}\left(\frac{\theta_\perp}{\theta_\parallel}-1\right)\left(a_{11}^{\alpha}\sin(\varphi-\eta)-a_{12}^{\alpha}\cos(\varphi-\eta)\right)$$

$$\times\left(a_{21}^{\alpha'}\cos(\varphi-\eta-\varsigma)+a_{22}^{\alpha'}\sin(\varphi-\eta-\varsigma)\right)$$

$$-\frac{i}{\omega_c}V_\parallel\left(K\eta+K_2^{\alpha'}\varsigma\right)\times$$

$$\times \Big\{ \left(a_{11}^{\alpha} \cos(\varphi-\eta) + a_{12}^{\alpha} \sin(\varphi-\eta) \right)$$

$$\cdot \Big[\left(1 - \frac{mv_\perp^2}{\theta_\perp}\right) \left(a_{21}^{\alpha'} \cos(\varphi-\eta-\zeta) + a_{22}^{\alpha'} \sin(\varphi-\eta-\zeta) \right)$$

$$- \frac{m}{\theta_\parallel} V_\parallel V_\perp a_{23}^{\alpha'} \Big]$$

$$- \frac{i\,\omega_2}{V_\perp \omega_c} \left(a_{11}^{\alpha} \sin(\varphi-\eta) - a_{12}^{\alpha} \cos(\varphi-\eta) \right)$$

$$\times \Big[V_\perp \left(a_{21}^{\alpha'} \cos(\varphi-\eta-\zeta) + a_{22}^{\alpha'} \sin(\varphi-\eta-\zeta) \right) + \frac{\theta_\perp}{\theta_\parallel} V_\parallel a_{23}^{\alpha'} \Big]$$

$$+ a_{13}^{\alpha} \Big[- \frac{m}{\theta_\parallel} V_\perp V_\parallel \left(a_{21}^{\alpha'} \cos(\varphi-\eta-\zeta) + a_{22}^{\alpha'} \sin(\varphi-\eta-\zeta) \right)$$

$$+ \frac{\theta_\perp}{\theta_\parallel} \left(1 - \frac{mV_\parallel^2}{\theta_\parallel} \right) a_{23}^{\alpha'} \Big] \Big\} \Big] \Big\}. \qquad \text{(I-3)}$$

We will illustrate the evaluation of the integrations in (I-3) by calculating the z-component. Integrating (I-3) over φ gives:

$$\Gamma_3^{(1)} = \frac{e\pi}{\theta_\perp \omega_c^2} \iint V_\parallel^2 V_\perp \, dV_\perp \, dV_\parallel \, f_0(V_\perp, V_\parallel) \int_\infty^0 d\eta \; e^{i \frac{(\omega + i\nu)}{\omega_c} \eta}$$

$$\times \Big\{ \left(a_{11}^{\alpha} a_{22}^{\alpha'} - a_{12}^{\alpha} a_{21}^{\alpha'} \right) \left(\frac{K_2^{\alpha'}}{\omega_2} \left(1 - \frac{\theta_\perp}{\theta_\parallel}\right) + \frac{K_1^{\alpha}}{\omega_1} \right)$$

$$+ \frac{i}{\omega_c} K V_\parallel \eta \left(a_{11}^{\alpha} a_{22}^{\alpha'} - a_{12}^{\alpha} a_{21}^{\alpha'} \right)$$

$$- \int_\infty^0 d\zeta \; e^{i \frac{(\omega_2 + i\nu)}{\omega_c} \zeta} \Big[\frac{V_\parallel K_2^{\alpha'}}{\omega_2} \Big\{ mV_\perp^2 \left(\frac{1}{\theta_\perp} - \frac{1}{\theta_\parallel} \right) + \left(\frac{\theta_\perp}{\theta_\parallel} - 1 \right) \Big\}$$

$$\times \Big\{ \left(a_{11}^{\alpha} a_{21}^{\alpha'} + a_{12}^{\alpha} a_{22}^{\alpha'} \right) \cos\zeta - \left(a_{11}^{\alpha} a_{22}^{\alpha'} - a_{12}^{\alpha} a_{21}^{\alpha'} \right) \sin\zeta \Big\}$$

$$+ \cdots$$

$$
\begin{aligned}
&-\frac{V_{\parallel} K_1^{\alpha}}{\omega_1}\left(1-\frac{mV_{\perp}^2}{\theta_{\perp}}\right)\Big\{\left(a_{11}^{\alpha} a_{21}^{\alpha'}+a_{12}^{\alpha} a_{22}^{\alpha'}\right)\cos\varsigma \\
&\qquad\qquad -\left(a_{11}^{\alpha} a_{22}^{\alpha'}-a_{12}^{\alpha} a_{21}^{\alpha'}\right)\sin\varsigma\Big\} \\
&+\frac{i V_{\parallel}}{\omega_c}\left(K_2^{\alpha'}+\frac{\omega_2}{\omega_1}K_1^{\alpha}\right)\Big\{\left(a_{11}^{\alpha} a_{21}^{\alpha'}+a_{12}^{\alpha} a_{22}^{\alpha'}\right)\sin\varsigma \\
&\qquad\qquad +\left(a_{11}^{\alpha} a_{22}^{\alpha'}-a_{12}^{\alpha} a_{21}^{\alpha'}\right)\cos\varsigma\Big\} \\
&-\frac{2i}{\omega_c} a_{13}^{\alpha} a_{23}^{\alpha'} K_2^{\alpha'} \varsigma \frac{\theta_{\perp}}{\theta_{\parallel}} V_{\parallel} \\
&-\frac{V_{\perp} K_1^{\alpha}}{\omega_1}\frac{m}{\theta_{\parallel}} V_{\perp} V_{\parallel}\Big\{\left(a_{11}^{\alpha} a_{21}^{\alpha'}+a_{12}^{\alpha} a_{22}^{\alpha'}\right)\cos\varsigma \\
&\qquad\qquad -\left(a_{11}^{\alpha} a_{22}^{\alpha'}-a_{12}^{\alpha} a_{21}^{\alpha'}\right)\sin\varsigma\Big\} \\
&-\frac{i}{\omega_c} V_{\parallel}\left(K\eta+K_2^{\alpha'}\varsigma\right)\left(1-\frac{mV_{\perp}^2}{\theta_{\perp}}\right) \\
&\quad\cdot\Big\{\left(a_{11}^{\alpha} a_{21}^{\alpha'}+a_{12}^{\alpha} a_{22}^{\alpha'}\right)\cos\varsigma-\left(a_{11}^{\alpha} a_{22}^{\alpha'}-a_{12}^{\alpha} a_{21}^{\alpha'}\right)\sin\varsigma\Big\} \\
&-\frac{\omega_2}{\omega_c^2} V_{\parallel}\left(K\eta+K_2^{\alpha'}\varsigma\right)\Big\{\left(a_{11}^{\alpha} a_{21}^{\alpha'}+a_{12}^{\alpha} a_{22}^{\alpha'}\right)\sin\varsigma \\
&\qquad\qquad +\left(a_{11}^{\alpha} a_{22}^{\alpha'}-a_{12}^{\alpha} a_{21}^{\alpha'}\right)\cos\varsigma\Big\} \\
&-\frac{2i}{\omega_c} V_{\parallel} a_{13}^{\alpha}\left(K\eta+K_2^{\alpha'}\varsigma\right)\frac{\theta_{\perp}}{\theta_{\parallel}}\left(1-\frac{mV_{\parallel}^2}{\theta_{\parallel}}\right) a_{23}^{\alpha'} \\
&-\frac{i}{\omega_c} V_{\parallel} K_2^{\alpha'}\left(\frac{\theta_{\perp}}{\theta_{\parallel}}-1\right)\Big\{\left(a_{11}^{\alpha} a_{21}^{\alpha'}+a_{12}^{\alpha} a_{22}^{\alpha'}\right)\sin\varsigma \\
&\qquad\qquad +\left(a_{11}^{\alpha} a_{22}^{\alpha'}-a_{12}^{\alpha} a_{21}^{\alpha'}\right)\cos\varsigma\Big\}\Big\}. \qquad \text{(I-4)}
\end{aligned}
$$

The remaining integrations can all be performed simultaneously and are all trivial. The result is (64). The integrations for the remaining cases are performed in an analogous fashion.

APPENDIX II

In this appendix we give the definitions of the quantities appearing in (65).

$$K_1^{(i)} \pm K_2^{(j)} \equiv K_{\pm r}^{(i,j)} + i K_{\pm i}^{(i,j)} \qquad K_{\pm r}^{(i,j)} \quad K_{\pm i}^{(i,j)} \text{ real}$$

$$K_j^{(i)} \equiv K_{jr}^{(i)} + i K_{ji}^{(i)} \qquad K_{jr}^{(i)} \quad K_{ji}^{(i)} \text{ real}$$

$$\frac{c}{\omega_\pm}\left(K_1^{(i)} \pm K_2^{(j)}\right) \equiv n_{\pm r}^{(i,j)} + i n_{\pm i}^{(i,j)} \qquad n_{\pm r}^{(i,j)} \quad n_{\pm i}^{(i,j)} \text{ real}$$

$$M_+^{(i,j)} \equiv \left(A_+^{(i,j)}\right)^2 + \left(B_+^{(i,j)}\right)^2$$

$$A_+^{(i,j)} \equiv 1 - \frac{X_\pm}{1+Z_\pm^2}\left[1 + \frac{3\beta_{\parallel}}{1+Z_\pm^2}\left\{\left[\left(n_{\pm r}^{(i,j)}\right)^2 - \left(n_{\pm i}^{(i,j)}\right)^2\right]\left(1-Z_\pm^2\right) + 4 Z_\pm n_{\pm r}^{(i,j)} n_{\pm i}^{(i,j)}\right\}\right]$$

$$B_+^{(i,j)} \equiv \frac{X_\pm}{1+Z_\pm^2}\left[Z_\pm - \frac{6\beta_{\parallel}}{1+Z_\pm^2}\left\{n_{\pm r}^{(i,j)} n_{\pm i}^{(i,j)}\left(1-Z_\pm^2\right) - Z_\pm\left[\left(n_{\pm r}^{(i,j)}\right)^2 - \left(n_{\pm i}^{(i,j)}\right)^2\right]\right\}\right]$$

$$P^{(i,j)} \equiv \left(Q^{(i,j)}\right)^2 + \left(S^{(i,j)}\right)^2$$

$$Q^{(i,j)} \equiv \left(K_{1r}^{(i)} + K_{01}\right)\left(K_{2r}^{(j)} + K_{02}\right)$$

$$S^{(i,j)} \equiv K_{1i}^{(i)}\left(K_{2r}^{(j)} + K_{02}\right) + K_{2i}^{(j)}\left(K_{1r}^{(i)} + K_{01}\right)$$

$$C^{(i,j)} \equiv \cos\left(K_{\pm r}^{(i,j)} z - \omega_{\pm} \tau\right)\left[\left(A_{1x} A_{2x} + A_{1y} A_{2y}\right) Q^{(i,j)} + \left(A_{1y} A_{2x} - A_{1x} A_{2y}\right) S^{(i,j)}\right]$$
$$- \sin\left(K_{\pm r}^{(i,j)} z - \omega_{\pm} \tau\right)\left[\left(A_{1y} A_{2x} - A_{1x} A_{2y}\right) Q^{(i,j)} - \left(A_{1x} A_{2x} + A_{1y} A_{2y}\right) S^{(i,j)}\right]$$

$$D^{(i,j)} \equiv \cos\left(K_{\pm r}^{(i,j)} z - \omega_{\pm} \tau\right)\left[\left(A_{1y} A_{2x} - A_{1x} A_{2y}\right) Q^{(i,j)} - \left(A_{1x} A_{2x} + A_{1y} A_{2y}\right) S^{(i,j)}\right]$$
$$+ \sin\left(K_{\pm r}^{(i,j)} z - \omega_{\pm} \tau\right)\left[\left(A_{1x} A_{2x} + A_{1y} A_{2y}\right) Q^{(i,j)} + \left(A_{1y} A_{2x} - A_{1x} A_{2y}\right) S^{(i,j)}\right]$$

$$G \equiv \frac{K_{1r}^{(2)} \left[(1+Y_2)(1-Y_2^2) + Z_2\left(Z_{\pm}(1-Y_2^2) + 2Z_2\right)\left(1 + \frac{1}{2}\frac{\beta_{\parallel}}{\beta_{\perp}}\left(1 - \frac{1}{Y_2}\right)\right)\right]}{\omega_1 \omega_2 (1-Y_2^2)^2 (1+Z_{\pm}^2)}$$
$$- \frac{Z_2 \frac{\beta_{\parallel}}{\beta_{\perp}} K_{\pm r}^{(2,3)} \left[Z_2(1-Z_{\pm}^2) - 2Z_{\pm}(1+Y_2)\right]}{2\omega_{\pm}^2 Y_{\pm}(1-Y_2^2)\left[(1-Z_{\pm}^2)^2 + 4Z_{\pm}^2\right]}$$

