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ON ASSOCIATED RADIAL HEAT EXPANSIONS

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1. Introduction. Let $\mu > 1$ and let Δ_μ denote the radial Laplacian operator $\Delta_\mu \equiv D_r^2 + \frac{\mu-1}{r} D_r$. Criterion has recently been given ([1],[3]) under which a classical solution $u(r,t)$ of the radial heat equation

$$(1.1) \qquad \frac{\partial}{\partial t} u(r,t) = \Delta_\mu u(r,t)$$

has valid series representations in terms of the set of radial heat polynomials $\{R_j^\mu(r,t)\}_{j=0}^\infty$ and the set of associated radial functions $\{\tilde{R}_j^\mu(r,t)\}_{j=0}^\infty$. These sets are defined by

$$(1.2) \qquad \begin{cases} \text{(a)} \quad R_j^\mu(r,t) = j!(4t)^j L_j^{\left(\frac{\mu}{2}-1\right)}(-r^2/4t) \\ \text{(b)} \quad \tilde{R}_j^\mu(r,t) = t^{-2j} S_\mu(r,t) R_j^\mu(r,-t) \end{cases} \qquad j = 0, 1, 2, \ldots$$

where $L_j^{(\mu/2-1)}(x)$ is the generalized Laguerre polynomial of degree j and index $\frac{\mu}{2}-1$ and $S_\mu(r,t)$ is the source solution $(4\pi t)^{-\mu/2} e^{-r^2/4t}$. It was shown that if a solution $u(r,t)$ of (1.1) exhibits the Huygens' property and satisfies the additional condition

$$(1.3) \qquad \int_0^\infty \xi^{\mu-1} e^{\xi^2/8t} \, | u(\xi,t) | \, d\xi < \infty$$

for $t > \sigma \geq 0$, then this $u(r,t)$ has an absolutely convergent expansion in terms of the set $\{\tilde{R}_j^\mu(r,t)\}_{j=0}^\infty$ for $t > \sigma \geq 0$. Moreover, the coefficients in the expansion

$$(1.4) \qquad u(r,t) = \sum_{j=0}^\infty a_j \tilde{R}_j^\mu(r,t)$$

are defined by the integral

$$(1.5) \qquad a_j = \frac{2\pi^{\mu/2}}{2^{4j} j! \Gamma(j+\frac{\mu}{2})} \int_0^{\infty} \xi^{\mu-1} R_j^{\mu}(\xi,-t)u(\xi,t)d\xi \, .$$

This series representation can be replaced by the integral representation

$$(1.6) \qquad u(r,t) = \int_0^{\infty} \zeta_{\mu}(r,\xi;t)\psi(\xi)d\xi \, .$$

In this, $\psi(\xi)$ is an entire function of growth $(1,\sigma)$ in ξ^2 and is given by $\psi(\xi) = \sum_{j=0}^{\infty} (-1)^j \, 4^j \, a_j \, \xi^{2j}$ and

$$(1.7) \qquad \zeta_{\mu}(r,\xi;t) = (2\pi)^{-\mu/2} \, r^{1-\mu/2} \, \xi^{\mu/2} \, J_{\frac{\mu}{2}-1}(r\xi)e^{-\xi^2 t} \, .$$

It is apparent that the construction of an associated radial heat expansion for a suitable $u(r,t)$ by means of (1.4) is not very efficient. The purpose of this note is to give a procedure for obtaining such a series representation that often avoids the usage of (1.4). The knowledge of this series will also lead directly to the appropriate choice for $\psi(\xi)$ in (1.6). From the fact that the $\widetilde{R}_j^{\mu}(r,t)$ are defined in terms of the Laguerre polynomials, one would expect to make use of some special property of them in this construction.

A procedure that involves the Laplace transform will prove to be effective in treating this problem. Expansions of functions in terms of Laguerre polynomials by means of the Laplace transform have received some attention. Doetsch [2] has examined this approach and has given a rather detailed proof for its validity (also see [4].

The method for evaluating the a_n along with the proof of its validity is given in Section 2. Examples of the use of this technique are given in Section 3 to obtain explicit expansions of the form (1.4) as well as integral representations of the form (1.6). A byproduct of this is that we obtain the evaluations of some rather complicated integrals.

2. A Coefficient Determination. Let $u(r,t)$ be a solution of (1.1) that satisfies the Huygens' property and (1.3) and let $x = r^2/4t$. Upon equating this u to the series (1.4) and using (1.2), we obtain

$$e^x\, u(2\sqrt{xt}\,,\, t) = \sum_{j=0}^{\infty} \frac{(-1)^j a_j\, j!\, 4^j}{(4\pi t)^{\mu/2} t^j} L_j^{(\mu/2-1)}(x)\,. \tag{2.1}$$

Now multiply both members of this by $x^{\mu/2-1}$ and form the Laplace transform of both members of this by selecting $\frac{1}{p}$ as the transforming variable for x. It follows that [2]

$$\left.\begin{aligned} &\int_0^{\infty} e^{-\left(\frac{1}{p}-1\right)x}\, u(2\sqrt{xt}\,,\, t)\, x^{\mu/2-1}\, dx \\ &= \sum_{j=0}^{\infty} \frac{(-1)^j a_j 4^j \Gamma(j+\mu/2)}{(4\pi t)^{\mu/2}\, t^j}\, p^{\mu/2}(1-p)^j . \end{aligned}\right\} \tag{2.2}$$

