COLLISIONAL INTERACTIONS OF ENERGETIC

TEST ELECTRONS WITH A PLASMA*

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by

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ABSTRACT

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An expression for the energy loss rate of an energetic test electron traversing a plasma is derived from a binary collision viewpoint. Energy loss due to the excitation of plasma oscillation is included through an ad hoc extension of the maximum impact parameter, $\lambda_D \frac{v}{w_t}$. The solution is generalized to include the effect of a translating plasma.

An expression for the distribution of energy losses of a beam of monoenergetic test electrons traversing a finite plasma is obtained through the solution of a Boltzman equation. The transmission of the beam is calculated using multiple scattering theory.

Various aspects of the electron- electron collision problem are discussed.

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CHAPTER I

INTRODUCTION

The problem of the interaction of a test particle with a plasma is a very old problem which has been treated extensively from many standpoints in the literature[1-16]. Interest in this problem has been stimulated primarily by the advent of the controlled thermonuclear program, especially the interaction of test electrons with a plasma through the interest in calculating the phenomenon of "run away" electrons[17,18]. A more basic interest stems from the relationship of this problem to the kinetic theory of ionized gases. Many authors have derived expressions for the energy loss[7-16] of an energetic test electron in a plasma.

Much of the theoretical and experimental work which has been done in connection with the charged-particle-plasma interaction has been concerned with the plasma interaction of a dense beam of electrons[19,20] rather than with single test electrons. The results are considerably different and of larger magnitude due to the coherent interaction of the beam particles among themselves. This paper will not be concerned with beam effects. Further complications arise due to applied magnetic fields[21,22]. An excellent review article which contains an extensive bibliography of the theoretical and experimental work on the interaction of charged particles and beams of charged particles with a plasma is presented in reference [19].

The object of the present report is to calculate, from a collisional viewpoint, the energy loss suffered by an energetic test electron traversing a plasma. The calculation is performed in considerable detail using the correct Mott-scattering cross section for electron-electron collisions. Some authors[23,24] have derived dynamical friction and diffusion coefficients based on simple inverse-square law interactions and have used those coefficients in a Fokker-Planck equation to study the thermalization of a group like particles, continuing the calculation from an initial state, $w >> w_t$, into a final state of complete equilibrium. As will be pointed out in a subsequent chapter, for test electron velocities close to the plasma thermal velocity, the maximum scattering angle becomes large enough so that the extra terms in the Mott cross section are significant and consequently the Fokker-Planck coefficients calculated from an inverse-square law interaction are incorrect in principle, especially as applied to the self thermalization of an electron stream.

The advantage of considering the energetic test electron interaction from a collisional viewpoint lies in obtaining a simple, physical interpretation of the results. The collisional picture facilitates a calculation for the distribution of energy losses of a beam of test electrons traversing a plasma; and also the transmission of the beam, through multiple scattering theory. An experimental investigation of test electron interactions would probably entail the use of a beam of test electrons, so that the distribution of losses and the transmission calculations are important.

CHAPTER II

AN ESSAY ON COLLECTIVE EFFECTS

In this paper the energy loss rate of an energetic test electron traversing a plasma is calculated from a purely binary collision viewpoint, and the collective energy losses are included through an extension of the maximum impact parameter. There seems to be some sort of paradox involved if, in fact, the binary collision treatment is conceptually correct. However, the paradox is resolved if one realizes that the apparent contradiction arises from a comparison of two mutually exclusive points of view; namely, the macroscopic and the microscopic. By its very nature a polarization effect belongs to a macroscopic viewpoint.

The problem arises in connection with the expressions derived for the energy loss of a fast particle traversing a uniform dielectric. The result is obtained in identical form whether one treats the problem as the passage of a charged body through a plasma which is considered to be a uniform dielectric or whether the problem is treated as a collisional phenomena between a charged particle and the constituent particles of the dielectric. In the former treatment the charged body suffers no sidewise deviation in its traversal, whereas in the latter a deflecting collision definitely occurs. How can the two results be reconciled?

From a macroscopic viewpoint, the charged particle interaction with the uniform dielectric is idealized as the interaction of a point charge with a uniform, continuous dielectric. The passage of the energetic point charge results in a uniform make of polarized media trailing the point charge.

On a microscopic scale the dielectric is composed of mobile negative charges embedded in a matrix of positive charges. On the average the mobile charges are arranged so that the electric field about any charge is the gradient of a screened coulomb potential. The test electron passing through the microscopic dielectric "sees" an electric field of this sort with which it "collides" and is scattered from its initial path by a small angle. In Chapter III a plausability argument* is advanced showing how energy can be transferred to a plasma through polarization by a fast electron when the electron velocity exceeds the plasma thermal velocity. This result is incorporated into the energy loss calculation by extending the normal minimum scattering angles in the electron collision, so that the polarization effects are accounted for by the smallest scattering angles in the collisions.

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*A more rigorous argument has been proposed by Rostoker[25].

We can return to the idealized viewpoint through the following argument. The charged particle suffers many small angle collisions in its traversal through the dielectric; the extremely small angle collisions being responsible for the polarization loss. From multiple scattering theory* the net deviation of the test electron from its original trajectory is distributed about the original trajectory equally on either side in the manner of a Gaussian law. Statistically, the test electron's path is predominantly along the original trajectory with a small probability of deviation symmetric about the original path. In this sense one can state that the test electron traversed the dielectric undeviated (most probably) from its original path.

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*See Chapter IV.

CHAPTER III

AVERAGE ENERGY LOSS RATE

3.1 Introduction

An expression for the average rate of energy loss of an energetic electron traversing a plasma in thermal equilibrium is derived on the assumption that the dominant mechanism for energy loss is the Coulomb interaction. It will further be assumed that the dominant Coulomb interactions for energy loss are electron-electron collisions, since an electron-ion or electron-neutral collision will degrade the electron's energy by a negligible amount. The average energy loss rate is derived in general for an arbitrary distribution of plasma electrons, $f(w)$. In order to obtain a numerical result $f(w)$ must be specified. An initial calculation is performed for a stationary plasma with a Maxwellian distribution of electron velocities. This calculation is then extended to the case of a translating plasma with a Maxwellian distribution of electron velocities (this is a model applicable to the plasma behind a moving shock front. The calculation for the average loss rate through a translating plasma indicates that, for laboratory conditions, the effect of the translation is negligible.

3.2 Average Energy Loss for an Energetic Test Electron in a Plasma

The collision between the test electron of velocity $\underline{v}$ and the plasma electron of velocity $\underline{w}$ can be analyzed easily in the center of mass coordinate system. For a completely general set of velocities, $\underline{v}$ and $\underline{w}$, the energy loss of the test electron is found to be

$$\Delta E = \frac{1}{2} m\left[\frac{1}{4}(|\underline{v}-\underline{w}|-|\underline{v}+\underline{w}|)^2+|\underline{v}-\underline{w}| \; |\underline{v}+\underline{w}|\sin^2 \frac{\theta_{cm}}{2} - w^2\right] \tag{3-1}$$

where θ_{cm} is the center of mass scattering angle. The energy loss is dependent upon the angle at which the test electron is scattered from its original trajectory. A relationship can also be derived which relates the center of mass scattering angle, θ_{cm}, to the laboratory scattering angle, θ_ℓ, which measures the actual deviation of the test particle from its original trajectory.

$$\cot\theta_\ell = \cot\theta_{cm} + \frac{|\underline{v}+\underline{w}|}{|\underline{v}-\underline{w}|} \cos\theta_{cm} \tag{3-2}$$

If the approximation is made that the test particle is energetic, i.e. $v \gg w$, the expression for the energy loss and the angle transformation can be simplified considerably, $\underline{v}+\underline{w} \approx \underline{v}-\underline{w}$, and

$$\theta_\ell = \frac{\theta_{cm}}{2} \quad . \tag{3-3}$$

Then, the energy loss of an energetic electron of velocity $\underline{v}$

after colliding with a plasma electron of velocity $\underline{w}$ is given by

$$\Delta E(\underline{v}, \underline{w}, \theta_\ell) = \frac{1}{2} m|\underline{v}-\underline{w}|^2 \sin^2\theta_\ell \tag{3-4}$$

where m is the electronic mass and θ_ℓ is the laboratory scattering angle.

The average energy loss per collision is obtained by averaging the typical loss $\Delta E(\underline{v}, \underline{w}, \theta_\ell)$ over the scattering angles θ_ℓ .

$$\Delta E(\underline{v}, \underline{w}) = \frac{1}{\sigma_t} \int_\theta \Delta E(\underline{v}, \underline{w}, \theta_\ell)\sigma(\theta_\ell)\, d\Omega_\ell \tag{3-5}$$

The weighting function is the Mott-scattering cross section*

$$\sigma(\theta_\ell)\ \Omega_\ell = \frac{1}{4}\left[\frac{e^2}{\frac{1}{2} m V^2}\right]^2\left[\frac{1}{\sin^4\theta_\ell} + \frac{1}{\cos^4\theta_\ell} - \frac{\cos[4\pi e^2/hV)\ell_n \tan\theta_\ell]}{\sin^2\theta_\ell \cos^2\theta_\ell}\right] 4\cos\theta_\ell d\,\Omega_\ell \tag{3-6}$$

where $V = |\underline{v}-\underline{w}|$

$d\Omega\ell = 2\pi \sin\theta_\ell d\theta_\ell$.

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*A comparison of the Mott and the Rutheford scattering cross-section is made in Appendix A .

The total cross section, σ_t is given by

$$\sigma_t = \int \sigma(\theta_\ell)\ \Omega_\ell$$

and the integration limits are from $\theta_{\ell_{min}}$ to $\theta_{\ell_{max}}$.

The result for $\Delta E(\underline{v}, \underline{w})$ is

$$\Delta E(\underline{v}, \underline{w}) = - \frac{2\pi}{\sigma_t} [\frac{1}{2} mV^2 (\frac{e^2}{\frac{1}{2} mV^2})^2][\ell_n \frac{\Lambda}{2} - \frac{1}{2}(\frac{\Lambda}{2})^{-2}$$

$$+ \ell_n(\sin\theta_{\ell_{max}}) + \ell_n(\cos^2 \theta_{\ell_{max}}) + \frac{1}{2} \tan^2\theta_{\ell_{max}}]$$

and $\Lambda = \frac{2}{\theta_{\ell_{min}}}$. (3-7)

The deBroglie wave length of the relative motion V is taken as the minimum impact parameter*.

$$b_{min} = \lambda = \frac{h}{m|\underline{v}-\underline{w}|} \qquad (3\text{-}8)$$

The maximum angle of deflection, $\theta_{\ell_{max}}$, will then be given by the scattering angle corresponding to b_{min} .

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*In section 3.4 this choice of minimum impact parameter is justified.

$$b_{min} = \frac{e^2}{\frac{1}{2}\, m|\underline{v}-\underline{w}|^2} \cot \theta_{\ell_{max}} \tag{3-9}$$

or

$$\theta_{\ell_{max}} = \cot^{-1}\left[\frac{h}{m|\underline{v}-\underline{w}|} \; \frac{1}{2} \; \frac{m}{e^2} |\underline{v}-\underline{w}|^2 \right] .$$

From the expression for $\theta_{\ell_{max}}$, $\sin \theta_{\ell_{max}}$, $\cos^2\theta_{\ell_{max}}$, and $\tan^2\theta_{\ell_{max}}$ can be found. Using these expressions the average energy loss per collision for velocities, $\overline{v}$ and $\overline{w}$, is given by

$$\Delta E(\underline{v},\underline{w}) = \frac{2\pi}{\sigma_t}\left[\frac{1}{2}\, mV^2 \left(\frac{e^2}{\frac{1}{2}\, mV^2}\right)^2\right]\left[\ell_n \Lambda - \frac{1}{2}\left(\frac{\Lambda}{2}\right)^{-2}\right.$$

$$\left. + \ell_n \frac{2e^2h^2V^2}{(h^2V^2+4e^4)^{3/2}} + 2\,\frac{e^4}{h^2V^2} \right] . \tag{3-10}$$

In a small time interval, δt, the number of collisions that the incident electron makes with plasma electrons having velocities in the range $d\underline{w}$ is given by

$$\sigma_t |\underline{v}-\underline{w}| \; \delta t \; f(\underline{w}) \; d\underline{w} \; .$$

The total average energy loss of the incident electron in a time, δt, due to many collisions with plasma electrons in the velocity range, $d\underline{w}$, is

$$\Delta E_{d\underline{w}} = \Delta E(\underline{v},\, \underline{w})\sigma_t|\underline{v}-\underline{w}|f(\underline{w})d\underline{w}\;\delta t \; . \tag{3-11}$$

Hence the rate of average energy loss becomes

$$\left(\frac{dE}{dt}\right)_{d\underline{w}} = \lim_{\delta t \to 0} \left(\frac{\Delta E}{\delta t}\right)_{d\underline{w}} .$$

The total energy loss rate for collisions with plasma electrons of all velocities is obtained by integrating over the plasma electron distribution function, thus

$$\left\langle \frac{dE}{dt} \right\rangle = \int_{\underline{w}} \left(\frac{dE}{dt}\right)_{d\underline{w}} f(\underline{w})\, d\underline{w} .$$

A Maxwellian distribution function is assumed.

$$f(\underline{w})\, d\underline{w} = \frac{n_e}{\pi^{3/2} w_t^3} \exp\left(-\frac{\underline{w} \cdot \underline{w}}{w_t^2}\right) 2\pi w^2 \sin\phi\, d\phi\, dw \qquad (3\text{-}12)$$

where ϕ is the velocity space polar angle* coordinate for a spherical coordinate system and

$$w_t^2 = \frac{2kT}{m} ,$$

T is the plasma electron temperature, m is the electronic mass, and

$$n_e = \int_{\underline{w}} f(\underline{w})\, d\underline{w} .$$

Carrying out the integration formally the result is obtained in terms of the dimensionless speeds α, β, defined below:

$$\frac{dE}{dt}(\beta) = \frac{-4\pi e^4}{m} \frac{n_e}{w_t} \frac{4}{\sqrt{\pi}} \left[G_1(\beta) + G_2(\beta) + \frac{2e^4}{w_t^2\, h^2} G_3(\beta)\right] \qquad (3\text{-}13)$$

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*See Fig. B in Appendix B .

$$\text{where} \quad \alpha = \frac{w}{w_t}$$

$$\beta = \frac{v}{w_t} \ ,$$

and

$$G_1(\beta) = \int_0^\infty \phi_1(\alpha,\beta)\alpha^2 e^{-\alpha^2}\, d\alpha \qquad (3\text{-}14a)$$

$$G_2(\beta) = \int_0^\infty \phi_2(\alpha,\beta)\alpha^2 e^{-\alpha^2}\, d\alpha \qquad (3\text{-}14b)$$

$$G_3(\beta) = \int_0^\infty \phi_3(\alpha,\beta)\alpha^2 e^{-\alpha^2}\, d\alpha \qquad (3\text{-}14c)$$

where the ϕ_i's* are the appropriate averages of the terms in equation (3-10) .

$$\phi_1(\alpha,\beta) = \frac{1}{2}\int_{-1}^{1} \frac{[\ln\Lambda(\alpha,\beta,\mu) - \frac{1}{2}\Lambda^{-2}]}{\sqrt{\beta^2 + 2\alpha\,\beta\,\mu + \alpha^2}}\, d\mu \ , \qquad (3\text{-}15a)$$

where $\mu = -\cos\phi$;

$$\phi_2(\alpha,\beta) = \frac{1}{2\alpha\beta}\Big\{(\beta+\alpha)\ln \frac{2e^2(\beta+\alpha)^2}{hw_t[(\beta+\alpha)^2 + \frac{4e^4}{h^2 w_t^2}]^{3/2}}$$

$$-|\beta-\alpha|\ln \frac{2e^2(\beta-\alpha)^2}{hw_t[(\beta-\alpha)^2 + \frac{4e^4}{h^2 w_t^2}]^{3/2}}$$

$$+ (\beta+\alpha) - |\beta-\alpha| - \frac{6e^2}{hw_t}\gamma\Big\} \ , \qquad (3\text{-}15b)$$

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*The ϕ_i functions are not to be confused with the polar angle ϕ .

$$\text{where}\quad \gamma = \tan^{-1} \frac{4e^2/hw_t}{|\beta^2 - \alpha^2| + \frac{4e^4}{h^2 w_t^2}} \cdot \begin{cases} \alpha, & \alpha < \beta \\ \beta, & \alpha > \beta \end{cases} ;$$

$$\text{and}\qquad \phi_3(\alpha,\beta) = \begin{cases} \frac{1}{\beta}\ \frac{1}{\beta^2 - \alpha^2}, & \alpha < \beta \\ \frac{1}{\alpha}\ \frac{1}{\alpha^2 - \beta^2}, & \alpha > \beta \end{cases} . \qquad (3\text{-}15c)$$

The integrations over velocity space angles, ϕ , ψ have been carried out to yield the ϕ_i functions (See Appendix B) .

