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Technical Report No. 32-761

Hadamard Matrices of Order 20

Marshall Hall, Jr.

JET PROPULSION LABORATORY
CALIFORNIA INSTITUTE OF TECHNOLOGY
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This Report studies Hadamard matrices of order 20. Proof is given that there are exactly three classes of such matrices; the work done to show this result is discussed, and the classes are found explicitly. *Class* refers to permutation of rows and columns, and changing the sign of entire rows and columns.

Author

I. INTRODUCTION

There are exactly three distinct classes of Hadamard matrices of order 20, if matrices equivalent under permutation of rows or columns or change of sign of rows or columns are considered in the same class. These three classes are exhibited in the Appendix, labeled Class Q, Class P, and Class N.

Class Q contains the matrix derivable from the quadratic residues modulo 19. Class P contains the matrix that Paley constructed from $GF(9)$ by first constructing a matrix of order 10 consisting of 0's, 1's and -1's and then forming one of order 20 by replacing each 0 by $\begin{bmatrix} 1, & -1 \\ -1, & -1 \end{bmatrix}$, each 1 by $\begin{bmatrix} 1, & 1 \\ 1, & -1 \end{bmatrix}$, and each -1 by $\begin{bmatrix} -1, & -1 \\ -1, & 1 \end{bmatrix}$. (For these constructions, see Paley's original paper, Ref. 1.) The matrix of order 20 constructed by Williamson (Ref. 2) turns out to belong to this same class. The class N is a new class not previously known. Each of the three classes possesses a symmetric representative, and so each is equivalent to its own transpose. This is exhibited in (6.14), (6.15), and (6.16).

The search for all Hadamard matrices of order 20 is enormously facilitated by the fact, proved in Theorem 2.1, that if any three rows of an H_{20} are put into the standard form (writing $-$ for -1 as a convenience)

$$\begin{array}{ccccc|ccccc|ccccc|ccccc} 1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1\\ 1&1&1&1&1&1&1&1&1&1&-&-&-&-&-&-&-&-&-&-\\ 1&1&1&1&1&-&-&-&-&-&1&1&1&1&1&-&-&-&-&- \end{array} \tag{1.1}$$

then there is exactly one further row that can be put in the form

$$\begin{array}{ccccc|ccccc|ccccc|ccccc} 1&-&-&-&-&-&1&1&1&1&-&1&1&1&1&1&-&-&-&- \end{array} \tag{1.2}$$

along with the three rows of (1.1). Also, any three of these four rows determine the fourth in the same manner. These row quadruples are very useful. The quadruples enable us to show (Theorem 2.4) that we may consider only three 8-row starts given in (2.16). Sections III, IV, and V are devoted to finding the completions of these three 8-row starts. Section VI deals with the complicated issue of finding all equivalences between the matrices so constructed and proving that the classes Q, P, and N include all matrices and are distinct classes.

These results are in contrast with those of Ref. 3, where it was shown that there are five classes of Hadamard matrices of order 16. Certain theoretical considerations make it plausible to expect more classes of Hadamard matrices of order n when $n \equiv 0 \pmod 8$ than when $n \equiv 4 \pmod 8$.

II. THE THREE MAIN SUBDIVISIONS

By changing the signs of columns appropriately, any row of an H_{20} can be made to consist entirely of $+1$'s. Having done this to the first row, we can arrange columns to take the first three rows in the standard form:

$$\begin{array}{ccccc|ccccc|ccccc|ccccc} 1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1\\ 1&1&1&1&1&1&1&1&1&1&-&-&-&-&-&-&-&-&-&-\\ 1&1&1&1&1&-&-&-&-&-&1&1&1&1&1&-&-&-&-&- \end{array} \tag{2.1}$$

In a further row, let x, y, z, and w be respectively the number of $+1$'s in the first, second, third, and fourth set of five columns. Then orthogonality of this row to the three above gives the equations

$$\begin{aligned} x-(5-x)+y-(5-y)+z-(5-z)+w-(5-w)&=0\\ x-(5-x)+y-(5-y)-z+(5-z)-w+(5-w)&=0\\ x-(5-x)-y+(5-y)+z-(5-z)-w+(5-w)&=0 \end{aligned} \tag{2.2}$$

These simplify to

$$\begin{aligned} x + y + z + w &= 10 \\ x + y - z - w &= 0 \\ x - y + z - w &= 0 \end{aligned} \qquad (2.3)$$

whence we find

$$\begin{aligned} y &= 5 - x \\ z &= 5 - x \\ w &= x \end{aligned} \qquad (2.4)$$

The possible values for (x, y, z, w) are therefore

$$\begin{aligned} &(0,5,5,0)\,, \quad (5,0,0,5) \\ &(1,4,4,1)\,, \quad (4,1,1,4) \\ &(2,3,3,2)\,, \quad (3,2,2,3) \end{aligned} \qquad (2.5)$$

Replacing a row by its negative replaces (x, y, z, w) by $(5 - x, 5 - y, 5 - z, 5 - w)$ and so the three types in the second column of (2.5) are merely the negatives of those in the first column.

Lemma 2.1. *There is no row of type (0, 5, 5, 0) or (5, 0, 0, 5).*

Proof: Suppose we have a row of type $(0, 5, 5, 0)$. The orthogonality of this with a row of type (x, y, z, w) gives

$$-x + (5 - x) + y - (5 - y) + z - (5 - z) - w + (5 - w) = 0 \qquad (2.6)$$

Using (2.4), this yields

$$2x = 5 \qquad (2.7)$$

an impossibility. Hence there is no row of type $(0, 5, 5, 0)$, and consequently also no row of the negative of this type $(5, 0, 0, 5)$.

Lemma 2.2. *There cannot be two different rows of type (1, 4, 4, 1) or (4, 1, 1, 4).*

Proof: Changing signs if necessary, take both rows to be of type $(1, 4, 4, 1)$ and arrange columns so that the first of these two rows is of the form

$$\begin{array}{ccccc|ccccc|ccccc|ccccc} \multicolumn{5}{c}{x} & \multicolumn{5}{c}{y} & \multicolumn{5}{c}{z} & \multicolumn{5}{c}{w} \\ 1 & - & - & - & - & - & 1 & 1 & 1 & 1 & - & 1 & 1 & 1 & 1 & 1 & - & - & - & - \end{array} \qquad (2.8)$$

Now in the first five columns, the second row has $1 - - - -$ in some order. If the 1 is in column one, the inner product for these five columns is $+5$. If it is in any other column, the inner product for these five columns is $+1$. Similarly, the inner product for the five columns in the y, z, w ranges is $+1$ or $+5$ in every case. As these are all positive, the inner product cannot be zero and the two rows are not orthogonal. This proves the lemma.

LEMMA 2.3. *Not all of rows 4 to 20 can be of type (2, 3, 3, 2) or (3, 2, 2, 3).*

Proof: Change the sign of the rows 4 to 20 so that the first column consists entirely of $+1$'s. Then, with proper rearrangement of rows 4 to 20, the first three columns of all 20 rows have the pattern given by (2.1) for the first three rows. In this case we have the pattern for the first five rows in the first five columns:

$$\begin{matrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & & \\ 1 & 1 & 1 & & \end{matrix} \tag{2.9}$$

Since the fourth and fifth rows have $x = 2$ or $x = 3$, this means that the fourth and fifth rows are of the form

$$\begin{matrix} 1 & 1 & 1 & - & - \\ 1 & 1 & 1 & - & - \end{matrix} \tag{2.10}$$

By changing signs in rows 4 through 20 to make a particular column consist entirely of $+1$'s, and taking two other columns, we find that there must be exactly two rows that are $+1$ in the three columns and -1 in the two remaining columns. This choice of sign in every case corresponds for each row to a choice of sign making $x = 3$, and gives exactly two -1's in two specified columns for precisely two rows. But we can choose two columns out of five in $\binom{5}{2} = 10$ ways and this requires 20 rows in addition to the first three to give all these patterns. This is a conflict, since only 17 rows remain to be added. Hence our lemma is proved.

Combination of Lemmas 2.1, 2, and 3 yields the following conclusion, which we state as a theorem:

THEOREM 2.1. *If the first three rows of an H_{20} are put in the form of (2.1) then in the remaining rows exactly one is of type (1, 4, 4, 1) or (4, 1, 1, 4) and the rest are of type (2, 3, 3, 2) or (3, 2, 2, 3).*

Proof: The preceding lemmas have ruled out all other possibilities.

Let us take the single row of type (1, 4, 4, 1) or (4, 1, 1, 4), with a sign so that it is of type (1, 4, 4, 1), and take it as the fourth row.

THEOREM 2.2. *For an* H_{20}, *if we put the first three rows in the form (2.1), then with an appropriate sign a row is determined as a fourth row with pattern (1, 4, 4, 1) and the patterns of the first five columns are fully determined as given in the following table:*

$$
\begin{array}{c|c|cccc|c|cccc|c|cccc|c|cccc}
 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 & 17 & 18 & 19 & 20 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
2 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & - & - & - & - & - & - & - & - & - & - \\
3 & 1 & 1 & 1 & 1 & 1 & - & - & - & - & - & 1 & 1 & 1 & 1 & 1 & - & - & - & - & - \\
4 & 1 & - & - & - & - & - & 1 & 1 & 1 & 1 & - & 1 & 1 & 1 & 1 & 1 & - & - & - & - \\
\cline{2-6}
5 & 1 & 1 & - & - & - \\
6 & 1 & - & 1 & - & - \\
7 & 1 & - & - & 1 & - \\
8 & 1 & - & - & - & 1 \\
\cline{2-6}
 & - & 1 & 1 & - & - \\
 & - & 1 & 1 & - & - \\
 & - & 1 & - & 1 & - \\
 & - & 1 & - & 1 & - \\
 & - & 1 & - & - & 1 \\
 & - & 1 & - & - & 1 \\
 & - & - & 1 & 1 & - \\
 & - & - & 1 & 1 & - \\
 & - & - & 1 & - & 1 \\
 & - & - & 1 & - & 1 \\
 & - & - & - & 1 & 1 \\
 & - & - & - & 1 & 1
\end{array}
\qquad (2.11)
$$

Proof: Taking the unique row of type (1, 4, 4, 1) or (4, 1, 1, 4) with a sign so that it is (1, 4, 4, 1), we can arrange the columns as above. If we change the sign of row 4 for the moment, the first five columns begin

$$
\begin{array}{ccccc}
1 & 2 & 3 & 4 & 5 \\
\hline
1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 \\
- & 1 & 1 & 1 & 1
\end{array}
\qquad (2.12)
$$

With an appropriate sign, there is exactly one more row with 1's in columns 3, 4, 5, because of the orthogonality of columns 3, 4, and 5. Because of the $(2,3,3,2)$ or $(3,2,2,3)$ pattern of the row, the other two values in columns 1 and 2 of this row must be -1's. This row, with the sign changed, is taken as row 5. In the same way rows 6, 7, and 8 are uniquely determined by the pattern of the first five columns. By the same argument there must be exactly two more rows with -1's for the following choices of columns: $1,2,3$; $1,2,4$; $1,2,5$; $1,3,4$; $1,3,5$; $1,4,5$; and $+1$'s in the other two columns. This gives the remainder of the pattern in (2.11).

As row 5 in (2.11) has the pattern $(2,3,3,2)$ from orthogonality with the first three rows, the orthogonality with row 4 gives the following form for row 5:

$$\begin{array}{c|c|cccc|c|cccc|c|cccc|c|cccc|}
 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 & 17 & 18 & 19 & 20 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
2 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & - & - & - & - & - & - & - & - & - & - \\
3 & 1 & 1 & 1 & 1 & 1 & - & - & - & - & - & 1 & 1 & 1 & 1 & 1 & - & - & - & - & - \\
4 & 1 & - & - & - & - & - & 1 & 1 & 1 & 1 & - & 1 & 1 & 1 & 1 & 1 & - & - & - & - \\
\cline{2-21}
5 & 1 & 1 & - & - & - & 1 & 1 & 1 & - & - & 1 & 1 & 1 & - & - & - & 1 & 1 & - & - \\
\cline{2-21}
\end{array} \qquad (2.13)$$

Here the values for columns 6, 11, and 16 are uniquely determined, but of course the placement of the two 1's and two -1's in columns 7–10, 12–15, and 17–20 is arbitrary. Hence there are $\binom{4}{2}^3 = 216$ choices for row 5 given the first four rows. For rows 6, 7, and 8, the columns 6 through 20 have exactly the same pattern as in row 5 above.

The same arguments used to establish the pattern of columns 1 through 5 may be applied to columns 16 through 20, and we conclude that there is a unique row that has the pattern $1\ 1\ -\ -\ -$ in columns 16 through 20. By the observation above, it is not one of rows 5 through 8. Let us call this row 9. Continuing in this way, we may determine a distinctive pattern for each of the rows. This we give as a theorem:

THEOREM 2.3. *With the first four rows of an H_{20} in the form as in (2.11) and the remaining rows with a sign chosen to be of type $(2,3,3,2)$, we have the following distinctive pattern: The ranges 2–5, 7–10, 12–15, 17–20 not specified below consist of two $+1$'s and two -1's in every case.*

$$
\begin{array}{r|c|cccc|c|cccc|c|cccc|c|cccc|}
 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 & 17 & 18 & 19 & 20 \\
\cline{2-21}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
2 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & - & - & - & - & - & - & - & - & - & - \\
3 & 1 & 1 & 1 & 1 & 1 & - & - & - & - & - & 1 & 1 & 1 & 1 & 1 & - & - & - & - & - \\
4 & 1 & - & - & - & - & - & 1 & 1 & 1 & 1 & - & 1 & 1 & 1 & 1 & 1 & - & - & - & - \\
\cline{2-21}
5 & 1 & 1 & - & - & - & 1 & & & & & 1 & & & & & - & & & & \\
6 & 1 & - & 1 & - & - & 1 & & & & & 1 & & & & & - & & & & \\
7 & 1 & - & - & 1 & - & 1 & & & & & 1 & & & & & - & & & & \\
8 & 1 & - & - & - & 1 & 1 & & & & & 1 & & & & & - & & & & \\
\cline{2-21}
9 & - & & & & & 1 & & & & & 1 & & & & & 1 & 1 & - & - & - \\
10 & - & & & & & 1 & & & & & 1 & & & & & 1 & - & 1 & - & - \\
11 & - & & & & & 1 & & & & & 1 & & & & & 1 & - & - & 1 & - \\
12 & - & & & & & 1 & & & & & 1 & & & & & 1 & - & - & - & 1 \\
\cline{2-21}
13 & - & & & & & 1 & & & & & - & 1 & 1 & 1 & - & - & & & & \\
14 & - & & & & & 1 & & & & & - & 1 & 1 & - & 1 & - & & & & \\
15 & - & & & & & 1 & & & & & - & 1 & - & 1 & 1 & - & & & & \\
16 & - & & & & & 1 & & & & & - & - & 1 & 1 & 1 & - & & & & \\
\cline{2-21}
17 & - & & & & & - & 1 & 1 & 1 & - & 1 & & & & & - & & & & \\
18 & - & & & & & - & 1 & 1 & - & 1 & 1 & & & & & - & & & & \\
19 & - & & & & & - & 1 & - & 1 & 1 & 1 & & & & & - & & & & \\
20 & - & & & & & - & - & 1 & 1 & 1 & 1 & & & & & - & & & & \\
\cline{2-21}
\end{array}
\tag{2.14}
$$

From here on, if we speak of row m, we shall mean that unique row numbered m in (2.14) that contains the distinctive pattern given there.