$$H \equiv \frac{K_{1i}^{(2)}(1+Y_2) + K_{1r}^{(2)}\left[Z_2\left(1 + \frac{1}{2}\frac{\beta_{\parallel}}{\beta_{\perp}}\left(1 - \frac{1}{Y_2}\right)\right) - Z_{\pm}(1-Y_2^2)(1+Y_2)\right]}{\omega_1 \omega_2 (1-Y_2^2)^2 (1+Z_{\pm}^2)}$$
$$+ \frac{Z_2 \frac{\beta_{\parallel}}{\beta_{\perp}}\left[K_{\pm r}^{(2,3)}(1+Y_2)(1-Z_{\pm}^2) + 2Z_{\pm}\left(K_{\pm r}^{(2,3)} Z_2 + K_{\pm i}^{(2,3)}(1+Y_2)\right)\right]}{2\omega_{\pm}^2 Y_{\pm}(1-Y_2^2)\left[(1-Z_{\pm}^2)^2 + 4Z_{\pm}^2\right]}$$

$$-G' \equiv \frac{K_{2r}^{(3)}}{\omega_1\omega_2(1+Y_1)(1+Z_\pm^2)} + Z_1\frac{\beta_\parallel}{\beta_\perp}\frac{\left[\left(K_{\pm r}^{(2,3)}Z_1 + K_{\pm i}^{(2,3)}(1-Y_1)\right)(1-Z_\pm^2) - 2K_{\pm r}^{(2,3)}Z_\pm(1-Y_1)\right]}{2\omega_\pm^2 Y_\pm(1-Y_1^2)\left[(1-Z_\pm^2)^2+4Z_\pm^2\right]}$$

$$-H' \equiv \frac{\left[K_{2r}^{(3)}Z_1\left(1+\frac{1}{2}\frac{\beta_\parallel}{\beta_\perp}\left(1+\frac{1}{Y_1}\right)\right) + K_{1i}^{(3)}(1-Y_1) \pm Z_\pm K_{2r}^{(3)}(1-Y_1)\right]}{\omega_1\omega_2(1-Y_1^2)(1+Z_\pm^2)}$$

$$- Z_1\frac{\beta_\parallel}{\beta_\perp}\frac{\left[K_{\pm r}^{(2,3)}(1-Y_1)(1-Z_\pm^2) + 2Z_\pm\left(K_{\pm r}^{(2,3)}Z_1 + K_{\pm i}^{(2,3)}(1-Y_1)\right)\right]}{2\omega_\pm^2 Y_\pm(1-Y_1^2)(1+Z_\pm^2)}$$

$$G'' \equiv \frac{K_{1r}^{(3)}(1-Y_2) + Z_\pm\left(K_{1i}^{(3)}(1-Y_2) + K_{1r}^{(3)}Z_2\left\{1-\frac{1}{2}\frac{\beta_\parallel}{\beta_\perp}\left(1-\frac{1}{Y_2}\right)\right\}\right)}{\omega_1\omega_2(1-Y_2^2)(1+Z_\pm^2)}$$

$$+ Z_2\frac{\beta_\parallel}{\beta_\perp}\frac{\left[(1-Z_\pm^2)\left(K_{\pm r}^{(3,2)}Z_2 + K_{\pm i}^{(3,2)}(1-Y_2)\right) - 2Z_\pm K_{\pm r}^{(3,2)}(1-Y_2)\right]}{2\omega_\pm^2 Y_\pm(1-Y_2^2)\left[(1-Z_\pm^2)^2+4Z_\pm^2\right]}$$

$$H'' \equiv \frac{\left[(1+Y_2)\left(K_{1i}^{(3)}(1-Y_2) + K_{1r}^{(3)}Z_2\left(1-\frac{1}{2}\frac{\beta_\parallel}{\beta_\perp}\left(1-\frac{1}{Y_2}\right)\right)\right) - K_{1r}^{(3)}\left(Z_\pm(1-Y_2^2)+2Z_2\right)\right]}{\omega_1\omega_2(1-Y_2^2)(1+Y_2)(1+Z_\pm^2)}$$

$$- \frac{Z_2\frac{\beta_\parallel}{\beta_\perp}K_{\pm r}^{(3,2)}\left[(1-Z_\pm^2)(1-Y_2) + 2Z_\pm Z_2\right]}{2\omega_\pm^2 Y_\pm(1-Y_2^2)\left[(1-Z_\pm^2)^2+4Z_\pm^2\right]}$$

$$G''' \equiv \frac{K_{2r}^{(2)}(1+Y_1) \pm Z_\pm\left(K_{2i}^{(2)}(1+Y_1) + K_{2r}^{(2)} Z_1\left(1+\frac{1}{2}\frac{\beta_{\parallel}}{\beta_{\perp}}\left(1-\frac{1}{Y_1}\right)\right)\right)}{\omega_1\omega_2(1-Y_1^2)(1+Z_\pm^2)}$$

$$\pm \frac{2 Z_1 Z_\pm \frac{\beta_{\parallel}}{\beta_{\perp}} K_{\pm r}^{(3,2)}}{2\omega_\pm^2 Y_\pm(1-Y_1)\left[(1-Z_\pm^2)^2 + 4Z_\pm^2\right]}$$

$$H''' \equiv \frac{\left[\left(K_{2i}^{(2)}(1+Y_1) + K_{2r}^{(2)} Z_1\left(1+\frac{1}{2}\frac{\beta_{\parallel}}{\beta_{\perp}}\left(1-\frac{1}{Y_2}\right)\right)\right)(1-Y_1) - K_{2r}^{(2)}\left(\pm Z_\pm(1-Y_1^2) + 2Z_1\right)\right]}{\omega_1\omega_2(1-Y_1^2)(1-Y_1)(1-Z_\pm^2)}$$

$$\pm \frac{Z_1 Z_\pm \frac{\beta_{\parallel}}{\beta_{\perp}}\left(K_{\pm r}^{(3,2)} Z_1 + K_{\pm i}^{(3,2)}(1+Y_1)\right)}{\omega_\pm^2 Y_\pm(1-Y_1^2)\left[(1-Z_\pm^2)^2 + 4Z_\pm^2\right]},$$

The quantities in (65) with the subscripts "$\pm$" are obtained from the quantities given above with the subscript "+" by means of the transformation,

$$Z_\pm \longrightarrow \pm Z_\pm .$$

In the course of deriving (65) we made some assumptions in the interest of simplicity. These assumptions are,

$$Z_{1,2}^2 << Y_{1,2}^2 \qquad \text{(II-1)}$$

which is a consequence of (3), and enabled us to neglect a considerable number of terms.

Assumptions (II-1) are satisfied in the F-region of the ionosphere which was the situation considered numerically in Section 5. For lower regions of the ionosphere, where (II-1) is not satisfied, it is a simple matter to derive modified forms of the quantities defined in this appendix.

APPENDIX III

In this appendix we give the definitions of the quantities appearing in (77).

The definitions of the propagation vectors and refractive indices are the same as defined in Appendix II.

$$A^{\alpha,\gamma} \equiv \left(1-\left(n_{\pm r}^{\alpha,\gamma}\right)^2\right)\cos\left(K_{\pm r}^{\alpha,\gamma} x - \omega_\pm \tau\right)$$

$$B^{\alpha,\gamma} \equiv \left(1-\left(n_{\pm r}^{\alpha,\gamma}\right)^2\right)\sin\left(K_{\pm r}^{\alpha,\gamma} x - \omega_\pm \tau\right)$$

$$\begin{aligned} C^{\alpha,\gamma} \equiv 1 - \left(n^{\alpha,\gamma}\right)^2 - X_\pm\Bigg[1 + \Big\{ & \left(n_{\pm r}^{\alpha,\gamma}\right)^2 \beta_{\parallel}\left(1-Y_\pm^2\right) \\ & + 2Z_\pm\left[2n_{\pm r}^{\alpha,\gamma} n_{\pm i}^{\alpha,\gamma}\beta_{\parallel} + Z_\pm\left(\beta_{\parallel}-\beta_{\perp}\right)\left(n_{\pm r}^{\alpha,\gamma}\right)^2\right] \\ & + Z_\pm\Big[\Big(2n_{\pm r}^{\alpha,\gamma} n_{\pm i}^{\alpha,\gamma}\beta_{\parallel} \\ & \qquad + Z_\pm\left(\beta_{\parallel}-\beta_{\perp}\right)\left(n_{\pm r}^{\alpha,\gamma}\right)^2\Big)\left(1-Y_\pm^2\right) \\ & \qquad - 2Z_\pm\left(n_{\pm r}^{\alpha,\gamma}\right)^2\beta_{\parallel}\Big]\Big\} \Big/ \left[\left(1-Y_\pm^2\right)^2 + 4Z_\pm^2\right]\Bigg] \end{aligned}$$

$$\begin{aligned} D^{\alpha,\gamma} \equiv & -2n_{\pm r}^{\alpha,\gamma} n_{\pm i}^{\alpha,\gamma} \\ & + X_\pm\left[\frac{2Z_\pm\left(n_{\pm r}^{\alpha,\gamma}\right)^2\beta_{\parallel}}{\left(1-Y_\pm^2\right)^2+4Z_\pm^2} + Z_\pm\left\{1 + \frac{\left(n_{\pm r}^{\alpha,\gamma}\right)^2\beta_{\parallel}\left(1-Y_\pm^2\right)}{\left(1-Y_\pm^2\right)^2+4Z_\pm^2}\right\}\right] \end{aligned}$$

$$E^{\alpha,\gamma} \equiv \left(K_{01} + K_{1r}^{(\alpha)}\right)\left(K_{02} + K_{2r}^{(\gamma)}\right)$$

$$F^{\alpha,\gamma} \equiv K_{1i}^{(\alpha)}\left(K_{02} + K_{2r}^{(\gamma)}\right) + K_{2i}^{(\gamma)}\left(K_{01} + K_{1r}^{(\alpha)}\right)$$

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$$A^{\alpha,\gamma} \equiv \left(1-(n_{\parallel}^{\alpha\gamma})^2\right)\cos\left(k_{\parallel}^{\alpha\gamma} x - \ldots z\right)$$

$$G \equiv \frac{Y_2 K_{2r}^{(3)}}{\omega_\pm \omega_2^2 (1-Y_2^2-X_2)} - \frac{Y_2 K_{1r}^{(1)}(1+X_2)}{\omega_\pm \omega_1 \omega_2 (1-Y_2^2)}$$

$$+ \frac{\left[K_{\pm r}^{1,3} Y_\pm (Y_2+Y_\pm)(1-Y_2^2)(1-Y_\pm^2) + \left(K_{\pm i}^{1,3} Y_\pm (Y_2+Y_\pm) + K_{\pm r}^{1,3}(Z_2-Z_\pm)\right) \times 2\left(Z_2(1-Y_\pm^2)+Z_\pm\right)\right]}{\omega_\pm \omega_2^2 Y_2 \left[(1-Y_2^2)^2(1-Y_\pm^2)^2 + 4Z_\pm^2\right]}$$

$$+ \frac{X_2 Y_2 \left[K_{\pm r}^{1,3}(\omega_2+\omega_\pm Y_\pm^2)\left\{(1-Y_2^2)(1-Y_\pm^2)(1-Y_2^2-X_2) - \left(Z_\pm(1-Y_2^2)+2Z_2\right)\left(2Z_\pm(1-Y_2^2-X_2)+Z_2(2-X_2)(1-Y_\pm^2)\right)\right\} + K_{\pm r}^{1,3}\omega_2(Z_\pm+Z_2)\left(2Z_\pm(1-Y_2^2-X_2)+Z_2(2-X_2)(1-Y_\pm^2)\right)(1-Y_2^2)\right]}{\omega_\pm^2 \omega_2^2 (1-Y_2^2)^2 \left\{2Z_\pm(1-Y_2^2-X_2)+Z_2(2-X_2)(1-Y_\pm^2)\right\}^2}$$

$$H \equiv \frac{X_2 Y_2 \left[K_{2i}^{(3)}(1-Y_2^2)(1-Y_2^2-X_2) - K_{2r}^{(3)}\left\{(Z_+(1-Y_2^2)+2Z_2)(1-Y_2^2-X_2)+Z_2(2-X_2)(1-Y_2^2)\right\}\right]}{\omega_+ \omega_2^2 (1-Y_2^2)^2 (1-Y_2^2-X_2)^2}$$