Many simplifications are possible in these expressions because of the high speeds of the incoming electrons and the form of the assumed plasma electron distribution function.

$$v >> w_t$$

The Maxwellian function $w^2 f(w)$ is a strong exponential which decreases rapidly with w^2 ; i.e., there are relatively few particles with speeds much greater than the thermal speed.* In particular the effects of collisions with particles of speed greater than $\sqrt{10}\ w_t$ can be neglected without an appreciable error. (This criterion is certainly ample, since at a speed of $\sqrt{10}\ w_t$ the probability of finding a plasma electron has decreased by exp(−8). In view of the argument presented above, the G functions can be approximated by

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*In fact, the same argument applies to any distribution function which does not contain a large population at high velocities.

setting the upper limit of the integrals at $\alpha_{max} = \dfrac{\sqrt{10}\, w_t}{w_t} = \sqrt{10}$.

For this calculation, $\beta \gtrsim \sqrt{1{,}000}$; therefore the condition $\beta^2 >> \alpha^2$ is satisfied and the integrands can be simplified accordingly. For $\beta^2 >> \alpha^2$,

$$\phi_2 \gtrsim \frac{1}{2}\frac{1}{\beta}\left(2\ln \frac{2e^2\ \beta^2}{hw_t\left[\beta^2 + \frac{4e^4}{h^2w_t^2}\right]^{3/2}}\right.$$

$$\left. + 2 - \frac{6e^2}{hw_t}\tan^{-1}\left[\frac{4e^2/hw_t}{\beta^2 + 4(e^2/hw_t)^2}\,\alpha\right]\right) , \qquad (3\text{-}16a)$$

and $$\phi_3 \gtrsim \frac{1}{\beta^3} \qquad (3\text{-}16b)$$

The evaluation of $G_1(\beta)$ requires a value for the minimum laboratory scattering angle in the collision process. This is equivalent to considering the maximum allowable impact parameter. Ordinarily, following Cohen, Spitzer and Routly[26], the maximum impact parameter would be the Debye radius. Physically the reason for this is that the charged particles in the plasma have adjusted themselves so that the Coulomb potential of the plasma electron has been shielded at distances greater than the Debye length. The result is that the incident electron feels no net potential unless it passes within a Debye length of the plasma electron. This procedure will give fairly accurate results when the incident electron has a velocity approximately equal to the plasma electron thermal velocity. However, in the present case, where the incident particle velocity exceeds the thermal velocity, the maximum impact parameter must be increased to include correctly the energy lost in polarizing the plasma with the resultant excitation of plasma oscillations. The effect is completely analogous to the Cerenkov effect.

A fast electron can transfer energy to a polarizable medium over a characteristic length which will be taken as the effective maximum impact parameter. The characteristic time of interaction of the fast electron with the plasma is $\frac{b_{max}}{v}$, where b_{max} is the maximum impact parameter. This time must be of the order of one period of plasma oscillation in order for energy to be transferred to the plasma through polarization. Thus

$$\frac{b}{v} \approx \frac{1}{\omega_p}$$

where ω_p = plasma "frequency". Since

$$\omega_p \approx \frac{w_t}{\lambda_d} ,$$

it follows that

$$b_{max} \approx \frac{v}{w_t} \lambda_d = \beta \lambda_d . \qquad (3\text{-}17)$$

The Debye length must be increased approximately by the factor β to include polarization effects.

The minimum scattering angle can now be written in terms of the maximum impact parameter calculated above. Thus

$$\theta_{\ell,min} = \frac{2}{w_t^3}\sqrt{\frac{32\pi}{m^3}}\,(e^3 n_e{}^{1/2})\,\frac{1}{\beta^2 + 2\alpha\beta\mu + \alpha^2}\,\frac{1}{\beta}\,. \tag{3-18}$$

The scattering angles, $\theta_{\ell,min}$, $\theta_{\ell,max}$, are functions of the relative velocity, $|\underline{v}-\underline{w}|$, and hence depend upon, $\mu = \cos(\underline{v}, \underline{w})$. Performing the integrations over angles, the result is

$$\phi_1 = \frac{1}{2\alpha\beta}\,[(\beta+\alpha)\ln A(\beta+\alpha)^2 - |\beta-\alpha|\ln A|\beta-\alpha|^2$$

$$-2(\beta+\alpha-|\beta-\alpha|)]\;, \tag{3-19}$$

where $A = \frac{1}{2}\,w_t^3\sqrt{\frac{m^3}{32\pi}}\,\frac{\beta}{e^3\,n_e{}^{1/2}}$. Again using the condition that $\beta^2 >> \alpha^2$,

$$\phi_1 \doteq \frac{1}{\beta}\,(\ln A\;\beta^2 - 2) \tag{3-20}$$

and

$$G_1 = \frac{\sqrt{\pi}}{4}\;\frac{1}{\beta}\,(\ln A\beta^2 - 2)\;. \tag{3-21}$$

For an energetic test electron, small terms can be neglected; then,

$$\frac{e^4}{h^2 w_t^2 \beta^2} << 1\;,$$

and

$$|\ln A\;\beta^2| >> 1\;.$$

The final result for the average energy loss rate becomes

$$<\frac{dE}{dt}> = -\,\frac{4\pi e^4}{mv}\,n_e\;\ln[\frac{1}{2}\sqrt{\frac{m^3}{8\pi}}\;\frac{v^2}{ehn_e{}^{1/2}}]\;. \tag{3-22}$$

If the energy loss in a distance, dx , is much smaller than the incident energy, E , the energy loss rate can be transformed as follows,

$$\frac{1}{v}\,\frac{dE}{dt} = \frac{dE}{dx}$$

and

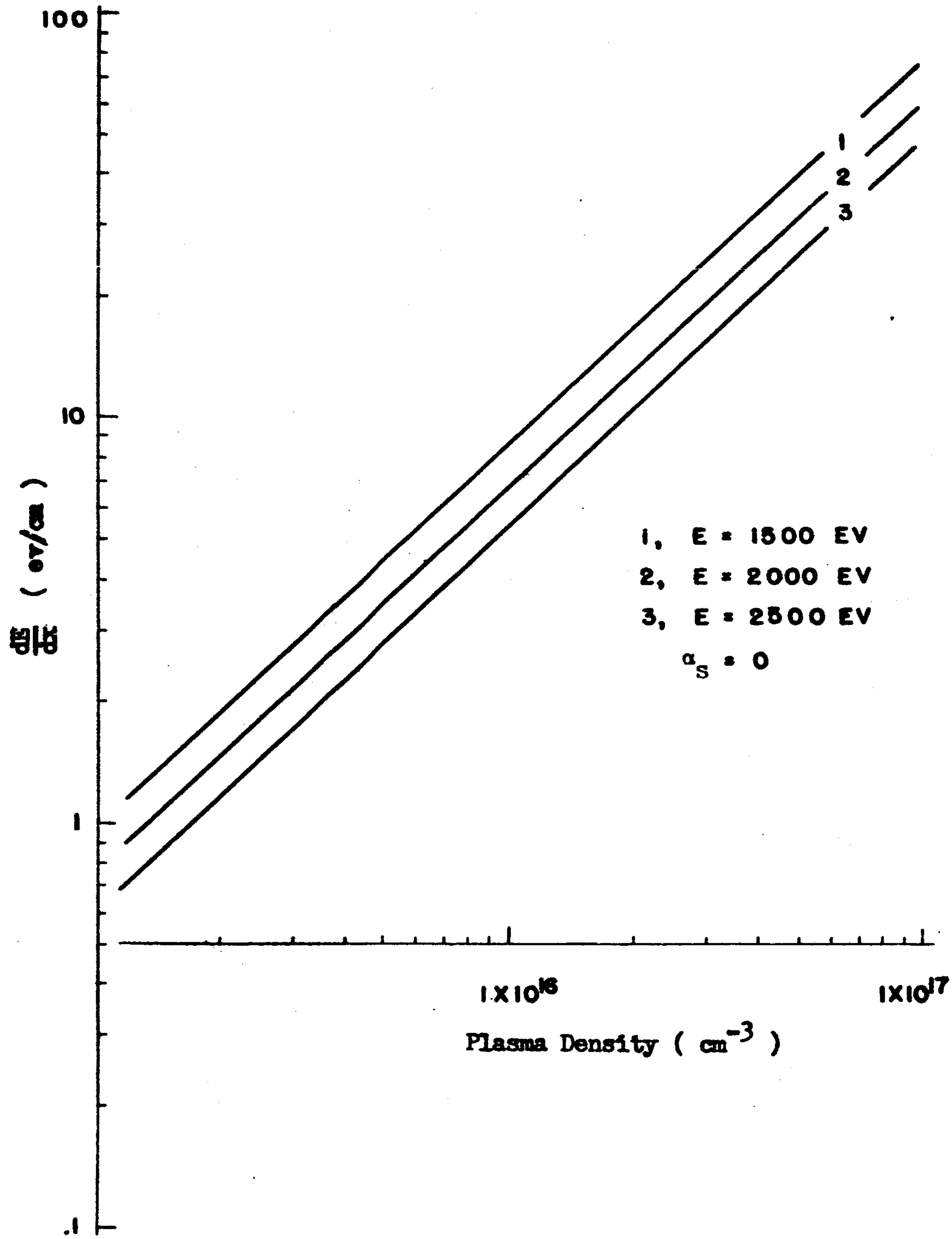
$$\left\langle \frac{dE}{dx} \right\rangle = -\,\frac{2\pi e^4}{E}\,n_e\,\ln\!\left[\frac{1}{\sqrt{8\pi^2}}\,\frac{E}{\hbar\omega_p}\right] \qquad (3\text{-}23)$$

where $E = \frac{1}{2}\,mV^2$ and $\omega_p = \left(\frac{4\pi ne^2}{m}\right)^{1/2}$. A plot of $\left\langle \frac{dE}{dx} \right\rangle$ is shown in Fig. 3.1 .

The result derived above for the average energy loss rate using a collisional theory through the ad hoc extension of the maximum impact parameter is identical to the results derived by various authors[7-16] . The results of a few different authors are shown in Table I . The expression for $\left\langle \frac{dE}{dx} \right\rangle$ shown in the table differ only in the choice of minimum impact parameter.

$\langle dE/dx \rangle$ For an Energetic Test Electron
Traversing a Plasma

Fig. 3.1



100
10
1
.1
$\frac{dE}{dx}$ (ev/cm)
1
2
3
1, E = 1500 EV
2, E = 2000 EV
3, E = 2500 EV
α_S = 0
1X10[16]
1X10[17]
Plasma Density (cm^{-3})

AUTHOR	dE/dx
Neufeld and Ritchie[8].........	$\frac{2\pi e^4 n}{E} \ln(2.246\sqrt{\frac{2}{m}}\ \frac{E^{3/2}}{e^2\omega_p})$
Klimontovich[9]	$\frac{2\pi e^4 n}{E} \ln(2\sqrt{\frac{1}{m}}\ \frac{E^{3/2}}{e^2\omega_p})$
Akhiezer[10]..................	$\frac{2\pi e^4 n}{E} \ln(2.246\sqrt{\frac{2}{m}}\ \frac{E^{3/2}}{e^2\omega_p})$
Pines and Bohm[12]............	$\frac{2\pi e^4 n}{E} \ln(2\sqrt{\frac{1}{m}}\ \frac{E^{3/2}}{e^2\omega_p})$
Larkin[13]....................	$\frac{2\pi e^4 n}{E} \ln(1.36\ \frac{E}{\hbar\ \omega_p})$
Smith (Present work).........	$\frac{2\pi e^4 n}{E} \ln(\frac{1}{\sqrt{8\pi^2}}\ \frac{E}{\hbar\omega_p})$

where n = plasma electron density

E = test particle energy

$\omega_p = \sqrt{\frac{4\pi n e^2}{m}}$ = plasma "frequency"

m = electronic mass

Table I.

3.3 Average Energy Loss Rate in a Translating Plasma

The average energy loss was calculated for a translating, Maxwellian plasma. The result indicates that the energy loss rate in a translating plasma is given by a function, $S(\alpha)_S$, of a dimensionless parameter, α_S , times the energy loss rate in a stationary plasma.

$$\frac{dE}{dx} (\alpha_S) = \frac{dE}{dx} (0) \; S \; (\alpha)_S \tag{3-24}$$

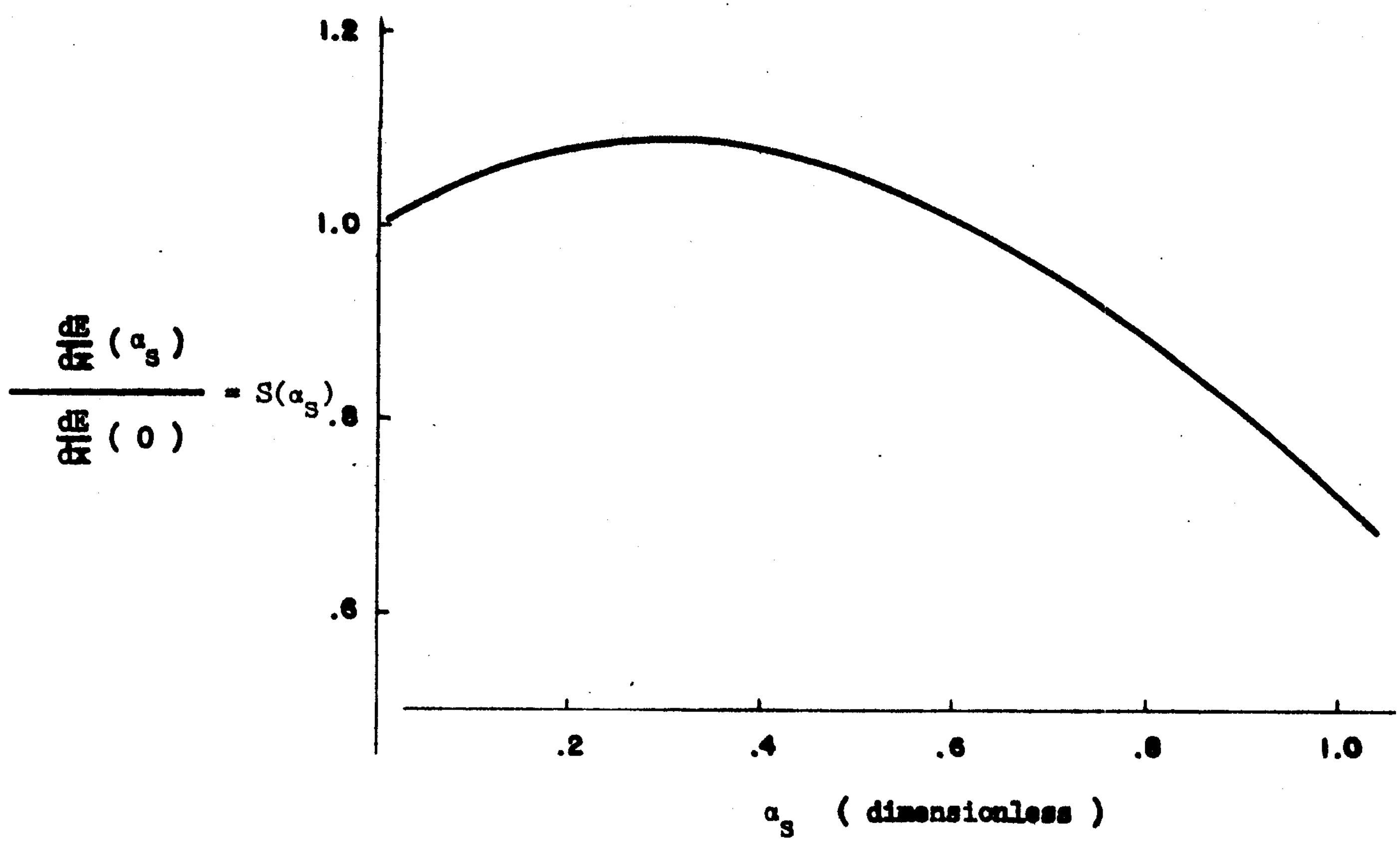
The parameter, α_S , is the dimensionless translation speed of the plasma.

$$\alpha_S = \frac{V_S}{w_t}$$

V_S is the translation velocity of the plasma and w_t is the plasma thermal velocity. The function $S(\alpha_S)$ is shown in Fig. 3.2 . For a complete derivation see Appendix C .