By appropriate permutation of rows 5 through 8 and columns $2, \cdots, 5$; $7, \cdots, 10$; $12, \cdots, 15$; $17, \cdots, 20$, rows 5 through 8 take one of the following forms:

$$
\begin{array}{c|c|cccc|c|cccc|c|cccc|c|cccc|}
\multicolumn{1}{c}{} & \multicolumn{1}{c}{1} & 2 & 3 & 4 & \multicolumn{1}{c}{5} & \multicolumn{1}{c}{6} & 7 & 8 & 9 & \multicolumn{1}{c}{10} & \multicolumn{1}{c}{11} & 12 & 13 & 14 & \multicolumn{1}{c}{15} & \multicolumn{1}{c}{16} & 17 & 18 & 19 & \multicolumn{1}{c}{20} \\
5 & 1 & 1 & - & - & - & 1 & 1 & 1 & - & - & 1 & 1 & 1 & - & - & - & 1 & 1 & - & - \\
6 & 1 & - & 1 & - & - & 1 & 1 & 1 & - & - & 1 & - & - & 1 & 1 & - & - & - & 1 & 1 \\
7 & 1 & - & - & 1 & - & 1 & - & - & 1 & 1 & 1 & 1 & 1 & - & - & - & - & - & 1 & 1 \\
8 & 1 & - & - & - & 1 & 1 & - & - & 1 & 1 & 1 & - & - & 1 & 1 & - & 1 & 1 & - & - \\
\multicolumn{21}{c}{} \\
5 & 1 & 1 & - & - & - & 1 & 1 & 1 & - & - & 1 & 1 & 1 & - & - & - & 1 & 1 & - & - \\
6 & 1 & - & 1 & - & - & 1 & 1 & 1 & - & - & 1 & - & - & 1 & 1 & - & - & - & 1 & 1 \\
7 & 1 & - & - & 1 & - & 1 & - & - & 1 & 1 & 1 & 1 & - & 1 & - & - & 1 & - & 1 & - \\
8 & 1 & - & - & - & 1 & 1 & - & - & 1 & 1 & 1 & - & 1 & - & 1 & - & - & 1 & - & 1 \\
\multicolumn{21}{c}{} \\
5 & 1 & 1 & - & - & - & 1 & 1 & 1 & - & - & 1 & 1 & 1 & - & - & - & 1 & 1 & - & - \\
6 & 1 & - & 1 & - & - & 1 & - & - & 1 & 1 & 1 & 1 & 1 & - & - & - & - & - & 1 & 1 \\
7 & 1 & - & - & 1 & - & 1 & 1 & - & 1 & - & 1 & - & - & 1 & 1 & - & 1 & - & 1 & - \\
8 & 1 & - & - & - & 1 & 1 & - & 1 & - & 1 & 1 & - & - & 1 & 1 & - & - & 1 & - & 1 \\
\multicolumn{21}{c}{} \\
5 & 1 & 1 & - & - & - & 1 & 1 & 1 & - & - & 1 & 1 & 1 & - & - & - & 1 & 1 & - & - \\
6 & 1 & - & 1 & - & - & 1 & - & - & 1 & 1 & 1 & - & - & 1 & 1 & - & 1 & 1 & - & - \\
7 & 1 & - & - & 1 & - & 1 & 1 & - & 1 & - & 1 & 1 & - & 1 & - & - & - & - & 1 & 1 \\
8 & 1 & - & - & - & 1 & 1 & - & 1 & - & 1 & 1 & - & 1 & - & 1 & - & - & - & 1 & 1 \\
\multicolumn{21}{c}{} \\
5 & 1 & 1 & - & - & - & 1 & 1 & 1 & - & - & 1 & 1 & 1 & - & - & - & 1 & 1 & - & - \\
6 & 1 & - & 1 & - & - & 1 & 1 & - & 1 & - & 1 & 1 & - & 1 & - & - & - & - & 1 & 1 \\
7 & 1 & - & - & 1 & - & 1 & - & - & 1 & 1 & 1 & - & 1 & - & 1 & - & 1 & - & 1 & - \\
8 & 1 & - & - & - & 1 & 1 & - & 1 & - & 1 & 1 & - & - & 1 & 1 & - & - & 1 & - & 1
\end{array}
\tag{2.15}
$$

If we replace columns in the following way:

$$
\begin{pmatrix}
1, 2, 3, 4, 5, 6, & 7, & 8, & 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20 \\
11, 2, 3, 4, 5, 1, & 12, & 13, & 14, 15, 6, 17, 18, 19, 20, 16, 7, 8, 9, 10
\end{pmatrix}
$$

and rows

$$\begin{pmatrix} 1, & 2,3, & 4,5,6,7,8 \\ 1, & -4,2, & -3,5,6,7,8 \end{pmatrix}$$

we permute the second, third, and fourth of these starts cyclically. Hence every H_{20} can be assumed to have a start involving the first, second, or fifth choice of rows 5–8 in (2.15).

This we state as a theorem, the chief result of this Section:

THEOREM 2.4. *Every H_{20} is equivalent to an H_{20} with one of the following three 8-row starts:*

		1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
1.	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
	2	1	1	1	1	1	1	1	1	1	1	−	−	−	−	−	−	−	−	−	−
	3	1	1	1	1	1	−	−	−	−	−	1	1	1	1	1	−	−	−	−	−
	4	1	−	−	−	−	−	1	1	1	1	−	1	1	1	1	1	−	−	−	−
	5	1	1	−	−	−	1	1	1	−	−	1	1	1	−	−	−	1	1	−	−
	6	1	−	1	−	−	1	1	1	−	−	1	−	−	1	1	−	−	−	1	1
	7	1	−	−	1	−	1	−	−	1	1	1	1	1	−	−	−	−	−	1	1
	8	1	−	−	−	1	1	−	−	1	1	1	−	−	1	1	−	1	1	−	−
2.	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
	2	1	1	1	1	1	1	1	1	1	1	−	−	−	−	−	−	−	−	−	−
	3	1	1	1	1	1	−	−	−	−	−	1	1	1	1	1	−	−	−	−	−
	4	1	−	−	−	−	−	1	1	1	1	−	1	1	1	1	1	−	−	−	−
	5	1	1	−	−	−	1	1	1	−	−	1	1	1	−	−	−	1	1	−	−
	6	1	−	1	−	−	1	1	1	−	−	1	−	−	1	1	−	−	−	1	1
	7	1	−	−	1	−	1	−	−	1	1	1	1	−	1	−	−	1	−	1	−
	8	1	−	−	−	1	1	−	−	1	1	1	−	1	−	1	−	−	1	−	1
3.	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
	2	1	1	1	1	1	1	1	1	1	1	−	−	−	−	−	−	−	−	−	−
	3	1	1	1	1	1	−	−	−	−	−	1	1	1	1	1	−	−	−	−	−
	4	1	−	−	−	−	−	1	1	1	1	−	1	1	1	1	1	−	−	−	−
	5	1	1	−	−	−	1	1	1	−	−	1	1	1	−	−	−	1	1	−	−
	6	1	−	1	−	−	1	1	−	1	−	1	1	−	1	−	−	−	−	1	1
	7	1	−	−	1	−	1	−	−	1	1	1	−	1	−	1	−	1	−	1	−
	8	1	−	−	−	1	1	−	1	−	1	1	−	−	1	1	−	−	1	−	1

(2.16)

III. COMPLETIONS OF THE FIRST START

Our first 8-row start is

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
2	1	1	1	1	1	1	1	1	1	1	−	−	−	−	−	−	−	−	−	−
3	1	1	1	1	1	−	−	−	−	−	1	1	1	1	1	−	−	−	−	−
4	1	−	−	−	−	−	1	1	1	1	−	1	1	1	1	1	−	−	−	−
5	1	1	−	−	−	1	1	1	−	−	1	1	1	−	−	−	1	1	−	−
6	1	−	1	−	−	1	1	1	−	−	1	−	−	1	1	−	−	−	1	1
7	1	−	−	1	−	1	−	−	1	1	1	1	1	−	−	−	−	−	1	1
8	1	−	−	−	1	1	−	−	1	1	1	−	−	1	1	−	1	1	−	−

(3.1)

There are 192 rows consistent with these. Let us use the notation (1−) to mean that the two positions are either 1− or −1. Here in any row the four (1−) combinations are independent and so each row below is really 16 rows.

Row	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
9	−	−	1	1	−	1	(1	−)	(1	−)	1	(1	−)	(1	−)	1	1	−	−	−
10	−	−	1	1	−	1	(1	−)	(1	−)	1	(1	−)	(1	−)	1	−	1	−	−
11	−	1	−	−	1	1	(1	−)	(1	−)	1	(1	−)	(1	−)	1	−	−	1	−
12	−	1	−	−	1	1	(1	−)	(1	−)	1	(1	−)	(1	−)	1	−	−	−	1
13	−	−	1	−	1	1	(1	−)	(1	−)	−	1	1	1	−	−	(1	−)	(1	−)
14	−	−	1	−	1	1	(1	−)	(1	−)	−	1	1	−	1	−	(1	−)	(1	−)
15	−	1	−	1	−	1	(1	−)	(1	−)	−	1	−	1	1	−	(1	−)	(1	−)
16	−	1	−	1	−	1	(1	−)	(1	−)	−	−	1	1	1	−	(1	−)	(1	−)
17	−	−	−	1	1	−	1	1	1	−	1	(1	−)	(1	−)	−	(1	−)	(1	−)
18	−	−	−	1	1	−	1	1	−	1	1	(1	−)	(1	−)	−	(1	−)	(1	−)
19	−	1	1	−	−	−	1	−	1	1	1	(1	−)	(1	−)	−	(1	−)	(1	−)
20	−	1	1	−	−	−	−	1	1	1	1	(1	−)	(1	−)	−	(1	−)	(1	−)

(3.2)

The group G_1 of automorphisms of the first eight rows is of order 3072. This consists of the permutations and sign changes of the 20 rows and eight columns taking it into itself. When applied to the 192 rows consistent with these, it is transitive on them. We do not count the trivial automorphism of simultaneously changing the sign of all rows and all columns.

Generating automorphisms are

$$\begin{array}{ll}
\alpha\left\{\begin{array}{ll}
\begin{array}{l}\text{Column}\\ \text{replacement}\end{array} & \begin{pmatrix} 1,2,3, & 4, & 5, & 6,7,8, & 9, & 10, & 11, & 12, & 13,14,15,16,17,18, & 19, & 20 \\ -5,6,1, & -16, & 11, & -4,8,7, & -9, & -10, & -3, & -14, & -15,13,12, \; 2,18,17, & -19, & -20 \end{pmatrix} \\
\begin{array}{l}\text{Row}\\ \text{replacement}\end{array} & \begin{pmatrix} 1, & 2, & 3,4,5,6,7, & 8 \\ -7, & -8, & -5,6,1,2,3, & -4 \end{pmatrix}
\end{array}\right. \\[2ex]
\beta\left\{\begin{array}{ll}
\text{Column} & \begin{pmatrix} 1,2,3,4,5, & 6, & 7, & 8, & 9,10,11,12,13,14,15,16,17,18,19,20 \\ 6,2,5,3,4, & 11, & 17, & 18, & 19,20, \; 1, \; 7, \; 8, \; 9,10,16,12,13,14,15 \end{pmatrix} \\
\text{Row} & \begin{pmatrix} 1,2, & 3, & 4,5,6,7,8 \\ 1,3, & -4, & -2,5,7,8,6 \end{pmatrix}
\end{array}\right. \\[2ex]
\gamma\left\{\begin{array}{ll}
\text{Column} & \begin{pmatrix} 1,2,3,4,5, & 6, & 7, & 8, & 9,10,11,12,13,14,15,16,17,18,19,20 \\ 1,2,4,3,5, & 11, & 12, & 13, & 14,15, \; 6, \; 7, \; 8, \; 9,10,16,17,18,19,20 \end{pmatrix} \\
\text{Row} & \begin{pmatrix} 1,2,3,4,5,6,7,8 \\ 1,3,2,4,5,7,6,8 \end{pmatrix}
\end{array}\right. \\[2ex]
\delta\left\{\begin{array}{ll}
\text{Column} & \begin{pmatrix} 1,2,3,4,5,6,7, & 8,9,10,11,12,13,14,15,16,17,18,19,20 \\ 1,5,4,3,2,6,9,10,7, & 8,11,14,15,12,13,16,17,18,19,20 \end{pmatrix} \\
\text{Row} & \begin{pmatrix} 1,2,3,4,5,6,7,8 \\ 1,2,3,4,8,7,6,5 \end{pmatrix}
\end{array}\right. \\[2ex]
\varepsilon\left\{\begin{array}{ll}
\text{Column} & \begin{pmatrix} 1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18,19,20 \\ 1,3,2,5,4,6,7,8,9,10,11,14,15,12,13,16,19,20,17,18 \end{pmatrix} \\
\text{Row} & \begin{pmatrix} 1,2,3,4,5,6,7,8 \\ 1,2,3,4,6,5,8,7 \end{pmatrix}
\end{array}\right.
\end{array} \tag{3.3}$$

Also we have the simple column transpositions where all rows and the remaining columns are fixed:

$$\zeta_1 = (7,8), \zeta_2 = (9,10), \zeta_3 = (12,13), \zeta_4 = (14,15), \zeta_5 = (17,18), \zeta_6 = (19,20) \tag{3.4}$$

Every row is equivalent to the particular choice for row 9

$$\begin{array}{lcccccccccccccccccccc}
 & 1, & 2, & 3, & 4, & 5, & 6, & 7, & 8, & 9, & 10, & 11, & 12, & 13, & 14, & 15, & 16, & 17, & 18, & 19, & 20 \\
\text{Row 9:} & - & \big|- & 1 & 1 & -\big| & 1 & \big|1 & - & 1 & - & \big|1 & \big|1 & - & 1 & - & \big|1 & \big|1 & - & - & -
\end{array} \tag{3.5}$$

This row 9 is fixed (i.e., taken into itself or its negative) by a subgroup H of order 16 generated by α, γ, δ, and ζ_6.

With row 9 given by (3.5), there is only one choice for row 10:

$$\text{Row 10:}\quad -\,\Big|\,-\;1\;1\;-\,\Big|\,1\,\Big|\,-\;1\;-\;1\,\Big|\,1\,\Big|\,-\;1\;-\;1\,\Big|\,1\,\Big|\,-\;1\;-\;- \tag{3.6}$$

The transposition $\zeta_6 = (19, 20)$ leaves all ten rows fixed and interchanges a row 11 with a row 12. Using this to give a 1 rather than a $-$ in row 11, column 7, gives exactly 6 completions:

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
9	−	−	1	1	−	1	1	−	1	−	1	1	−	1	−	1	1	−	−	−
10	−	−	1	1	−	1	−	1	−	1	1	−	1	−	1	1	−	1	−	−
11	−	1	−	−	1	1	1	−	1	−	1	−	1	−	1	1	−	−	1	−
12	−	1	−	−	1	1	−	1	−	1	1	1	−	1	−	1	−	−	−	1
13	−	−	1	−	1	1	1	−	−	1	−	1	1	1	−	−	−	1	1	−
14	−	−	1	−	1	1	−	1	1	−	−	1	1	−	1	−	1	−	−	1
15	−	1	−	1	−	1	−	1	1	−	−	1	−	1	1	−	−	1	1	−
16	−	1	−	1	−	1	1	−	−	1	−	−	1	1	1	−	1	−	−	1
17	−	−	−	1	1	−	1	1	1	−	1	−	1	1	−	−	−	1	−	1
18	−	−	−	1	1	−	1	1	−	1	1	1	−	−	1	−	1	−	1	−
19	−	1	1	−	−	−	1	−	1	1	1	1	−	−	1	−	−	1	−	1
20	−	1	1	−	−	−	−	1	1	1	1	−	1	1	−	−	1	−	1	−

(A.1)

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
9	−	−	1	1	−	1	1	−	1	−	1	1	−	1	−	1	1	−	−	−
10	−	−	1	1	−	1	−	1	−	1	1	−	1	−	1	1	−	1	−	−
11	−	1	−	−	1	1	1	−	1	−	1	−	1	−	1	1	−	−	1	−
12	−	1	−	−	1	1	−	1	−	1	1	1	−	1	−	1	−	−	−	1
13	−	−	1	−	1	1	−	1	1	−	−	1	1	1	−	−	−	1	1	−
14	−	−	1	−	1	1	1	−	−	1	−	1	1	−	1	−	1	−	−	1
15	−	1	−	1	−	1	1	−	−	1	−	1	−	1	1	−	−	1	1	−
16	−	1	−	1	−	1	−	1	1	−	−	−	1	1	1	−	1	−	−	1
17	−	−	−	1	1	−	1	1	1	−	1	1	−	−	1	−	−	1	−	1
18	−	−	−	1	1	−	1	1	−	1	1	−	1	1	−	−	1	−	1	−
19	−	1	1	−	−	−	1	−	1	1	1	−	1	1	−	−	−	1	−	1
20	−	1	1	−	−	−	−	1	1	1	1	1	−	−	1	−	1	−	1	−