$$+ \frac{Y_2 \left[K_{2i}^{(3)}(1-Y_2^2) - K_{2r}^{(3)}\left((Z_++Z_2)(1-Y_2^2)+2Z_2\right)\right]}{\omega_+ \omega_2^2 (1-Y_2^2)^2}$$

$$- \frac{X_2 Y_2 \left[\left(K_{1i}^{(1)} + K_{1r}^{(1)} Z_2\left(1-\frac{\beta_\parallel}{2\beta_\perp}\right)\right)(1-Y_2^2)(1-Y_2^2-X_2) - K_{1r}^{(1)}\left\{(Z_+(1-Y_2^2)+2Z_2)(1-Y_2^2-X_2)+Z_2(2-X_2)(1-Y_2^2)\right\}\right]}{\omega_+ \omega_1 \omega_2 (1-Y_2^2)^2 (1-Y_2^2-X_2)^2} + \cdots$$

$$- \frac{Y_2 \left[\left(K_{1i}^{(1)} - K_{1r}^{(1)} \frac{\beta_\parallel}{2\beta_\perp} \frac{Z_2}{Y_2^2} \right)(1-Y_2^2) - K_{1r}^{(1)} \left(Z_+ (1-Y_2^2) + 2 Z_2 \right) \right]}{\omega_+ \, \omega_1 \, \omega_2 \, (1-Y_2^2)^2}$$

$$+ \frac{\left[\left(K_{+i}^{1,3} Y_+ (Y_2 + Y_+) + K_{+r}^{1,3} (Z_2 - Z_+) \right)(1-Y_2^2)(1-Y_+^2) - 2 K_{+r}^{1,3} Y_+ (Y_2 + Y_+)\left(Z_2 (1-Y_+^2) + Z_+ \right) \right]}{\omega_+ \omega_2^2 Y_2 \left[(1-Y_2^2)^2 (1-Y_+^2)^2 + 4 Z_+^2 \right]}$$

$$+ \frac{X_2 Y_2 \left[(1-Y_2^2)(1-Y_+^2)(1-Y_2^2 - X_2) \left\{ K_{+i}^{1,3} (\omega_2 + \omega_+ Y_+^2) + K_{+r}^{1,3} \omega_2 (Z_+ + Z_2) \right\} - K_{+r}^{1,3} (\omega_2 + \omega_+ Y_+^2) \left\{ \left(Z_+ (1-Y_2^2) + 2 Z_2 \right)(1-Y_+^2)(1-Y_2^2 - X_2) + (1-Y_2^2)\left(2 Z_+ (1-Y_2^2 - X_2) + Z_2 (2-X_2)(1-Y_+^2) \right) \right\} \right]}{\omega_+^2 \omega_2^2 (1-Y_2^2)^2 (1-Y_2^2 - X_2)^2 \left[(1-Y_+^2)^2 + 4 Z_+^2 \right]}$$

$$G' \equiv \frac{Z_2 X_1 Y_1 \left[\left(K_{2r}^{(1)} Z_i - K_{2i}^{(1)} \frac{\beta_\parallel}{\beta_\perp} \right)(1-Y_2^2)(1-Y_1^2 - X_1) + K_{2r}^{(1)} \frac{\beta_\parallel}{\beta_\perp} \left\{ \left(Z_+ (1-Y_2^2) + 2 Z_2 \right)(1-Y_1^2 - X_1) + Z_1 (2-X_1)(1-Y_2^2) \right\} \right]}{(1-Y_2^2)^2 (1-Y_1^2 - X_1)^2}$$

$$+ \frac{Z_2 \left[\left\{ K_{2r}^{(1)} Z_2 \left(1 + \frac{\beta_\parallel}{\beta_\perp} \right) + \frac{\beta_\parallel}{\beta_\perp} K_{2i}^{(1)} \right\}(1-Y_2^2) - K_{2r}^{(1)} \frac{\beta_\parallel}{\beta_\perp} Z_2 (3-Y_2^2) \right]}{2 \omega_+ \omega_2^2 Y_2 (1-Y_2^2)^2}$$

$$+ \cdots$$

$$-\frac{Z_2\left(K_{1i}^{(3)}-K_{1r}^{(3)}(Z_+ + Z_2)\right)}{\omega_+\omega_1\omega_2 Y_2}+\frac{K_{+r}^{3,1} X_1 Y_1}{\omega_+^2\omega_2\left[(1-Y_+^2)^2+4Z_+^2\right]}$$

$$+\frac{\left[K_{+r}^{3,1} Y_+ Y_2 (1-Y_+^2-3Z_+^2-2Z_+Z_2) + \left(K_{+i}^{3,1} Y_+ Y_2 - Z_2 K_{+r}^{3,1}\right)\left((Z_+ + Z_2)(1-Y_+^2)+2Z_+\right)\right]}{\omega_+^2\omega_2 Y_2\left[(1-Y_+^2)^2+4Z_+^2\right]}$$

$$H' \equiv \frac{Z_2 X_1 Y_1 K_{2r}^{(1)} \frac{\beta_{\parallel}}{\beta_{\perp}}}{(1-Y_2^2)(1-Y_1^2-X_1)} - \frac{Z_2 K_{2r}^{(1)} \frac{\beta_{\parallel}}{\beta_{\perp}}}{2\omega_+\omega_2^2 Y_2(1-Y_2^2)} + \frac{Z_2 K_{1r}^{(3)}}{\omega_+\omega_1\omega_2 Y_2}$$

$$+\frac{X_1 Y_1\left(2K_{+i}^{3,1}+K_{+r}^{3,1} Z_+\right)}{2\omega_+^2\omega_2\left[(1-Y_+^2)^2+4Z_+^2\right]}$$

$$+\frac{\left[\left(K_{+i}^{3,1} Y_+ Y_2 - Z_2 K_{+r}^{3,1}\right)(1-Y_+^2) - \left((Z_+ + Z_2)(1-Y_+^2)+2Z_+\right)\left(K_{+r}^{3,1} Y_+ Y_2 + K_{+i}^{3,1} Z_2\right)\right]}{\omega_+^2\omega_2 Y_2\left[(1-Y_+^2)^2+4Z_+^2\right]}$$

In the derivation of the quantities given in this appendix use has again been made of (II-1). As discussed at the end of Appendix II, it is a simple matter to derive the corresponding expressions for the general case.

APPENDIX IV

In this appendix we list the coefficients γ_i appearing in (79) and (80).

$$\gamma_1 \equiv \frac{i\left[U_+ U_2 (U_2 - 1) - Y_2 (U_+ Y_2 + Y_+)\right]}{Y_2\, \omega_1 \omega_2 \omega_+ (U_+^2 - Y_+^2)(U_2^2 - Y_2^2)}$$

$$\gamma_2 \equiv \frac{\omega_2 \left(U_2 (U_2 - 1) - Y_2^2\right) - \omega_+ U_+}{\omega_1 \omega_2 \omega_+^2 (U_+^2 - Y_+^2)(U_2^2 - Y_2^2)}$$

$$\gamma_3 \equiv \frac{(U_2 - 1)\,\omega_2 - 2 U_+ \omega_+}{2\,\omega_1 \omega_2 \omega_+^2 U_2 (U_+^2 - Y_+^2)}$$

$$\gamma_4 \equiv \frac{i\left[\omega_2 \left(U_2 (U_2 - 1) - Y_2^2\right) + 9\,\omega_+ Y_+ Y_2 U_+^2 - Y_+^2 \left(10\,\omega_2 \left(U_2 (U_2 - 1) + Y_2^2\right) + 3\,\omega_+ U_+\right]}{4 Y_+ \omega_1 \omega_2 \omega_+^2 (U_+^2 - Y_+^2)(U_2^2 - Y_2^2)(U_+^2 - 4 Y_+^2)}$$

$$\gamma_5 \equiv \frac{\left[U_+^2 \left(3 U_+ \omega_+^2 - \frac{3}{4} \omega_+ \omega_2 (U_2 - 1)\right) + Y_+^2 \left(-\frac{39}{4} \omega_+^2 U_+ + \frac{15}{2} \omega_+ \omega_2 U_2 - 3\,\omega_+ \omega_2\right) - \frac{3}{4} U_+ \omega_2^2 (U_2^2 - Y_2^2)\right]}{\omega_2^2 \omega_+^3 (U_2^2 - Y_2^2)(U_+^2 - Y_+^2)(U_+^2 - 4 Y_+^2)}$$

$$Y_6 \equiv -\frac{\left[U_+^2\left(\frac{3}{2}\omega_+^2 U_+ + \frac{3}{4}\omega_+\omega_2 U_2 - \frac{1}{4}\omega_2\omega_+\right) + Y_+^2\omega_+\left(-\frac{15}{4}\omega_+ U_+ + \frac{3}{2}\omega_2 U_2 + \omega_2\right) - \frac{3}{4}U_+\omega_2^2\left(U_2^2 - Y_2^2\right)\right]}{\omega_2^2\omega_+^3\left(U_2^2-Y_2^2\right)\left(U_+^2-4Y_+^2\right)\left(U_+^2-Y_+^2\right)}$$

$$Y_7 \equiv i\,\frac{\left[-\frac{3}{4}\omega_2^2 U_2(U_2-1) - \frac{5}{2}U_+^2 Y_2^2\omega_2^2 + Y_+^2\left\{\omega_+\omega_2\left(\frac{9}{4}U_+U_2 + \frac{35}{2}Y_+Y_2\right) + \frac{3}{2}\omega_2^2\left(U_2(U_2-1) - Y_2^2\right)\right\}\right]}{Y_1\omega_1\omega_2^2\omega_+^2\left(U_2^2-Y_2^2\right)\left(U_+^2-Y_+^2\right)\left(U_+^2-4Y_+^2\right)}$$

$$Y_8 \equiv -\frac{1}{4}\frac{\left[U_2^2\left(5\omega_2U_2 + \omega_+U_+ + \omega_2\right) + 4Y_2^2\left(-\frac{7}{2}\omega_2U_2 + 2\omega_+U_+ - \omega_2\right)\right]}{\omega_2^2\omega_+^2\left(U_+^2-Y_+^2\right)\left(U_2^2-Y_2^2\right)\left(U_2^2-4Y_2^2\right)}$$

$$Y_9 \equiv -\frac{\left[-\frac{\omega_+}{4}U_+\left(5U_2^2-8Y_2^2\right) - \frac{1}{2}\omega_2U_2Y_2^2 + \frac{U_2^3}{4}\left(-5\omega_+U_+ + 7\omega_2U_2\right) - 3\omega_2U_2^4 + Y_2^2\left(-2\omega_2 + \frac{27}{4}\omega_2U_2^2 + \frac{13}{2}\omega_+U_+U_2 + 4\omega_2Y_2^2\right)\right]}{\omega_+^2\omega_2^2U_2\left(U_+^2-Y_+^2\right)\left(U_2^2-4Y_2^2\right)}$$

$$Y_{10} \equiv \frac{i}{4}\frac{\left[3Y_2\omega_2U_2 + Y_+\omega_+\left(U_2^2-16Y_2^2\right)\right]}{\omega_+^2\omega_2^2\left(U_+^2-Y_+^2\right)\left(U_2^2-Y_2^2\right)\left(U_2^2-4Y_2^2\right)}$$

$$Y_{11} \equiv \frac{i\left[\omega_+ U_+ U_2^3 (U_2 - 1) + Y_+ Y_2 \omega_+ U_2 \left(U_2^2 + 8Y_2^2\right) + Y_2^2\left(6\omega_+ U_+ U_2^2 - 16\omega_+ U_+ Y_2^2 - \omega_2\left(U_2^2 + 8Y_2^2\right)\right)\right]}{4 Y_1 \omega_1 \omega_2 \omega_+^2 U_2 \left(U_+^2 - Y_+^2\right)\left(U_2^2 - Y_2^2\right)\left(U_2^2 - 4Y_2^2\right)}$$

$$Y_{12} \equiv \frac{-\omega_2\left(U_2(U_2 - 1) - Y_2^2\right) + \omega_+ U_+}{\omega_1 \omega_2 \omega_+^2 \left(U_+^2 - Y_+^2\right)\left(U_2^2 - Y_2^2\right)}$$

$$Y_{13} \equiv \frac{i\left[U_+ U_2 (U_2 - 1) - Y_2\left(U_+ Y_2 + Y_+\right)\right]}{\omega_+ \omega_1 \omega_2 Y_2 \left(U_+^2 - Y_+^2\right)\left(U_2^2 - Y_2^2\right)}$$