$S(\alpha_S)$ For the Effect of the Plasma Translation on the Energy Loss

Fig. 3.2



1.2
1.0
.8
.6
$\frac{\frac{dE}{dx}(\alpha_s)}{\frac{dE}{dx}(0)} = S(\alpha_s)$
.2
.4
.6
.8
1.0
α_s (dimensionless)

3.4 Minimum Impact Parameter

If the electron test particle speed is large enough to excite collective, longitudinal oscillations in the plasma the minimum impact parameter must be taken to be the deBroglie wavelength of the test electron. This can be shown to be true for usual laboratory plasmas. The classical distance of closest approach, which is the impact parameter corresponding to a right angle deflection in the center of mass coordinate system, is given by $b = e^2/E$, where E is the relative energy of approach. The deBroglie wave length of the test electron of momentum mv is $\lambdabar = \hbar/mv$. The ratio of these two lengths is given by

$$\frac{b}{\lambdabar} = \frac{e^2}{\hbar c}\,\frac{c}{v} = \frac{\alpha}{\beta}$$

where α is the fine structure constant, 1/137, and $\beta = v/c$. The ratio is unity for $\beta \sim .01$ which corresponds to an energy of the test electron of about 20 electron volts. The criterion for longitudinal plasma oscillations to be induced by the test electron is that $v \gg w_{th}$, where w_{th} is the thermal speed of the plasma electrons. Specifically, let a lower limit be chosen so that $v/w_{th} \gtrsim 10$. Then it follows that for $v/w_{th} \gtrsim 10$, the following inequality must be satisfied for the test electron energy,

$$E \gtrsim 100\ kT\ ,$$

since $v/w_{th} = \left(\frac{E}{kT}\right)^{1/2}$. Laboratory plasmas might cover a range of temperatures from 1/4 ev for Cesium up to 4 or 5 ev for a shock heated plasma. Using these values as typical, it is evident that, for the test particle to excite longitudinal oscillations in the plasma, it must have an energy of 25 to 500 electron volts or greater. At these energies the deBroglie wave length certainly exceeds the classical distance of closest approach.

3.5 Validity of the Classical Collision Model

From the Heisenberg uncertainty principle, a relationship can be derived for the minimum uncertainty in the impact parameter in a Coulomb force field collision.

$$\left(\frac{\Delta p}{p}\right)_{min} \approx \frac{137\ \beta}{2\ Z\ z}$$

In order for classical collision theory to hold, we would require that the uncertainty in the impact parameter be negligible so that an impact parameter *per se* would have physical meaning. The minimum uncertainty in p is small when

$$\frac{2\ Z\ z}{137\ \beta} >> 1 \ .$$

In terms of the collision diameter, b (classical distance of closest approach), and the rationalized deBroglie wave length for relative motion, this expression can be rewritten, for $z = Z = 1$

$$\frac{b}{\lambda\!\!\!^{-}} >> 1 \ .$$

This result would seem to indicate that a collision between an energetic test electron and a plasma electron could not be treated using classical collision theory. (In fact, it was shown that, for energetic test electrons, $\frac{b}{\lambdabar} << 1$.) Fortunately, W. Gordon[27] has obtained an exact quantum mechanical treatment of the Coulomb scattering problem and it is identical[3] with the classical Rutherford result for all values of the parameter $\frac{2\ Z\ z}{137\ \beta}$. E. J. Williams[28] has pointed out a reason for the plausibility of this agreement. From a dimensional argument he has shown that if the force law of the collision varies as r^n , then the scattering cross section will vary as h^{4+2n} , where h is Planck's constant. Only for an inverse-square force law interaction does the dependence on h vanish and certainly the vanishing of h is a necessary condition for equality between a classical and a quantum mechanical theory.

CHAPTER IV

DISTRIBUTION OF ENERGY LOSSES

When energetic electrons traverse a layer, x , of plasma, they will lose on the average an energy $< \Delta >$. However there will be a statistical fluctuation of energy losses so that on traversal of the slab of plasma an initially monoenergetic electron beam will emerge with a distribution of energy losses which will be denoted by $f(x, \Delta)$, where f is normalized so that

$$\int_{\Delta} f(x,\Delta)d\Delta = 1 .$$

Let $w(E, \varepsilon)$ be the probability per unit path length that an incident electron of energy E will suffer a loss ε . Throughout this calculation it will be assumed that $\varepsilon << E$ so that E will be taken to be constant, E_o .

A kinetic equation for $f(x,\Delta)$ is obtained by equating the change in f along a path length, dx , to the change produced by collisions. The collision integral expresses the difference in the number of particles which acquire, due to collisional losses, an energy, E , and the number of particles which leave the volume in energy space. This is essentially a steady flow Boltzmann equation with no externally applied forces.

$$\frac{\partial f}{\partial x} = \int_0^\infty w(E_o, \varepsilon)[f(x,\Delta- \varepsilon)-f(x,\Delta)]\, d\varepsilon \qquad (4\text{-}1)$$

The probability of losing an energy greater than E_o is zero* so that the upper limit of the integral can be increased to infinity.

Since the kinetic equation does not contain the independent variables x and Δ explicitly, a solution can be obtained by using the Laplace transform technique. The transform of the kinetic equation is given by,

$$\frac{\partial \varphi(x,s)}{\partial x} = - \varphi(x,s) \int_0^\infty w(\varepsilon)[1 - e^{-s\varepsilon}]\, d\varepsilon \quad . \qquad (4\text{-}2)$$

The initial condition on $f(x, \Delta)$ is that $f(0, \Delta) = \delta(0)$, i.e., the beam of incident particles is monoenergetic at the entrance to the slab. This inital condition gives $\varphi(0, s) = 1$, so that equation 4-2 can be integrated to obtain

$$\varphi(x,s) = e^{-x \int_0^\infty w(\varepsilon)[1-e^{-s\varepsilon}]d\varepsilon} \quad .$$

Finally, by substituting this expression into the inverse transform for $f(x, \Delta)$, the formal solution is given by

$$f(x,\Delta) = \frac{1}{2\pi i} \int_{-i\infty+\sigma}^{i\infty+\sigma} e^{-s\Delta - x \int_0^\infty w(\varepsilon)[1-e^{-s\varepsilon}]d\varepsilon}\, ds \qquad (4\text{-}3)$$

where s is the Laplace transform variable. The inverse transform is carried out along a line parallel to the imaginary axis of the complex plane and shifted to the right by $\sigma > 0$.

*In this theory energy gain mechanisms are not allowed.

To complete the solution for f the integral can be simplified for various ranges of ε. Let ε_{min} be a small energy loss so that $\varepsilon_{min} \ll \varepsilon_{max}$ where ε_{max} is the largest possible loss. It is assumed that only those values of s are important for which

$$s\varepsilon_{min} \ll 1 \; ; \; s\varepsilon_{max} \gg 1 \; .$$

Another energy ε_1 will be introduced such that $\varepsilon_1 > \varepsilon_{min}$ and $s\varepsilon_1 \ll 1$. The integral over ε can be split into two integrals with limits from 0 to ε_1 and from ε_1 to ∞.

$$\int_0^\infty w(\varepsilon)[1-e^{-s\varepsilon}]d\varepsilon = \int_0^\infty \varepsilon w(\ )d\ \ + \int_{\varepsilon_1}^\infty w(\varepsilon)[1-e^{-s\varepsilon}]d\varepsilon \; ,$$

where the first integral has been approximated by the relation

$$e^{-s\varepsilon} \approx 1-s\varepsilon \quad \text{for} \quad s\varepsilon_1 \ll 1 \; .$$

The probability of energy loss, $w(\varepsilon)$, can be derived from the cross section for energy loss[3]. The differential cross section for an energy loss, ε, is given by

$$d\sigma = \frac{\pi e^4}{E} \frac{d\varepsilon}{\varepsilon^2} \qquad (4\text{-}4)$$

where ε is the energy loss of the test electron of energy, E, . It is assumed that the test electron is considerably more energetic than the plasma electron. The probability of losing an energy between ε and $\varepsilon+d\varepsilon$ per collision is

$$\frac{d\sigma}{\sigma_t} = \frac{1}{\sigma_t}\,\frac{\pi e^4}{E}\,\frac{d\varepsilon}{\varepsilon^2} \tag{4-4}$$

where σ_t is the total cross section for energy loss and is given by

$$\sigma_t = \frac{\pi e^4}{E}\int_{\varepsilon_{min}}^{\varepsilon_{max}} \varepsilon^{-2}\, d\varepsilon \;.$$

The number of collisions per unit length traversed by the test electron is $\sigma_t n_e$; so that the probability of losing an energy between ε and $\varepsilon + d\varepsilon$ per unit length of travel is

$$w(\varepsilon)d\varepsilon = \frac{\pi e^4 n_e}{E}\,\frac{d\varepsilon}{\varepsilon^2} \quad . \tag{4-5}$$

Using this expression to continue the solution for f , integrals of the following type must be evaluated,

$$\int_{\varepsilon_1}^{\infty} \frac{1-e^{-s\varepsilon}}{\varepsilon^2}\, d\varepsilon = \frac{1}{\varepsilon_1}(1-e^{-s\varepsilon_1}) + s\int_{\varepsilon_1}^{\infty} \frac{e^{-s\varepsilon}}{\varepsilon}\, d\varepsilon \;.$$

It was stipulated that $s\varepsilon_1 \ll 1$ so that the integral is simplified to

$$\int_{\varepsilon_1}^{\infty} \frac{1}{\varepsilon^2}(1-e^{-s\varepsilon})d\varepsilon = s[1 + \int_{s\varepsilon_1}^{\infty} \frac{e^{-u}}{u}\, du]$$

The integral on the RHS of this equation can be rewritten as follows.

$$\int_{s\varepsilon_1}^{\infty} \frac{e^{-u}}{u}\, du = \int_{s\varepsilon_1}^{1} \frac{e^{-u}-1}{u}\, du + \int_{s\varepsilon_1}^{1} \frac{du}{u} + \int_{1}^{\infty} \frac{e^{-u}}{u}\, du$$

$$= \int_{0}^{1} \frac{e^{-u}-1}{u}\, du - \int_{0}^{s\varepsilon_1} \frac{e^{-u}-1}{u}\, du + \int_{s\varepsilon_1}^{1} \frac{du}{u} + \int_{1}^{\infty} \frac{e^{-u}}{u}\, du$$

for

$$s\varepsilon_1 << 1 \; ; \int_0^{s\varepsilon_1} \frac{e^{-u}-1}{u} du \sim s\varepsilon_1$$

then

$$\frac{1}{s} \int_{\varepsilon_1}^{\infty} \frac{1-e^{-s\varepsilon}}{\varepsilon^2} d\varepsilon = 1 + \int_{s\varepsilon_1}^{1} \frac{du}{u} + \int_0^1 \frac{e^{-u}-1}{u} du + \int_1^{\infty} \frac{e^{-u}}{u} du$$

$$= 1 - \ln s\varepsilon_1 - C$$

where $C = 0.5777 \ldots$ is Euler's constant. The result is

$$\int_{\varepsilon_1}^{\infty} \frac{1-e^{-s\varepsilon}}{\varepsilon^2} d\varepsilon = s(1-C-\ln s\varepsilon_1) \qquad (4\text{-}6)$$

The second integral to be evaluated can be integrated directly to yield

$$s \int_0^{\varepsilon_1} \varepsilon w(\varepsilon) d\varepsilon = s \frac{\pi e^4 n}{E} \ln \frac{\varepsilon_1}{\varepsilon_{min}} , \qquad (4\text{-}7)$$

where ε_{min} is given by the minimum energy loss corresponding to the minimum scattering angle θ_{min}. The complete expression can now be written as

$$x \int_0^{\infty} w(\varepsilon)[1-e^{-s\varepsilon}]d\varepsilon = \xi s[1-C-\ln s\varepsilon_{min}] .$$

Performing the integrations, the solution for $f(x,\Delta)$ reduces to

$$f(x,\Delta) = \frac{1}{\xi} \phi(\lambda) \qquad (4\text{-}8)$$

where

$$\phi(\lambda) = \frac{1}{2\pi i} \int_{-i\infty+\sigma}^{i\infty+\sigma} e^{v \ln v + \lambda v} dv \qquad (4\text{-}9)$$

and $$v = \xi s$$

$$\lambda = \frac{\Delta - \xi(\ln \xi/\varepsilon_{min} + 1 - c)}{\xi} \qquad (4\text{-}10)$$

$$\xi = x \frac{\pi e^4 n_e}{E} \qquad (4\text{-}11)$$

and $c = 0.577$ is Euler's constant.

The distribution of energy losses, f , is found to be $\frac{1}{\xi}$ times a universal function of a dimensionless parameter λ . The function $\phi(\lambda)$, shown in Fig. 4.1 , is given by Landau in his paper[29]. The function has a maximum at $\lambda = -.05$ so that the most probably energy loss is given by (see Fig. 4.2).

$$\Delta_o = \xi(\ln \frac{\xi}{\varepsilon_{min}} + 0.37) \ . \qquad (4\text{-}12)$$

A plot of $f(x,\Delta)$ as a function of plasma electron density, electron beam energy, and distance traversed is shown in Fig. 4.3 , and Fig. 4.4 .

The solution which has been obtained describes the interaction of an initially monoenergetic beam of energetic test electrons traversing plasma. It is important to note that the spreading in energy of the test particles does not depend upon a temperature for the test particles since, in fact, they have been assumed to enter with zero temperature (delta function initial condition). The spreading in energy emerges from the solution of the Boltzmann equation and is due physically to the binary collisions which were assumed as the model.

In each binary collision there is a possible variation of energy losses due to collisions at different angles. Most of the collisions occur at very small scattering angles so that the spread in energy losses should be small and, indeed, the solution indicates that this is the case (see Fig. 4.4).

The probability of an energy loss between Δ and $\Delta + d\Delta$ is given by

$$f(x,\Delta)d\Delta = \frac{1}{\xi}\varphi(\lambda)d\Delta \ ,$$

$$d\lambda = \frac{1}{\xi} d\Delta \ ,$$

so that $f(x,\Delta)d\Delta = \varphi(\lambda)d\lambda$.