(A.2)

$$
\begin{array}{r|c|cccc|c|cccc|c|cccc|c|cccc|}
 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 & 17 & 18 & 19 & 20 \\
9 & - & - & 1 & 1 & - & 1 & 1 & - & 1 & - & 1 & 1 & - & 1 & - & 1 & 1 & - & - & - \\
10 & - & - & 1 & 1 & - & 1 & - & 1 & - & 1 & 1 & - & 1 & - & 1 & 1 & - & 1 & - & - \\
11 & - & 1 & - & - & 1 & 1 & 1 & - & - & 1 & 1 & 1 & - & - & 1 & 1 & - & - & 1 & - \\
12 & - & 1 & - & - & 1 & 1 & - & 1 & 1 & - & 1 & - & 1 & 1 & - & 1 & - & - & - & 1 \\
\cline{2-21}
13 & - & - & 1 & - & 1 & 1 & 1 & - & - & 1 & - & 1 & 1 & 1 & - & - & - & 1 & - & 1 \\
14 & - & - & 1 & - & 1 & 1 & - & 1 & 1 & - & - & 1 & 1 & - & 1 & - & 1 & - & 1 & - \\
15 & - & 1 & - & 1 & - & 1 & - & 1 & - & 1 & - & 1 & - & 1 & 1 & - & 1 & - & - & 1 \\
16 & - & 1 & - & 1 & - & 1 & 1 & - & 1 & - & - & - & 1 & 1 & 1 & - & - & 1 & 1 & - \\
\cline{2-21}
17 & - & - & - & 1 & 1 & - & 1 & 1 & 1 & - & 1 & 1 & - & - & 1 & - & - & 1 & - & 1 \\
18 & - & - & - & 1 & 1 & - & 1 & 1 & - & 1 & 1 & - & 1 & 1 & - & - & 1 & - & 1 & - \\
19 & - & 1 & 1 & - & - & - & 1 & - & 1 & 1 & 1 & - & 1 & - & 1 & - & 1 & - & - & 1 \\
20 & - & 1 & 1 & - & - & - & - & 1 & 1 & 1 & 1 & 1 & - & 1 & - & - & - & 1 & 1 & -
\end{array}
\tag{A.3}
$$

$$
\begin{array}{r|c|cccc|c|cccc|c|cccc|c|cccc|}
 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 & 17 & 18 & 19 & 20 \\
9 & - & - & 1 & 1 & - & 1 & 1 & - & 1 & - & 1 & 1 & - & 1 & - & 1 & 1 & - & - & - \\
10 & - & - & 1 & 1 & - & 1 & - & 1 & - & 1 & 1 & - & 1 & - & 1 & 1 & - & 1 & - & - \\
11 & - & 1 & - & - & 1 & 1 & 1 & - & - & 1 & 1 & 1 & - & - & 1 & 1 & - & - & 1 & - \\
12 & - & 1 & - & - & 1 & 1 & - & 1 & 1 & - & 1 & - & 1 & 1 & - & 1 & - & - & - & 1 \\
\cline{2-21}
13 & - & - & 1 & - & 1 & 1 & - & 1 & - & 1 & - & 1 & 1 & 1 & - & - & 1 & - & 1 & - \\
14 & - & - & 1 & - & 1 & 1 & 1 & - & 1 & - & - & 1 & 1 & - & 1 & - & - & 1 & - & 1 \\
15 & - & 1 & - & 1 & - & 1 & - & 1 & 1 & - & - & 1 & - & 1 & 1 & - & - & 1 & 1 & - \\
16 & - & 1 & - & 1 & - & 1 & 1 & - & - & 1 & - & - & 1 & 1 & 1 & - & 1 & - & - & 1 \\
\cline{2-21}
17 & - & - & - & 1 & 1 & - & 1 & 1 & 1 & - & 1 & - & 1 & - & 1 & - & 1 & - & 1 & - \\
18 & - & - & - & 1 & 1 & - & 1 & 1 & - & 1 & 1 & 1 & - & 1 & - & - & - & 1 & - & 1 \\
19 & - & 1 & 1 & - & - & - & 1 & - & 1 & 1 & 1 & - & 1 & 1 & - & - & - & 1 & 1 & - \\
20 & - & 1 & 1 & - & - & - & - & 1 & 1 & 1 & 1 & 1 & - & - & 1 & - & 1 & - & - & 1
\end{array}
\tag{A.4}
$$

$$
\begin{array}{r|c|cccc|c|cccc|c|cccc|c|cccc|}
\multicolumn{1}{r}{} & \multicolumn{1}{c}{1} & \multicolumn{1}{c}{2} & 3 & 4 & \multicolumn{1}{c}{5} & \multicolumn{1}{c}{6} & \multicolumn{1}{c}{7} & 8 & 9 & \multicolumn{1}{c}{10} & \multicolumn{1}{c}{11} & \multicolumn{1}{c}{12} & 13 & 14 & \multicolumn{1}{c}{15} & \multicolumn{1}{c}{16} & \multicolumn{1}{c}{17} & 18 & 19 & \multicolumn{1}{c}{20} \\
9 & - & - & 1 & 1 & - & 1 & 1 & - & 1 & - & 1 & 1 & - & 1 & - & 1 & 1 & - & - & - \\
10 & - & - & 1 & 1 & - & 1 & - & 1 & - & 1 & 1 & - & 1 & - & 1 & 1 & - & 1 & - & - \\
11 & - & 1 & - & - & 1 & 1 & 1 & - & - & 1 & 1 & - & 1 & 1 & - & 1 & - & - & 1 & - \\
12 & - & 1 & - & - & 1 & 1 & - & 1 & 1 & - & 1 & 1 & - & - & 1 & 1 & - & - & - & 1 \\
\cline{2-21}
13 & - & - & 1 & - & 1 & 1 & - & 1 & 1 & - & - & 1 & 1 & 1 & - & - & - & 1 & 1 & - \\
14 & - & - & 1 & - & 1 & 1 & 1 & - & - & 1 & - & 1 & 1 & - & 1 & - & 1 & - & - & 1 \\
15 & - & 1 & - & 1 & - & 1 & - & 1 & - & 1 & - & 1 & - & 1 & 1 & - & 1 & - & 1 & - \\
16 & - & 1 & - & 1 & - & 1 & 1 & - & 1 & - & - & - & 1 & 1 & 1 & - & - & 1 & - & 1 \\
\cline{2-21}
17 & - & - & - & 1 & 1 & - & 1 & 1 & 1 & - & 1 & - & 1 & - & 1 & - & 1 & - & 1 & - \\
18 & - & - & - & 1 & 1 & - & 1 & 1 & - & 1 & 1 & 1 & - & 1 & - & - & - & 1 & - & 1 \\
19 & - & 1 & 1 & - & - & - & 1 & - & 1 & 1 & 1 & 1 & - & - & 1 & - & - & 1 & 1 & - \\
20 & - & 1 & 1 & - & - & - & - & 1 & 1 & 1 & 1 & - & 1 & 1 & - & - & 1 & - & - & 1 \\
\end{array}
\tag{A.5}
$$

$$
\begin{array}{r|c|cccc|c|cccc|c|cccc|c|cccc|}
\multicolumn{1}{r}{} & \multicolumn{1}{c}{1} & \multicolumn{1}{c}{2} & 3 & 4 & \multicolumn{1}{c}{5} & \multicolumn{1}{c}{6} & \multicolumn{1}{c}{7} & 8 & 9 & \multicolumn{1}{c}{10} & \multicolumn{1}{c}{11} & \multicolumn{1}{c}{12} & 13 & 14 & \multicolumn{1}{c}{15} & \multicolumn{1}{c}{16} & \multicolumn{1}{c}{17} & 18 & 19 & \multicolumn{1}{c}{20} \\
9 & - & - & 1 & 1 & - & 1 & 1 & - & 1 & - & 1 & 1 & - & 1 & - & 1 & 1 & - & - & - \\
10 & - & - & 1 & 1 & - & 1 & - & 1 & - & 1 & 1 & - & 1 & - & 1 & 1 & - & 1 & - & - \\
11 & - & 1 & - & - & 1 & 1 & 1 & - & - & 1 & 1 & - & 1 & 1 & - & 1 & - & - & 1 & - \\
12 & - & 1 & - & - & 1 & 1 & - & 1 & 1 & - & 1 & 1 & - & - & 1 & 1 & - & - & - & 1 \\
\cline{2-21}
13 & - & - & 1 & - & 1 & 1 & - & 1 & - & 1 & - & 1 & 1 & 1 & - & - & 1 & - & - & 1 \\
14 & - & - & 1 & - & 1 & 1 & 1 & - & 1 & - & - & 1 & 1 & - & 1 & - & - & 1 & 1 & - \\
15 & - & 1 & - & 1 & - & 1 & 1 & - & - & 1 & - & 1 & - & 1 & 1 & - & - & 1 & - & 1 \\
16 & - & 1 & - & 1 & - & 1 & - & 1 & 1 & - & - & - & 1 & 1 & 1 & - & 1 & - & 1 & - \\
\cline{2-21}
17 & - & - & - & 1 & 1 & - & 1 & 1 & 1 & - & 1 & - & 1 & 1 & - & - & - & 1 & - & 1 \\
18 & - & - & - & 1 & 1 & - & 1 & 1 & - & 1 & 1 & 1 & - & - & 1 & - & 1 & - & 1 & - \\
19 & - & 1 & 1 & - & - & - & 1 & - & 1 & 1 & 1 & - & 1 & - & 1 & - & 1 & - & - & 1 \\
20 & - & 1 & 1 & - & - & - & - & 1 & 1 & 1 & 1 & 1 & - & 1 & - & - & - & 1 & 1 & - \\
\end{array}
\tag{A.6}
$$

IV. COMPLETIONS OF THE SECOND START

Here our first eight rows are

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
2	1	1	1	1	1	1	1	1	1	1	–	–	–	–	–	–	–	–	–	–
3	1	1	1	1	1	–	–	–	–	–	1	1	1	1	1	–	–	–	–	–
4	1	–	–	–	–	–	1	1	1	1	–	1	1	1	1	1	–	–	–	–
5	1	1	–	–	–	1	1	1	–	–	1	1	1	–	–	–	1	1	–	–
6	1	–	1	–	–	1	1	1	–	–	1	–	–	1	1	–	–	–	1	1
7	1	–	–	1	–	1	–	–	1	1	1	1	–	1	–	–	1	–	1	–
8	1	–	–	–	1	1	–	–	1	1	1	–	1	–	1	–	–	1	–	1

(4.1)

There are 160 rows consistent with this start, 16 possibilities for each of rows 9 through 16 and 8 possibilities for each of rows 17 through 20. The group G_2 of automorphisms of these eight rows is of order 512.

Generators for the group G_2 of automorphisms of these eight rows are

$$\alpha_2 \left\{ \begin{array}{ll} \text{Column replacement} & \begin{pmatrix} 1,2,3, & 4,\ 5, & 6,7,8, & 9, & 10, & 11, & 12,13, & 14,15,16, & 17,18, & 19,20 \\ -5,1,6, & -16,11, & -4,7,8, & -10, & -9, & -3, & -20,12, & -15,17,\ 2, & -19,13, & -14,18 \end{pmatrix} \\ \text{Row replacement} & \begin{pmatrix} 1, & 2, & 3,4,5,6,7, & 8 \\ -7, & -8, & -6,5,1,2,3, & -4 \end{pmatrix} \end{array} \right.$$

$$\beta_2 \left\{ \begin{array}{ll} \text{Column} & \begin{pmatrix} 1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18,19,20 \\ 6,2,3,4,5,1,7,8,9,10,11,17,18,19,20,16,12,13,14,15 \end{pmatrix} \\ \text{Row} & \begin{pmatrix} 1,2, & 3, & 4,5,6,7,8 \\ 1,2, & -4, & -3,5,6,7,8 \end{pmatrix} \end{array} \right.$$

$$\gamma_2 \left\{ \begin{array}{ll} \text{Column} & \begin{pmatrix} 1,2,3,4,5,6,7,\ 8,9,10,11,12,13,14,15,16,17,18,19,20 \\ 1,4,5,2,3,6,9,10,7,\ 8,11,12,14,13,15,16,17,19,18,20 \end{pmatrix} \\ \text{Row} & \begin{pmatrix} 1,2,3,4,5,6,7,8 \\ 1,2,3,4,7,8,5,6 \end{pmatrix} \end{array} \right. \tag{4.2}$$

$$\delta_2 \left\{ \begin{array}{ll} \text{Column} & \begin{pmatrix} 1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18,19,20 \\ 1,2,3,5,4,6,7,8,9,10,11,13,12,15,14,16,18,17,20,19 \end{pmatrix} \\ \text{Row} & \begin{pmatrix} 1,2,3,4,5,6,7,8 \\ 1,2,3,4,5,6,8,7 \end{pmatrix} \end{array} \right.$$

Transpositions: Columns (7, 8)
Columns (9, 10)

There are 160 rows consistent with the second 8-row start. There are 16 possibilities for each of rows 9, · · · , 16 and eight for each of rows 17, 18, 19, 20. The possibilities for rows 9, · · · , 16 are permuted in one orbit by G_2, and those for rows 17, 18, 19, and 20 in another orbit. We give here the choices for row 9 and row 17, where in row 9 the combinations for columns 7, 8, 9, and 10 written (1 –) (1 –) stand for the four choices 1 – 1 –, 1 – – 1, – 1 1 –, and – 1 – 1. Other rows are easily constructed from these by the automorphisms.

$$\begin{array}{c|c|cccc|c|cccc|c|cccc|c|cccc|}
\text{Row 9} & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 & 17 & 18 & 19 & 20 \\
 & - & - & 1 & 1 & - & 1 & (1 & -) & (1 & -) & 1 & - & 1 & - & 1 & 1 & 1 & - & - & - \\
 & - & 1 & - & - & 1 & 1 & (1 & -) & (1 & -) & 1 & - & - & 1 & 1 & 1 & 1 & - & - & - \\
 & - & - & 1 & - & 1 & 1 & (1 & -) & (1 & -) & 1 & 1 & - & - & 1 & 1 & 1 & - & - & - \\
 & - & - & 1 & - & 1 & 1 & (1 & -) & (1 & -) & 1 & - & 1 & 1 & - & 1 & 1 & - & - & -
\end{array}$$

Row 17

$$\begin{array}{c|c|cccc|c|cccc|c|cccc|c|cccc|}
1 & - & - & - & 1 & 1 & - & 1 & 1 & 1 & - & 1 & - & - & 1 & 1 & - & 1 & 1 & - & - \\
2 & - & - & - & 1 & 1 & - & 1 & 1 & 1 & - & 1 & - & 1 & - & 1 & - & 1 & - & 1 & - \\
3 & - & - & - & 1 & 1 & - & 1 & 1 & 1 & - & 1 & - & 1 & 1 & - & - & 1 & - & - & 1 \\
4 & - & - & - & 1 & 1 & - & 1 & 1 & 1 & - & 1 & - & 1 & 1 & - & - & - & 1 & 1 & - \\
5 & - & - & - & 1 & 1 & - & 1 & 1 & 1 & - & 1 & 1 & - & - & 1 & - & 1 & - & - & 1 \\
6 & - & - & - & 1 & 1 & - & 1 & 1 & 1 & - & 1 & 1 & - & - & 1 & - & - & 1 & 1 & - \\
7 & - & - & - & 1 & 1 & - & 1 & 1 & 1 & - & 1 & 1 & - & 1 & - & - & - & 1 & - & 1 \\
8 & - & - & - & 1 & 1 & - & 1 & 1 & 1 & - & 1 & 1 & 1 & - & - & - & - & - & 1 & 1
\end{array} \tag{4.3}$$

The automorphism α_2 moves the eight possibilities for row 17 in the following way:

$$\alpha_2\text{: } (1, 5, 2, 6, 8, 4, 7, 3) \tag{4.4}$$

Hence without loss of generality we may take the first choice for row 17 in finding completions. For this choice of row 17 there is a unique choice for row 18, and, using the transposition of columns 7 and 8, just three ways of adding rows 19 and 20. Each of these three choices for rows 17, 18, 19, and 20 can be completed in exactly two ways. This gives us a set of six matrices arising from one second start.