$$Y_{14} \equiv \frac{i\left[U_+ (U_2 - 1) - 2Y_+ Y_2\right]}{2\omega_+ \omega_1 \omega_2 Y_2 U_2 \left(U_+^2 - Y_+^2\right)}$$

$$Y_{15} \equiv -\frac{\left[\omega_c^2\left(U_+^2 - 24Y_+^2\right) - \omega_2^2 U_2^2\left(U_+^2 + 8Y_+^2\right) + 3\omega_+ \omega_2 U_+ U_2\left(U_+^2 + 2Y_+^2\right) + \omega_+ \omega_2 U_+\left(U_+^2 - 4Y_+^2\right) - 2U_2 \omega_2^2\left(U_+^2 - 4Y_+^2\right) + Y_+^2\left(15\omega_+^2 U_+^2 + 8\omega_2^2 Y_2^2\right)\right]}{4\omega_+^3 \omega_2^2 U_+ \left(U_+^2 - Y_+^2\right)\left(U_+^2 - 4Y_+^2\right)\left(U_2^2 - Y_2^2\right)}$$

$$Y_{16} \equiv -\frac{\left[\left(\frac{3}{4}\omega_+ \omega_2 U_+ U_2 + 2\omega_c^2\right)\left(U_+^2 + 2Y_+^2\right) - \frac{\omega_2^2 U_2^2}{4}\left(5U_+^2 - 8Y_+^2\right) + \frac{\omega_2^2 U_2}{2}\left(U_+^2 - 4Y_+^2\right) + \frac{\omega_2 \omega_+}{4} U_+^3 - U_+ \omega_2^2 Y_+ Y_2\right]}{\omega_+^3 \omega_2^2 U_+ \left(U_+^2 - Y_+^2\right)\left(U_+^2 - 4Y_+^2\right)\left(U_2^2 - Y_2^2\right)}$$

$$Y_{17} \equiv \frac{i\left[-8\omega_+ U_2 Y_+^2(2U_+^2-3Y_+^2) - 3U_+ Y_+ Y_2(U_+^2+2Y_+^2) - \omega_2 U_+ U_2(U_+^2+2Y_+^2) + \omega_2 U_+(U_2^2-Y_2^2)(U_+^2+2Y_+^2) + 8\omega_+ Y_+^2(U_+^2-Y_+^2) + 9\omega_+ U_+^2 Y_+^2(U_2-1)\right]}{4Y_1\omega_1\omega_2\omega_+^2 U_+(U_+^2-Y_+^2)(U_+^2-4Y_+^2)(U_2^2-Y_2^2)}$$

$$Y_{18} \equiv \frac{i\left[-16\omega_+ U_2 Y_+^2(U_+^2-Y_+^2) + 3\omega_+ U_+ Y_+ Y_2(U_+^2-2Y_+^2) + \omega_2 U_+ U_2(U_+^2+2Y_+^2) - \omega_2 U_+^2(U_2^2-Y_2^2) + 8\omega_+ Y_+^2(U_+^2-Y_+^2 U_2) - \omega_+ U_+^2(3-38U_2)Y_+^2 - Y_+^2\left(11\omega_+ U_+ U_2 + 2(U_2^2-Y_2^2)\right)\right]}{4Y_1\omega_1\omega_2\omega_+^2 U_+(U_+^2-Y_+^2)(U_+^2-4Y_+^2)(U_2^2-Y_2^2)}$$

$$Y_{19} \equiv \frac{i\left[-\omega_+ Y_+ Y_2(7U_2^2-34Y_2^2) - \omega_+ U_2(U_+^2-5Y_2^2) + \omega_+ U_+ U_2(U_2^2-Y_2^2) + 3\omega_2 U_2 Y_2^2\right]}{4Y_1\omega_1\omega_2\omega_+^2(U_+^2-Y_+^2)(U_2^2-Y_2^2)(U_2^2-4Y_2^2)}$$

$$Y_{20} \equiv \frac{i\left[-2\omega_+ U_+ Y_2^2(U_2^2+2Y_2^2) - \frac{\omega_+ U_2 Y_+ Y_2}{4}(5U_2^2-8Y_2^2) - \frac{\omega_+ U_+ U_2^2}{4}(U_2^2-10Y_2^2) + 2\omega_2 Y_2^2(U_2^2-Y_2^2) - \frac{Y_2^2\omega_2 U_2}{2}(4-U_2) + \frac{\omega_+ U_+}{4}(U_2-4Y_2^2)\right]}{Y_1\omega_1\omega_2\omega_+^2 U_2(U_2^2-Y_2^2)(U_+^2-Y_+^2)(U_2^2-4Y_2^2)}$$

$$Y_{21} \equiv - \frac{\left[\omega_+ U_+ U_2 (U_2^2 - 10 Y_2^2) - 8 \omega_2 Y_2^2 (U_2^2 + 2 Y_2^2) + \omega_2 U_2^2 (U_2^2 + 5 Y_2^2) - \omega_2 U_2 (U_2^2 + 4 Y_2^2) + 2 \omega_+ U_+ (U_2^2 - 4 Y_2^2) \right]}{4 \omega_+^2 \omega_2^2 U_2 (U_+^2 - Y_+^2)(U_2^2 - Y_2^2)(U_2^2 - 4 Y_2^2)}$$

$$Y_{22} \equiv - \frac{\left[\omega_+ U_+ (U_2^2 - 22 Y_2^2) - 3 \omega_2 U_2 (U_2^2 - Y_2^2) - 12 \omega_2 Y_2^2 \right]}{4 \omega_2^2 \omega_+^2 (U_+^2 - Y_+^2)(U_2^2 - Y_2^2)(U_2^2 - 4 Y_2^2)}$$

APPENDIX V

In this appendix we give the definitions of the quantities appearing in (81) and (83).

The definitions of the propagation vectors and refractive indices are the same as defined in Appendices II and III.

$$A_\alpha \equiv \cos\left(K_{\pm r}^{\alpha} x - \omega_\pm \tau\right)\left(K_{01} + K_{1r}^{(\alpha)}\right)\left(K_{02} + K_{2r}^{(\alpha)}\right)$$

$$B_\alpha \equiv \sin\left(K_{\pm r}^{\alpha} x - \omega_\pm \tau\right)\left(K_{01} + K_{1r}^{(\alpha)}\right)$$

$$E_\alpha \equiv 1 - \frac{X_\pm\left(1 - Y_\pm^2 + Z_\pm^2\right)}{\left(1 - Y_\pm^2\right)^2 + 4Z_\pm^2}$$

$$- \frac{3\beta_\perp X_\pm}{\left[\left(1 - Y_\pm^2\right)^2 + 4Z_\pm^2\right]\left[\left(1 - 4Y_\pm^2\right)^2 + 4Z_\pm^2\right]}\left[\left(1 - Y_\pm^2\right)\left(n_{\pm r}^{\alpha}\right)^2\left(1 - 4Y_\pm^2\right) + 2Z_\pm\left(1 + Y_\pm^2\right)\left\{n_{\pm r}^{\alpha} n_{\pm i}^{\alpha}\left(1 - 4Y_\pm^2\right) - Z_\pm\left(n_{\pm r}^{\alpha}\right)^2\right\}\right]$$

$$F_\alpha \equiv \frac{X_\pm Z_\pm\left(1 + Y_\pm^2\right)}{\left(1 - Y_\pm^2\right)^2 + 4Z_\pm^2}$$

$$+ \frac{3\beta_\perp X_\pm}{\left[\left(1 - Y_\pm^2\right)^2 + 4Z_\pm^2\right]\left[\left(1 - 4Y_\pm^2\right)^2 + 4Z_\pm^2\right]}\left[Z_\pm\left(1 + Y_\pm^2\right)\left\{\left(n_{\pm r}^{\alpha}\right)^2\left(1 - 4Y_\pm^2 - Z_\pm^2\right) + 4Z_\pm n_{\pm r}^{\alpha} n_{\pm i}^{\alpha}\right\} + 2\left(1 - Y_\pm^2\right) Z_\pm\left(n_{\pm r}^{\alpha}\right)^2\right]$$

$$I_\alpha \equiv \frac{X_\pm Y_\pm\left(1 - Y_\pm^2 - Z_\pm^2\right)}{\left(1 - Y_\pm^2\right)^2 + 4Z_\pm^2}$$

$$- \frac{6\beta_\perp X_\pm Y_\pm}{\left[\left(1 - Y_\pm^2\right)^2 + 4Z_\pm^2\right]\left[\left(1 - 4Y_\pm^2\right)^2 + 4Z_\pm^2\right]}\left[\left(1 - Y_\pm^2\right)\left\{\left(n_{\pm r}^{\alpha}\right)^2\left(1 - 4Y_\pm^2\right) + 4Z_\pm n_{\pm r}^{\alpha} n_{\pm i}^{\alpha}\right\} + 4Z_\pm\left\{n_{\pm r}^{\alpha} n_{\pm i}^{\alpha}\left(1 - 4Y_\pm^2\right) - Z_\pm\left(n_{\pm r}^{\alpha}\right)^2\right\}\right]$$

$$J_\alpha \equiv -\frac{2 Z_\pm X_\pm Y_\pm}{(1-Y_\pm^2)^2 + 4 Z_\pm^2}$$

$$+ \frac{12 \beta_\perp X_\pm Y_\pm}{\left[(1-Y_\pm^2)^2 + 4Z_\pm^2\right]\left[(1-4Z_\pm^2)^2 + 4Z_\pm^2\right]} \Big[- Z_\pm \left\{ (n_{\pm r}^\alpha)^2 (1-4Y_\pm^2) + 4 Z_\pm n_{\pm r}^\alpha n_{\pm i}^\alpha \right\}$$
$$+ (1-Y_\pm^2) \left\{ n_{\pm r}^\alpha n_{\pm i}^\alpha (1-4Y_\pm^2) - Z_\pm (n_{\pm r}^\alpha)^2 \right\} \Big]$$

$$M_\alpha \equiv 1 - (n_{\pm r}^\alpha)^2 - \frac{X_\pm (1 - Y_\pm^2 + 2 Z_\pm^2)}{(1-Y_\pm^2)^2 + 4Z_\pm^2}$$

$$- \frac{\beta_\perp X_\pm}{\left[(1-Y_\pm^2)^2 + 4Z_\pm^2\right]\left[(1-4Y_\pm^2)^2 + 16 Z_\pm^2 (1-2Y_\pm^2)^2\right]} \Big[(1+8Y_\pm^2)(n_{\pm r}^\alpha)^2 (1-4Y_\pm^2) + 8 Z_\pm (1-2Y_\pm^2) \left\{ Z_\pm (n_{\pm r}^\alpha)^2 + n_{\pm r}^\alpha n_{\pm i}^\alpha (1+8Y_\pm^2) \right\}$$
$$+ 2 Z_\pm (1+Y_\pm^2) \left\{ \left(Z_\pm (n_{\pm r}^\alpha)^2 + n_{\pm r}^\alpha n_{\pm i}^\alpha (1+8Y_\pm^2) \right)(1-4Y_\pm^2) - 2(1-2Y_\pm^2)(1+8Y_\pm^2)(n_{\pm r}^\alpha)^2 \right\} \Big]$$

$$N_\alpha \equiv -2 n_{\pm r}^\alpha n_{\pm i}^\alpha + \frac{X_\pm Z_\pm (1+Y_\pm^2)}{(1-Y_\pm^2)^2 + 4Z_\pm^2}$$

$$- \frac{\beta_\perp X_\pm}{\left[(1-Y_\pm^2)^2 + 4Z_\pm^2\right]\left[(1-4Y_\pm^2)^2 + 4Z_\pm^2\right]} \Big[- Z_\pm (1+Y_\pm^2)(1+8Y_\pm^2)(n_{\pm r}^\alpha)^2 (1-4Y_\pm^2)$$
$$+ 8 Z_\pm (1-2Y_\pm^2) \left\{ Z_\pm (n_{\pm r}^\alpha)^2 + n_{\pm r}^\alpha n_{\pm i}^\alpha (1+8Y_\pm^2) \right\}$$
$$+ 2(1-Y_\pm^2) \left\{ \left(Z_\pm (n_{\pm r}^\alpha)^2 + n_{\pm r}^\alpha n_{\pm i}^\alpha (1+8Y_\pm^2) \right)(1-4Y_\pm^2) - 2 Z_\pm (1-2Y_\pm^2)(1+8Y_\pm^2)(n_{\pm r}^\alpha)^2 \right\} \Big]$$

$$C_\alpha \equiv E_\alpha M_\alpha - F_\alpha N_\alpha - I_\alpha^2 + J_\alpha^2$$

$$D_\alpha \equiv F_\alpha M_\alpha + E_\alpha N_\alpha - 2 I_\alpha J_\alpha$$

$$Q_\alpha \equiv \left(K_{01} + K_{1r}^{(\alpha)}\right)^2 \left(K_{02} + K_{2r}^{(\alpha)}\right)^2$$