The integral here exists for p near 1 by the condition (1.3). Next, multiply both members of this by $p^{-\mu/2}$ and evaluate the n^{th} derivative of both members of this with respect to p at $p = 1$. We get

$$(2.3) \qquad a_n = \frac{(4\pi t)^{\mu/2}\ t^n}{4^n\ n!\ \Gamma(\frac{\mu}{2}+n)} \left(\frac{d}{dp}\right)^n \left\{ p^{-\mu/2} \int_0^\infty e^{-\left(\frac{1}{p}-1\right)x} x^{\frac{\mu}{2}-1} u(2\sqrt{xt}, t)\, dx \right\}_{p=1} .$$

For the purpose of showing that (2.3) reduces to (1.5), let us observe that (see [5], p.84)

$$p^{-\mu/2} e^{-(\frac{1}{p}-1)x} = [1-(1-p)]^{-\mu/2} e^{-x(1-p)/[1-(1-p)]}$$

$$= \sum_{j=0}^{\infty} L_j^{(\mu/2-1)}(x)(1-p)^j .$$

It results that

$$\left(\frac{d}{dp}\right)^n \left\{ p^{-\mu/2} e^{-\{\frac{1}{p}-1\}x} \right\}_{p=1} = (-1)^n\ n!\ L_n^{(\frac{\mu}{2}-1)}(x) .$$

Finally, substituting this last into (2.3) gives

$$(2.4) \quad a_n = \frac{(-1)^n (4\pi t)^{\mu/2}\ t^n}{4^n\ \Gamma(\frac{\mu}{2}+n)} \int_0^\infty L_n^{(\mu/2-1)}(x) x^{\mu/2-1}\ u(2\sqrt{xt}, t)\, dx .$$

The reduction of the right member of this to the right member of (1.5) now follows by reintroducing the change of variables $x = r^2/4t$ (with $dx = \frac{r}{2t}\, dr$) into this and replacing $L_n^{(\frac{\mu}{2}-1)}(r^2/4t)$ in terms of $R_n^{\mu}(r,-t)$ by means of (1.2a).

It is clear from this that if we have an explicit evaluation of the left member of (2.2), then we can determine the coefficients a_n in (1.4) without resorting to the integrals (1.5). In these cases, we can then formulate other representations, both series and integral, rather readily.

3. Some Examples. We now indicate the applicability of the above procedure to specific functions. One of the principle points of interest here is the connection between various integral representations for the same function. The 'good' examples make note of this.

Example 1. Let $a > 0$ and let $u(r,t) = (1+4at)^{-\mu/2} e^{-a r^2/(1+4at)}$. Then the condition (1.3) is satisfied by choosing $\sigma = 1/4a$. The term in brackets in the right member of (2.3) has the evaluation

$\Gamma(\mu/2)(1+4at-p)^{-\mu/2}$. It follows by (2.3) that $a_n = (\frac{\pi}{a})^{\mu/2} \frac{1}{(16a)^n n!}$.

We then have the pair of representations

$$u(r,t) = \begin{cases} (\frac{\pi}{a})^{\mu/2} \sum_{n=0}^{\infty} \frac{1}{n!} \frac{1}{(16a)^n} \tilde{R}^{\mu}_{n}(r,t) \\ \text{or} \\ \frac{1}{(2a)}\,\mu/2 \int_0^{\infty} r^{1-\mu/2} \xi^{\mu/2} J_{\frac{\mu}{2}-1}(r\xi)\, e^{-\xi^2(t+\frac{1}{4a})} d\xi . \end{cases}$$

Compare the latter with the second formula on page 35 of [5].

Example 2. $\mu = 2$ and

$$u(r,t) = (1+16t^2)^{-1} e^{-4r^2 t/(1+16t^2)} \left\{ \sin \frac{r^2}{1+16t^2} + 4t \cos \frac{r^2}{1+16t^2} \right\}.$$

In this case, the bracketed term in the right member of (2.3) is just $4t[16t^2 + (p-1)^2]^{-1}$. It then follows in this case that

$$u(r,t) = \begin{cases} \pi \displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{4^{4n}(2n)!} \tilde{R}^2_{2n}(r,t) \\ \text{or} \\ \frac{1}{2} \displaystyle\int_0^{\infty} \xi \, J_0(r\xi) \, e^{-\xi^2 t} \cos(\xi^2/4) d\xi . \end{cases}$$

These representations converge absolutely if $t > 1/4$. The given function $u(r,t)$ here is a solution of (1.1) corresponding to the initial data $u(r,0) = \sin(r^2)$. From [1], it follows that we also have the following integral representation for this same $u(r,t)$:

$$u(r,t) = \frac{e^{-r^2/4t}}{2t} \int_0^{\infty} \xi \, I_0 \left(\frac{r\xi}{2t}\right) e^{-\xi^2/4t} \sin(\xi^2) d\xi .$$

<u>Example 3</u>. $\mu = 2$ and $u(r,t) = e^{-t} J_0(r)$. In this situation the criterion (1.3) fails and so no expansion of the type (1.4) exists.

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