$$
\begin{array}{r|c|cccc|c|cccc|c|cccc|c|cccc|}
 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 & 17 & 18 & 19 & 20 \\
9 & - & - & 1 & - & 1 & 1 & - & 1 & - & 1 & 1 & 1 & - & - & 1 & 1 & 1 & - & - & - \\
10 & - & - & 1 & 1 & - & 1 & 1 & - & - & 1 & 1 & - & 1 & 1 & - & 1 & - & 1 & - & - \\
11 & - & 1 & - & - & 1 & 1 & - & 1 & 1 & - & 1 & - & 1 & 1 & - & 1 & - & - & 1 & - \\
12 & - & 1 & - & 1 & - & 1 & 1 & - & 1 & - & 1 & 1 & - & - & 1 & 1 & - & - & - & 1 \\
\cline{2-21}
13 & - & - & 1 & - & 1 & 1 & 1 & - & 1 & - & - & 1 & 1 & 1 & - & - & 1 & - & - & 1 \\
14 & - & - & 1 & 1 & - & 1 & - & 1 & 1 & - & - & 1 & 1 & - & 1 & - & - & 1 & 1 & - \\
15 & - & 1 & - & - & 1 & 1 & 1 & - & - & 1 & - & 1 & - & 1 & 1 & - & - & 1 & 1 & - \\
16 & - & 1 & - & 1 & - & 1 & - & 1 & - & 1 & - & - & 1 & 1 & 1 & - & 1 & - & - & 1 \\
\cline{2-21}
17 & - & - & - & 1 & 1 & - & 1 & 1 & 1 & - & 1 & - & - & 1 & 1 & - & 1 & 1 & - & - \\
18 & - & - & - & 1 & 1 & - & 1 & 1 & - & 1 & 1 & 1 & 1 & - & - & - & - & - & 1 & 1 \\
19 & - & 1 & 1 & - & - & - & 1 & - & 1 & 1 & 1 & - & 1 & - & 1 & - & 1 & - & 1 & - \\
20 & - & 1 & 1 & - & - & - & - & 1 & 1 & 1 & 1 & 1 & - & 1 & - & - & - & 1 & - & 1
\end{array}
\tag{B.1}
$$

$$
\begin{array}{r|c|cccc|c|cccc|c|cccc|c|cccc|}
 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 & 17 & 18 & 19 & 20 \\
9 & - & - & 1 & - & 1 & 1 & - & 1 & - & 1 & 1 & - & 1 & 1 & - & 1 & 1 & - & - & - \\
10 & - & - & 1 & 1 & - & 1 & 1 & - & - & 1 & 1 & 1 & - & - & 1 & 1 & - & 1 & - & - \\
11 & - & 1 & - & - & 1 & 1 & - & 1 & 1 & - & 1 & 1 & - & - & 1 & 1 & - & - & 1 & - \\
12 & - & 1 & - & 1 & - & 1 & 1 & - & 1 & - & 1 & - & 1 & 1 & - & 1 & - & - & - & 1 \\
\cline{2-21}
13 & - & - & 1 & - & 1 & 1 & 1 & - & 1 & - & - & 1 & 1 & 1 & - & - & - & 1 & 1 & - \\
14 & - & - & 1 & 1 & - & 1 & - & 1 & 1 & - & - & 1 & 1 & - & 1 & - & 1 & - & - & 1 \\
15 & - & 1 & - & - & 1 & 1 & 1 & - & - & 1 & - & 1 & - & 1 & 1 & - & 1 & - & - & 1 \\
16 & - & 1 & - & 1 & - & 1 & - & 1 & - & 1 & - & - & 1 & 1 & 1 & - & - & 1 & 1 & - \\
\cline{2-21}
17 & - & - & - & 1 & 1 & - & 1 & 1 & 1 & - & 1 & - & - & 1 & 1 & - & 1 & 1 & - & - \\
18 & - & - & - & 1 & 1 & - & 1 & 1 & - & 1 & 1 & 1 & 1 & - & - & - & - & - & 1 & 1 \\
19 & - & 1 & 1 & - & - & - & 1 & - & 1 & 1 & 1 & - & 1 & - & 1 & - & 1 & - & 1 & - \\
20 & - & 1 & 1 & - & - & - & - & 1 & 1 & 1 & 1 & 1 & - & 1 & - & - & - & 1 & - & 1
\end{array}
\tag{B.2}
$$

$$
\begin{array}{r|c|cccc|c|cccc|c|cccc|c|cccc|}
 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 & 17 & 18 & 19 & 20 \\
9 & - & - & 1 & 1 & - & 1 & - & 1 & - & 1 & 1 & - & 1 & - & 1 & 1 & 1 & - & - & - \\
10 & - & - & 1 & - & 1 & 1 & 1 & - & - & 1 & 1 & 1 & - & 1 & - & 1 & - & 1 & - & - \\
11 & - & 1 & - & - & 1 & 1 & - & 1 & 1 & - & 1 & - & 1 & 1 & - & 1 & - & - & 1 & - \\
12 & - & 1 & - & 1 & - & 1 & 1 & - & 1 & - & 1 & 1 & - & - & 1 & 1 & - & - & - & 1 \\
\hline
13 & - & - & 1 & 1 & - & 1 & - & 1 & 1 & - & - & 1 & 1 & 1 & - & - & - & 1 & - & 1 \\
14 & - & - & 1 & - & 1 & 1 & 1 & - & 1 & - & - & 1 & 1 & - & 1 & - & 1 & - & 1 & - \\
15 & - & 1 & - & - & 1 & 1 & - & 1 & - & 1 & - & 1 & - & 1 & 1 & - & 1 & - & - & 1 \\
16 & - & 1 & - & 1 & - & 1 & 1 & - & - & 1 & - & - & 1 & 1 & 1 & - & - & 1 & 1 & - \\
\hline
17 & - & - & - & 1 & 1 & - & 1 & 1 & 1 & - & 1 & - & - & 1 & 1 & - & 1 & 1 & - & - \\
18 & - & - & - & 1 & 1 & - & 1 & 1 & - & 1 & 1 & 1 & 1 & - & - & - & - & - & 1 & 1 \\
19 & - & 1 & 1 & - & - & - & 1 & - & 1 & 1 & 1 & - & 1 & 1 & - & - & 1 & - & - & 1 \\
20 & - & 1 & 1 & - & - & - & - & 1 & 1 & 1 & 1 & 1 & - & - & 1 & - & - & 1 & 1 & -
\end{array}
\tag{B.3}
$$

$$
\begin{array}{r|c|cccc|c|cccc|c|cccc|c|cccc|}
 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 & 17 & 18 & 19 & 20 \\
9 & - & - & 1 & - & 1 & 1 & 1 & - & - & 1 & 1 & 1 & - & - & 1 & 1 & 1 & - & - & - \\
10 & - & - & 1 & 1 & - & 1 & - & 1 & - & 1 & 1 & - & 1 & 1 & - & 1 & - & 1 & - & - \\
11 & - & 1 & - & 1 & - & 1 & 1 & - & 1 & - & 1 & - & 1 & - & 1 & 1 & - & - & 1 & - \\
12 & - & 1 & - & - & 1 & 1 & - & 1 & 1 & - & 1 & 1 & - & 1 & - & 1 & - & - & - & 1 \\
\hline
13 & - & - & 1 & - & 1 & 1 & 1 & - & 1 & - & - & 1 & 1 & 1 & - & - & - & 1 & 1 & - \\
14 & - & - & 1 & 1 & - & 1 & - & 1 & 1 & - & - & 1 & 1 & - & 1 & - & 1 & - & - & 1 \\
15 & - & 1 & - & 1 & - & 1 & 1 & - & - & 1 & - & 1 & - & 1 & 1 & - & - & 1 & - & 1 \\
16 & - & 1 & - & - & 1 & 1 & - & 1 & - & 1 & - & - & 1 & 1 & 1 & - & 1 & - & 1 & - \\
\hline
17 & - & - & - & 1 & 1 & - & 1 & 1 & 1 & - & 1 & - & - & 1 & 1 & - & 1 & 1 & - & - \\
18 & - & - & - & 1 & 1 & - & 1 & 1 & - & 1 & 1 & 1 & 1 & - & - & - & - & - & 1 & 1 \\
19 & - & 1 & 1 & - & - & - & 1 & - & 1 & 1 & 1 & - & 1 & 1 & - & - & 1 & - & - & 1 \\
20 & - & 1 & 1 & - & - & - & - & 1 & 1 & 1 & 1 & 1 & - & - & 1 & - & - & 1 & 1 & -
\end{array}
\tag{B.4}
$$

$$
\begin{array}{r|c|cccc|c|cccc|c|cccc|c|cccc|}
 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 & 17 & 18 & 19 & 20 \\
9 & - & - & 1 & 1 & - & 1 & 1 & - & - & 1 & 1 & - & 1 & - & 1 & 1 & 1 & - & - & - \\
10 & - & - & 1 & - & 1 & 1 & - & 1 & - & 1 & 1 & 1 & - & 1 & - & 1 & - & 1 & - & - \\
11 & - & 1 & - & - & 1 & 1 & 1 & - & 1 & - & 1 & 1 & - & - & 1 & 1 & - & - & 1 & - \\
12 & - & 1 & - & 1 & - & 1 & - & 1 & 1 & - & 1 & - & 1 & 1 & - & 1 & - & - & - & 1 \\
\hline
13 & - & - & 1 & - & 1 & 1 & 1 & - & 1 & - & - & 1 & 1 & 1 & - & - & 1 & - & - & 1 \\
14 & - & - & 1 & 1 & - & 1 & - & 1 & 1 & - & - & 1 & 1 & - & 1 & - & - & 1 & 1 & - \\
15 & - & 1 & - & 1 & - & 1 & 1 & - & - & 1 & - & 1 & - & 1 & 1 & - & - & 1 & - & 1 \\
16 & - & 1 & - & - & 1 & 1 & - & 1 & - & 1 & - & - & 1 & 1 & 1 & - & 1 & - & 1 & - \\
\hline
17 & - & - & - & 1 & 1 & - & 1 & 1 & 1 & - & 1 & - & - & 1 & 1 & - & 1 & 1 & - & - \\
18 & - & - & - & 1 & 1 & - & 1 & 1 & - & 1 & 1 & 1 & 1 & - & - & - & - & - & 1 & 1 \\
19 & - & 1 & 1 & - & - & - & 1 & - & 1 & 1 & 1 & - & 1 & 1 & - & - & - & 1 & 1 & - \\
20 & - & 1 & 1 & - & - & - & - & 1 & 1 & 1 & 1 & 1 & - & - & 1 & - & 1 & - & - & 1
\end{array}
\tag{B.5}
$$

$$
\begin{array}{r|c|cccc|c|cccc|c|cccc|c|cccc|}
 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 & 17 & 18 & 19 & 20 \\
9 & - & - & 1 & - & 1 & 1 & - & 1 & - & 1 & 1 & - & 1 & 1 & - & 1 & 1 & - & - & - \\
10 & - & - & 1 & 1 & - & 1 & 1 & - & - & 1 & 1 & 1 & - & - & 1 & 1 & - & 1 & - & - \\
11 & - & 1 & - & 1 & - & 1 & - & 1 & 1 & - & 1 & - & 1 & - & 1 & 1 & - & - & 1 & - \\
12 & - & 1 & - & - & 1 & 1 & 1 & - & 1 & - & 1 & 1 & - & 1 & - & 1 & - & - & - & 1 \\
\hline
13 & - & - & 1 & 1 & - & 1 & - & 1 & 1 & - & - & 1 & 1 & 1 & - & - & - & 1 & - & 1 \\
14 & - & - & 1 & - & 1 & 1 & 1 & - & 1 & - & - & 1 & 1 & - & 1 & - & 1 & - & 1 & - \\
15 & - & 1 & - & - & 1 & 1 & - & 1 & - & 1 & - & 1 & - & 1 & 1 & - & - & 1 & 1 & - \\
16 & - & 1 & - & 1 & - & 1 & 1 & - & - & 1 & - & - & 1 & 1 & 1 & - & 1 & - & - & 1 \\
\hline
17 & - & - & - & 1 & 1 & - & 1 & 1 & 1 & - & 1 & - & - & 1 & 1 & - & 1 & 1 & - & - \\
18 & - & - & - & 1 & 1 & - & 1 & 1 & - & 1 & 1 & 1 & 1 & - & - & - & - & - & 1 & 1 \\
19 & - & 1 & 1 & - & - & - & 1 & - & 1 & 1 & 1 & - & 1 & 1 & - & - & - & 1 & 1 & - \\
20 & - & 1 & 1 & - & - & - & - & 1 & 1 & 1 & 1 & 1 & - & - & 1 & - & 1 & - & - & 1
\end{array}
\tag{B.6}
$$

V. COMPLETIONS OF THE THIRD START

Our third start is

$$\begin{array}{c|c|cccc|c|cccc|c|cccc|c|cccc|}
 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 & 17 & 18 & 19 & 20 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
2 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & - & - & - & - & - & - & - & - & - & - \\
3 & 1 & 1 & 1 & 1 & 1 & - & - & - & - & - & 1 & 1 & 1 & 1 & 1 & - & - & - & - & - \\
4 & 1 & - & - & - & - & - & 1 & 1 & 1 & 1 & - & 1 & 1 & 1 & 1 & 1 & - & - & - & - \\
\cline{2-21}
5 & 1 & 1 & - & - & - & 1 & 1 & 1 & - & - & 1 & 1 & 1 & - & - & - & 1 & 1 & - & - \\
6 & 1 & - & 1 & - & - & 1 & 1 & - & 1 & - & 1 & 1 & - & 1 & - & - & - & - & 1 & 1 \\
7 & 1 & - & - & 1 & - & 1 & - & - & 1 & 1 & 1 & - & 1 & - & 1 & - & 1 & - & 1 & - \\
8 & 1 & - & - & - & 1 & 1 & - & 1 & - & 1 & 1 & - & - & 1 & 1 & - & - & 1 & - & 1
\end{array} \tag{5.1}$$

The group G_3 of automorphisms of this start is of order 192. Generating automorphisms of G_3 are

$$\begin{array}{ll}
\alpha_3 \left\{ \begin{array}{ll} \begin{array}{l}\text{Column}\\ \text{replacement}\end{array} & \begin{pmatrix} 1,2,3,4,5, & 6, & 7, & 8, & 9,10,11,12,13,14,15,16,17,18,19,20 \\ 1,2,3,5,4,11,12,13,14,15, & 6, & 7, & 8, & 9,10,16,18,17,20,19 \end{pmatrix} \\ \begin{array}{l}\text{Row}\\ \text{replacement}\end{array} & \begin{pmatrix} 1,2,3,4,5,6,7,8 \\ 1,3,2,4,5,6,8,7 \end{pmatrix} \end{array} \right. \\
\\
\beta_3 \left\{ \begin{array}{ll} \text{Column} & \begin{pmatrix} 1,2,3,4,5,6,7,8, & 9,10,11,12,13,14,15,16,17,18,19,20 \\ 6,3,4,5,2,1,9,7,10, & 8,11,19,20,17,18,16,14,12,15,13 \end{pmatrix} \\ \text{Row} & \begin{pmatrix} 1,2, & 3, & 4,5,6,7,8 \\ 1,2, & -4, & -3,8,5,6,7 \end{pmatrix} \end{array} \right. \\
\\
\gamma_3 \left\{ \begin{array}{ll} \text{Column} & \begin{pmatrix} 1, & 2,3, & 4,5, & 6, & 7, & 8, & 9,10, & 11, & 12, & 13,14,15,16, & 17, & 18,19,20 \\ -4, & -16,1,11,6, & -2, & -17, & -13,14,20, & -3, & -19, & -9, & 8,18, & 5,-7, & -12,15,10 \end{pmatrix} \\ \text{Row} & \begin{pmatrix} 1, & 2, & 3,4,5,6, & 7,8 \\ -5, & -7, & -8,6,3,2, & -4,1 \end{pmatrix} \end{array} \right. \\
\\
\delta_3 \left\{ \begin{array}{ll} \text{Column} & \begin{pmatrix} 1,2,3,4,5, & 6, & 7, & 8, & 9,10,11,12,13,14,15,16,17,18,19,20 \\ 1,3,2,4,5,11,12,14,13,15, & 6, & 7, & 9, & 8,10,16,19,20,17,18 \end{pmatrix} \\ \text{Row} & \begin{pmatrix} 1,2,3,4,5,6,7,8 \\ 1,3,2,4,6,5,7,8 \end{pmatrix} \end{array} \right.
\end{array} \tag{5.2}$$

There are 144 rows consistent with our start of eight rows, there being 12 possibilities for each of rows 9 through 20. We give here the possibilities for rows 9, 10, 11, and 12.