In the quantities defined above α assumes the values 1 and 3. For brevity we have also used the definition,

$$K_{\pm r}^{\alpha} \equiv K_{\pm r}^{\alpha\alpha}$$

$$n_{\pm r}^{\alpha} \equiv n_{\pm r}^{\alpha\alpha}$$

$$n_{\pm i}^{\alpha} \equiv n_{\pm i}^{\alpha\alpha} .$$

$$G \equiv - \frac{\left[2\left(1 - Y_\pm^2 - Z_\pm^2 - 2 Z_2 Z_\pm\right) + Z_\pm\left(Z_2(1 - Y_\pm^2) + 2 Z_\pm\right)\right]}{\omega_\pm \left[(1 - Y_\pm^2)^2 + 4 Z_\pm^2\right]}$$

$$H \equiv - \frac{\left[Z_\pm\left(1 - Y_\pm^2 - Z_\pm^2 - 2 Z_2 Z_\pm\right) - 2\left(Z_2(1 - Y_\pm^2) + 2 Z_\pm\right)\right]}{\omega_\pm \left[(1 - Y_\pm^2)^2 + 4 Z_\pm^2\right]}$$

$$K \equiv - \frac{\left[Z_2\left(1 - Y_\pm^2\right) + 2 Y_\pm Y_2 \left(Z_2(1 - Y_\pm^2) + 2 Z_\pm\right)\right]}{\omega_\pm \omega_2 Y_2 \left[(1 - Y_\pm^2)^2 + 4 Z_\pm^2\right]}$$

$$L \equiv \frac{\left[-2 Y_\pm Y_2 \left(1 - Y_\pm^2 - Z_\pm^2 - 2 Z_2 Z_\pm\right) + Z_2\left(Z_2(1 - Y_\pm^2) + 2 Z_\pm\right)\right]}{\omega_\pm \omega_2 Y_2 \left[(1 - Y_\pm^2)^2 + 4 Z_\pm^2\right]}$$

$$G_1 \equiv \frac{X_2 \Big[-Y_2 (Y_2 + Y_\pm)\{(1-Y_\pm^2 - Z_\pm^2)(1-Y_2^2) - 4 Z_\pm Z_2\} + 2(Z_2 - Z_\pm Y_2^2)\big(Z_\pm (1-Y_2^2) + Z_2 (1-Y_\pm^2)\big)\Big]}{\omega_1 \omega_2 \omega_\pm \big[(1-Y_\pm^2)^2 + 4 Z_\pm^2\big](1-Y_2^2)^2}$$

$$- \frac{(\omega_2 Y_2^2 + \omega_\pm)\big[(1-Y_\pm^2 - Z_\pm^2)(1-Y_2^2) - 4 Z_\pm Z_2\big]}{\omega_1 \omega_2 \omega_\pm^2 \big[(1-Y_\pm^2)^2 + 4 Z_\pm^2\big](1-Y_2^2)^2}$$

$$H_1 \equiv \frac{X_2 \big[(Z_2 - Z_\pm Y_2^2)(1-Y_\pm^2) + 2(Z_\pm + Z_2) Y_2 (Y_2 + Y_\pm)\big]}{\omega_1 \omega_2 \omega_\pm (1-Y_2^2)\big[(1-Y_\pm^2)^2 + 4 Z_\pm^2\big]}$$

$$+ \frac{2(\omega_2 Y_2^2 + \omega_\pm)\big[Z_\pm (1-Y_2^2) + Z_2 (1-Y_\pm^2)\big]}{\omega_1 \omega_2 \omega_\pm^2 (1-Y_2^2)^2 \big[(1-Y_\pm^2)^2 + 4 Z_\pm^2\big]}$$

$$G_+ \equiv \frac{\begin{aligned} X_1 Y_1 \Big[&\{8 \omega_\pm Y_\pm Y_2 + 10 \omega_2 Y_\pm^2 Y_2^2 + 3 \omega_\pm\} \\ &\cdot \{(1-Y_\pm^2)(1-Y_2^2)(1-4Y_\pm^2)(1-Y_1^2 - X_1) \\ &\quad - 4\big(Z_\pm (1-Y_2^2) + Z_2 (1-Y_\pm^2)\big)\big(2 Z_\pm (1-Y_1^2 - X_1) + Z_1 (2-X_1)(1-4Y_\pm^2)\big)\} \\ &+ \omega_2 Z_2 (9 Y_\pm Y_2 - 5 Y_\pm^2 + 2)\{2\big(Z_\pm (1-Y_2^2) + Z_2 (1-Y_\pm^2)\big)(1-4Y_\pm^2)(1-Y_1^2 - X_1) \\ &\quad + (1-Y_\pm^2)(1-Y_2^2)\big(2 Z_\pm (1-Y_1^2 - X_1) + Z_1 (2-X_1)(1-4Y_\pm^2)\big)\} \Big] \end{aligned}}{4 Y_\pm \omega_1 \omega_2 \omega_\pm^2 (1-Y_2^2)^2 (1-Y_1^2 - X_1)^2 \big[(1-Y_\pm^2)^2 (1-4Y_\pm^2)^2 + 4 Z_\pm^2 (2-5Y_\pm^2)^2\big]} + \cdots$$

$$+\frac{\begin{aligned}&\Big[\{\omega_\pm^2(12-39Y_\pm^2)+18\,\omega_\pm\omega_2 Y_\pm^2-3\,\omega_2^2(1-Y_2^2)\}\\&\quad\times\{(1-Y_2^2)(1-Y_\pm^2)(1-4Y_\pm^2)-4Z_\pm\left(Z_2(1-Y_\pm^2)+Z_\pm(1-Y_2^2)\right)\}\\&\quad+2\,\omega_2 Z_2\{\omega_\pm(33-9Y_\pm^2)-3\,\omega_2(3-Y_2^2)\}\\&\quad\times\{Z_\pm(1-Y_2^2)(2-5Y_\pm^2)\}\Big]\end{aligned}}{4\,\omega_2^2\,\omega_\pm^3(1-Y_2^2)^2\left[(1-Y_\pm^2)^2(1-4Y_\pm^2)^2+4Z_\pm^2(2-5Y_\pm^2)^2\right]}$$

$$+\frac{\begin{aligned}&X_1Y_1X_2Y_2\Big[\{3\,\omega_\pm^2(2-5Y_\pm^2)+2\,\omega_\pm\omega_2(1+5Y_\pm^2)-3\,\omega_2^2(1-Y_2^2)\}\\&\quad\times\{(1-Y_1^2-X_1)(1-Y_2^2-X_2)\big[(1-Y_2^2)\left((1-Y_\pm^2)(1-4Y_\pm^2)-Z_\pm^2(2-5Y_\pm^2)\right)-4Z_2Z_\pm(1-Y_\pm^2)\\&\qquad\qquad-4Z_\pm\left(Z_2(1-4Y_\pm^2)+Z_\pm(1-Y_2^2)\right)\big]\\&\quad-3Z_\pm(1-Y_2^2)(1-2Y_\pm^2)\left[Z_1(2-X_1)(1-Y_2^2-X_2)+Z_2(2-X_2)(1-Y_1^2-X_1)\right]\}\\&\quad+3Z_\pm(1-Y_2^2)(1-2Y_\pm^2)(1-Y_1^2-X_1)(1-Y_2^2-X_2)\,\omega_2\{-2\omega_2Z_2+3\omega_\pm Z_2(7-3Y_\pm^2)\\&\qquad\qquad-3\,\omega_2 Z_\pm(1-Y_2^2)\}\Big]\end{aligned}}{4\,\omega_2^2\,\omega_\pm^3(1-Y_1^2-X_1)(1-Y_2^2-X_2)(1-Y_2^2)\left[(1-Y_\pm^2)^2(1-4Y_\pm^2)^2+9Z_\pm^2(1-2Y_\pm^2)^2\right]}$$

$$+\frac{\begin{aligned}&X_2Y_2\Big[Y_2^2\{-10\,\omega_2+\omega_\pm(9+64\,Y_\pm Y_2)\}\\&\quad\times\{(1-Y_2^2-X_2)(1-Y_2^2)(1-Y_\pm^2)(1-4Y_\pm^2)-2Z_\pm Z_2(2-5Y_\pm^2)\left(2(1-Y_2^2-X_2)+(2-X_2)(1-Y_2^2)\right)\}\\&\quad+\{-3\,\omega_2Z_2-20\,Z_\pm Y_2^2\omega_2+3\,Y_2^2Z_2(3\omega_\pm+5\omega_2)\}\\&\quad\times\{Z_2\left(2(1-Y_2^2-X_2)+(2-X_2)(1-Y_2^2)\right)(1-Y_\pm^2)(1-4Y_\pm^2)\\&\qquad+2Z_\pm(2-5Y_\pm^2)(1-Y_2^2-X_2)(1-Y_2^2)\}\Big]\end{aligned}}{4Y_1\,\omega_1\omega_2\,\omega_\pm^2(1-Y_2^2-X_2)^2(1-Y_2^2)^2\left[(1-Y_\pm^2)^2(1-4Y_\pm^2)^2+4Z_\pm^2(2-5Y_\pm^2)^2\right]}$$

$$
\begin{aligned}
H_+ \equiv \frac{\begin{aligned} X_1 Y_1 \Big[& -\{8\omega_\pm Y_\pm Y_2 + 10\omega_2 Y_\pm^2 Y_2^2 + 3\omega_\pm\} \\
& \times \{2\left(Z_\pm(1-Y_2^2) + Z_2(1-Y_\pm^2)\right)(1-4Y_\pm^2)(1-Y_1^2) \\
& \quad + \left(2Z_\pm(1-Y_1^2-X_1) + Z_1(2-X_1)(1-4Y_\pm^2)\right)(1-Y_\pm^2)(1-Y_2^2)\} \\
& + 2\omega_2 Z_2 (9Y_\pm Y_2 - 5Y_\pm^2 + 2)(1-Y_\pm^2)(1-Y_2^2)(1-4Y_\pm^2)(1-Y_1^2-X_1)\Big] \end{aligned}}{\begin{aligned} 4Y_\pm \omega_1\omega_2\omega_\pm^2(1-Y_2^2)^2 \Big[& (1-Y_\pm^2)^2(1-4Y_\pm^2)^2(1-Y_1^2-X_1)^2 + 4Z_\pm^2\{(1-4Y_\pm^2)(1-Y_1^2) \\
& + (1-Y_1^2-X_1)(1-Y_\pm^2)\}^2\Big] \end{aligned}}
\end{aligned}
$$

$$
+ \frac{\begin{aligned} \Big[& -2\{\omega_\pm^2(12-39Y_\pm^2) + 18\omega_\pm\omega_2 Y_\pm^2 - 3\omega_2^2(1-Y_2^2)\} \\
& \times \{Z_\pm(1-Y_2^2)(1-Y_\pm^2) + (1-4Y_\pm^2)\left(Z_2(1-Y_\pm^2) + Z_\pm(1-Y_2^2)\right)\} \\
& + \omega_2 Z_2\{\omega_\pm(33-9Y_\pm^2) - 3\omega_2(3-Y_2^2)\}(1-Y_2^2)(1-Y_\pm^2)(1-4Y_\pm^2)\Big] \end{aligned}}{4\omega_2^2\omega_\pm^3\left[(1-Y_\pm^2)^2(1-4Y_\pm^2)^2 + 4Z_\pm^2(2-5Y_\pm^2)^2\right](1-Y_2^2)^2}
$$

$$
+ \frac{\begin{aligned} X_2 Y_2 \Big[& \{-3\omega_2 Z_2 - 20 Z_\pm Y_2^2\omega_2 + 3Y_2^2 Z_2(3\omega_\pm + 5\omega_2)\} \\
& \times (1-Y_2^2-X_2)(1-Y_2^2)(1-Y_\pm^2)(1-4Y_\pm^2) \\
& + Y_2^2\{10\omega_2 - \omega_\pm(9+64Y_\pm Y_2)\} \\
& \times \{Z_2\left(2(1-Y_2^2-X_2) + (2-X_2)(1-Y_2^2)\right)(1-Y_\pm^2)(1-4Y_\pm^2) \\
& \quad + 2Z_\pm(2-5Y_\pm^2)(1-Y_2^2-X_2)(1-Y_2^2)\}\Big] \end{aligned}}{4Y_1\omega_1\omega_2\omega_\pm^2(1-Y_2^2-X_2)^2(1-Y_2^2)^2\left[(1-Y_\pm^2)^2(1-4Y_\pm^2)^2 + 4Z_\pm^2(2-5Y_\pm^2)^2\right]}
$$