In terms of $\Delta-\Delta_o$ we can write

$$\lambda = \frac{\Delta-\Delta_o - .05\xi}{\xi} \tag{4-13}$$

and for conditions considered in this paper

$$.05\xi \ll \Delta_o \ ,$$

so that

$$\lambda = \frac{\Delta-\Delta_o}{\xi} \ ,$$

and then

$$f(x,\Delta)d\Delta = \phi\left(\frac{\Delta-\Delta_o}{\xi}\right)d\left(\frac{\Delta-\Delta_o}{\xi}\right) \ . \tag{4-14}$$

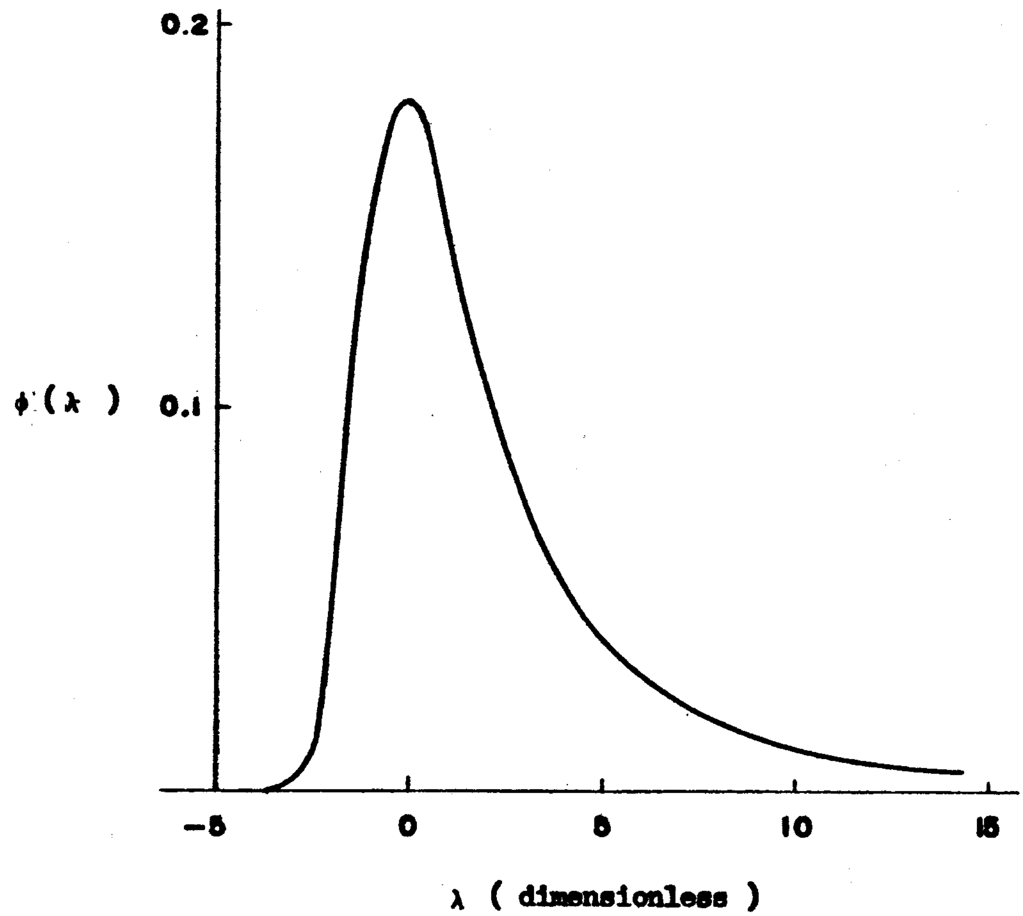
The minimum energy loss in an average collision is given in terms of the minimum scattering angle by the following

$$\varepsilon_{min} = \frac{1}{2} m \langle|\underline{v}-\underline{w}|^2\rangle \sin^2\theta_{min} \ . \tag{4-15}$$

For an energetic test electron the average relative energy can be

Landau's Universal Function, $\phi(\lambda)$

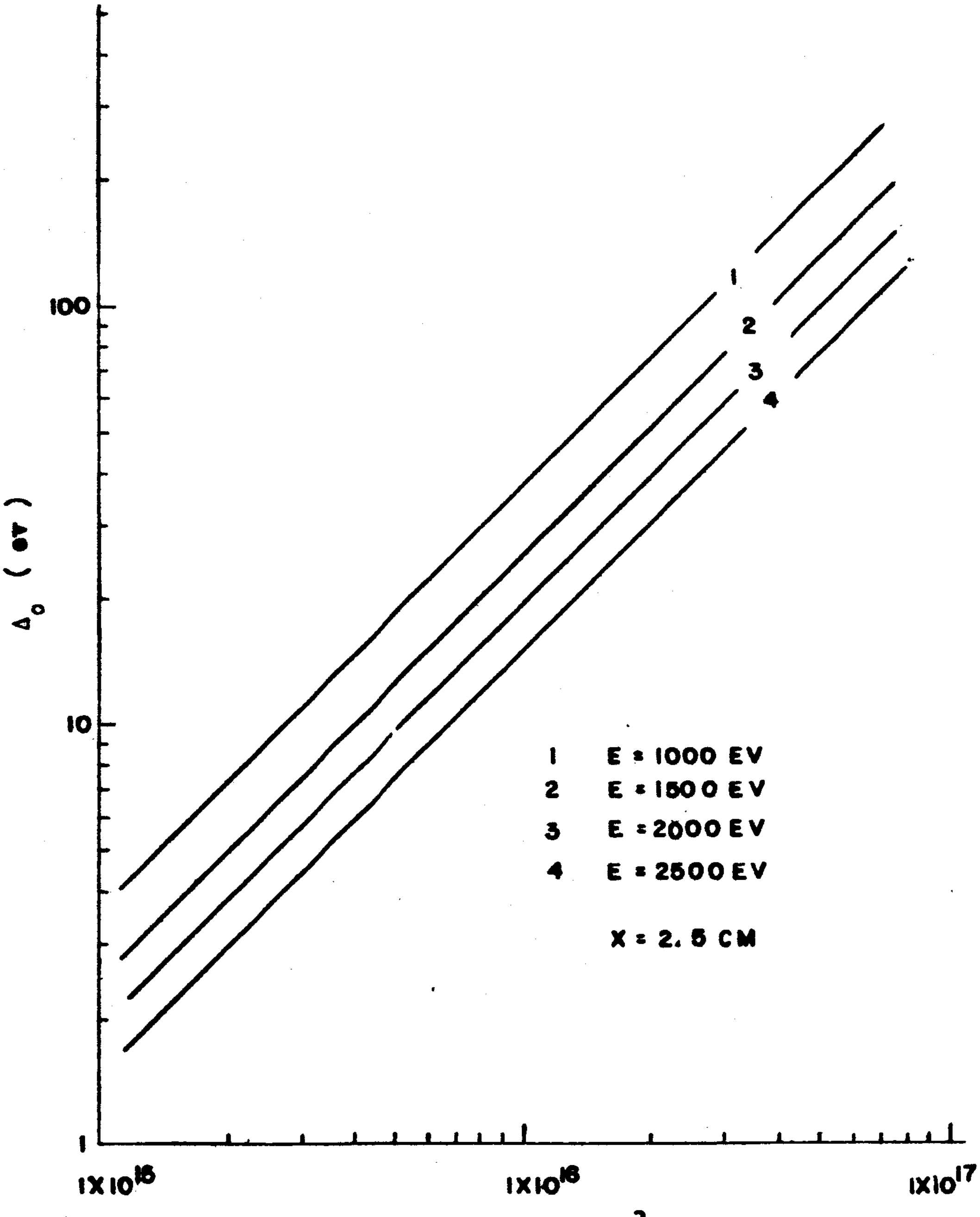
Fig. 4.1



0.2
0.1
ϕ(λ)
−5
0
5
10
15
λ (dimensionless)

Most Probable Energy Loss as a Function
of Beam Energy and Plasma Density

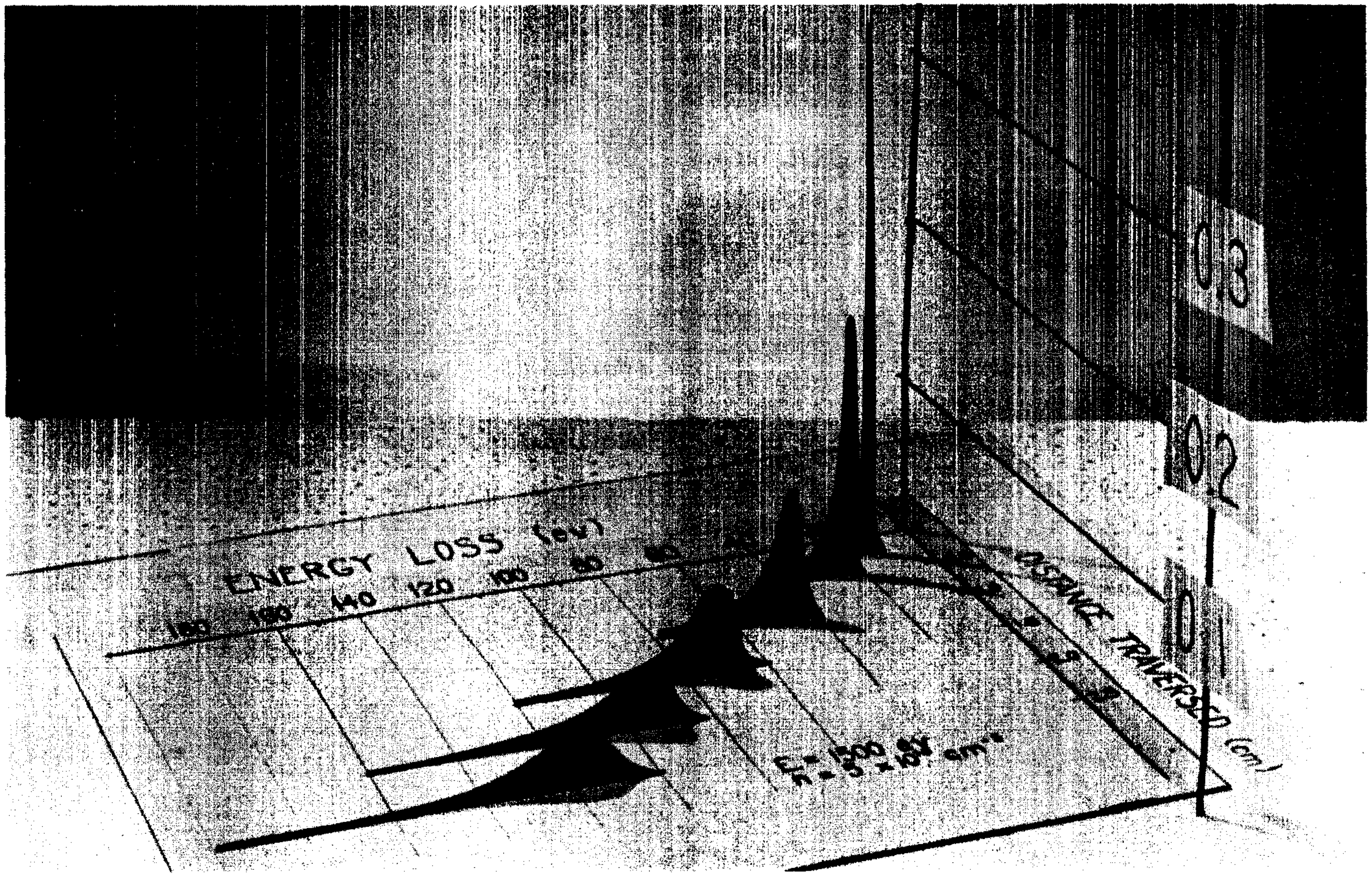
Fig. 4.2



100
10
1
Δ_o (eV)
1X10^15
1X10^16
1X10^17
Plasma Density (cm^-3)
1
2
3
4
1 E = 1000 EV
2 E = 1500 EV
3 E = 2000 EV
4 E = 2500 EV
X = 2.5 CM

Energy Relaxation of an Energetic Test
Electron Beam in a Plasma

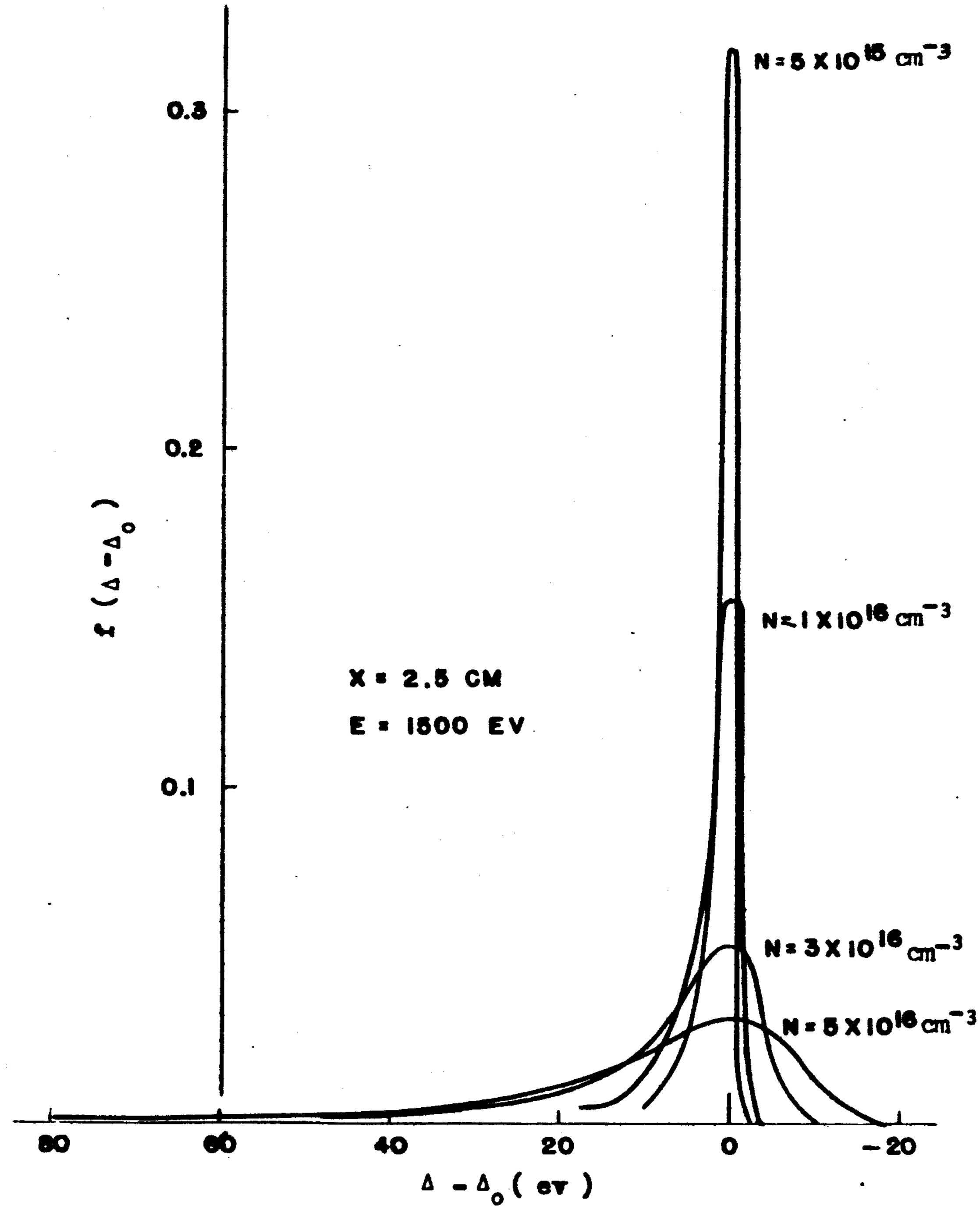
Fig. 4.3



ENERGY LOSS (eV)
180 160 140 120 100 80 60
DISTANCE TRAVERSED (cm)
0.3
0.2
0.1

Distribution of Energy Losses as a Function of Electron Density for 1500 ev Test Electrons

Fig. 4.4



N = 5 X 10^15 cm^-3
N ≈ 1 X 10^16 cm^-3
N = 3 X 10^16 cm^-3
N = 5 X 10^16 cm^-3
X = 2.5 CM
E = 1500 EV
f (Δ - Δo)
0.3
0.2
0.1
80
60
40
20
0
-20
Δ - Δo (ev)

approximated as

$$\frac{1}{2} m <|\underline{v}-\underline{w}|^2> \stackrel{\sim}{\sim} E \quad ,$$

where E is the energy of the test particle only. Using small angle approximations for $\sin^2\theta_{min}$,

$$\sin^2\theta_{min} \stackrel{\sim}{\sim} \left(\frac{2}{\Lambda}\right)^2 \quad , \qquad (4\text{-}16)$$

so that

$$\varepsilon_{min} = \frac{4E}{\Lambda^2} \quad .$$

Inserting the expression for Λ found previously,

$$\Lambda^2 = \frac{E^3}{4\pi e^6 n} \quad ,$$

and

$$\varepsilon_{min} = \frac{16\pi e^6}{E^2} n \quad .$$

The ratio of $\frac{\xi}{\varepsilon_{min}}$ is a quantity of importance and it is given by

$$\frac{\xi}{\varepsilon_{min}} = \frac{1}{16} \frac{E}{e^2} x \quad .$$

Typically the values for the various parameters are E = 1500 ev, $n = 5x10^{16}$, $\xi = 5.73$ and Δ_o = 72 ev, so that the statement that $.05\xi << \Delta_o$ is in general valid, verifying that

$$f(x,\Delta) = \frac{1}{\xi}\varphi\left(\frac{\Delta-\Delta_o}{\xi}\right) \quad .$$