Row 9	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
1	–	1	1	–	–	1	1	–	–	1	1	–	–	1	1	1	1	–	–	–
2	–	1	1	–	–	1	–	1	1	–	1	–	–	1	1	1	1	–	–	–
3	–	1	–	–	1	1	–	–	1	1	1	1	–	1	–	1	1	–	–	–
4	–	1	–	–	1	1	1	–	1	–	1	–	–	1	1	1	1	–	–	–
5	–	–	1	1	–	1	1	1	–	–	1	–	–	1	1	1	1	–	–	–
6	–	–	1	1	–	1	–	1	–	1	1	1	–	1	–	1	1	–	–	–
7	–	–	1	–	1	1	1	–	–	1	1	1	–	–	1	1	1	–	–	–
8	–	–	1	–	1	1	1	–	–	1	1	–	1	1	–	1	1	–	–	–
9	–	–	1	–	1	1	–	1	1	–	1	1	–	–	1	1	1	–	–	–
10	–	–	1	–	1	1	–	1	1	–	1	–	1	1	–	1	1	–	–	–
11	–	–	–	1	1	1	1	–	–	1	1	1	–	1	–	1	1	–	–	–
12	–	–	–	1	1	1	–	1	1	–	1	1	–	1	–	1	1	–	–	–

(5.3)

Row 10	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
1	–	1	1	–	–	1	–	–	1	1	1	1	–	–	1	1	–	1	–	–
2	–	1	1	–	–	1	–	–	1	1	1	–	1	1	–	1	–	1	–	–
3	–	1	–	1	–	1	1	–	1	–	1	–	–	1	1	1	–	1	–	–
4	–	1	–	1	–	1	–	–	1	1	1	1	–	1	–	1	–	1	–	–
5	–	–	1	–	1	1	–	–	1	1	1	1	1	–	–	1	–	1	–	–
6	–	–	1	–	1	1	1	–	1	–	1	–	1	–	1	1	–	1	–	–
7	–	–	1	1	–	1	1	–	–	1	1	1	–	–	1	1	–	1	–	–
8	–	–	1	1	–	1	–	1	1	–	1	1	–	–	1	1	–	1	–	–
9	–	–	1	1	–	1	1	–	–	1	1	–	1	1	–	1	–	1	–	–
10	–	–	1	1	–	1	–	1	1	–	1	–	1	1	–	1	–	1	–	–
11	–	–	–	1	1	1	1	–	1	–	1	1	–	–	1	1	–	1	–	–
12	–	–	–	1	1	1	1	–	1	–	1	–	1	1	–	1	–	1	–	–

(5.4)

Row 11	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
1	–	1	1	–	–	1	–	1	–	1	1	1	–	–	1	1	–	–	1	–
2	–	1	1	–	–	1	–	1	–	1	1	–	1	1	–	1	–	–	1	–
3	–	–	1	–	1	1	1	1	–	–	1	–	1	–	1	1	–	–	1	–
4	–	–	1	–	1	1	–	1	–	1	1	1	1	–	–	1	–	–	1	–
5	–	1	–	1	–	1	–	1	–	1	1	1	–	1	–	1	–	–	1	–
6	–	1	–	1	–	1	1	1	–	–	1	–	–	1	1	1	–	–	1	–
7	–	1	–	–	1	1	1	–	–	1	1	1	–	–	1	1	–	–	1	–
8	–	1	–	–	1	1	–	1	1	–	1	1	–	–	1	1	–	–	1	–
9	–	1	–	–	1	1	1	–	–	1	1	–	1	1	–	1	–	–	1	–
10	–	1	–	–	1	1	–	1	1	–	1	–	1	1	–	1	–	–	1	–
11	–	–	–	1	1	1	1	1	–	–	1	1	–	–	1	1	–	–	1	–
12	–	–	–	1	1	1	1	1	–	–	1	–	1	1	–	1	–	–	1	–

(5.5)

Row 12	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
1	–	1	1	–	–	1	1	–	–	1	1	–	1	–	1	1	–	–	–	1
2	–	1	1	–	–	1	–	1	1	–	1	–	1	–	1	1	–	–	–	1
3	–	–	1	1	–	1	–	1	–	1	1	1	1	–	–	1	–	–	–	1
4	–	–	1	1	–	1	1	1	–	–	1	–	1	–	1	1	–	–	–	1
5	–	1	–	–	1	1	1	–	1	–	1	–	1	–	1	1	–	–	–	1
6	–	1	–	–	1	1	–	–	1	1	1	1	1	–	–	1	–	–	–	1
7	–	1	–	1	–	1	1	–	–	1	1	1	–	–	1	1	–	–	–	1
8	–	1	–	1	–	1	1	–	–	1	1	–	1	1	–	1	–	–	–	1
9	–	1	–	1	–	1	–	1	1	–	1	1	–	–	1	1	–	–	–	1
10	–	1	–	1	–	1	–	1	1	–	1	–	1	1	–	1	–	–	–	1
11	–	–	–	1	1	1	1	–	–	1	1	1	1	–	–	1	–	–	–	1
12	–	–	–	1	1	1	–	1	1	–	1	1	1	–	–	1	–	–	–	1

(5.6)

The numbering of the 12 possibilities in each case has been chosen so that rows numbered *i* are interchanged between the row 9 and row 10 possibilities by α_3, and α_3 also interchanges the row 11 and row 12 possibilities. The automorphism δ_3 interchanges the *i*th possibility for row 9 and row 11. By similar applications of automorphisms, it is easy to list the possibilities for the remaining rows.

The group G_3 moves the 144 possibilities in two orbits, one of 48, containing numbers 1, 2, 11, and 12 in each of the lists above, the other of 96, containing numbers 3, 4, 5, 6, 7, 8, 9, and 10 in each of the lists above. We easily find that every completion involves using rows from both orbits, and so without loss of generality we may use choice 1 for row 9 in every case. The automorphism γ_3 is of order 4 and generates the subgroup of G_3, fixing choice 1 of row 9.

With choice 1 for row 9, there are exactly 10 completions for our 8-row start.

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
9	–	1	1	–	–	1	1	–	–	1	1	–	–	1	1	1	1	–	–	–
10	–	–	–	1	1	1	1	–	1	–	1	1	–	–	1	1	–	1	–	–
11	–	–	1	–	1	1	–	1	–	1	1	1	1	–	–	1	–	–	1	–
12	–	1	–	1	–	1	–	1	1	–	1	–	1	1	–	1	–	–	–	1
13	–	–	–	1	1	1	1	–	–	1	–	1	1	1	–	–	1	–	–	1
14	–	1	1	–	–	1	–	–	1	1	–	1	1	–	1	–	–	1	–	1
15	–	1	–	–	1	1	–	1	1	–	–	1	–	1	1	–	1	–	1	–
16	–	–	1	1	–	1	1	1	–	–	–	–	1	1	1	–	–	1	1	–
17	–	–	1	–	1	–	1	1	1	–	1	–	1	–	1	–	1	–	–	1
18	–	1	–	1	–	–	1	1	–	1	1	1	–	–	1	–	–	–	1	1
19	–	1	–	–	1	–	1	–	1	1	1	–	1	1	–	–	–	1	1	–
20	–	–	1	1	–	–	–	1	1	1	1	1	–	1	–	–	1	1	–	–

(C.1)

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
9	–	1	1	–	–	1	1	–	–	1	1	–	–	1	1	1	1	–	–	–
10	–	–	–	1	1	1	1	–	1	–	1	1	–	–	1	1	–	1	–	–
11	–	1	–	–	1	1	–	1	1	–	1	–	1	1	–	1	–	–	1	–
12	–	–	1	1	–	1	–	1	–	1	1	1	1	–	–	1	–	–	–	1
13	–	–	1	–	1	1	–	–	1	1	–	1	1	1	–	–	1	1	–	–
14	–	1	–	–	1	1	1	–	–	1	–	1	1	–	1	–	–	–	1	1
15	–	1	–	1	–	1	–	1	1	–	–	1	–	1	1	–	1	–	–	1
16	–	–	1	1	–	1	1	1	–	–	–	–	1	1	1	–	–	1	1	–
17	–	–	1	–	1	–	1	1	1	–	1	–	1	–	1	–	1	–	–	1
18	–	–	–	1	1	–	1	1	–	1	1	1	–	1	–	–	1	–	1	–
19	–	1	–	1	–	–	1	–	1	1	1	–	1	1	–	–	–	1	–	1
20	–	1	1	–	–	–	–	1	1	1	1	1	–	–	1	–	–	1	1	–

(C.2)

$$
\begin{array}{r|c|cccc|c|cccc|c|cccc|c|cccc|}
\multicolumn{1}{r}{} & \multicolumn{1}{c}{1} & \multicolumn{1}{c}{2} & 3 & 4 & \multicolumn{1}{c}{5} & \multicolumn{1}{c}{6} & \multicolumn{1}{c}{7} & 8 & 9 & \multicolumn{1}{c}{10} & \multicolumn{1}{c}{11} & \multicolumn{1}{c}{12} & 13 & 14 & \multicolumn{1}{c}{15} & \multicolumn{1}{c}{16} & \multicolumn{1}{c}{17} & 18 & 19 & \multicolumn{1}{c}{20} \\
9 & - & 1 & 1 & - & - & 1 & 1 & - & - & 1 & 1 & - & - & 1 & 1 & 1 & 1 & - & - & - \\
10 & - & - & 1 & 1 & - & 1 & - & 1 & 1 & - & 1 & - & 1 & 1 & - & 1 & - & 1 & - & - \\
11 & - & 1 & - & - & 1 & 1 & - & 1 & 1 & - & 1 & 1 & - & - & 1 & 1 & - & - & 1 & - \\
12 & - & - & - & 1 & 1 & 1 & 1 & - & - & 1 & 1 & 1 & 1 & - & - & 1 & - & - & - & 1 \\
\cline{2-21}
13 & - & - & 1 & - & 1 & 1 & - & 1 & - & 1 & - & 1 & 1 & 1 & - & - & 1 & - & 1 & - \\
14 & - & 1 & 1 & - & - & 1 & - & - & 1 & 1 & - & 1 & 1 & - & 1 & - & - & 1 & - & 1 \\
15 & - & - & - & 1 & 1 & 1 & 1 & - & 1 & - & - & 1 & - & 1 & 1 & - & 1 & 1 & - & - \\
16 & - & 1 & - & 1 & - & 1 & 1 & 1 & - & - & - & - & 1 & 1 & 1 & - & - & - & 1 & 1 \\
\cline{2-21}
17 & - & - & 1 & - & 1 & - & 1 & 1 & 1 & - & 1 & - & 1 & - & 1 & - & 1 & - & - & 1 \\
18 & - & - & 1 & 1 & - & - & 1 & 1 & - & 1 & 1 & 1 & - & - & 1 & - & - & 1 & 1 & - \\
19 & - & 1 & - & - & 1 & - & 1 & - & 1 & 1 & 1 & - & 1 & 1 & - & - & - & 1 & 1 & - \\
20 & - & 1 & - & 1 & - & - & - & 1 & 1 & 1 & 1 & 1 & - & 1 & - & - & 1 & - & - & 1 \\
\end{array}
\tag{C.3}
$$

$$
\begin{array}{r|c|cccc|c|cccc|c|cccc|c|cccc|}
\multicolumn{1}{r}{} & \multicolumn{1}{c}{1} & \multicolumn{1}{c}{2} & 3 & 4 & \multicolumn{1}{c}{5} & \multicolumn{1}{c}{6} & \multicolumn{1}{c}{7} & 8 & 9 & \multicolumn{1}{c}{10} & \multicolumn{1}{c}{11} & \multicolumn{1}{c}{12} & 13 & 14 & \multicolumn{1}{c}{15} & \multicolumn{1}{c}{16} & \multicolumn{1}{c}{17} & 18 & 19 & \multicolumn{1}{c}{20} \\
9 & - & 1 & 1 & - & - & 1 & 1 & - & - & 1 & 1 & - & - & 1 & 1 & 1 & 1 & - & - & - \\
10 & - & - & 1 & 1 & - & 1 & - & 1 & 1 & - & 1 & - & 1 & 1 & - & 1 & - & 1 & - & - \\
11 & - & - & - & 1 & 1 & 1 & 1 & 1 & - & - & 1 & 1 & - & - & 1 & 1 & - & - & 1 & - \\
12 & - & 1 & - & - & 1 & 1 & - & - & 1 & 1 & 1 & 1 & 1 & - & - & 1 & - & - & - & 1 \\
\cline{2-21}
13 & - & - & 1 & - & 1 & 1 & - & 1 & - & 1 & - & 1 & 1 & 1 & - & - & 1 & - & 1 & - \\
14 & - & - & 1 & 1 & - & 1 & 1 & - & - & 1 & - & 1 & 1 & - & 1 & - & - & 1 & - & 1 \\
15 & - & 1 & - & 1 & - & 1 & - & 1 & 1 & - & - & 1 & - & 1 & 1 & - & 1 & - & - & 1 \\
16 & - & 1 & - & - & 1 & 1 & 1 & - & 1 & - & - & - & 1 & 1 & 1 & - & - & 1 & 1 & - \\
\cline{2-21}
17 & - & - & 1 & - & 1 & - & 1 & 1 & 1 & - & 1 & - & 1 & - & 1 & - & 1 & - & - & 1 \\
18 & - & 1 & - & 1 & - & - & 1 & 1 & - & 1 & 1 & - & 1 & 1 & - & - & - & - & 1 & 1 \\
19 & - & - & - & 1 & 1 & - & 1 & - & 1 & 1 & 1 & 1 & - & 1 & - & - & 1 & 1 & - & - \\
20 & - & 1 & 1 & - & - & - & - & 1 & 1 & 1 & 1 & 1 & - & - & 1 & - & - & 1 & 1 & - \\
\end{array}
\tag{C.4}
$$

$$\begin{array}{r|c|cccc|c|cccc|c|cccc|c|cccc|}
 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 & 17 & 18 & 19 & 20 \\
9 & - & 1 & 1 & - & - & 1 & 1 & - & - & 1 & 1 & - & - & 1 & 1 & 1 & 1 & - & - & - \\
10 & - & - & 1 & 1 & - & 1 & - & 1 & 1 & - & 1 & 1 & - & - & 1 & 1 & - & 1 & - & - \\
11 & - & 1 & - & - & 1 & 1 & - & 1 & 1 & - & 1 & - & 1 & 1 & - & 1 & - & - & 1 & - \\
12 & - & - & - & 1 & 1 & 1 & 1 & - & - & 1 & 1 & 1 & 1 & - & - & 1 & - & - & - & 1 \\
\cline{2-21}
13 & - & - & 1 & - & 1 & 1 & - & - & 1 & 1 & - & 1 & 1 & 1 & - & - & 1 & 1 & - & - \\
14 & - & 1 & 1 & - & - & 1 & - & 1 & - & 1 & - & 1 & 1 & - & 1 & - & - & - & 1 & 1 \\
15 & - & - & - & 1 & 1 & 1 & 1 & 1 & - & - & - & 1 & - & 1 & 1 & - & 1 & - & 1 & - \\
16 & - & 1 & - & 1 & - & 1 & 1 & - & 1 & - & - & - & 1 & 1 & 1 & - & - & 1 & - & 1 \\
\cline{2-21}
17 & - & - & 1 & - & 1 & - & 1 & 1 & 1 & - & 1 & - & 1 & - & 1 & - & 1 & - & - & 1 \\
18 & - & - & 1 & 1 & - & - & 1 & 1 & - & 1 & 1 & - & 1 & 1 & - & - & - & 1 & 1 & - \\
19 & - & 1 & - & - & 1 & - & 1 & - & 1 & 1 & 1 & 1 & - & - & 1 & - & - & 1 & 1 & - \\
20 & - & 1 & - & 1 & - & - & - & 1 & 1 & 1 & 1 & 1 & - & 1 & - & - & 1 & - & - & 1
\end{array} \tag{C.5}$$

$$\begin{array}{r|c|cccc|c|cccc|c|cccc|c|cccc|}
 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 & 17 & 18 & 19 & 20 \\
9 & - & 1 & 1 & - & - & 1 & 1 & - & - & 1 & 1 & - & - & 1 & 1 & 1 & 1 & - & - & - \\
10 & - & - & 1 & 1 & - & 1 & - & 1 & 1 & - & 1 & 1 & - & - & 1 & 1 & - & 1 & - & - \\
11 & - & - & - & 1 & 1 & 1 & 1 & 1 & - & - & 1 & - & 1 & 1 & - & 1 & - & - & 1 & - \\
12 & - & 1 & - & - & 1 & 1 & - & - & 1 & 1 & 1 & 1 & 1 & - & - & 1 & - & - & - & 1 \\
\cline{2-21}
13 & - & - & 1 & 1 & - & 1 & - & 1 & - & 1 & - & 1 & 1 & 1 & - & - & 1 & - & - & 1 \\
14 & - & - & 1 & - & 1 & 1 & 1 & - & - & 1 & - & 1 & 1 & - & 1 & - & - & 1 & 1 & - \\
15 & - & 1 & - & - & 1 & 1 & - & 1 & 1 & - & - & 1 & - & 1 & 1 & - & 1 & - & 1 & - \\
16 & - & 1 & - & 1 & - & 1 & 1 & - & 1 & - & - & - & 1 & 1 & 1 & - & - & 1 & - & 1 \\
\cline{2-21}
17 & - & - & 1 & - & 1 & - & 1 & 1 & 1 & - & 1 & - & 1 & - & 1 & - & 1 & - & - & 1 \\
18 & - & 1 & - & 1 & - & - & 1 & 1 & - & 1 & 1 & 1 & - & - & 1 & - & - & - & 1 & 1 \\
19 & - & - & - & 1 & 1 & - & 1 & - & 1 & 1 & 1 & 1 & - & 1 & - & - & 1 & 1 & - & - \\
20 & - & 1 & 1 & - & - & - & - & 1 & 1 & 1 & 1 & - & 1 & 1 & - & - & - & 1 & 1 & -
\end{array} \tag{C.6}$$