$$+ \cdots$$

$$+X_1Y_1X_2Y_2\Big[-\{3\omega_\pm^2(2-5Y_\pm^2)+2\omega_\pm\omega_2(1+5Y_\pm^2)-3\omega_2^2(1-Y_2^2)\}$$

$$\cdot\{3z_\pm(1-Y_2^2)(1-Y_1^2-X_1)(1-Y_2^2-X_2)(1-2Y_\pm^2)$$

$$+[z_1(2-X_1)(1-Y_2^2-X_2)+z_2(2-X_2)(1-Y_1^2-X_1)](1-Y_\pm^2)(1-Y_2^2)(1-4Y_\pm^2)\}$$

$$+\{\omega_\pm z_\pm(4\omega_2+21\omega_\pm-9\omega_\pm Y_\pm^2)-3\omega_2^2(z_\pm(1-Y_2^2)+2z_2)\}$$

$$\cdot(1-Y_1^2-X_1)(1-Y_2^2-X_2)(1-Y_\pm^2)(1-4Y_\pm^2)(1-Y_2^2)\Big]$$

$$\overline{4\omega_2^2\omega_\pm^3(1-Y_1^2-X_1)^2(1-Y_2^2-X_2)^2(1-Y_2^2)^2[(1-Y_\pm^2)^2(1-4Y_\pm^2)^2+9z_\pm^2(1-2Y_\pm^2)^2]}$$

$$G_2\equiv X_1Y_1X_2Y_2\Big[\{6\omega_2(1-3Y_2^2)+\omega_\pm(1+8Y_2^2)\}$$

$$\cdot\{[(1-4Y_2^2)((1-Y_\pm^2-z_\pm^2)(1-Y_2^2)-4z_\pm z_2)-4z_2z_\pm(1-Y_2^2)]$$

$$\cdot(1-Y_1^2-X_1)(1-Y_2^2-X_2)$$

$$-2[z_1(2-X_1)(1-Y_2^2-X_2)+z_2(2-X_2)(1-Y_1^2-X_1)]$$

$$\cdot[z_\pm(1-Y_2^2)(1-4Y_2^2)+z_2(1-Y_\pm^2)(2-5Y_2^2)]\}$$

$$+\{\omega_2z_2(17-14Y_2^2)+\omega_\pm(z_\pm(1+8Y_\pm^2)+2z_2)\}$$

$$\cdot\{2[z_\pm(1-Y_2^2)(1-4Y_2^2)+z_2(1-Y_\pm^2)(2-5Y_2^2)](1-Y_1^2-X_1)(1-Y_2^2-X_2)$$

$$+[z_1(2-X_1)(1-Y_2^2-X_2)+z_2(2-X_2)(1-Y_1^2-X_1)]$$

$$\cdot[(1-4Y_2^2)(1-Y_\pm^2)(1-Y_2^2)-4z_2(z_\pm(1-Y_2^2)+z_2(1-Y_\pm^2))]\}\Big]$$

$$\overline{4\omega_2^2\omega_\pm^2(1-Y_2^2)^2(1-Y_1^2-X_1)^2(1-Y_2^2-X_2)^2[(1-4Y_2^2)^2(1-Y_\pm^2)^2+4z_\pm^2]}$$

$$+\cdots$$

$$
+\frac{\begin{aligned}&\Big[\big\{2\omega_\pm(5-17Y_2^2)+\omega_2(5-17Y_2^2-16Y_2^4)\big\}(1-Y_\pm^2)(1-4Y_2^2)\\
&\quad+\big\{\omega_\pm[2Z_\pm(5-17Y_2^2)+25Z_2-26Z_2Y_2^2]\\
&\qquad+\omega_2(4Z_2(5-13Y_2^2)-16Y_2^4)\big\}\\
&\qquad\cdot\big\{Z_2(1-Y_\pm^2)(3-4Y_2^2)+2Z_\pm(1-4Y_2^2)\big\}\Big]\end{aligned}}{\omega_\pm^2\omega_2^2(1-4Y_2^2)^2\left[(1-Y_\pm^2)^2+4Z_\pm^2\right]}
$$

$$
+\frac{\begin{aligned}&X_2Y_2^2\Big[4\big\{\big((1-Y_2^2-X_2)(1-Y_\pm^2-Z_\pm^2)-2Z_\pm Z_2(2-X_2)\big)(1-Y_2^2)(1-4Y_2^2)\\
&\qquad-4Z_2Z_\pm(2-5Y_2^2)(1-Y_2^2-X_2)\big\}\\
&\qquad+10Z_2Z_\pm(1-4Y_2^2)(1-Y_2^2-X_2)\Big]\end{aligned}}{4\omega_\pm^2\omega_2(1-Y_2^2)(1-4Y_2^2)^2(1-Y_2^2-X_2)^2\left[(1-Y_\pm^2)^2+4Z_\pm^2\right]}
$$

$$
+\frac{\begin{aligned}&X_1\Big[\big\{\omega_\pm[6Y_2^2-16Y_2^4+Y_\pm Y_2(1+8Y_2^2)]-\omega_2Y_2^2(1+8Y_2^2)\big\}\\
&\quad\times\big\{(1-Y_1^2-X_1)\big((1-Y_\pm^2-Z_\pm^2)(1-Y_2^2)-4Z_\pm Z_2\big)-2Z_1(2-X_1)Z_\pm(1-Y_2^2)(1-4Y_2^2)\\
&\qquad-2Z_2Z_\pm(3-4Y_2^2)(1-Y_2^2)(1-Y_1^2-X_1)\big\}\\
&\quad+\big\{\omega_\pm[Z_2+6Y_2^2(Z_\pm+2Z_2)-16Z_\pm Y_2^4+Y_\pm Y_2Z_2(3+8Y_2^2)]\\
&\qquad-2\omega_1Z_2Y_2^2\big\}\,2Z_\pm(1-Y_2^2)(1-Y_1^2-X_1)(1-4Y_2^2)\Big]\end{aligned}}{4\omega_1\omega_2\omega_\pm^2(1-Y_1^2-X_1)^2(1-Y_2^2)^2(1-4Y_2^2)^2\left[(1-Y_\pm^2)^2+4Z_\pm^2\right]}
$$

$$
\begin{aligned}
H_2 \equiv\; & X_1 Y_1 X_2 Y_2 \frac{\begin{aligned}\Big[& -\{6\omega_2(1-3Y_2^2)+\omega_\pm(1+8Y_2^2)\} \\ & \times\{2(1-Y_1^2-X_1)(1-Y_2^2-X_2)[Z_\pm(1-Y_2^2)(1-4Y_2^2)+Z_2(1-Y_\pm^2)(2-5Y_2^2)] \\ & +(1-4Y_2^2)(1-Y_\pm^2)(1-Y_2^2)[Z_1(2-X_1)(1-Y_2^2-X_2)+Z_2(2-X_2)(1-Y_1^2-X_1)]\} \\ & +(1-4Y_2^2)(1-Y_\pm^2)(1-Y_2^2)\{6Z_\pm(3-Y_2^2)+2Z_2\}\omega_\pm \Big]\end{aligned}}{4\omega_2^2\omega_\pm^2(1-Y_2^2)^2(1-4Y_2^2)^2[(1-Y_\pm^2)^2+4Z_\pm^2]} \\
& + \frac{\begin{aligned}\Big[& -\{2\omega_\pm(5-17Y_2^2)+\omega_2(5-17Y_2^2-16Y_2^4)\} \\ & \cdot\{Z_2(1-Y_\pm^2)(3-4Y_2^2)+2Z_\pm(1-4Y_2^2)\} \\ & +(1-Y_\pm^2)(1-4Y_2^2)\{\omega_\pm[5Z_\pm(5-17Y_2^2)+Z_2(25-26Y_2^2)] \\ & \qquad +4\omega_2(Z_2(5-13Y_2^2)-4Y_2^4)\} \Big]\end{aligned}}{\omega_\pm^2\omega_2^2(1-4Y_2^2)^2[(1-Y_\pm^2)^2+4Z_\pm^2]} \\
& + X_2 Y_2^2 \frac{\begin{aligned}\Big[& -4\{(1-Y_2^2)(1-4Y_2^2)(Z_2(2-X_2)(1-Y_\pm^2)+2Z_\pm(1-Y_2^2-X_2)) \\ & +2Z_2(2-5Y_2^2)(1-Y_2^2-X_2)(1-Y_\pm^2)\}+5Z_2(1-Y_2^2-X_2)(1-Y_\pm^2)(1-4Y_2^2)\Big]\end{aligned}}{4\omega_\pm^2\omega_2(1-Y_2^2-X_2)^2(1-Y_2^2)(1-4Y_2^2)^2[(1-Y_\pm^2)^2+4Z_\pm^2]} \\
& + \cdots
\end{aligned}
$$

$$+ X_1 \Big[- \{ \omega_\pm [6Y_2^2 - 16 Y_2^4 + Y_\pm Y_2 (1+8Y_2^2)] - \omega_2 Y_2^2 (1+8Y_2^2) \}$$

$$\times \{ (1-4Y_2^2) [Z_1 (2-X_1)(1-Y_\pm^2)(1-Y_2^2) + 2(1-Y_1^2-X_1)\big(Z_\pm(1-Y_2^2)+Z_2(1-Y_\pm^2)\big)]$$

$$+ Z_2 (3-4Y_2^2)(1-Y_1^2-X_1)(1-Y_\pm^2)(1-Y_2^2) \}$$

$$+ (1-Y_1^2-X_1)(1-Y_\pm^2)(1-Y_2^2)(1-4Y_2^2)$$

$$\times \{ \omega_\pm [Z_2 + 6Y_2^2(Z_\pm + 2Z_2) - 16 Z_\pm Y_2^4 + Y_\pm Y_2 Z_2 (3+8Y_2^2)] - 2\omega_2 Z_2 Y_2^2 \} \Big]$$

$$\overline{4\omega_1\omega_2\omega_\pm^2 (1-Y_2^2)^2 (1-4Y_2^2)^2 (1-Y_1^2-X_1)^2 [(1-Y_\pm^2)^2 + 4Z_\pm^2]}$$

$$K_1 \equiv \frac{X_2 Y_2 (\omega_2 Y_2^2 + \omega_\pm) \Big[(1-Y_2^2-X_2)\big((1-Y_\pm^2-Z_\pm^2)(1-Y_2^2) - 4Z_\pm Z_2\big) - 2Z_2(2-X_2)\big(Z_\pm(1-Y_2^2) + Z_2(1-Y_\pm^2)\big) \Big]}{\omega_1\omega_2\omega_\pm^2 (1-Y_2^2-X_2)^2 (1-Y_2^2)^2 [(1-Y_\pm^2)^2 + 4Z_\pm^2]}$$

$$+ \frac{\Big[Y_2 (Y_2 + Y_\pm)\big((1-Y_\pm^2-Z_\pm^2)(1-Y_2^2) - 4Z_\pm Z_2\big) - 2(Z_2 - Y_2^2 Z_\pm)\big(Z_\pm(1-Y_2^2) + Z_2(1-Y_\pm^2)\big) \Big]}{\omega_\pm \omega_1 \omega_2 Y_2 (1-Y_2^2)^2 [(1-Y_\pm^2)^2 + 4Z_\pm^2]}$$

$$L_1 \equiv \frac{- X_2 Y_2 (\omega_2 Y_2^2 + \omega_\pm) \Big\{ Z_2 (2-X_2)\big((1-Y_\pm^2-Z_\pm^2)(1-Y_2^2) - 4Z_\pm Z_2\big) + 2(1-Y_2^2-X_2)\big(Z_\pm(1-Y_2^2) + Z_2(1-Y_\pm^2)\big) \Big\}}{\omega_1\omega_2\omega_\pm^2 (1-Y_2^2-X_2)^2 (1-Y_2^2)^2 [(1-Y_\pm^2)^2 + 4Z_\pm^2]} + \cdots$$

$$
\frac{-\left[2Y_2(Y_2+Y_\pm)\left(Z_\pm(1-Y_2^2)+Z_2(1-Y_\pm^2)\right)+(Z_2-Y_2^2Z_\pm)(1-Y_\pm^2)(1-Y_2^2)\right]}{\omega_\pm\omega_1\omega_2 Y_2(1-Y_2^2)^2\left[(1-Y_\pm^2)^2+4Z_\pm^2\right]}
$$