Assumptions were made in obtaining the solution that

$$s\varepsilon_{min} << 1 \; ; \; s\varepsilon_{max} >> 1 \quad .$$

In the expression for $\varphi(\lambda)$, the variable of integration is significant in the region $v \stackrel{\sim}{\sim} 1$. The assumptions that were made are equivalent to the conditions that

$$\xi >> \varepsilon_{min}$$

$$\xi << \varepsilon_{max} \quad , \quad \text{since} \quad v = s\xi \ .$$

The first condition is satisfied easily since typical values for ξ/ε_{min} are $\sim 10^9$. This result if feasible because the observed energies ξ are the accumulation of numerous multiple scatterings of extremely small energy losses ε_i . The results of this calculation are limited by the second condition imposed above, $\xi << \varepsilon_{max}$. Since the collisions of the energetic test electron result in small energy losses, $\varepsilon_{max} << E_o$, where ε_{max} corresponds to the energy loss in an average collision at the maximum scattering angle, $\theta_{\ell_{max}}$.

$$\theta_{\ell_{max}} = \cot^{-1} hE/mve^2 \ ,$$

and
$$\varepsilon_{max} = \frac{1}{2} m <|\underline{v}-\underline{w}|^2> \sin^2\theta_{max} \ ,$$

$$\varepsilon_{max} \stackrel{\sim}{\sim} E(\tan^{-1} mve^2/hE)^2 \ ,$$

so that
$$\xi << E(\tan^{-1} mve^2/hE)^2 \ . \qquad (4\text{-}17)$$

In order to understand the restriction imposed on the solution by equn. (4-17), the inequality is rewritten in terms of average energy loss and the definition of ξ . For a small energy loss,

$$\Delta E = 2\pi e^4 \frac{nx}{E} \ln \frac{1}{\sqrt{8\pi^2}} \frac{E}{\hbar w p} \ ,$$

where x is the distance traversed through the plasma.

$$\xi = \pi e^4 \frac{nx}{E}$$

Then,

$$\xi = \frac{\Delta E}{E} \frac{1}{2\ell n \dfrac{1}{\sqrt{8\pi^2}} \dfrac{E}{\hbar \omega_p}} \tag{4-18}$$

The fact that the test electron is energetic implies that

$$\frac{2e^2}{hv} << 1 ,$$

and using a small angle approximation,

$$\tan^{-1}(\frac{mve^2}{hE}) = \tan^{-1}(\frac{2e^2}{hv}) \approx \frac{2e^2}{hv} .$$

Then the condition in eqn. (4-17) becomes,

$$\xi << E(\frac{2e^2}{hv})^2 ,$$

or

$$\frac{\Delta E}{E} << 8 \frac{e^4}{h^2 v^2} \ell n \frac{1}{\sqrt{8\pi^2}} \frac{E}{\hbar \omega_p} . \tag{4-19}$$

This inequality imposes a modest restriction on the relative energy loss. The stipulation was that

$$\frac{e^2}{h^2 v^2} << 1 .$$

Taking a reasonable valve,

$$\frac{e^2}{h^2 v^2} \lesssim .1$$

and

$$\frac{e^4}{h^2 v^2} \lesssim .01 .$$

The logarithm term is of the order 10, 20 that the inequality is

$$\frac{\Delta E}{E} << .8$$

or

$$\frac{\Delta E}{E} \lesssim \frac{1}{10} .8 = 8\% . \qquad (4\text{-}20)$$

The solution for $f(\Delta)$ should be valid, conservatively, for energy losses less than 9 per cent of the initial beam energy.

CHAPTER V

TRANSMISSION OF THE ELECTRON BEAM

When an energetic test electron traverses a slab of dense plasma, the test electron experiences a large number of small angle scatterings resulting in a net angular deviation from its original trajectory. Since the individual scatterings are at small angles, $\theta_i \ll 1$, the problem can be treated statistically in a reasonably simple manner*.[7] For small scattering angles, multiple scattering theory predicts an accumulated deflection angle, θ, distributed about $\theta=0$ according to a Gaussian law. θ is measured from the entrance plane of the beam. Thus

$$p(\theta)\, d\theta = \text{const.} \exp(-\frac{\theta^2}{\langle\theta^2\rangle})\, d\theta$$

where $p(\theta)\, d\theta$ is the probability of realizing a net deflection angle between θ and $\theta+d\theta$, and $\langle\theta^2\rangle$ is the average squared, accumulated deflection angle. This is equivalent to the statement that a beam of independently interacting test particles will have an emergent current distribution (provided the energy change of the particles is small, i.e., $\Delta E \ll E_o$) given by

$$j(\theta)d\theta = \text{const.} \exp(-\frac{\theta^2}{\langle\theta^2\rangle})\, d\theta \qquad (5\text{-}1)$$

*See Appendix D for a simple treatment of elementary, multiple scattering theory.

where $j d\theta$ is the beam current emerging between an angle of θ and $\theta + d\theta$. The value of the constant can be determined by equating the total current emerging at all angles to the incident beam current.

$$j_{bo} = \int_0^{\pi/2} \text{const. } \exp(-\frac{\theta^2}{\langle\theta^2\rangle})\, d\theta \tag{5-2}$$

Solving for the constant,

$$j(\theta)\, d\theta = \frac{J_o}{(\frac{\pi}{2}\langle\theta^2\rangle)^{1/2} \operatorname{erf}(\frac{\pi}{2}/\langle\theta^2\rangle^{1/2})} \exp(-\frac{\theta^2}{\langle\theta^2\rangle})\, d\theta \ , \tag{5-3}$$

where J_o is the incident beam current and

$$\int_0^{\pi/2} e^{-\frac{\theta^2}{\langle\theta^2\rangle}}\, d\theta = (\frac{\pi}{2}\langle\theta^2\rangle)^{1/2} \operatorname{erf}[\frac{\pi}{2}(\frac{1}{\langle\theta^2\rangle})^{1/2}] \ .$$

The transmission of the beam through an exit aperture defined by θ_o (see Fig. 5.1) is calculated by taking the ratio of the beam current exiting through θ_o to the total incident beam current, J_o . A diagram of the transmission is shown in Fig. 5.1 .

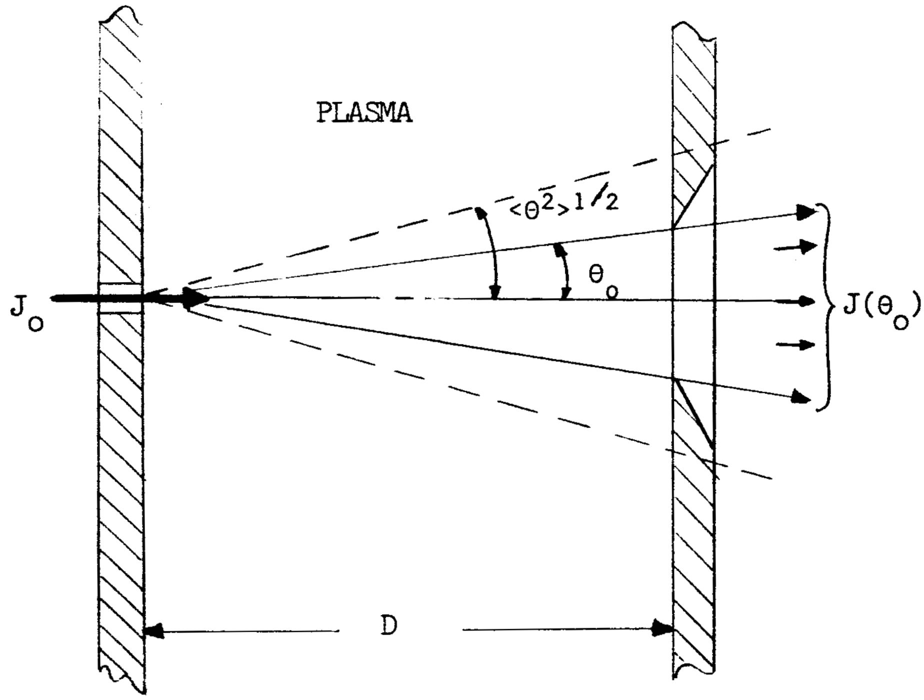


Fig. 5.1 Diagram of Beam Transmission

Then the transmission is given by

$$T = \frac{J(\theta_o)}{J_o} = \frac{\mathrm{erf}(\theta_o/<\theta^2>^{1/2})}{\mathrm{erf}(\pi/2/<\theta^2>^{1/2})} \tag{5-4}$$

where the total current J_o is given by

$$J_o = J(\pi/2) \ . \tag{5-5}$$

A plot of the transmission is shown in Fig. 5.3 .

The average squared, accumulated deflection angle, $<\Theta^2>$ is equal to the average squared deflection angle per collision times the total number of collisions. Thus,

$$<\Theta^2> = P <\theta^2> \ , \tag{5-6}$$

where P is the average number of collisions in a single traversal of the plasma slab and $<\theta^2>$ is the average squared angle of deflection per collision. For small angle collisions $P = \sigma_t Dn$, here σ_t is the total scattering cross section and is given in terms of the impact parameter by

$$\sigma_t = \int_{x_{min}}^{x_{max}} 2\pi \, x \, dx \, .$$

D is the distance traversed, and n is the plasma electron density. Performing the integration,

$$P = 2\pi \, Dn \, \frac{1}{2}(x_{max}^2 - x_{min}^2) \, . \qquad (5\text{-}7)$$

The maximum impact parameter is the dominant term and $x_{max}^2 = (\lambda_D \frac{v}{w_t})^2$, where λ_D is the Debye radius, v is the test electron speed and w_t is the most probable plasma electron speed. Substituting the expression for λ_D abd w_t ,

$$P = DE/2e^2 \, , \qquad (5\text{-}8)$$

where $E = \frac{1}{2} mv^2$.

The value for $<\theta^2>$ per collision is calculated by averaging θ^2 over the scattering cross section for θ . The Rutherford scattering cross section is used.*

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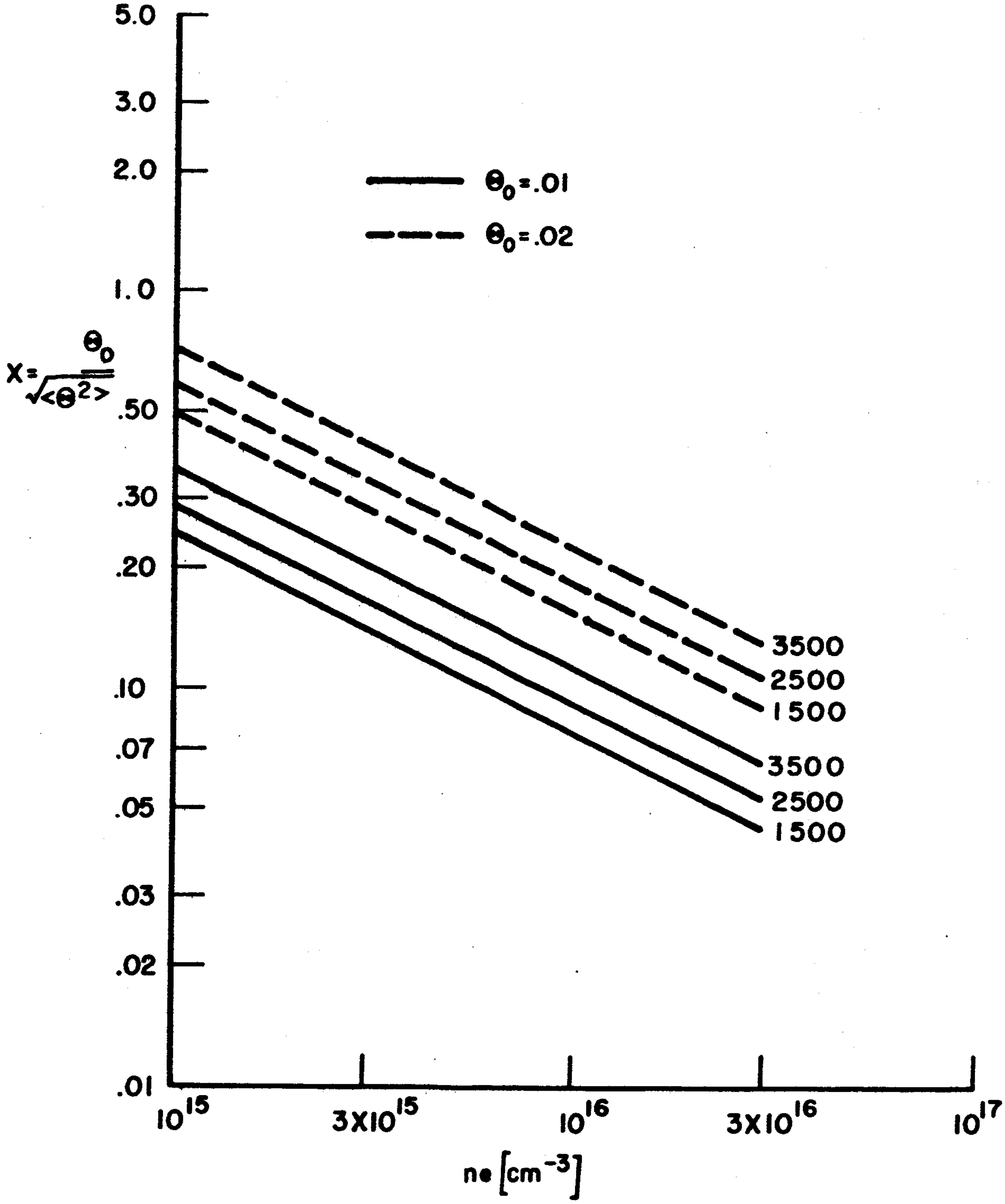
*In the limit of small angle collisions, the Rutherford cross-section is identical to the Mott scattering cross-section.

Argument of the Error Function

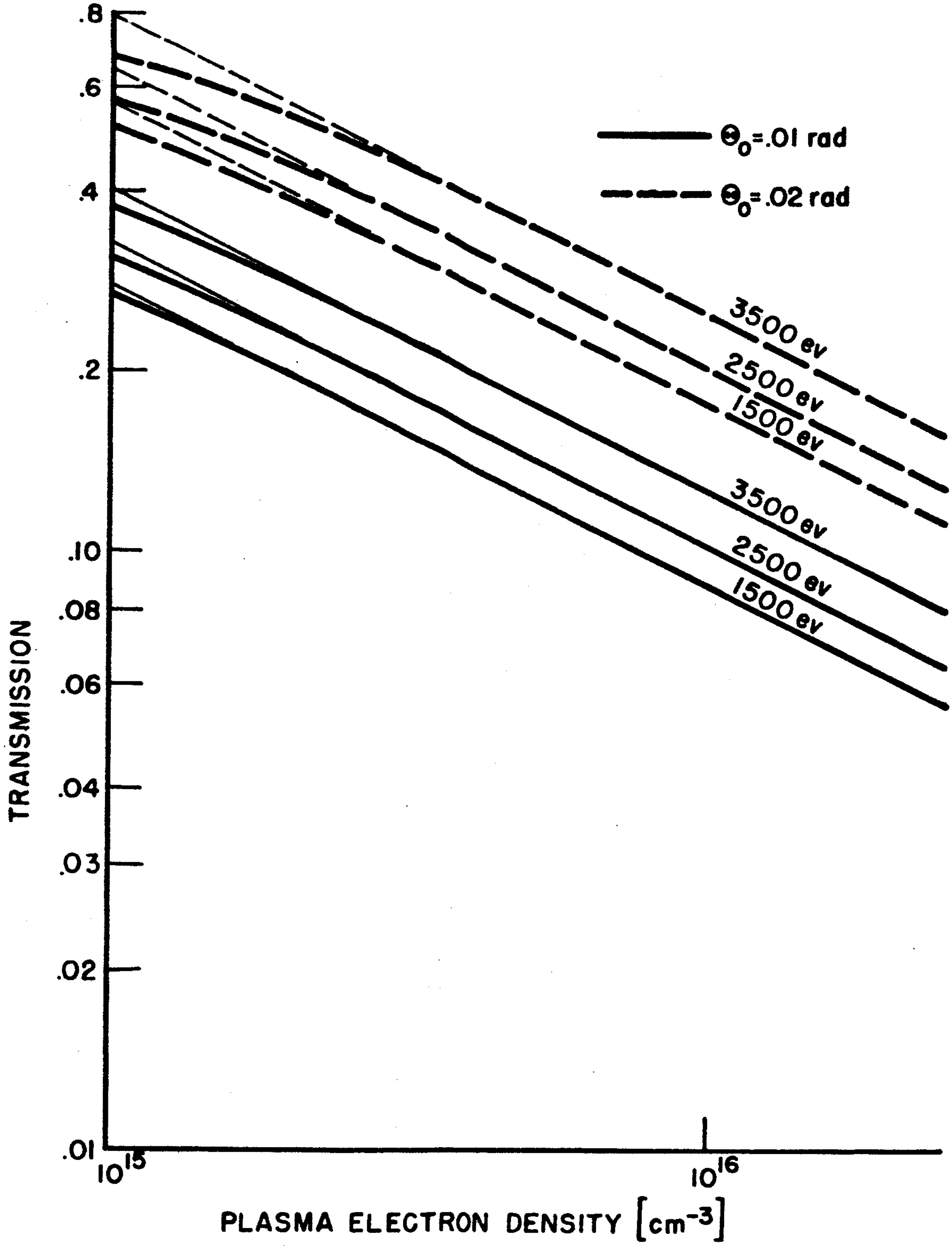
Fig. 5. 2



5.0
3.0
2.0
1.0
.50
.30
.20
.10
.07
.05
.03
.02
.01
$X = \frac{\Theta_0}{\sqrt{\langle\Theta^2\rangle}}$
$\Theta_0 = .01$
$\Theta_0 = .02$
3500
2500
1500
3500
2500
1500
10^{15}
$3X10^{15}$
10^{16}
$3X10^{16}$
10^{17}
ne $[cm^{-3}]$