$$\begin{array}{r|c|cccc|c|cccc|c|cccc|c|cccc|}
 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 & 17 & 18 & 19 & 20 \\
9 & - & 1 & 1 & - & - & 1 & 1 & - & - & 1 & 1 & - & - & 1 & 1 & 1 & 1 & - & - & - \\
10 & - & - & 1 & - & 1 & 1 & - & - & 1 & 1 & 1 & 1 & 1 & - & - & 1 & - & 1 & - & - \\
11 & - & - & - & 1 & 1 & 1 & 1 & 1 & - & - & 1 & - & 1 & 1 & - & 1 & - & - & 1 & - \\
12 & - & 1 & - & 1 & - & 1 & - & 1 & 1 & - & 1 & 1 & - & - & 1 & 1 & - & - & - & 1 \\
\cline{2-21}
13 & - & - & 1 & 1 & - & 1 & - & 1 & - & 1 & - & 1 & 1 & 1 & - & - & 1 & - & - & 1 \\
14 & - & 1 & - & - & 1 & 1 & 1 & - & - & 1 & - & 1 & 1 & - & 1 & - & - & - & 1 & 1 \\
15 & - & - & - & 1 & 1 & 1 & 1 & - & 1 & - & - & 1 & - & 1 & 1 & - & 1 & 1 & - & - \\
16 & - & 1 & 1 & - & - & 1 & - & 1 & 1 & - & - & - & 1 & 1 & 1 & - & - & 1 & 1 & - \\
\cline{2-21}
17 & - & - & 1 & - & 1 & - & 1 & 1 & 1 & - & 1 & - & 1 & - & 1 & - & 1 & - & - & 1 \\
18 & - & - & 1 & 1 & - & - & 1 & 1 & - & 1 & 1 & 1 & - & - & 1 & - & - & 1 & 1 & - \\
19 & - & 1 & - & 1 & - & - & 1 & - & 1 & 1 & 1 & - & 1 & 1 & - & - & - & 1 & - & 1 \\
20 & - & 1 & - & - & 1 & - & - & 1 & 1 & 1 & 1 & 1 & - & 1 & - & - & 1 & - & 1 & -
\end{array} \tag{C.7}$$

$$\begin{array}{r|c|cccc|c|cccc|c|cccc|c|cccc|}
 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 & 17 & 18 & 19 & 20 \\
9 & - & 1 & 1 & - & - & 1 & 1 & - & - & 1 & 1 & - & - & 1 & 1 & 1 & 1 & - & - & - \\
10 & - & - & 1 & - & 1 & 1 & - & - & 1 & 1 & 1 & 1 & 1 & - & - & 1 & - & 1 & - & - \\
11 & - & - & - & 1 & 1 & 1 & 1 & 1 & - & - & 1 & 1 & - & - & 1 & 1 & - & - & 1 & - \\
12 & - & 1 & - & 1 & - & 1 & - & 1 & 1 & - & 1 & - & 1 & 1 & - & 1 & - & - & - & 1 \\
\cline{2-21}
13 & - & - & - & 1 & 1 & 1 & 1 & - & - & 1 & - & 1 & 1 & 1 & - & - & 1 & - & - & 1 \\
14 & - & 1 & 1 & - & - & 1 & - & 1 & - & 1 & - & 1 & 1 & - & 1 & - & - & - & 1 & 1 \\
15 & - & - & 1 & 1 & - & 1 & - & 1 & 1 & - & - & 1 & - & 1 & 1 & - & 1 & 1 & - & - \\
16 & - & 1 & - & - & 1 & 1 & 1 & - & 1 & - & - & - & 1 & 1 & 1 & - & - & 1 & 1 & - \\
\cline{2-21}
17 & - & - & 1 & - & 1 & - & 1 & 1 & 1 & - & 1 & - & 1 & - & 1 & - & 1 & - & - & 1 \\
18 & - & - & 1 & 1 & - & - & 1 & 1 & - & 1 & 1 & - & 1 & 1 & - & - & - & 1 & 1 & - \\
19 & - & 1 & - & 1 & - & - & 1 & - & 1 & 1 & 1 & 1 & - & - & 1 & - & - & 1 & - & 1 \\
20 & - & 1 & - & - & 1 & - & - & 1 & 1 & 1 & 1 & 1 & - & 1 & - & - & 1 & - & 1 & -
\end{array} \tag{C.8}$$

$$
\begin{array}{r|c|cccc|c|cccc|c|cccc|c|cccc|}
 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 & 17 & 18 & 19 & 20 \\
9 & - & 1 & 1 & - & - & 1 & 1 & - & - & 1 & 1 & - & - & 1 & 1 & 1 & 1 & - & - & - \\
10 & - & - & - & 1 & 1 & 1 & 1 & - & 1 & - & 1 & - & 1 & 1 & - & 1 & - & 1 & - & - \\
11 & - & 1 & - & - & 1 & 1 & - & 1 & 1 & - & 1 & 1 & - & - & 1 & 1 & - & - & 1 & - \\
12 & - & - & 1 & 1 & - & 1 & - & 1 & - & 1 & 1 & 1 & 1 & - & - & 1 & - & - & - & 1 \\
\cline{2-21}
13 & - & 1 & - & - & 1 & 1 & - & - & 1 & 1 & - & 1 & 1 & 1 & - & - & 1 & - & - & 1 \\
14 & - & - & 1 & - & 1 & 1 & 1 & - & - & 1 & - & 1 & 1 & - & 1 & - & - & 1 & 1 & - \\
15 & - & - & 1 & 1 & - & 1 & - & 1 & 1 & - & - & 1 & - & 1 & 1 & - & 1 & 1 & - & - \\
16 & - & 1 & - & 1 & - & 1 & 1 & 1 & - & - & - & - & 1 & 1 & 1 & - & - & - & 1 & 1 \\
\cline{2-21}
17 & - & - & 1 & - & 1 & - & 1 & 1 & 1 & - & 1 & - & 1 & - & 1 & - & 1 & - & - & 1 \\
18 & - & - & - & 1 & 1 & - & 1 & 1 & - & 1 & 1 & 1 & - & 1 & - & - & 1 & - & 1 & - \\
19 & - & 1 & - & 1 & - & - & 1 & - & 1 & 1 & 1 & 1 & - & - & 1 & - & - & 1 & - & 1 \\
20 & - & 1 & 1 & - & - & - & - & 1 & 1 & 1 & 1 & - & 1 & 1 & - & - & - & 1 & 1 & -
\end{array}
\tag{C.9}
$$

$$
\begin{array}{r|c|cccc|c|cccc|c|cccc|c|cccc|}
 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 & 17 & 18 & 19 & 20 \\
9 & - & 1 & 1 & - & - & 1 & 1 & - & - & 1 & 1 & - & - & 1 & 1 & 1 & 1 & - & - & - \\
10 & - & - & - & 1 & 1 & 1 & 1 & - & 1 & - & 1 & - & 1 & 1 & - & 1 & - & 1 & - & - \\
11 & - & - & 1 & - & 1 & 1 & - & 1 & - & 1 & 1 & 1 & 1 & - & - & 1 & - & - & 1 & - \\
12 & - & 1 & - & 1 & - & 1 & - & 1 & 1 & - & 1 & 1 & - & - & 1 & 1 & - & - & - & 1 \\
\cline{2-21}
13 & - & 1 & - & - & 1 & 1 & - & - & 1 & 1 & - & 1 & 1 & 1 & - & - & 1 & - & - & 1 \\
14 & - & - & 1 & 1 & - & 1 & 1 & - & - & 1 & - & 1 & 1 & - & 1 & - & - & 1 & - & 1 \\
15 & - & - & - & 1 & 1 & 1 & 1 & 1 & - & - & - & 1 & - & 1 & 1 & - & 1 & - & 1 & - \\
16 & - & 1 & 1 & - & - & 1 & - & 1 & 1 & - & - & - & 1 & 1 & 1 & - & - & 1 & 1 & - \\
\cline{2-21}
17 & - & - & 1 & - & 1 & - & 1 & 1 & 1 & - & 1 & - & 1 & - & 1 & - & 1 & - & - & 1 \\
18 & - & 1 & - & 1 & - & - & 1 & 1 & - & 1 & 1 & - & 1 & 1 & - & - & - & - & 1 & 1 \\
19 & - & 1 & - & - & 1 & - & 1 & - & 1 & 1 & 1 & 1 & - & - & 1 & - & - & 1 & 1 & - \\
20 & - & - & 1 & 1 & - & - & - & 1 & 1 & 1 & 1 & 1 & - & 1 & - & - & 1 & 1 & - & -
\end{array}
\tag{C.10}
$$

VI. THE THREE CLASSES OF EQUIVALENT MATRICES

Under equivalence the H-matrices of order 20 fall into three classes. We have already shown that every H-matrix of order 20 is equivalent to one of the 22 matrices we have listed, six with the first start, six with the second, and ten with the third. We must find the equivalences between these 22 matrices, and show that we have found all such equivalences.

These H_{20}'s fall into three equivalence classes, which we designate by Q, P, and N, since Q is the class determined by quadratic residues modulo 19; P is the class determined by Paley, where since $20 = 2(9 + 1)$, we may use his construction based on $GF(3^2)$; and N is a new class. The 22 H_{20}'s fall into these three classes in the following way:

$$\begin{aligned}\text{Class } Q &= \text{C.2, C.6, C.8, C.10}\\ \text{Class } P &= \text{A.3, A.4, A.5, A.6, B.1, B.2, B.5, B.6, C.3}\\ \text{Class } N &= \text{A.1, A.2, B.3, B.4, C.1, C.4, C.5, C.7, C.9}\end{aligned} \tag{6.1}$$

We take (C.2) as our representative of class Q. Its group of automorphisms $G(Q)$ is of order $20 \cdot 19 \cdot 9 = 3520$ and is generated by the following automorphisms:

Automorphisms $G(Q)$

$$\rho\left\{\begin{array}{ll}\text{Row} & \begin{pmatrix}1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20\\ -3, -1, 2, 4, 15, 16, 13, 14, 9, 12, 10, 11, 17, 20, 18, 19, 7, 5, 6, 8\end{pmatrix}\\ \begin{array}{l}\text{Column}\\ \text{inverse}\end{array} & \begin{pmatrix}1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20\\ -6, -9, -8, -10, -7, 11, 14, 12, 13, 15, -1, -3, -2, -5, -4, 16, 17, 19, 20, 18\end{pmatrix}\end{array}\right.$$

$$\sigma\left\{\begin{array}{ll}\text{Row} & \begin{pmatrix}1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20\\ 1, 18, -2, -6, -3, -11, -20, 17, 14, 19, -13, -7, 10, -15, 16, 4, 12, 8, -5, -9\end{pmatrix}\\ \begin{array}{l}\text{Column}\\ \text{inverse}\end{array} & \begin{pmatrix}1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20\\ 19, 16, 20, 9, 10, 17, 6, 8, 18, 7, 5, 3, 15, 4, 14, 13, 1, 12, 2, 11\end{pmatrix}\end{array}\right. \tag{6.2}$$

$$\tau\left\{\begin{array}{ll}\text{Row} & \begin{pmatrix}1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20\\ 1, 2, 10, 20, 4, -16, -18, -14, -15, -17, -13, -19, 3, -11, 5, 8, -6, 12, 9, -7\end{pmatrix}\\ \begin{array}{l}\text{Column}\\ \text{inverse}\end{array} & \begin{pmatrix}1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20\\ 6, 9, 10, 5, 3, 4, 2, 8, 1, 7, 14, 12, 17, 18, 13, 11, 20, 15, 16, 19\end{pmatrix}\end{array}\right.$$

The following column mappings take (C.6), (C.8), and (C.10) into (C.2). The column mappings completely determine the row mappings, which are not given.

$$\begin{array}{l}\text{Col. C.6}\\ \text{Col. C.2}\end{array}\begin{pmatrix}1, & 2, & 3, & 4, & 5, & 6, & 7, & 8, & 9, & 10, & 11, & 12, & 13, & 14, & 15, & 16, & 17, & 18, & 19, & 20\\ 6, & 10, & 9, & 8, & 7, & 1, & 2, & 4, & 3, & 5, & 16, & 18, & 19, & 20, & 17, & 11, & 12, & 14, & 15, & 13\end{pmatrix},$$

$$\begin{array}{l}\text{Col. C.8}\\ \text{Col. C.2}\end{array}\begin{pmatrix}1, & 2, & 3, & 4, & 5, & 6, & 7, & 8, & 9, & 10, & 11, & 12, & 13, & 14, & 15, & 16, & 17, & 18, & 19, & 20\\ 16, & 15, & 13, & 14, & 12, & 11, & 19, & 20, & 18, & 17, & 1, & 7, & 8, & 10, & 9, & 6, & 4, & 2, & 3, & 5\end{pmatrix},$$

$$\begin{array}{l}\text{Col. C.10}\\ \text{Col. C.2}\end{array}\begin{pmatrix}1, & 2, & 3, & 4, & 5, & 6, & 7, & 8, & 9, & 10, & 11, & 12, & 13, & 14, & 15, & 16, & 17, & 18, & 19, & 20\\ 6, & 3, & 4, & 5, & 2, & 1, & 9, & 7, & 10, & 8, & 11, & 19, & 20, & 17, & 18, & 16, & 14, & 12, & 15, & 13\end{pmatrix}. \tag{6.3}$$

The following matrix is that derived from quadratic residues modulo 19:

$$Q19$$

$$\begin{array}{r|cccccccccccccccccccc}
 & \infty & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 & 17 & 18\\ \hline
\infty & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1\\
0 & 1 & - & 1 & - & - & 1 & 1 & 1 & 1 & - & 1 & - & 1 & - & - & - & - & 1 & 1 & -\\
1 & 1 & - & - & 1 & - & - & 1 & 1 & 1 & 1 & - & 1 & - & 1 & - & - & - & - & 1 & 1\\
2 & 1 & 1 & - & - & 1 & - & - & 1 & 1 & 1 & 1 & - & 1 & - & 1 & - & - & - & - & 1\\
3 & 1 & 1 & 1 & - & - & 1 & - & - & 1 & 1 & 1 & 1 & - & 1 & - & 1 & - & - & - & -\\
4 & 1 & - & 1 & 1 & - & - & 1 & - & - & 1 & 1 & 1 & 1 & - & 1 & - & 1 & - & - & -\\
5 & 1 & - & - & 1 & 1 & - & - & 1 & - & - & 1 & 1 & 1 & 1 & - & 1 & - & 1 & - & -\\
6 & 1 & - & - & - & 1 & 1 & - & - & 1 & - & - & 1 & 1 & 1 & 1 & - & 1 & - & 1 & -\\
7 & 1 & - & - & - & - & 1 & 1 & - & - & 1 & - & - & 1 & 1 & 1 & 1 & - & 1 & - & 1\\
8 & 1 & 1 & - & - & - & - & 1 & 1 & - & - & 1 & - & - & 1 & 1 & 1 & 1 & - & 1 & -\\
9 & 1 & - & 1 & - & - & - & - & 1 & 1 & - & - & 1 & - & - & 1 & 1 & 1 & 1 & - & 1\\
10 & 1 & 1 & - & 1 & - & - & - & - & 1 & 1 & - & - & 1 & - & - & 1 & 1 & 1 & 1 & -\\
11 & 1 & - & 1 & - & 1 & - & - & - & - & 1 & 1 & - & - & 1 & - & - & 1 & 1 & 1 & 1\\
12 & 1 & 1 & - & 1 & - & 1 & - & - & - & - & 1 & 1 & - & - & 1 & - & - & 1 & 1 & 1\\
13 & 1 & 1 & 1 & - & 1 & - & 1 & - & - & - & - & 1 & 1 & - & - & 1 & - & - & 1 & 1\\
14 & 1 & 1 & 1 & 1 & - & 1 & - & 1 & - & - & - & - & 1 & 1 & - & - & 1 & - & - & 1\\
15 & 1 & 1 & 1 & 1 & 1 & - & 1 & - & 1 & - & - & - & - & 1 & 1 & - & - & 1 & - & -\\
16 & 1 & - & 1 & 1 & 1 & 1 & - & 1 & - & 1 & - & - & - & - & 1 & 1 & - & - & 1 & -\\
17 & 1 & - & - & 1 & 1 & 1 & 1 & - & 1 & - & 1 & - & - & - & - & 1 & 1 & - & - & 1\\
18 & 1 & 1 & - & - & 1 & 1 & 1 & 1 & - & 1 & - & 1 & - & - & - & - & 1 & 1 & - & -
\end{array} \tag{6.4}$$

A column mapping taking Q into (C.2) is

$$\begin{array}{l}\text{Col. } Q \\ \text{Col. C.2}\end{array}\begin{pmatrix}\infty, 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18 \\ 8, 1, 19, 2, 16, 13, 15, 14, 4, 9, 18, 12, 3, 20, 11, 5, 10, 7, 6, 17\end{pmatrix} \tag{6.5}$$

For the class P we take (A.3) as representative.