$$
\begin{aligned}
K_+ \equiv -X_1Y_1 \Big[& \left\{\omega_c^2(1-24Y_\pm^2)+2\omega_\pm\omega_2(2+Y_\pm^2)+15\omega_\pm^2Y_2^2-\omega_2^2(3-8Y_2^4)\right\} \\
& \cdot\left\{(1-Y_1^2-X_1)\left[(1-Y_2^2)\left((1-Y_\pm^2)(1-4Y_\pm^2)-Z_\pm^2(6-5Y_\pm^2)\right)-4Z_2Z_\pm(2-5Y_\pm^2)\right]\right. \\
& -2\left(Z_\pm(1-Y_1^2-X_1)+Z_1(2-X_1)\right) \\
& \left.\cdot\left[Z_2(1-Y_\pm^2)(1-4Y_\pm^2)+Z_\pm(2-5Y_\pm^2)(1-Y_2^2)\right]\right\} \\
& +\left\{2\omega_c^2Z_\pm-2\omega_2^2\left(Z_2(1+8Y_\pm^2)+Z_\pm\right)\right. \\
& +\omega_\pm\omega_2\left(3\left((Z_\pm+Z_2)(1+2Y_\pm^2)+2Z_\pm\right)+2Z_\pm(1-2Y_\pm^2)\right) \\
& \left.-2\omega_2^2\left(Z_2(1-4Y_\pm^2)+2Z_\pm\right)+30Z_\pm\omega_\pm^2Y_2^2\right\} \\
& \cdot 2\left\{Z_2(1-Y_\pm^2)(1-4Y_\pm^2)+Z_\pm(2-5Y_\pm^2)(1-Y_2^2)\right\}\Big]
\end{aligned}
$$

$$
\overline{4\omega_\pm^3\omega_2^2(1-Y_2^2)^2\left[(1-Y_1^2-X_1)^2(1-Y_\pm^2)(1-4Y_\pm^2)^2+4Z_\pm^2(2-5Y_\pm^2)^2\right]}
$$

$$
\begin{aligned}
-X_2Y_2\Big[& \left\{2\omega_\pm\omega_2(2+3Y_\pm^2)+16\omega_c^2Y_\pm^2-\omega_2^2(3+4Y_\pm Y_2)\right\} \\
& \cdot\left\{(1-Y_2^2-X_2)\left[(1-Y_2^2)\left((1-Y_\pm^2)(1-4Y_\pm^2)-Z_\pm^2(6-5Y_\pm^2)\right)-4Z_2Z_\pm(2-5Y_\pm^2)\right]\right. \\
& \left.-2\left(Z_2(1-Y_\pm^2)(1-4Y_\pm^2)+Z_\pm(2-5Y_\pm^2)(1-Y_2^2)\right)\left(Z_\pm(1-Y_2^2-X_2)+Z_2(2-X_2)\right)\right\} \\
& +\left\{\omega_\pm\omega_2\left(6Z_\pm(2+Y_2^2)+3Z_2(1+2Y_2^2)\right)+16\omega_c^2Z_\pm-2\omega_2^2\left(3(Z_2+Z_\pm)+2Y_\pm Y_2Z_\pm\right)\right\} \\
& \cdot\left\{\left(Z_\pm(1-Y_2^2-X_2)+Z_2(2-X_2)\right)(1-Y_2^2)(1-Y_\pm^2)(1-4Y_\pm^2)\right. \\
& \left.+2(1-Y_2^2-X_2)\left[Z_2(1-Y_\pm^2)(1-4Y_\pm^2)+Z_\pm(2-5Y_\pm^2)(1-Y_2^2)\right]\right\}\Big]
\end{aligned}
$$

$$
\overline{4\omega_\pm^3\omega_2^2(1-Y_2^2)^2(1-Y_2^2-X_2)^2\left[(1-Y_\pm^2)^2(1-4Y_\pm^2)^2+4Z_\pm^2(2-5Y_\pm^2)^2\right]} + \cdots
$$

$$
\begin{aligned}
&+ X_1 X_2 Y_2 \Big[\big\{ -8\omega_\pm Y_2^2 (1-2Y_\pm^2) - 3\omega_\pm Y_\pm Y_2 (1+2Y_\pm^2) \\
&\qquad + \omega_2 \big((1-Y_2^2)(2+Y_\pm^2) - (1+2Y_\pm^2) \big) \big\} \\
&\times \Big\{ (1-Y_1^2-X_1)(1-Y_2^2-X_2) \Big[(1-Y_\pm^2)(1-4Y_\pm^2) - 2Z_\pm (3-Y_\pm^2)\big(Z_\pm (1-Y_2^2) + Z_2 (1-4Y_\pm^2)\big) \Big] \\
&- \big[Z_1 (2-X_1)(1-Y_2^2-X_2) + Z_2 (2-X_2)(1-Y_1^2-X_1) \big] \\
&\quad \big[Z_\pm (3-Y_\pm^2)(1-4Y_\pm^2)(1-Y_2^2) + 2(1-Y_2^2)\big(Z_\pm (1-Y_2^2) + Z_2 (1-4Y_\pm^2)\big) \big] \Big\} \\
&+ \big\{ -8\omega_\pm Y_2^2 \big(2Z_\pm + Z_2 (2-3Y_\pm^2)\big) - 3\omega_\pm Y_\pm Y_2 \big(Z_2 (1+2Y_\pm^2) + 2Z_\pm\big) \\
&\quad + \omega_2 \big[Z_2 (1+2Y_\pm^2) + Z_\pm \big(1-Y_\pm^2 - Y_2^2 (4+Y_\pm^2)\big) \big] \big\} \\
&\times (1-Y_\pm^2)(1-4Y_\pm^2)(1-Y_2^2) \big[Z_1 (2-X_1)(1-Y_2^2-X_2) + Z_2 (2-X_2)(1-Y_1^2-X_1) \big] \Big]
\end{aligned}
$$

$$
\overline{4\omega_1 \omega_2 \omega_\pm^2 (1-Y_1^2-X_1)^2 (1-Y_2^2-X_2)^2 \big[(1-Y_\pm^2)(1-4Y_\pm^2)^2 + 4Z_\pm^2 (1-Y_2^2)^4 \big]}
$$

$$
\begin{aligned}
&- \Big[\big\{ 8\omega_\pm Y_\pm^2 (2+Y_\pm^2) + \omega_2 \big(Y_\pm^2 + Y_2^2 (3-4Y_\pm^2) \big) \big\} \\
&\quad \times \big\{ (1-Y_\pm^2) \big((1-4Y_\pm^2)(1-Y_2^2) - (Z_\pm^2 (1-Y_2^2) + 4 Z_2 Z_\pm) \big) \\
&\qquad - 2Z_\pm (3-Y_\pm^2) \big(Z_2 (1-4Y_\pm^2) + Z_\pm (1-Y_2^2) \big) \big\} \\
&\quad + \big\{ \omega_\pm Y_\pm^2 Z_2 (11+8Y_\pm^2) + Z_\pm \big[\omega_2 (1+2Y_\pm^2+2Y_2^2) + \omega_\pm \big(9Y_\pm Y_2 + Y_\pm^2 (41-6Y_\pm Y_2) \big) \big] \big\} \\
&\quad \times \big\{ Z_\pm (3-Y_\pm^2)(1-4Y_\pm^2)(1-Y_2^2) + 2(1-Y_\pm^2)\big(Z_2 (1-4Y_\pm^2) + Z_\pm (1-Y_2^2) \big) \big\} \Big]
\end{aligned}
$$

$$
\overline{4Y_1 \omega_1 \omega_2 \omega_\pm^2 (1-Y_2^2)^2 \big[(1-Y_\pm^2)^2 (1-4Y_\pm^2)^2 + Z_\pm^2 \big\{ (3-Y_\pm^2)(1-4Y_\pm^2) + 2(1-Y_\pm^2) \big\}^2 \big]}
$$

$$
L_+ \equiv -X_1 Y_1 \frac{\begin{aligned}\Big[& -\big\{\omega_2^2(1-24Y_\pm^2) + 2\omega_\pm \omega_2 (2+Y_\pm^2) + 15\omega_\pm^2 Y_2^2 \\ & \qquad - \omega_2^2(3-8Y_2^4)\big\} \\ & \times \big\{ \big(Z_\pm(1-Y_1^2-X_1) + Z_1(2-X_1)\big)(1-Y_\pm^2)(1-4Y_\pm^2)(1-Y_2^2) \\ & + 2(1-Y_1^2-X_1)\left[Z_2(1-Y_\pm^2)(1-4Y_\pm^2) + Z_\pm(2-5Y_\pm^2)(1-Y_2^2)\right]\big\} \\ & + (1-Y_1^2-X_1)(1-Y_2^2)(1-Y_\pm^2)(1-4Y_\pm^2)\big\{2Y_2^2 Z_\pm(\omega_2^2+15\omega_\pm^2) \\ & - 2\omega_2^2\big(2Z_2(1+2Y_\pm^2)+3Z_\pm\big) + \omega_\pm\omega_2\Big(3\big((Z_\pm+Z_2)(1+2Y_\pm^2)+2Z_\pm\big) \\ & \qquad + 2Z_\pm(1-2Y_\pm^2)\Big)\big\}\Big]\end{aligned}}{4\omega_\pm^3 \omega_2^2 (1-Y_1^2-X_1)^2(1-Y_2^2)^2\left[(1-Y_\pm^2)^2(1-4Y_\pm^2)^2 + 4Z_\pm^2(2-5Y_\pm^2)^2\right]}
$$

$$
-X_2 Y_2 \frac{\begin{aligned}\Big[& -\omega_2\big\{2\omega_\pm(2+3Y_\pm^2) + \omega_2(16Y_2^2Y_\pm^2 - 3 - 4Y_\pm Y_2)\big\} \\ & \times \big\{(1-Y_2^2)(1-Y_\pm^2)(1-4Y_\pm^2)\big(Z_\pm(1-Y_2^2-X_2) + Z_2(2-X_2)\big) \\ & + 2(1-Y_2^2-X_2)\left[Z_2(1-Y_\pm^2)(1-4Y_\pm^2) + Z_\pm(2-5Y_\pm^2)(1-Y_2^2)\right]\big\} \\ & + (1-Y_2^2-X_2)(1-Y_2^2)(1-Y_\pm^2)(1-4Y_\pm^2) \\ & \times \omega_2\big\{\omega_\pm\big(6Z_\pm(2+Y_2^2) + Z_2(1+2Y_2^2)\big) + \omega_2\big(16Y_2^2 Z_\pm - 6(Z_2+Z_\pm) - 4Y_\pm Y_2 Z_\pm\big)\big\}\Big]\end{aligned}}{4\omega_\pm^3 \omega_2^2 (1-Y_2^2-X_2)^2(1-Y_2^2)^2\left[(1-Y_\pm^2)^2(1-4Y_\pm^2)^2 + 4Z_\pm^2(2-5Y_\pm^2)^2\right]}
$$

$$+ \cdots$$

$$
\frac{\begin{aligned}
&+ X_1 X_2 Y_2 \Big[-\big\{ -8\omega_\pm Y_2^2 (1-2Y_\pm^2) - 3\omega_\pm Y_\pm Y_2 (1+2Y_\pm^2) \\
&\qquad + \omega_2 (1 - Y_\pm^2 - Y_2^2 - Y_\pm^2 Y_2^2) \big\} \\
&\times \big\{ [Z_1 (2-X_1)(1-Y_2^2-X_2) + Z_2 (2-X_2)(1-Y_1^2-X_1)] (1-Y_\pm^2)(1-4Y_\pm^2)(1-Y_2^2) \\
&+ (1-Y_1^2-X_1)(1-Y_2^2-X_2)(1-Y_2^2) [Z_\pm (3-Y_\pm^2)(1-4Y_\pm^2) + 2 (Z_\pm (1-Y_2^2) + Z_2 (1-4Y_\pm^2))] \big\} \\
&+ \big\{ -8\omega_\pm Y_2^2 (2Z_\pm + Z_2 (2-3Y_\pm^2)) + \omega_2 Z_\pm (1 - Y_\pm^2 - Y_2^2 (4+Y_\pm^2)) \\
&+ \omega_2 Z_2 (1+2Y_\pm^2) - 3\omega_\pm Y_\pm Y_2 (Z_2 (1+2Y_\pm^2) + 2Z_\pm) \big\} \\
&\times (1-Y_1^2-X_1)(1-Y_2^2-X_2)(1-Y_\pm^2)(1-4Y_\pm^2)(1-Y_2^2) \Big]
\end{aligned}}{4\omega_1\omega_2\omega_\pm^2 (1-Y_1^2-X_1)^2 (1-Y_2^2-X_2)^2 (1-Y_2^2)^2 \big[(1-Y_\pm^2)^2 (1-4Y_\pm^2)^2 + \{ 2Z_\pm (1-Y_2^2) + (1-4Y_\pm^2)(2Z_2 + Z_\pm (3-Y_\pm^2)) \}^2 \big]}
$$