Beam Transmission

Fig. 5. 3



TRANSMISSION
.8
.6
.4
.2
.10
.08
.06
.04
.03
.02
.01
Θ_0=.01 rad
Θ_0=.02 rad
3500 ev
2500 ev
1500 ev
3500 ev
2500 ev
1500 ev
10^{15}
10^{16}
PLASMA ELECTRON DENSITY [cm^{-3}]

$$<\theta^2> = \frac{1}{\sigma_t}\int_{\theta_{min}}^{\theta_{max}} \theta^2 d\sigma(\theta) = \frac{1}{\sigma_t}\frac{2\pi e^4}{E^2}\int_{\theta_{min}}^{\theta_{max}} \frac{\theta^2}{\sin^4\theta}\sin\theta\cos\theta d\theta \tag{5-9}$$

Making a small angle approximation, $\sin\theta \approx \theta$, the result is

$$<\theta^2> \approx \frac{1}{\sigma_t}\frac{2\pi e^4}{E^2}\int_{\theta_{min}}^{\theta_{max}} \frac{d\theta}{\theta} = \frac{1}{\sigma_t}\frac{2\pi e^4}{E^2}\ln\frac{\theta_{max}\Lambda}{2} \tag{5-10}$$

where $\theta_{max} = \frac{2e^2}{hv}$

$\theta_{min} \equiv 2/\Lambda$.

The total cross section* is related to the average number of collisions in one traversal of the slab by

$$\sigma_t = \frac{P}{nD} = \frac{1}{2}\frac{E}{n_e^2} \; , \tag{5-11}$$

and

$$<\theta^2> = \frac{4\pi e^6}{E^3} n\ln(\frac{1}{2\pi\sqrt{2}} \frac{E}{\hbar\omega_p}) \; . \tag{5-12}$$

*The total cross section for an inverse-square law interaction potential as derived above has a peculiar dependence on energy and density. In fact, the dependence is the reciprocal of what one would expect, $\sigma_t \sim E/n$. The reason for this peculiar dependence is that the total Coulomb interaction cross section owes its magnitude to the small angle encounters and these are more numerous for a tenuous plasma and a high velocity test particle.

Finally, the average squared, accumulated deflection angle is given by

$$\langle\theta^2\rangle = P \langle\theta^2\rangle = \frac{2\pi e^4}{E^2} nD\ln\left(\frac{1}{2\pi\sqrt{2}} \frac{E}{\hbar\omega_p} \right. . \qquad (5\text{-}13)$$

A calculation of the scattering due to ions proceeds in complete analogy to the electron scattering calculation. The result is that the ions contribute another equal contribution to the scattering of the beam. Since the multiple scattering profile is Gaussian in shape, the effect of the ions is included in the beam transmission result by increasing the average squared, accumulated deflection angle by the factor two.

The average squared, accumulated deflection angle is related to the average energy loss rate in a simple manner. Using the expressions derived for $\langle\frac{dE}{dx}\rangle$ and for $\langle\theta^2\rangle$, the following relation is obtained,

$$\frac{\langle\frac{dE}{dx}\rangle}{\langle\theta^2\rangle} = \frac{E}{2D} , \qquad (5\text{-}14)$$

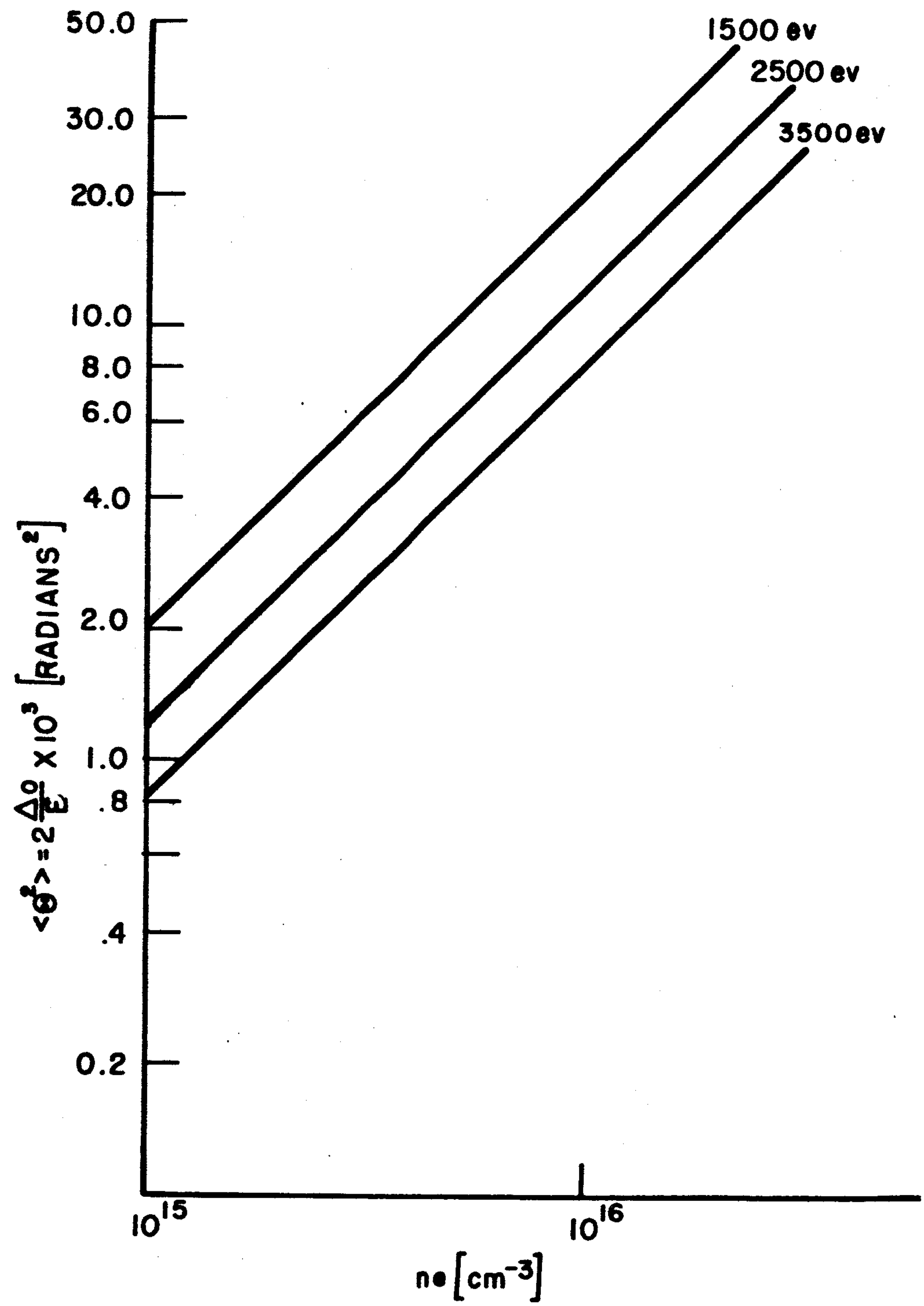
where the scattering due to the ions is included in $\langle\theta^2\rangle$. This result is independent of plasma electron density and depends only upon the energy of the test electron beam and the distance traversed through the plasma. A similar expression can be derived in terms of the most probable energy loss.

$$\langle\theta^2\rangle = 2 \frac{\Delta_o}{E} \qquad (5\text{-}15)$$

A plot of this last expression is given in Fig. 5.4 .

Average Squared, Accumulated Deflection
Angle

Fig. 5. 4



50.0
30.0
20.0
10.0
8.0
6.0
4.0
2.0
1.0
.8
.4
0.2
1500 ev
2500 ev
3500 ev
$<\Theta^2> = 2\frac{\Delta_0}{E} \times 10^3$ [RADIANS^2]
10^{15}
10^{16}
ne [cm^{-3}]

APPENDIX A

ELECTRON-ELECTRON COLLISION ASPECTS

A.1 Comparison of Mott and Rutherford Cross Sections

The calculation of the energy loss rate, dE/dx, yields the same result for $v \gg w_t$ regardless of the choice of cross-section. The reason for this is evident from Fig. A.1. The difference in dE/dx using the Rutherford cross-section as compared to that using the correct, Mott cross-section will arise in the computation of $\langle \Delta E \rangle$ per collision. We can write $\langle \Delta E \rangle$ as follows.

$$\langle \Delta E \rangle_i = \frac{2\pi e^4}{E_o \sigma_t} \int_{\theta min}^{\theta_{max}} F_i(\theta_\ell) d\theta_\ell \qquad \text{(A-1)}$$

$F_i(\theta_\ell)$ refers to either the Rutherford cross-section times the solid angle function, or the Mott cross-section times the solid angle function.

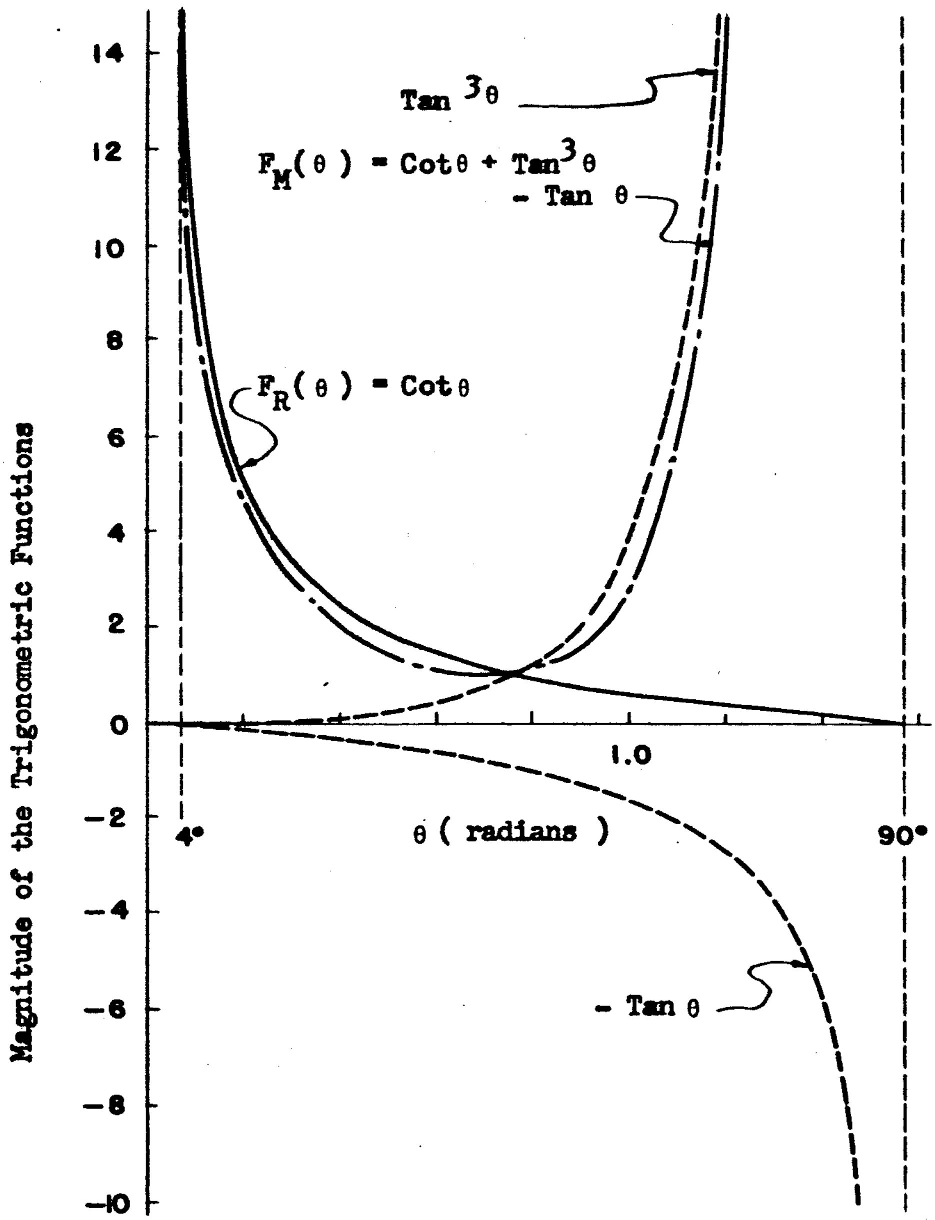
$$F_M(\theta_\ell) = \cot\theta_\ell - \tan^3\theta_\ell - \tan\theta_\ell \qquad \text{(A-2a)}$$

$$F_R(\theta_\ell) = \cot\theta_\ell \qquad \text{(A-2b)}$$

It is evident from Fig. A.1 that for small θ_{max}, $F_M = F_R$ and hence either cross-section will yield the same result. As the velocity of the test electron decreases the maximum scattering angle, θ_{max}, increases because the minimum impact parameter, which is the

Comparison of the Rutherford and the Mott
Scattering Function

Fig. A.1



Tan $^3\theta$
$F_M(\theta) = \text{Cot}\,\theta + \text{Tan}^3\theta - \text{Tan}\,\theta$
$F_R(\theta) = \text{Cot}\,\theta$
$-$ Tan θ
14
12
10
8
6
4
2
0
−2
−4
−6
−8
−10
1.0
4°
90°
θ (radians)
Magnitude of the Trigonometric Functions

deBroglie wave length, decreases. Therefore, for lower energy test electrons it would be incorrect to use the Rutherford cross-section.