Column mappings taking other matrices of class P into (A.3) are

$$\begin{array}{l}\text{Col. A.4} \\ \text{Col. A.3}\end{array}\begin{pmatrix}1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20 \\ 1, 5, 4, 3, 2, 6, 9, 10, 7, 8, 11, 14, 15, 12, 13, 16, 17, 18, 20, 19\end{pmatrix}$$

$$\begin{array}{l}\text{Col. A.5} \\ \text{Col. A.3}\end{array}\begin{pmatrix}1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20 \\ 6, 5, 2, 4, 3, 11, 18, 17, 20, 19, 1, 10, 9, 7, 8, 16, 14, 15, 13, 12\end{pmatrix}$$

$$\begin{array}{l}\text{Col. A.6} \\ \text{Col. A.3}\end{array}\begin{pmatrix}1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20 \\ 6, 5, 4, 2, 3, 1, 10, 9, 7, 8, 11, 18, 17, 20, 19, 16, 14, 15, 12, 13\end{pmatrix}$$

$$\begin{array}{l}\text{Col. B.1} \\ \text{Col. A.3}\end{array}\begin{pmatrix}1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20 \\ 2, 4, 14, 10, 20, 15, 8, 17, 6, 12, 16, 11, 1, 7, 18, 13, 19, 9, 5, 3\end{pmatrix}$$

$$\begin{array}{l}\text{Col. B.2} \\ \text{Col. A.3}\end{array}\begin{pmatrix}1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20 \\ 15, 4, 14, 10, 20, 2, 17, 8, 12, 6, 16, 19, 9, 5, 3, 13, 11, 1, 7, 18\end{pmatrix}$$

$$\begin{array}{l}\text{Col. B.5} \\ \text{Col. A.3}\end{array}\begin{pmatrix}1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20 \\ 5, 3, 2, 8, 7, 4, 9, 10, 6, 1, 16, 14, 15, 18, 17, 11, 19, 20, 12, 13\end{pmatrix}$$

$$\begin{array}{l}\text{Col. B.6} \\ \text{Col. A.3}\end{array}\begin{pmatrix}1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20 \\ 5, 2, 3, 8, 7, 4, 10, 9, 1, 6, 16, 18, 17, 14, 15, 11, 12, 13, 19, 20\end{pmatrix}$$

$$\begin{array}{l}\text{Col. C.3} \\ \text{Col. A.3}\end{array}\begin{pmatrix}1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20 \\ 16, 18, 15, 14, 17, 5, 7, 3, 2, 8, 11, 19, 13, 12, 20, 4, 9, 1, 6, 10\end{pmatrix} \tag{6.6}$$

The Class *P* includes the Paley type constructed from *GF* (9) given in (6.7) and the Williamson type given in (6.8).

Paley type

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
1	1	–	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
2	–	–	1	–	1	–	1	–	1	–	1	–	1	–	–	–	1	–	1	–
3	1	1	1	–	1	1	–	–	1	1	–	–	1	1	–	–	1	1	–	–
4	1	–	–	–	1	–	–	1	1	–	–	1	1	–	–	1	1	–	–	1
5	1	1	1	1	1	–	–	–	–	–	–	–	1	1	1	1	–	–	1	1
6	1	–	1	–	–	–	–	1	–	1	–	1	1	–	1	–	–	1	1	–
7	1	1	–	–	–	–	1	–	1	1	1	1	1	1	–	–	–	–	1	1
8	1	–	–	1	–	1	–	–	1	–	1	–	1	–	–	1	–	1	1	–
9	1	1	1	1	–	–	1	1	1	–	–	–	–	–	–	–	1	1	1	1
10	1	–	1	–	–	1	1	–	–	–	–	1	–	1	–	1	1	–	1	–
11	1	1	–	–	–	–	1	1	–	–	1	–	1	1	1	1	1	1	–	–
12	1	–	–	1	–	1	1	–	–	1	–	–	1	–	1	–	1	–	–	1
13	1	1	1	1	1	1	1	1	–	–	1	1	1	–	–	–	–	–	–	–
14	1	–	1	–	1	–	1	–	–	1	1	–	–	–	–	1	–	1	–	1
15	1	1	–	–	1	1	–	–	–	–	1	1	–	–	1	–	1	1	1	1
16	1	–	–	1	1	–	–	1	–	1	1	–	–	1	–	–	1	–	1	–
17	1	1	1	1	–	–	–	–	1	1	1	1	–	–	1	1	1	–	–	–
18	1	–	1	–	–	1	–	1	1	–	1	–	–	1	1	–	–	–	–	1
19	1	1	–	–	1	1	1	1	1	1	–	–	–	–	1	1	–	–	1	–
20	1	–	–	1	1	–	1	–	1	–	–	1	–	1	1	–	–	1	–	–

(6.7)

Williamson type

$$\begin{array}{r|ccccc|ccccc|ccccc|ccccc|}
 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 & 17 & 18 & 19 & 20 \\
\cline{2-21}
1 & 1 & - & 1 & 1 & - & 1 & 1 & - & - & 1 & 1 & - & - & - & - & 1 & - & - & - & - \\
2 & - & 1 & - & 1 & 1 & 1 & 1 & 1 & - & - & - & 1 & - & - & - & - & 1 & - & - & - \\
3 & 1 & - & 1 & - & 1 & - & 1 & 1 & 1 & - & - & - & 1 & - & - & - & - & 1 & - & - \\
4 & 1 & 1 & - & 1 & - & - & - & 1 & 1 & 1 & - & - & - & 1 & - & - & - & - & 1 & - \\
5 & - & 1 & 1 & - & 1 & 1 & - & - & 1 & 1 & - & - & - & - & 1 & - & - & - & - & 1 \\
\cline{2-21}
6 & - & - & 1 & 1 & - & 1 & - & 1 & 1 & - & - & 1 & 1 & 1 & 1 & 1 & - & - & - & - \\
7 & - & - & - & 1 & 1 & - & 1 & - & 1 & 1 & 1 & - & 1 & 1 & 1 & - & 1 & - & - & - \\
8 & 1 & - & - & - & 1 & 1 & - & 1 & - & 1 & 1 & 1 & - & 1 & 1 & - & - & 1 & - & - \\
9 & 1 & 1 & - & - & - & 1 & 1 & - & 1 & - & 1 & 1 & 1 & - & 1 & - & - & - & 1 & - \\
10 & - & 1 & 1 & - & - & - & 1 & 1 & - & 1 & 1 & 1 & 1 & 1 & - & - & - & - & - & 1 \\
\cline{2-21}
11 & - & 1 & 1 & 1 & 1 & 1 & - & - & - & - & 1 & - & 1 & 1 & - & - & - & 1 & 1 & - \\
12 & 1 & - & 1 & 1 & 1 & - & 1 & - & - & - & - & 1 & - & 1 & 1 & - & - & - & 1 & 1 \\
13 & 1 & 1 & - & 1 & 1 & - & - & 1 & - & - & 1 & - & 1 & - & 1 & 1 & - & - & - & 1 \\
14 & 1 & 1 & 1 & - & 1 & - & - & - & 1 & - & 1 & 1 & - & 1 & - & 1 & 1 & - & - & - \\
15 & 1 & 1 & 1 & 1 & - & - & - & - & - & 1 & - & 1 & 1 & - & 1 & - & 1 & 1 & - & - \\
\cline{2-21}
16 & - & 1 & 1 & 1 & 1 & - & 1 & 1 & 1 & 1 & 1 & 1 & - & - & 1 & 1 & - & 1 & 1 & - \\
17 & 1 & - & 1 & 1 & 1 & 1 & - & 1 & 1 & 1 & 1 & 1 & 1 & - & - & - & 1 & - & 1 & 1 \\
18 & 1 & 1 & - & 1 & 1 & 1 & 1 & - & 1 & 1 & - & 1 & 1 & 1 & - & 1 & - & 1 & - & 1 \\
19 & 1 & 1 & 1 & - & 1 & 1 & 1 & 1 & - & 1 & - & - & 1 & 1 & 1 & 1 & 1 & - & 1 & - \\
20 & 1 & 1 & 1 & 1 & - & 1 & 1 & 1 & 1 & - & 1 & - & - & 1 & 1 & - & 1 & 1 & - & 1 \\
\cline{2-21}
\end{array} \qquad (6.8)$$

Mappings of rows and columns taking these into (A.3) are

$$\left.\begin{array}{l}
\begin{array}{l}\text{Paley rows}\\ \text{A.3 rows}\end{array}
\begin{pmatrix}
1, & 2, & 3, & 4, & 5, & 6, & 7, & 8, & 9, & 10, & 11, & 12, & 13, & 14, & 15, & 16, & 17, & 18, & 19, & 20 \\
1, & -5, & 15, & -14, & 3, & -7, & 20, & -17, & 19, & -18, & -10, & 12, & 16, & -13, & -4, & -8, & -9, & 11, & 2, & -6
\end{pmatrix} \\
\begin{array}{l}\text{Paley columns}\\ \text{A.3 columns}\end{array}
\begin{pmatrix}
1, & 2, & 3, & 4, & 5, & 6, & 7, & 8, & 9, & 10, & 11, & 12, & 13, & 14, & 15, & 16, & 17, & 18, & 19, & 20 \\
2, & -16, & 15, & 13, & 4, & 6, & 9, & 7, & 10, & 8, & 19, & 18, & 14, & 12, & 5, & 1, & 20, & 17, & 3, & 11
\end{pmatrix}
\end{array}\right\} \quad (6.9)$$

$$\left.\begin{array}{l}
\begin{array}{l}\text{Williamson rows}\\ \text{A.3 rows}\end{array}
\begin{pmatrix}
1, & 2, & 3, & 4, & 5, & 6, & 7, & 8, & 9, & 10, & 11, & 12, & 13, & 14, & 15, & 16, & 17, & 18, & 19, & 20 \\
-4, & -2, & 1, & -19, & 15, & -8, & 6, & -5, & 18, & -14, & 16, & 3, & -20, & -9, & -10, & -13, & -7, & 17, & 11, & 12
\end{pmatrix} \\
\begin{array}{l}\text{Williamson columns}\\ \text{A.3 columns}\end{array}
\begin{pmatrix}
1, & 2, & 3, & 4, & 5, & 6, & 7, & 8, & 9, & 10, & 11, & 12, & 13, & 14, & 15, & 16, & 17, & 18, & 19, & 20 \\
5, & -3, & 2, & -10, & 15, & -1, & 11, & 16, & 8, & -13, & -12, & -6, & 7, & -17, & -18, & -14, & -4, & 9, & -20, & -19
\end{pmatrix}
\end{array}\right\} \quad (6.10)$$

The group $G(P)$ of automorphisms of class P as represented by (A.3) is of order $20 \cdot 2 \cdot 9 \cdot 8 = 2880$ and is generated by

$$\alpha_P \left\{ \begin{array}{ll} \text{Row} & \begin{pmatrix} 1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18,19,20 \\ 1,3,2,4,5,7,6,8,9,10,11,12,17,18,19,20,13,14,15,16 \end{pmatrix} \\ \begin{array}{l}\text{Column}\\ \text{inverse}\end{array} & \begin{pmatrix} 1,2,3,4,5,\ 6,\ 7,\ 8,\ 9,10,11,12,13,14,15,16,17,18,19,20 \\ 1,2,4,3,5,11,12,13,14,15,\ 6,\ 7,\ 8,\ 9,10,16,17,18,19,20 \end{pmatrix} \end{array} \right.$$

$$\beta_P \left\{ \begin{array}{ll} \text{Row} & \begin{pmatrix} 1,2,\ 3,\ 4,5,6,7,8,\ 9,\ 10,\ 11,\ 12,\ 13,\ 14,\ 15,\ 16,17,18,19,20 \\ 1,2,-4,-3,5,6,8,7,-16,-15,-13,-14,-11,-12,-10,-9,18,17,20,19 \end{pmatrix} \\ \begin{array}{l}\text{Column}\\ \text{inverse}\end{array} & \begin{pmatrix} 1,2,3,4,5,6,7,8,\ 9,10,11,12,13,14,15,16,17,18,19,20 \\ 6,2,3,5,4,1,8,7,10,\ 9,11,17,18,20,19,16,12,13,15,14 \end{pmatrix} \end{array} \right.$$

$$\gamma_P \left\{ \begin{array}{ll} \text{Rows} & \begin{pmatrix} 1,\ 2,\ 3,\ 4,\ 5,\ 6,7,\ 8,9,10,\ 11,\ 12,13,\ 14,15,\ 16,17,18,\ 19,20 \\ 1,-16,-2,-9,-5,13,6,-11,3,20,-7,-17,14,-12,10,-15,\ 8,18,-19,\ 4 \end{pmatrix} \\ \text{Columns} & \begin{pmatrix} 1,\ 2,3,\ 4,5,\ 6,\ 7,\ 8,\ 9,10,11,12,13,14,15,16,17,18,19,20 \\ 9,16,6,17,7,20,19,10,18,\ 8,\ 4,\ 3,14,11,12,\ 2,\ 5,15,13,\ 1 \end{pmatrix} \end{array} \right. \tag{6.11}$$

$$\delta_P \left\{ \begin{array}{ll} \text{Rows} & \begin{pmatrix} 1,2,\ 3,\ 4,\ 5,\ 6,7,\ 8,\ 9,\ 10,\ 11,12,\ 13,\ 14,15,\ 16,17,18,\ 19,\ 20 \\ 20,3,-2,16,-17,-7,6,13,19,-15,-18,14,-8,-12,10,-4,\ 5,11,-9,-1 \end{pmatrix} \\ \text{Columns} & \begin{pmatrix} 1,2,3,\ 4,\ 5,\ 6,\ 7,\ 8,\ 9,10,11,12,\ 13,14,\ 15,\ 16,\ 17,18,19,\ 20 \\ -14,7,6,-11,-12,-3,-2,20,-16,17,\ 4,\ 5,-19,\ 1,-18,-9,-10,15,13,-8 \end{pmatrix} \end{array} \right.$$

$$\varepsilon_P \left\{ \begin{array}{ll} \text{Rows} & \begin{pmatrix} 1,2,3,\ 4,\ 5,\ 6,\ 7,8,\ 9,10,\ 11,\ 12,\ 13,\ 14,15,16,\ 17,\ 18,19,20 \\ 5,6,7,-8,-1,-2,-3,4,11,12,-9,-10,-16,-15,14,13,-20,-19,18,17 \end{pmatrix} \\ \text{Columns} & \begin{pmatrix} 1,\ 2,\ 3,\ 4,\ 5,6,7,\ 8,\ 9,\ 10,11,12,13,\ 14,\ 15,\ 16,17,18,\ 19,\ 20 \\ 5,16,-11,-6,-1,4,9,10,-7,-8,\ 3,14,15,-12,-13,-2,20,19,-18,-17 \end{pmatrix} \end{array} \right.$$

$G(P)$ is transitive on the rows of (A.3) but is imprimitive, moving them in the pairs (1,5), (2,6), (3,7), (4,8), (9,11), (10,12), (13,16), (14,15), (17,20), (18,19) corresponding to subdivisions in the Paley form (6.7). The element ε_P interchanges the rows of these pairs. The element γ_P fixes row 1 and takes row 5 into its negative. The elements α_P, β_P, and γ_P generate the subgroup H_P, fixing row 1, and also taking row 5 into itself or its negative. H_P is doubly transitive on rows 2, 3, 4, 9, 10, 15, 16, 19, and 20 and simultaneously doubly transitive on the rows paired with these.