$$
-\frac{\begin{aligned}
&\Big[-\big\{ 2\omega_\pm Y_\pm^2 (8 + Y_\pm (4Y_\pm - 3Y_2)) + \omega_2 Y_\pm^2 (1+2Y_2^2) + 3\omega_\pm Y_\pm Y_2 \big\} \\
&\times \big\{ 2Z_2 (1-4Y_\pm^2) + Z_\pm (1-Y_2^2)(4(1+Y_\pm^4) - 14Y_\pm^2) \big\} \\
&+ \big\{ \omega_\pm Y_\pm^2 Z_2 (11+8Y_\pm^2) + 41\omega_\pm Y_\pm^2 Z_\pm + \omega_2 (Z_\pm (1+2Y_\pm^2+2Y_2^2) - Z_2) \\
&\qquad + 3\omega_\pm Y_\pm Y_2 Z_\pm (3-2Y_\pm^2) \big\} (1-Y_\pm^2)(1-4Y_\pm^2)(1-Y_2^2) \Big]
\end{aligned}}{4Y_1 \omega_1 \omega_2 \omega_\pm^2 (1-Y_2^2)^2 \big[(1-Y_\pm^2)^2 (1-4Y_\pm^2)^2 + Z_\pm^2 \{ (3-Y_\pm^2)(1-4Y_\pm^2) + 1 - Y_\pm^2 \}^2 \big]}
$$

$$
\begin{aligned}
K_2 \equiv X_1 X_2 Y_2 \Bigg[& \frac{\begin{aligned} & Y_2 \{ -\omega_\pm Y_\pm (7-34Y_2^2) + 9\omega_\pm Y_2 + 3\omega_2 Y_2 \} \\ & \times \{ (1-Y_1^2-X_1)(1-Y_2^2-X_2) [(1-4Y_2^2)((1-Y_\pm^2-Z_\pm^2)(1-Y_2^2) - 4Z_\pm Z_2) \\ & \qquad - 4Z_2(Z_\pm(1-Y_2^2) + Z_2(1-Y_\pm^2))] \\ & \quad - 2[Z_1(2-X_1)(1-Y_2^2-X_2) + Z_2(2-X_2)(1-Y_1^2-X_1)] \\ & \quad \times [Z_2(1-Y_\pm^2)(2-5Y_2^2) + Z_\pm(1-Y_2^2)(1-4Y_2^2)] \} \\ & + \{ -14\omega_\pm Y_\pm Y_2 Z_2 + 3\omega_2 Y_2^2 Z_2 + \omega_\pm [-Z_\pm(1+Y_2^2) + Z_2(2+9Y_2^2)] \} \\ & \times \{ [Z_1(2-X_1)(1-Y_2^2-X_2) + Z_2(2-X_2)(1-Y_1^2-X_1)](1-4Y_2^2)(1-Y_\pm^2) \\ & \qquad \times (1-Y_2^2) \\ & + 2(1-Y_1^2-X_1)(1-Y_2^2-X_2)[Z_2(1-Y_\pm^2)(2-5Y_2^2) + Z_\pm(1-Y_2^2)(1-4Y_2^2)] \} \end{aligned}}{4\omega_1\omega_2\omega_\pm^2 (1-Y_1^2-X_1)^2 (1-Y_2^2-X_2)^2 (1-Y_2^2)^2 (1-4Y_2^2)^2 [(1-Y_\pm^2)^2 + 4Z_\pm^2]} \\
& - \frac{\begin{aligned} & [\{ -2\omega_\pm(4+5Y_2^2) - 2\omega_\pm Y_\pm Y_2(5-2Y_2^2) + 2\omega_2 Y_2^2(1-4Y_2^2) \} \\ & \times \{ (1-Y_2^2)((1-Y_\pm^2-Z_\pm^2)(1-4Y_2^2) - 4Z_\pm Z_2) - 2Z_2(3-Y_2^2)(Z_\pm(1-4Y_2^2) + Z_2(1-Y_\pm^2)) \} \\ & + \{ \omega_\pm [- Z_2(27-4Y_2^2) + 10Z_\pm Y_2^2 - 2Y_\pm Y_2 Z_2(15-2Y_2^2)] + 12\omega_2 Z_2 Y_2^2 \} \\ & \times \{ 2(Z_\pm(1-4Y_2^2) + Z_2(1-Y_\pm^2))(1-Y_2^2) + Z_2(3-Y_2^2)(1-Y_\pm^2)(1-4Y_2^2) \}] \end{aligned}}{4Y_1\omega_1\omega_2\omega_\pm^2 (1-4Y_2^2)^2 [(1-Y_2^2)^2(1-Y_\pm^2) + 4Z_\pm^2]} \\
& + \cdots
\end{aligned}
$$

$$
\frac{\begin{aligned}
&-X_1 Y_1 \Big[\big\{ 3\omega_\pm (1-6Y_2^2) - \omega_2 Y_2^2 (7+16Y_2^2) \big\} \\
&\quad \times \Big\{ (1-Y_1^2-X_1)\Big[(1-4Y_2^2)\big((1-Y_\pm^2-Z_\pm^2)(1-Y_2^2) - 4Z_\pm Z_2\big) - 4Z_2\big(Z_\pm(1-Y_2^2)+Z_2(1-Y_\pm^2)\big)\Big] \\
&\quad -2\big[Z_2(1-Y_1^2-X_1) + Z_1(2-X_1)\big] \\
&\quad \times \big[Z_2(1-Y_\pm^2)(2-5Y_2^2) + Z_\pm(1-Y_2^2)(1-4Y_2^2)\big]\Big\} \\
&\quad + \omega_\pm \big\{ Z_2(7-10Y_2^2) + 4Z_\pm(1-7Y_2^2) \big\} \\
&\quad \times \Big\{ \big(Z_2(1-Y_1^2-X_1) + Z_1(2-X_1)\big)(1-4Y_2^2)(1-Y_\pm^2)(1-Y_2^2) \\
&\quad + 2(1-Y_1^2-X_1)\big[Z_2(1-Y_\pm^2)(2-5Y_2^2) + Z_\pm(1-Y_2^2)(1-4Y_2^2)\big]\Big\} \Big]
\end{aligned}}{4\omega_\pm^2 \omega_2^2 (1-Y_1^2-X_1)^2 (1-Y_2^2)^2 (1-4Y_2^2)^2 \big[(1-Y_\pm^2)^2 + 4Z_\pm^2\big]}
$$

$$
\frac{\begin{aligned}
&-X_2 Y_2 \Big[\big\{ \omega_\pm (1-22Y_2^2) - 3\omega_2 (1+3Y_2^2) \big\} \\
&\quad \times \Big\{ \big[(1-Y_\pm^2-Z_\pm^2)(1-Y_2^2-X_2) - 2Z_\pm Z_2(2-X_2)\big](1-Y_2^2)(1-4Y_2^2) \\
&\quad - 2Z_2(2-5Y_2^2)\big(2Z_\pm(1-Y_2^2-X_2) + Z_2(2-X_2)(1-Y_\pm^2)\big)\Big\} \\
&\quad + \omega_\pm \big\{ -Z_\pm(8+19Y_2^2) + 2Z_2 \big\} \big\{ 2Z_2(2-5Y_2^2)(1-Y_\pm^2)(1-Y_2^2-X_2) \\
&\quad + (1-Y_2^2)(1-4Y_2^2)\big(2Z_\pm(1-Y_2^2-X_2) + Z_2(2-X_2)(1-Y_\pm^2)\big) \big\} \Big]
\end{aligned}}{4\omega_2^2 \omega_\pm^2 (1-Y_2^2)^2 (1-4Y_2^2)^2 (1-Y_2^2-X_2)^2 \big[(1-Y_\pm^2)^2 + 4Z_\pm^2\big]}
$$

$$L_2 \equiv X_1 X_2 Y_2 \Big[\{ \omega_\pm Y_\pm Y_2 (7 - 34 Y_2^2) - 3 Y_2^2 (3\omega_\pm + \omega_2) \}$$

$$\times \{ [Z_1 (2 - X_1)(1 - Y_2^2 - X_2) + Z_2 (2 - X_2)(1 - Y_1^2 - X_1)](1 - 4Y_2^2)(1 - Y_\pm^2) \cdot (1 - Y_2^2)$$

$$+ 2 [Z_2 (1 - Y_\pm^2)(2 - 5Y_2^2) + Z_\pm (1 - Y_2^2)(1 - 4Y_2^2)](1 - Y_1^2 - X_1)(1 - Y_2^2 - X_2) \}$$

$$+ \{ -11 \omega_\pm Y_\pm Y_2 Z_2 + \omega_\pm (- Z_\pm (1 + Y_2^2) + Z_2 (2 + 9 Y_2^2)) \}$$

$$\times (1 - Y_1^2 - X_1)(1 - Y_2^2 - X_2)(1 - Y_2^2)(1 - 4Y_2^2)(1 - Y_\pm^2) \Big]$$

$$\overline{4 \omega_1 \omega_2 \omega_\pm^2 (1 - Y_1^2 - X_1)^2 (1 - Y_2^2 - X_2)^2 (1 - Y_2^2)^2 (1 - 4Y_2^2)^2 [(1 - Y_\pm^2)^2 + 4 Z_\pm^2]}$$

$$- \Big[-2 \{ -\omega_\pm (4 + 5Y_2^2) + \omega_2 Y_2^2 (1 - 4Y_2^2) - \omega_\pm Y_\pm Y_2 (5 - 2Y_2^2) \}$$

$$\times \{ Z_2 (1 - Y_\pm^2)(5(1 - 3Y_2^2) + 4 Y_2^4) + 2 Z_\pm (1 - 4Y_2^2)(1 - Y_2^2) \}$$

$$+ \omega_\pm (1 - Y_2^2)(1 - Y_\pm^2)(1 - 4Y_2^2) \{ Z_2 [-27 + 4Y_2^2 - 2 Y_\pm Y_2 (15 - 2Y_2^2)]$$

$$+ 18 Z_\pm Y_2^2 \} \Big]$$

$$\overline{4 Y_1 \omega_1 \omega_2 \omega_\pm^2 (1 - Y_2^2)^2 (1 - 4Y_2^2)^2 [(1 - Y_\pm^2)^2 + 4 Z_\pm^2]}$$

$+ \cdots$

$$- X_1 Y_1 \Big[- \{ 3\omega_\pm (1-6Y_2^2) - \omega_2 Y_2^2 (7+16Y_2^2) \}$$
$$\times \{ [Z_2 (1-Y_1^2 - X_1) + Z_1 (2-X_1)] (1-4Y_2^2)(1-Y_\pm^2)(1-Y_2^2)$$
$$+ 2(1-Y_1^2 - X_1)[Z_2(1-Y_\pm^2)(2-5Y_2^2) + Z_\pm (1-Y_2^2)(1-4Y_2^2)] \}$$
$$+ \omega_\pm (1-Y_1^2 - X_1)(1-Y_2^2)(1-4Y_2^2)(1-Y_\pm^2) \Big]$$
$$\overline{4\omega_\pm^2 \omega_2^2 (1-Y_1^2-X_1)^2 (1-Y_2^2)^2 (1-4Y_2^2)^2 [(1-Y_\pm^2)^2 + 4Z_\pm^2]}$$

$$- X_2 Y_2 \Big[- \{ \omega_\pm (1-22Y_2^2) - 3\omega_2 (1+3Y_2^2) \}$$
$$\times \{ 2Z_2 (2-5Y_2^2)(1-Y_\pm^2)(1-Y_2^2 - X_2)$$
$$+ (1-Y_2^2)(1-4Y_2^2)(2Z_\pm(1-Y_2^2-X_2) + Z_2(2-X_2)(1-Y_\pm^2)) \}$$
$$+ \omega_\pm (1-Y_\pm^2)(1-Y_2^2-X_2)(1-Y_2^2)(1-4Y_2^2)\{-Z_\pm(8+19Y_2^2) + 2Z_2\} \Big]$$
$$\overline{4\omega_2^2 \omega_\pm^2 (1-Y_2^2-X_2)^2 (1-Y_2^2)^2 (1-4Y_2^2)^2 [(1-Y_\pm^2)^2 + 4Z_\pm^2]}$$

We have again used assumption (II-1) in the derivation of the quantities given in this appendix. As mentioned at the end of Appendix II, it is a simple matter to derive expressions for the general case.

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