A. 2 Maximum Scattering Angle

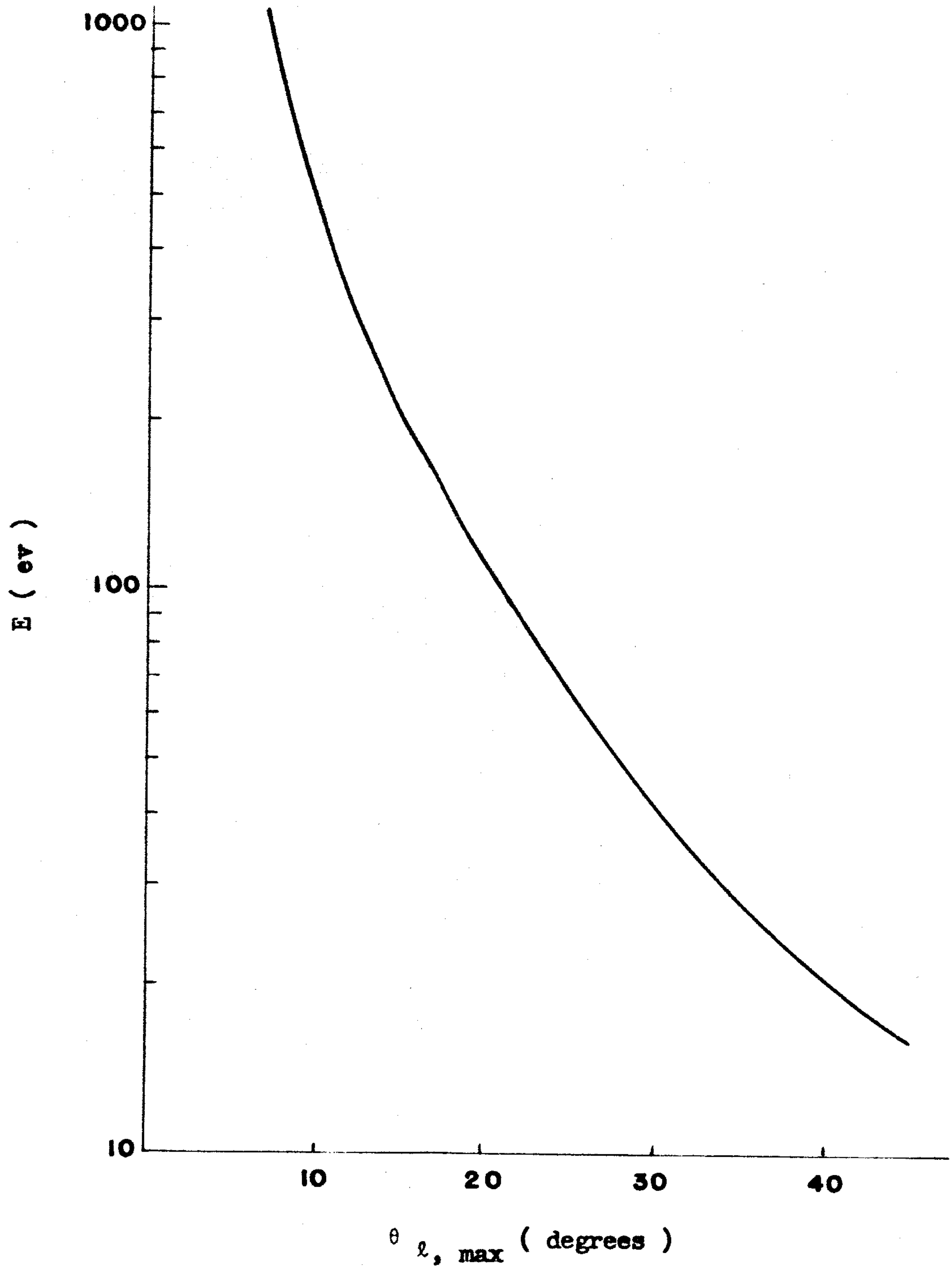
The maximum scattering angle for a single collision event is found to be of the order of 4° for the situation described in this paper, $v >> w_t$. From the collision kinematic equations, the maximum laboratory scattering angle is related to the minimum impact parameter, $\theta_{max} = \tan^{-1} \frac{2e^2}{hv} \approx \tan^{-1} \alpha/\beta$, where α is the fine structure constant 1/137, and $\beta = \frac{v}{c}$. A typical value for β is $\beta = .1$ and $\therefore\ \theta_{max} \approx \tan^{-1} \frac{1}{137 x .1} \approx 4^{\circ}$. The maximum scattering angle is limited to a small value because of the limit placed on the minimum impact parameter, namely that it be equal to the deBroglie wave length of the test electron.

Because of the smallness of the maximum scattering angle, the results obtained using the correct, Mott scattering cross-section are identical to those obtained using the Rutherford cross-section. For lower test particle energies the maximum scattering angle can become much larger, e.g., $\beta = .01$, $\theta_{max} = 36^{\circ}$, and for these angles the effect of the extra terms introduced by the Mott cross-section begins to be appreciable.

A plot of the maximum scattering angle as a function of test electron energy is given in Fig. A.2 .

Maximum Scattering Angle Per Collision for Electron-Electron Scattering with Relative Energy Approach, E .

Fig. A. 2



1000
100
10
E (ev)
10
20
30
40
θ ℓ, max (degrees)

The maximum scattering angle of interest to this calculation is approximately 37°, corresponding to the lower limit of approximately 25 ev set on the test particle energy so that collective oscillations would be excited. From Fig. A.2 it is evident that any calculations made for test particle energy losses from binary collisions in the regime $v >> w_t$ would be insensitive to the choice of scattering cross-section.

A.3 Total Cross Section for Electron-Electron Scattering

The total cross section for a Coulomb law interaction, in general, diverges due to the long range nature of the Coulomb potential. In a plasma there are natural, physical cut-offs which remove the divergence of the total scattering cross section. The larger scattering angles are restricted by quantum mechanical effects for an energetic test particle, in that the minimum impact parameter is limited to the deBroglie wave length of the energetic test particle. The small angle scatterings are limited by the natural screening of the microscopic electric fields which exist within a plasma. The electric fields about a charged particle in a plasma are, on the average, screened Coulomb fields and they extend approximately only as far as a Debye radius.

In terms of the scattering angles, the total cross section is

$$\sigma_t = \frac{2\pi e^4}{E^2} \int_{\theta_{min}}^{\theta_{max}} \left(\frac{1}{\sin^4\theta} + \frac{1}{\cos^4\theta} - \frac{1}{\sin^2\theta\cos^2\theta}\right)\sin\theta\cos\theta\,d\theta \quad \text{(A-3)}$$

The integrand is the Mott scattering function for energetic electrons and E is the energy associated with the relative velocity of the colliding electrons. It was shown in Sec. A.1 that only the first term of the integrand is significant for energetic particles, so that the expression reduces to the Rutherford formula

$$\sigma_t \approx \frac{2\pi e^4}{E^2} \int_{\theta_{min}}^{\theta_{max}} \frac{1}{\sin^4\theta} \sin\theta\cos\theta d\theta \qquad \text{(A-4)}$$

The total cross section can also be written in terms of the impact parameters. Written in this manner, the physical meaning of the cross section as a collision area is more apparent.

$$\sigma_t = \int_{P_{min}}^{P_{max}} 2\pi p dp \qquad \text{(A-5)}$$

Either form will yield the same numerical result since there is a one to one correspondence between the impact parameter and the scattering angle. Performing the integration over impact parameters we find that

$$\sigma_t = \pi(P_{max}^2 - P_{min}^2) .$$

Substituting the expressions, for P_{max} and P_{min} ,

$$P_{max} = \lambda_D \frac{v}{w_t}$$

$$P_{min} << P_{max}$$

$$\therefore \quad \sigma_t = \frac{1}{2} \frac{E}{ne^2} .$$

APPENDIX B

CALCULATION OF THE ϕ_i FUNCTIONS IN VELOCITY SPACE

The expression for the energy loss rate must be integrated over the velocity space angles ψ, and ϕ defined in the diagram below.

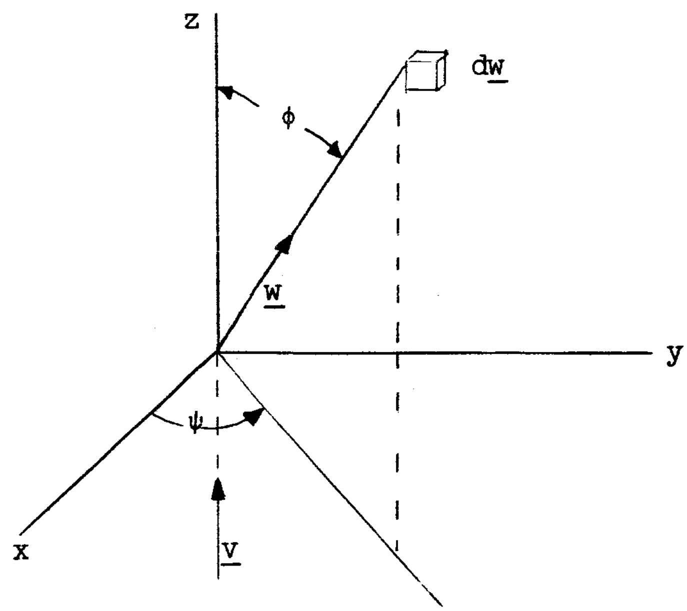


Fig.B.1 Velocity Space Coordinate System

The z-direction is oriented in the direction of the incoming electron velocity, $\underline{v}$. The distribution function $f(\underline{w})$ is assumed to be isotropic in velocity space. This condition is manifested in the result that all functions are independent of the angle ψ, and the integration over ψ yields 2π. Integration over the polar angle, ϕ, involves integrals of the following type.

$$\int \frac{\ell n\Lambda}{|v - w|} \sin\phi d\phi \tag{B-1}$$

where $\ell n\Lambda = \ell n\ A(\beta^2 + 2\alpha\beta\mu + \alpha^2)$.

The relative speed is rewritten,

$$|\underline{v} - \underline{w}| = \sqrt{\beta^2 + 2\alpha\beta\mu + \alpha^2} \ ,$$

where $\mu = -\cos\phi$.

Integrating by parts after making the substitution

$$u = \ell n\ A(\beta^2 + 2\alpha\beta\mu + \alpha^2) \ ,$$

and $$dv = \frac{du}{\sqrt{\beta^2 + 2\alpha\beta\mu + \alpha^2}} \ ,$$

$$\int \frac{\ell n\ A \sin\phi d\phi}{|\underline{v} - \underline{w}|} = \frac{1}{w_t}\left[\frac{1}{\alpha\beta}\sqrt{\beta^2 + 2\alpha\beta\mu + \alpha^2}\ \ell n\ A(\beta^2 + 2\alpha\beta\mu + \alpha^2) - \frac{2}{\alpha\beta}(\beta + \alpha - |\beta - \alpha|)\right]_{-1}^{1}$$

$$= \frac{1}{\alpha\beta w_t}[(\beta+\alpha)\ell nA(\beta+\alpha)^2 - |\beta-\alpha|\ell nA|\beta-\alpha|^2 - 2(\beta+\alpha-|\beta-\alpha|)] \ . \qquad \text{(B-2)}$$

Another integral of interest is

$$\int_0^\pi \frac{\sin\phi d\phi}{|\underline{v} - \underline{w}|^3} . \qquad \text{(B-3)}$$

Making the substitution $\mu = -\cos\phi$,

$$\int_0^\pi \frac{\sin\phi d\phi}{|\underline{v} - \underline{w}|^3} = \int_{-1}^{1} \frac{d\mu}{(v^2 + w^2 + 2vw\mu)^{3/2}}$$

$$= -\frac{1}{vw}(v^2 + w^2 + 2vw\mu)^{-1/2}\Big|_{-1}^{1}$$

$$= 2. \begin{cases} \frac{1}{v}\frac{1}{v^2 - w^2}, & w < v \\ \frac{1}{w}\frac{1}{w^2 - v^2}, & w > v \end{cases} \tag{B-4}$$

A more complicated integral of interest involves the logarithmic function,

$$\int_0^{\pi} \frac{1}{|\underline{v} - \underline{w}|} \ln \frac{2e^2h^2|\underline{v} - \underline{w}|^2}{(\hbar^2|\underline{v} - \underline{w}|^2 + 4e^4)^{3/2}} \sin\phi d\phi \quad . \tag{B-5}$$

We write the integrand in terms of $\mu = -\cos\phi$ and then make another substitution

$$u = v^2 + w^2 + 2vw\mu \ .$$

Then we have transformed to a simpler integral,

$$\frac{1}{2vw} \int u^{-1/2} \ln \frac{2e^2h^2u}{[h^2u + 4e^4]^{3/2}} du \ . \tag{B-6}$$

Integrating by parts with the following arrangement,

let $x = \ln \dfrac{2e^2h^2u}{[h^2u + 4e^4]^{3/2}}$, and $dy = u^{-1/2} du$,

then, $\int ydx = - \int [\dfrac{u^{1/2}}{u + 4e^4/h^2} - \dfrac{8e^4}{h^2}\dfrac{u^{-1/2}}{u + 4e^4/h^2}] du$.

With the substitution $v = u^{1/2}$, the $\int ydx$ is transformed into trigonometric forms, so that

$$\int \frac{u^{1/2}}{u + 4e^4/h^2} du = 2[u^{1/2} - \frac{2e^2}{h}\tan^{-1}\frac{u^{1/2}}{2e^2/h}] \ , \tag{B-7a}$$

and $$\int \frac{u^{-1/2}}{u + 4e^4/h^2}\, du = \frac{1}{e^2/h} \tan^{-1} \frac{u^{1/2}}{2e^2/h} \,. \qquad \text{(B-7b)}$$

With these preliminary results, the original integral can be evaluated.

$$\int_0^\pi \frac{1}{|\underline{v}-\underline{w}|} \ln \frac{2e^2h^2|\underline{v}-\underline{w}|^2}{(h^2|\underline{v}-\underline{w}|^2 + 4e^4)^{3/2}} \sin\phi d\phi$$

$$= \frac{1}{vw} \{ |v-w| \ln \frac{2e^2h^2(v+w)^2}{[h^2(v+w)^2 + 4e^4]^{3/2}} -$$

$$- |v-w| \ln \frac{2e^2h^2(v+w)^2}{[h^2(v-w)^2 + 4e^4]^{3/2}} + (v+w) - |v-w| - 6\frac{e^2}{n}\gamma \},$$

(B-8)

where $\gamma \equiv \tan^{-1} \frac{(v+w)}{2e^2/h} - \tan^{-1} \frac{|v-w|}{2e^2/h}$.

The angle γ can be simplified by means of the mnemonic triangle below.

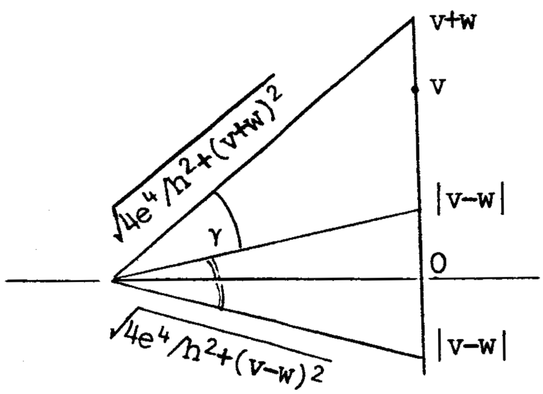


Fig. B.2 Mnemonic Triangle for the Angle γ .

$$\cos\gamma = \begin{cases} \dfrac{v^2 - w^2 + 4e^4/h^2}{\sqrt{(v^2 - w^2)^2 + 2(4e^4/h^2)(v^2 + w^2) + (4e^4/h^2)^2}} , & w < v \\[2ex] \dfrac{w^2 - v^2 + 4e^4/h^2}{\sqrt{(w^2 - v^2)^2 + 2(4e^4/h^2)(w^2 + v^2) + (4e^4/h^2)^2}} , & w > v \end{cases} \tag{B-9}$$

Finally we can solve for γ in a simpler form

$$\gamma = \tan^{-1} \begin{cases} \dfrac{(4e^2/h)\, v}{w^2 - v^2 + 4e^4/h^2} , & w > v \\[2ex] \dfrac{(4e^2/h)\, w}{v^2 - w^2 + 4e^4/h^2} , & w < v \end{cases} \tag{B-10}$$

APPENDIX C

DERIVATION OF $S(\alpha_S)$ FOR A TRANSLATING PLASMA

An expression for the average energy loss rate of an energetic test electron traversing a plasma was derived in Chapter III. The final result was obtained by assuming a Maxwellian distribution of plasma electron velocities. However, the formalism of the loss rate calculation allows the result to be obtained, in principle, for any distribution of plasma electron velocities provided there is no high energy tail present. In particular, it will prove interesting to choose a distribution function which is isotropic in a translating reference frame. Such a distribution function describes the plasma electrons behind a moving shock front.

In the following treatment, it will be assumed that the plasma electrons, behind a moving shock front, can be described as having a Maxwellian distribution in a frame of reference which is moving with a constant velocity, $\underline{V}_S$. The coordinate system is indicated in Fig. C.1 .