We take (A.1) as the representative of class N. Column mappings taking other matrices of N into (A.1) are as follows:

$$\begin{array}{l}\text{Col. A.2}\\ \text{Col. A.1}\end{array}\begin{pmatrix}1,&2,&3,&4,&5,&6,&7,&8,&9,&10,&11,&12,&13,&14,&15,&16,&17,&18,&19,&20\\ 1,&2,&4,&3,&5,&11,&12,&13,&14,&15,&6,&7,&8,&9,&10,&16,&17,&18,&20,&19\end{pmatrix}$$

$$\begin{array}{l}\text{Col. B.3}\\ \text{Col. A.1}\end{array}\begin{pmatrix}1,&2,&3,&4,&5,&6,&7,&8,&9,&10,&11,&12,&13,&14,&15,&16,&17,&18,&19,&20\\ 15,&12,&14,&19,&18,&13,&20,&17,&11,&16,&6,&9,&8,&2,&4,&1,&5,&3,&7,&10\end{pmatrix}$$

$$\begin{array}{l}\text{Col. B.4}\\ \text{Col. A.1}\end{array}\begin{pmatrix}1,&2,&3,&4,&5,&6,&7,&8,&9,&10,&11,&12,&13,&14,&15,&16,&17,&18,&19,&20\\ 13,&12,&14,&19,&18,&15,&17,&20,&16,&11,&6,&5,&3,&7,&10,&1,&9,&8,&2,&4\end{pmatrix}$$

$$\begin{array}{l}\text{Col. C.1}\\ \text{Col. A.1}\end{array}\begin{pmatrix}1,&2,&3,&4,&5,&6,&7,&8,&9,&10,&11,&12,&13,&14,&15,&16,&17,&18,&19,&20\\ 11,&9,&5,&19,&16,&3,&12,&18,&1,&8,&15,&2,&7,&13,&6,&17,&20,&10,&14,&4\end{pmatrix}$$

$$\begin{array}{l}\text{Col. C.4}\\ \text{Col. A.1}\end{array}\begin{pmatrix}1,&2,&3,&4,&5,&6,&7,&8,&9,&10,&11,&12,&13,&14,&15,&16,&17,&18,&19,&20\\ 6,&16,&7,&12,&4,&20,&1,&18,&15,&5,&3,&14,&9,&11,&17,&10,&13,&2,&8,&19\end{pmatrix}$$

$$\begin{array}{l}\text{Col. C.5}\\ \text{Col. A.1}\end{array}\begin{pmatrix}1,&2,&3,&4,&5,&6,&7,&8,&9,&10,&11,&12,&13,&14,&15,&16,&17,&18,&19,&20\\ 16,&18,&12,&13,&17,&2,&10,&4,&5,&9,&11,&14,&19,&20,&15,&3,&7,&1,&6,&8\end{pmatrix}$$

$$\begin{array}{l}\text{Col. C.7}\\ \text{Col. A.1}\end{array}\begin{pmatrix}1,&2,&3,&4,&5,&6,&7,&8,&9,&10,&11,&12,&13,&14,&15,&16,&17,&18,&19,&20\\ 13,&9,&11,&4,&20,&1,&12,&10,&6,&19,&8,&7,&14,&18,&5,&3,&16,&15,&17,&2\end{pmatrix}$$

$$\begin{array}{l}\text{Col. C.9}\\ \text{Col. A.1}\end{array}\begin{pmatrix}1,&2,&3,&4,&5,&6,&7,&8,&9,&10,&11,&12,&13,&14,&15,&16,&17,&18,&19,&20\\ 10,&7,&5,&19,&12,&17,&15,&4,&11,&8,&14,&6,&13,&18,&3,&16,&2,&20,&9,&1\end{pmatrix} \tag{6.12}$$

The group $G(N)$ of automorphisms of (A.1) is of order $20 \cdot 3 \cdot 2 \cdot 2 \cdot 4 \cdot 2 = 1920$. It is generated by automorphisms

$$
\begin{array}{ll}
\alpha_N \left\{ \begin{array}{ll}
\text{Row} & \left(\begin{array}{cccccccccccccccccccc}
1, & 2, & 3, & 4, & 5, & 6, & 7, & 8, & 9, & 10, & 11, & 12, & 13, & 14, & 15, & 16, & 17, & 18, & 19, & 20 \\
1, & 4, & -2, & -3, & -14, & -15, & -16, & -13, & 5, & 7, & 6, & 8, & -12, & -9, & -11, & -10, & 18, & 19, & 17, & 20
\end{array}\right) \\
\begin{array}{l}\text{Column}\\ \text{inverse}\end{array} & \left(\begin{array}{cccccccccccccccccccc}
1, & 2, & 3, & 4, & 5, & 6, & 7, & 8, & 9, & 10, & 11, & 12, & 13, & 14, & 15, & 16, & 17, & 18, & 19, & 20 \\
6, & 17, & 19, & 18, & 20, & 16, & 7, & 9, & 10, & 8, & 11, & 4, & 3, & 2, & 5, & 1, & 14, & 12, & 13, & 15
\end{array}\right)
\end{array}\right. \\
\\
\beta_N \left\{ \begin{array}{ll}
\text{Row} & \left(\begin{array}{cccccccccccccccccccc}
1, & 2, & 3, & 4, & 5, & 6, & 7, & 8, & 9, & 10, & 11, & 12, & 13, & 14, & 15, & 16, & 17, & 18, & 19, & 20 \\
-8, & -6, & -7, & 5, & 1, & 3, & 2, & -4, & -11, & -12, & 9, & 10, & 18, & 17, & -19, & -20, & 13, & 14, & -16, & -15
\end{array}\right) \\
\begin{array}{l}\text{Column}\\ \text{inverse}\end{array} & \left(\begin{array}{cccccccccccccccccccc}
1, & 2, & 3, & 4, & 5, & 6, & 7, & 8, & 9, & 10, & 11, & 12, & 13, & 14, & 15, & 16, & 17, & 18, & 19, & 20 \\
-5, & 1, & 11, & 6, & -16, & -3, & 13, & 12, & -14, & -15, & -4, & 7, & 8, & -10, & -9, & 2, & -20, & -19, & 18, & 17
\end{array}\right)
\end{array}\right. \\
\\
\gamma_N \left\{ \begin{array}{ll}
\text{Row} & \left(\begin{array}{cccccccccccccccccccc}
1, & 2, & 3, & 4, & 5, & 6, & 7, & 8, & 9, & 10, & 11, & 12, & 13, & 14, & 15, & 16, & 17, & 18, & 19, & 20 \\
1, & 2, & -3, & -4, & -17, & -18, & -19, & -20, & -16, & -13, & -15, & -14, & -12, & -10, & -9, & -11, & -7, & -8, & -5, & -6
\end{array}\right) \\
\begin{array}{l}\text{Column}\\ \text{inverse}\end{array} & \left(\begin{array}{cccccccccccccccccccc}
1, & 2, & 3, & 4, & 5, & 6, & 7, & 8, & 9, & 10, & 11, & 12, & 13, & 14, & 15, & 16, & 17, & 18, & 19, & 20 \\
6, & 8, & 7, & 10, & 9, & 1, & 5, & 4, & 3, & 2, & 16, & 17, & 19, & 20, & 18, & 11, & 13, & 14, & 12, & 15
\end{array}\right)
\end{array}\right.
\end{array}
\tag{6.13}
$$

$$
\delta_N \left\{ \begin{array}{ll}
\text{Rows} & (5,6)\,(7,8)\,(9,12)\,(10,11)\,(13,15)\,(14,16)\,(17,18)\,(19,20) \\
\\
\text{Columns} & (2,3)\,(4,5)\,(7,\ 8)\,(\ 9,10)\,(12,14)\,(13,15)\,(17,20)\,(18,19)
\end{array}\right.
$$

$$
\varepsilon_N \left\{ \begin{array}{ll}
\text{Rows} & (5,7)\,(6,8)\,(9,11)\,(10,12)\,(13,14)\,(15,16)\,(17,19)\,(18,20) \\
\\
\text{Columns} & (2,4)\,(3,5)\,(7,\ 9)\,(\ 8,10)\,(12,13)\,(14,15)\,(17,19)\,(18,20)
\end{array}\right.
$$

$$
\zeta_N \left\{ \begin{array}{ll}
\text{Rows} & (9,10)\,(11,12)\,(13,14)\,(15,16)\,(17,18)\,(19,20) \\
\\
\text{Columns} & (7,\ 8)\,(\ 9,10)\,(12,13)\,(14,15)\,(17,18)\,(19,20)
\end{array}\right.
$$

There remains the troublesome problem of showing that the three classes Q, P, and N are in fact distinct classes. This we shall show using a certain invariance property and a knowledge of the automorphisms. If we were certain that we had found all automorphisms of Q, P, and N, our task would be finished, since the groups $G(Q)$, $G(P)$, and $G(N)$ as given here are of different orders.

We may choose any row of an H_{20} as the first row if we then change column signs to make this row consist entirely of $+1$'s. Having done this, any two further rows or their negatives may be used as a second and third row, and, with columns appropriately arranged, we have the standard form (2.1). Theorem 2.1 tells us that there is a unique row among the remaining ones, which when given an appropriate sign may be taken as row 4 in (2.11). Thus any three rows of an H_{20} uniquely determine a fourth. Furthermore, an easy calculation shows that, of such a quadruple of rows, any three uniquely determine the fourth. A quadruple of rows may be characterized by the property that there exist four columns for which these rows are identical. In (2.14), if the four rows are $1, 2, 3, 4$, the four columns may be taken as any one of the sets $2, 3, 4, 5$; $7, 8, 9, 10$; $12, 13, 14, 15$; or $17, 18, 19, 20$. Thus the pattern of (2.14) shows that the remaining 16 rows are divided into four disjoint sets of quadruples determined by columns 1, 6, 11, and 16: the rows $(5, 6, 7, 8)$ form a quadruple set; also $(9, 10, 11, 12)$, $(13, 14, 15, 16)$, and $(17, 18, 19, 20)$. We showed in (2.16) that, by appropriate equivalences on rows $1, 2, 3, 4$, each of the further quadruple sets can be put into the form of rows $5, 6, 7, 8$ in either type 1, 2, or 3, as given there. These three types are distinguishable by the fact that in type 1 our eight rows have, as pairs of identical columns, 7 and 8, 9 and 10, 12 and 13, 14 and 15, 17 and 18, and 19 and 20. Type 2 has identical columns 7 and 8, 9 and 10, and no others, while type 3 has no pair of identical columns. Taking a quadruple as rows $1, 2, 3, 4$, any automorphism taking these four into themselves may permute the row quadruples $(5, 6, 7, 8)$, $(9, 10, 11, 12)$, $(13, 14, 15, 16)$, and $(17, 18, 19, 20)$ among themselves but will not alter the aggregate of types. Thus in (A.1) the types are all 1's, and so (A.1) may be assigned the type characteristic 1^4. (A.3) has types $1, 2, 2$, and 2 or $1, 2^3$, while (C.2) has types $3, 3, 3, 3$, or 3^4. If rows a, b, c, d of an H_{20} form a quadruple set, then, taking these as rows $1, 2, 3, 4$, we find a type characteristic associated with these rows. Thus, taking rows $1, -6, -2, -5$ of (A.1) as the first four rows on an H_{20}, then (A.1) is transformed into (C.5), which has $2^2, 3^2$ as its type characteristic. Hence for (A.1) the quadruple $1, 2, 5, 6$ has type characteristic $2^2, 3^2$, and, similarly, any four rows a, b, c, d that may be obtained from $1, 2, 5, 6$ by any automorphism of (A.1) will have the same characteristic.

For class Q, every quadruple of rows is equivalent to rows $1, 2, 3, 4$ and so every type characteristic for every quadruple of Q is 3^4. Hence, as P has a representation of type $1, 2^3$, and N has a representation of 1^4, the class Q cannot be equivalent to either class P or N. In class N, application of the automorphisms shows that every quadruple is equivalent to one of $1, 2, 3, 4$ of type 1^4; $1, 2, 5, 6$ of type $2^2, 3^2$; $1, 2, 9, 12$ of type $2^2, 3^2$; $1, 5, 9, 20$ of type 3^4; and $1, 5, 14, 19$ of type 3^4. Since no one of these is of type $1, 2^3$, that of (A.3), we conclude that class N is not equivalent to class P. This is sufficient to show that the three classes are inequivalent.

Each of the three classes is equivalent to its own transpose. For class Q, with (C.2) as representative, if we take the rows as columns in the order

$$2, 7, 6, 5, 8, 3, -11, -9, -10, -12, -4, 16, 15, 13, 14, -1, 19, 18, 20, -7 \tag{6.14}$$

then the columns become rows in exactly this same order. In this form the matrix is symmetric. In the same way with class P, taking (A.3) as the representative, if we take the rows as columns in the order

$$-8, 1, 2, 3, -4, -7, -18, -17, 19, 20, -6, -14, -13, 15, 16, 5, 12, 11, -9, -10 \tag{6.15}$$

then the resulting matrix is symmetric and we have the columns becoming rows in exactly the same order. For type N as represented by (A.1), if we take the rows as columns in the order

$$-8, 1, 2, 3, -4, -7, -18, -17, 19, 20, -6, -13, -14, 16, 15, 5, 12, 11, -9, 10 \tag{6.16}$$

the columns become rows in exactly the same order and the matrix in this form is symmetric.

APPENDIX

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
2	1	1	1	1	1	1	1	1	1	1	—	—	—	—	—	—	—	—	—	—
3	1	1	1	1	1	—	—	—	—	—	1	1	1	1	1	—	—	—	—	—
4	1	—	—	—	—	—	1	1	1	1	—	1	1	1	1	1	—	—	—	—
5	1	1	—	—	—	1	1	1	—	—	1	1	1	—	—	—	1	1	—	—
6	1	—	1	—	—	1	1	—	1	—	1	1	—	1	—	—	—	—	1	1
7	1	—	—	1	—	1	—	—	1	1	1	—	1	—	1	—	1	—	1	—
8	1	—	—	—	1	1	—	1	—	1	1	—	—	1	1	—	—	1	—	1
9	—	1	1	—	—	1	1	—	—	1	1	—	—	1	1	1	1	—	—	—
10	—	—	—	1	1	1	1	—	1	—	1	1	—	—	1	1	—	1	—	—
11	—	1	—	—	1	1	—	1	1	—	1	—	1	1	—	1	—	—	1	—
12	—	—	1	1	—	1	—	1	—	1	1	1	1	—	—	1	—	—	—	1
13	—	—	1	—	1	1	—	—	1	1	—	1	1	1	—	—	1	1	—	—
14	—	1	—	—	1	1	1	—	—	1	—	1	1	—	1	—	—	—	1	1
15	—	1	—	1	—	1	—	1	1	—	—	1	—	1	1	—	1	—	—	1
16	—	—	1	1	—	1	1	1	—	—	—	—	1	1	1	—	—	1	1	—
17	—	—	1	—	1	—	1	1	1	—	1	—	1	—	1	—	1	—	—	1
18	—	—	—	1	1	—	1	1	—	1	1	1	—	1	—	—	1	—	1	—
19	—	1	—	1	—	—	1	—	1	1	1	—	1	1	—	—	—	1	—	1
20	—	1	1	—	—	—	—	1	1	1	1	1	—	—	1	—	—	1	1	—

Fig. A-1. Class Q (represented by C.2)

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
2	1	1	1	1	1	1	1	1	1	1	–	–	–	–	–	–	–	–	–	–
3	1	1	1	1	1	–	–	–	–	–	1	1	1	1	1	–	–	–	–	–
4	1	–	–	–	–	–	1	1	1	1	–	1	1	1	1	1	–	–	–	–
5	1	1	–	–	–	1	1	1	–	–	1	1	1	–	–	–	1	1	–	–
6	1	–	1	–	–	1	1	1	–	–	1	–	–	1	1	–	–	–	1	1
7	1	–	–	1	–	1	–	–	1	1	1	1	1	–	–	–	–	–	1	1
8	1	–	–	–	1	1	–	–	1	1	1	–	–	1	1	–	1	1	–	–
9	–	–	1	1	–	1	1	–	1	–	1	1	–	1	–	1	1	–	–	–
10	–	–	1	1	–	1	–	1	–	1	1	–	1	–	1	1	–	1	–	–
11	–	1	–	–	1	1	1	–	–	1	1	1	–	–	1	1	–	–	1	–
12	–	1	–	–	1	1	–	1	1	–	1	–	1	1	–	1	–	–	–	1
13	–	–	1	–	1	1	1	–	–	1	–	1	1	1	–	–	–	1	–	1
14	–	–	1	–	1	1	–	1	1	–	–	1	1	–	1	–	1	–	1	–
15	–	1	–	1	–	1	–	1	–	1	–	1	–	1	1	–	1	–	–	1
16	–	1	–	1	–	1	1	–	1	–	–	–	1	1	1	–	–	1	1	–
17	–	–	–	1	1	–	1	1	1	–	1	1	–	–	1	–	–	1	–	1
18	–	–	–	1	1	–	1	1	–	1	1	–	1	1	–	–	1	–	1	–
19	–	1	1	–	–	–	1	–	1	1	1	–	1	–	1	–	1	–	–	1
20	–	1	1	–	–	–	–	1	1	1	1	1	–	1	–	–	–	1	1	–

Fig. A-2. Class *P* (represented by A.3)

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
2	1	1	1	1	1	1	1	1	1	1	–	–	–	–	–	–	–	–	–	–
3	1	1	1	1	1	–	–	–	–	–	1	1	1	1	1	–	–	–	–	–
4	1	–	–	–	–	–	1	1	1	1	–	1	1	1	1	1	–	–	–	–
5	1	1	–	–	–	1	1	1	–	–	1	1	1	–	–	–	1	1	–	–
6	1	–	1	–	–	1	1	1	–	–	1	–	–	1	1	–	–	–	1	1
7	1	–	–	1	–	1	–	–	1	1	1	1	1	–	–	–	–	–	1	1
8	1	–	–	–	1	1	–	–	1	1	1	–	–	1	1	–	1	1	–	–
9	–	–	1	1	–	1	1	–	1	–	1	1	–	1	–	1	1	–	–	–
10	–	–	1	1	–	1	–	1	–	1	1	–	1	–	1	1	–	1	–	–
11	–	1	–	–	1	1	1	–	1	–	1	–	1	–	1	1	–	–	1	–
12	–	1	–	–	1	1	–	1	–	1	1	1	–	1	–	1	–	–	–	1
13	–	–	1	–	1	1	1	–	–	1	–	1	1	1	–	–	–	1	1	–
14	–	–	1	–	1	1	–	1	1	–	–	1	1	–	1	–	1	–	–	1
15	–	1	–	1	–	1	–	1	1	–	–	1	–	1	1	–	–	1	1	–
16	–	1	–	1	–	1	1	–	–	1	–	–	1	1	1	–	1	–	–	1
17	–	–	–	1	1	–	1	1	1	–	1	–	1	1	–	–	–	1	–	1
18	–	–	–	1	1	–	1	1	–	1	1	1	–	–	1	–	1	–	1	–
19	–	1	1	–	–	–	1	–	1	1	1	1	–	–	1	–	–	1	–	1
20	–	1	1	–	–	–	–	1	1	1	1	–	1	1	–	–	1	–	1	–

Fig. A-3. Class *N* (represented by A.1)

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