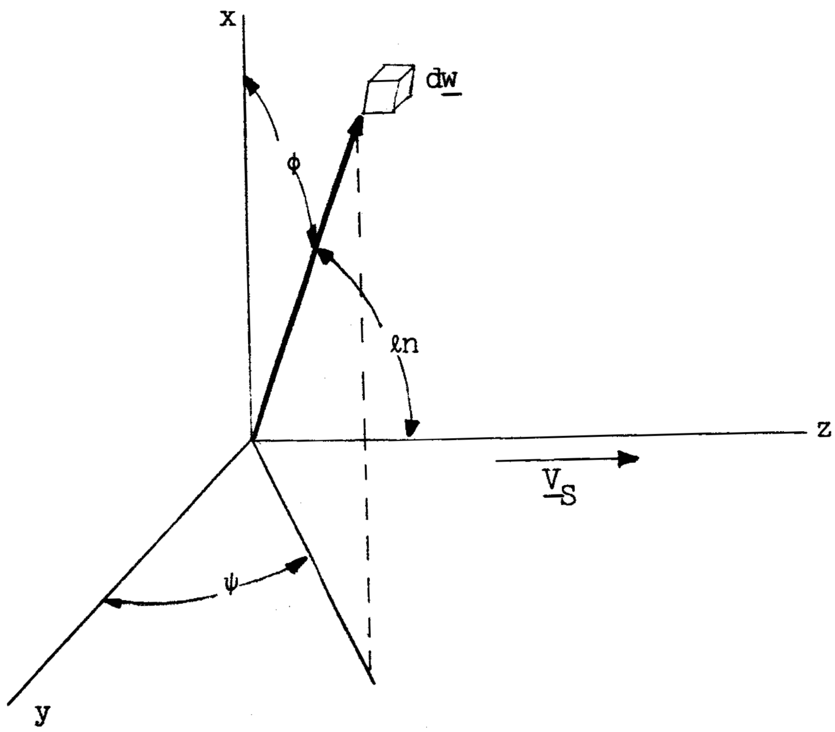


Fig. C.1 Spherical Coordinate System in the Translating Frame

The z-axis is taken along the direction of propagation of the plane shock. In the unprimed, translating frame, the plasma electrons have a Maxwellian distribution,

$$f(\underline{w})d\underline{w} = \frac{n_e}{\pi^{3/2} w_t^3} \; e^{\frac{-\underline{w}\cdot\underline{w}}{2kT/m}} \, w^2 \sin\phi \, d\phi \, d\psi \, dw \; . \qquad \text{(C-1)}$$

The transformation from the unprimed, translating frame, to the primed, laboratory frame is given by,

$$\underline{w} = \underline{w}' - \underline{V}_S \; . \qquad \text{(C-2)}$$

Making the transformation,

$$\underline{w}\cdot\underline{w} = (\underline{w}'-\underline{V}_S)\cdot(\underline{w}'-\underline{V}_S)$$

and since $\underline{V}_S = \hat{k}V_S$

$$\underline{w}\cdot\underline{w} = w_x'^2 + w_y'^2 + w_z'^2 + V_S^2 - 2w_z'V_S$$

where
$$w_x = w\cos\phi$$
$$w_y = w\sin\phi\cos\psi$$
$$w_z = w\sin\phi\sin\psi\ .$$

Then, in terms of ϕ, ψ,

$$\underline{w}\cdot\underline{w} = w^2 - 2wV_S\sin\phi\sin\psi + V_S^2\ . \qquad \text{(C-3)}$$

In order to evaluate the energy loss rate, the collision function must be averaged over the asymmetric distribution function. This averaging involves integrations in velocity space of the following types:

$$\int e^{\frac{V_S w}{kT/m}\sin\phi\sin\psi}\,d\psi \qquad \text{(C-4)}$$

From the integral representation of the Bessel function, J_o,

$$J_o(kr) = \frac{1}{2\pi}\int_0^{2\pi} e^{ikr\cos\theta}\,d\theta\ , \qquad \text{(C-5)}$$

and making the substitution that $\cos\theta = \sin\psi$, we can evaluate the integral.

$$\int_0^{2\pi} e^{\left(\frac{V_S w}{kT/m}\sin\phi\right)\sin\psi}\,d\psi = 2\pi\, I_o\!\left(\frac{V_S w}{kT/m}\sin\phi\right)\ , \qquad \text{(C-6)}$$

where
$$I_o(k) = J_o(ik) = J_o(-ik)\ . \qquad \text{(C-7)}$$

The integration over the polar angle ϕ involves another Bessel function integral,

$$\int_0^\pi I_o(k \sin\phi) \sin\phi d\phi \quad . \tag{C-8}$$

An integral identity appropariate for this calculation is

$$J_\mu(z)J_\nu(z) = \frac{2}{\pi}\int_0^{\pi/2} J_{\mu+\nu}(2z \cos\theta) \cos(\mu-\nu)\theta d\theta \quad . \tag{C-9}$$

With appropriate changes of variables, we can write

$$-\int_{-\pi/2}^{0} I_o(k \sin x) \sin x \, dx = \frac{\pi}{2} I_{1/2}(\tfrac{k}{2}) \, I_{-1/2}(\tfrac{k}{2}) \quad .$$

Making the change of variables, $x = -y$, and noting that the integrand is symmetric about $y = \pi/2$, we can evaluate the initial integral. The result is

$$\int_0^\pi I_o(k \sin y) \sin y \, dy = \pi \, I_{1/2}(\tfrac{k}{2}) \, I_{-1/2}(\tfrac{k}{2}) \quad . \tag{C-10}$$

A final integration must be resolved involving an integral of the following type,

$$\int_0^\infty y^2 I_{1/2}(y) \, I_{-1/2}(y) \, e^{-A^2y^2} dy \quad . \tag{C-11}$$

The $I_{1/2}$, $I_{-1/2}$ are defined in terms of elementary functions.

$$I_{1/2}(y) = J_{1/2}(iy) = \sqrt{\frac{2}{\pi}} \, \frac{\sin i \, y}{\sqrt{i \, y}} \quad , \tag{C-12a}$$

and $$I_{-1/2}(y) = J_{-1/2}(iy) = \sqrt{\frac{2}{\pi}} \, \frac{\cos i \, y}{\sqrt{i \, y}} \quad ; \tag{C-12b}$$

then $I_{1/2}\, I_{-1/2} = \frac{i}{\pi y} \sin(2\, i\, y)$. (C-13)

Using the exponential definition of sin (2iy) , this can be written as

$$y^2 I_{1/2}(y)\, I_{-1/2}(y) = -\frac{y}{2\pi}(e^{-2y} - e^{2y}) ,$$

then

$$\int_0^\infty y^2 I_{1/2} I_{-1/2} e^{-Ay^2} dy = \frac{1}{2\pi} \int_0^\infty y[e^{-(Ay^2-2y)} - e^{-(Ay^2+2y)}]dy$$

$$\equiv F(A). \tag{C-14}$$

Formally, the result for the average energy loss rate is given by,

$$\frac{dE}{dt} = -\frac{4\pi e^4}{m} \frac{4}{\sqrt{\pi}} \frac{n_e}{w_t^3}[w_t^2\, g_1 + w_t^2\, g_2] , \tag{C-15}$$

where,

$$g_1 = \frac{1}{2} \int_0^\pi \int_0^\infty \frac{\ell n\Lambda}{\beta}\, e^{-\alpha_s^2}\, \alpha^2 e^{-\alpha^2} I_o(\alpha_s \alpha \sin\phi) \sin\phi\, d\phi\, d\alpha, \tag{C-16a}$$

$$g_2 = [\frac{1}{2} \int_0^\pi \int_0^\infty \frac{1}{\beta}\, \ell n[\frac{2e^2\, \beta^2}{w_t h(\beta^2 + \frac{4e^4}{w_t^2 h^2})^{3/2}}] e^{-\alpha_s^2} \alpha^2 e^{-\alpha^2}]$$

$$\times\, [I_o \sin\phi\, d\phi\, d\alpha] \tag{C-16b}$$

and

$$\alpha_s \equiv \frac{v_s}{w_t} .$$

The result obtained for the energy loss rate must reduce to the expression obtained previously for the case of no shock velocity in the distribution function. We can see that this is indeed the case.

$$\lim_{\alpha_s \to o} g_1 = \frac{1}{2} \int_o^{\pi} \int_o^{\infty} \frac{\ell n\Lambda}{\beta} \alpha^2 e^{-\alpha^2} \sin\phi d\phi d\alpha$$

$$= \int_o^{\infty} \frac{1}{\beta} [\ell n\Lambda] \alpha^2 e^{-\alpha^2} d\alpha$$

$$\therefore g_1 = G_1 \quad \text{for} \quad \beta^2 >> \alpha^2$$

$$\lim_{\alpha_s \to o} g_2 = \frac{1}{2} \int_o^{\pi} \int_o^{\infty} \frac{1}{\beta} \ell n\left[\frac{2e^2\beta^2}{w_t h\left(\beta^2 + \frac{4e^4}{w_t^2 h^2}\right)^{3/2}}\right] \alpha^2 e^{-\alpha^2} \sin\phi d\phi d\alpha$$

$$= \int_o^{\infty} \frac{1}{\beta} \ell n\left[\frac{2e^2\beta^2}{w_t h\left(\beta^2 + \frac{4e^4}{w_t^2 h^2}\right)^{3/2}}\right] \alpha^2 e^{-\alpha^2} d\alpha$$

$$\therefore g_2 = G_2 \quad \text{for} \quad \beta^2 >> \alpha^2$$

We can simplify g_1 and g_2 by using the integrals which were evaluated previously,

$$g_1 = \frac{1}{4}\left(\frac{2}{\alpha_s}\right)^3 \frac{1}{\beta} \ell n\Lambda e^{-\alpha_s^2} F(\alpha_s) \;, \qquad \text{(C-17a)}$$

$$g_2 = \frac{1}{4}\left(\frac{2}{\alpha_s}\right)^3 \frac{1}{\beta} \ell n\left[\frac{2e^2\beta^2}{w_t h\left(\beta^2 + \frac{4e^4}{w_t^2 h^2}\right)^{3/2}}\right] e^{-\alpha_s^2} F(\alpha_s) \;, \qquad \text{(C-17b)}$$

where $$F(\alpha_s) = \int_0^\infty \alpha'[e^{-(A^2\alpha'^2 - 2\alpha')} - e^{-(A^2\alpha'^2 + 2\alpha')}]d\alpha' \ , \qquad \text{(C-18)}$$

$$A = \frac{2}{\alpha_s} \ ,$$

and $$\alpha' = \frac{\alpha_s \alpha}{2}$$

By completing the square in the exponent of the $F(\alpha_s)$ function, the integral can be evaluated.

$$F(\alpha_s) = \sqrt{\pi}\left(\frac{\alpha_s}{2}\right)^3 [\mathrm{erf}(\frac{\alpha_s}{2}) + 1] e^{\left(\frac{\alpha_s}{2}\right)^2} \qquad \text{(C-19)}$$

The function $S(\alpha_s)$ is defined to be the effect of the translation.

$$S(\alpha_s) \equiv \frac{dE/dt(\alpha_s)}{dE/dt(0)} \qquad \text{(C-20)}$$

Comparing these two expressions, the result is

$$S(\alpha_s) = [\mathrm{erf}(\frac{\alpha_s}{2}) + 1] \exp(-\frac{3}{4}\alpha_s^2) \ . \qquad \text{(C-21)}$$

In this expression we can note that the plasma temperature (related to w_t) appears only in the $\alpha_s = \frac{v_s}{w_t}$ term, which is negligible for small shock velocities. Therefore, the energy loss is not a function of the plasma temperature for low shock velocities. This result is not surprising because in the derivation we have neglected the plasma electron velocity in comparison with the beam electron velocity. This corresponds to a zero temperature approximation. The plasma temperature appears in connection with the shock velocity because here the plasma electron velocity enters as a cross product term giving rise to the Bessel function rather than as a difference and hence is not neglected. For shock conditions normally found in experimental apparatus, $V_s \gtrsim 10^6$ cm/sec and $T \gtrsim 40{,}000^\circ K$ so that $\alpha_s \gtrsim 10^{-2}$ which implies that the stationary calculations are adequate for laboratory conditions.

APPENDIX D

ELEMENTARY MULTIPLE SCATTERING THEORY

The interaction of a beam of energetic test electrons traversing a finite plasma slab can be described statistically by multiple scattering theory. Multiple scattering theory is valid because the energetic test electrons make numerous, small angle collision during the traversal and the resulting energy loss is a small perturbation on the initial energy.

Consider the i^{th} collision of the electron which occurs at a point y_i measured from the exit plane of the plasma slab.

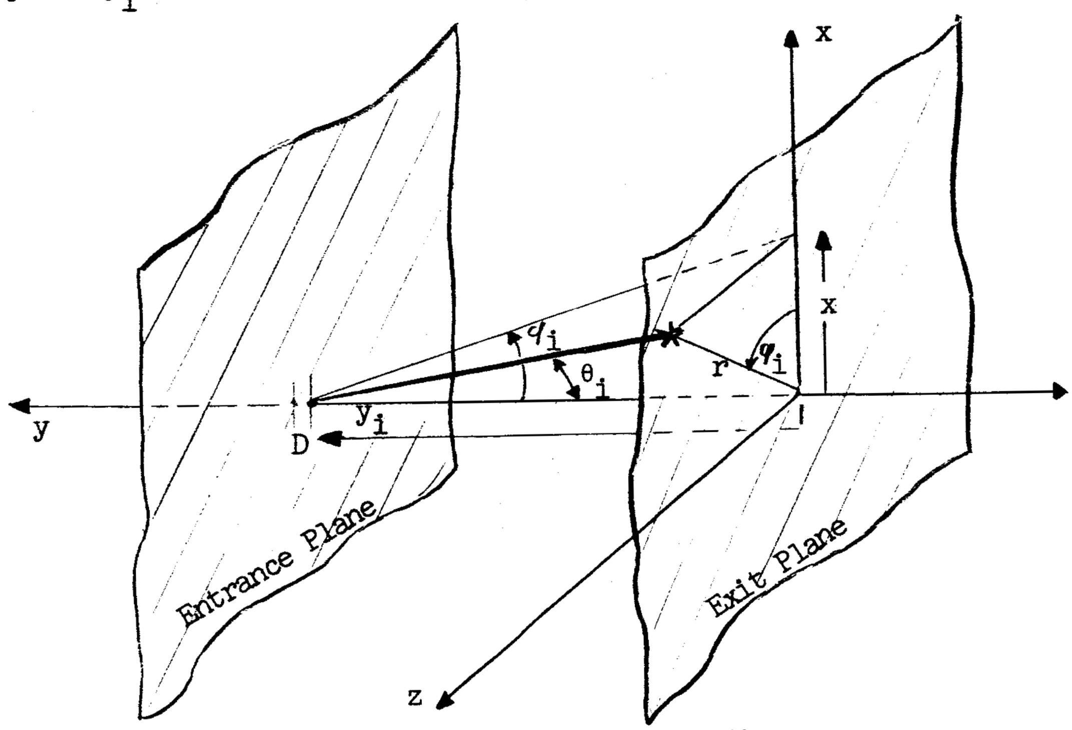


Fig. D.1 Coordinate System for the i^{th} Multiple Scattering

The i^{th} collision results in a scattering angle, θ_i , and the projected path of the scattered test electron on the exit plane is given by the polar coordinates r, φ_i on the x-z plane. After multiple collisions, the accumulated, projected scattering angle on the x-y plane is $\psi = \sum_i \theta_i \cos\varphi_i$, for small θ_i . The projected displacement is

$$x = \sum_i y_i \theta_i \cos\varphi_i \ ,$$

$$x^2 = \sum_{ij} y_i y_j \theta_i \theta_j \cos\varphi_i \cos\varphi_j \ .$$

Since y_i and θ_i are independent, and φ_i is random ,

$$\overline{\cos\varphi_i \cos\varphi_j} = \frac{1}{2}\delta_{ij} \ ,$$

and

$$\overline{x^2} = \frac{1}{2}\sum_i \overline{y_i^2 \ \theta_i^2} \ .$$

The average squared value of y_i is simply the location of the moment of inertia of the slab.

$$\overline{y_i^2} = \overline{y^2} = 1/D \int_o^D y^2 dy = 1/3D^2.$$

For $\theta_i << 1$ it can be shown that $\sum_i \overline{\theta_i^2} = <\theta^2> P \equiv <\theta^2>$ where $<\theta^2>$ is the average squared angle of deflection per collision and P is the average number of collisions in one traversal of the slab.

$$\therefore \quad \overline{x^2} = 1/6D^2 <\theta^2> P \ .$$

The actual deviation from the original trajectory is $\sqrt{\overline{r^2}}$

$$\overline{x^2} = \overline{r^2} \sum_{ij} \overline{\cos\varphi_i \cos\varphi_j}$$

$$\overline{x^2} = \overline{r^2}\ \overline{\cos^2\varphi}\ ,$$ since r and φ are independent.

Also

$$\overline{\cos^2\varphi} = 1/2\pi \int_0^{2\pi} \cos^2\varphi d\varphi = \frac{1}{2}$$

$$\therefore \quad \overline{r^2} = \overline{2x^2}\ .$$

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