Effect of Spacewise Variations in a Random Load Field on the Response of a Linear System

by

Ferdinand P. Beer
Robert J. Ravera

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Institute of Research
Lehigh University
Bethlehem, Pennsylvania

ABSTRACT

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Spacewise variations in a random load field are often neglected in computing the statistical properties of the dynamic response of a structure to that field. A method is given for determining whether such an approximation is conservative or not, and for computing the correction which should be used to take into account the spacewise variations in the input field. The proposed method is based on the determination of the response of the structure to a few simple deterministic loadings and does not require the use of multidimensional power spectral analysis. It yields a corrected value for the power spectral density of the response, from which values may be obtained for the mean square of the response, as well as for other of its statistical characteristics.

Author

NOMENCLATURE

a	= distance from center of beam to spring support
b	= half-length of beam
$f(\underset{\sim}{r},t)$	= loading at point $\underset{\sim}{r}$ and time t
$f_{i1}(t)$, $f_{i2}(t)$	= deterministic loadings defined in Eq.(22)
$F_i(\omega)$	= function defined in Eq.(14). See also Eqs. (26) and (27).
$G_i(\omega)$	= second moment of $\Phi_f(\underset{\sim}{k},\omega)$. See Eq.(15)
$H(\omega)$	= transfer function for coherent loading
$H(\underset{\sim}{k},\omega)$	= Multidimensional transfer function
H_{i1}, H_{i2}	= Complex amplitude of responses to loadings f_{i1}, f_{i2}
$\underset{\sim}{k}$	= vector of components k_i
k_i	= wave numbers
L	= correlation length or scale of random field
M	= mass of beam
$q(t)$	= response of system
$\overline{q}^2$	= mean square of response
$\overline{q}_o^2$	= mean square of response (simplified analysis)
$\underset{\sim}{r}$, $\underset{\sim}{s}$	= positions vectors
r_i, s_i	= coordinates
u	= $(\omega/\omega_o)^2$
v	= $(a/b)^2$

Δ	= correction
θ, θ_{12}	= phase angles
$\underset{\sim}{\kappa}$	= L $\underset{\sim}{k}$
κ_i	= L k_i
μ	= radius of gyration
ρ	= position vector
φ_f	= correlation fucntion of load field
Φ_f	= power spectral density of load field
Φ_q	= power spectral density of response
Φ_{q_o}	= power spectral density of response (simplified analysis)
σ, τ	= time intervals
ω	= frequency
ω_o	= natural frequency in translation

1. Introduction

It is known that the power spectral density $\Phi_q(\omega)$ of the response q(t) of a linear system to a stationary random load f(t) may be expressed as

$$\Phi_q(\omega) = |H(\omega)|^2 \; \Phi_f(\omega) \tag{1}$$

where $\Phi_f(\omega)$ is the power spectral density of the load and $H(\omega)$ is the transfer function of the system. When the system is subjected simultaneously at several points to a stationary homogeneous random load field $f(\underset{\sim}{r}, t)$ which varies in space as well as in time, Eq.(1) must be replaced[1,2] by

$$\Phi_q(\omega) = \frac{1}{(2\pi)^3} \int_{-\infty}^{\infty} |H(\underset{\sim}{k}, \omega)|^2 \; \Phi_f(\underset{\sim}{k}, \omega) \; d^3\underset{\sim}{k} \tag{2}$$

where H and Φ_f are functions of the wave numbers k_1, k_2, k_3 defining the vector $\underset{\sim}{k}$ and of the frequency ω. Whether Eq.(1) or Eq. (2) is used to determine $\phi_q(\omega)$, the mean-square value of the response may be expressed as

$$\overline{q}^2 = \frac{1}{2\pi} \int_{-\infty}^{\infty} \Phi_q(\omega) \; d\omega \tag{3}$$

If the load field is not perfectly random spacewise, it is generally possible to define a correlation length or scale

L different from zero. When the largest dimension of the system is small compared to L, as it occurs in many cases of practical interest (e.g. the response of an airplane to atmospheric turbulence), it is often assumed that the spacewise variations in the load field may be neglected and Eq.(1) is used. Such an assumption leads to a conservative estimate of the probability of survival of a structure subjected to a random load field when the critical response q(t) depends chiefly upon the even-order modes of vibration of the structure. Indeed, this assumption is equivalent to that of complete coherence of the load field at all input points and results in a high estimate of $\overline{q}^2$. On the other hand, when the response q(t) depends in a large measure on the odd-order modes of vibration, this assumption may well lead to a low estimate of $\overline{q}^2$ since it does not make any allowance for the possibility of excitation of such modes.

We shall present in this paper a method for determining whether the simplified analysis based on Eq.(1) leads to a conservative estimate of the probability of survival of a given structure. We shall also indicate how the power spectral density obtained from Eq.(1) and the corresponding value of the mean square of the response should be corrected to take into account the effect of spacewise variations in the load field. The method presented may be easily applied to the analysis of the response of a large airframe to atmospheric turbulence.

2. Approximate Expressions for the Corrections to the Power Spectral Density and the Mean Square of the Response

Denoting respectively by $\Phi_{q_o}(\omega)$ and $\overline{q_o^2}$ the power spectral density and the mean square of the response obtained from the simplified analysis based on Eq. (1), and by $\Phi_q(\omega)$ and $\overline{q^2}$ the exact expressions obtained from Eq.(2), we define the corrections

$$\Delta \Phi_q(\omega) = \Phi_q(\omega) - \Phi_{q_o}(\omega) \qquad (4)$$

and

$$\Delta \overline{q^2} = \overline{q^2} - \overline{q_o^2} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \Delta \Phi_q(\omega) \, d\omega \qquad (5)$$

The object of this section is to find approximate expressions for these two corrections.

We first recall[2] that the transfer function $H(\underset{\sim}{k}, \omega)$ used in Eq.(2) is defined as the multiple Fourier transform

$$H(\underset{\sim}{k}, \omega) = \int_{-\infty}^{\infty} h(\underset{\sim}{s}, \sigma) \, e^{-i(\underset{\sim}{k} \cdot \underset{\sim}{s} + \omega\sigma)} \, d^3\underset{\sim}{s} \, d\sigma \qquad (6)$$

of the response h of the system to a unit impulse applied at point $\underset{\sim}{r}'$ and time t'. The vector $\underset{\sim}{s}$ and the scalar σ represent, respectively, the differences $\underset{\sim}{s} = \underset{\sim}{r} - \underset{\sim}{r}'$ and $\sigma = t - t'$, where $\underset{\sim}{r}$ defines a reference point fixed in the system and t the time at which the response is measured. It may be verified that $H(\underset{\sim}{k},\omega)$ represents the complex amplitude of the response of the system to the loading

$$f(\underset{\sim}{r},t) = e^{i(\underset{\sim}{k}\circ\underset{\sim}{r}+\omega t)} \tag{7}$$

corresponding to a train of normal plane sinusoidal waves of wave length $2\pi/|\underset{\sim}{k}|$ moving in the direction $-\underset{\sim}{k}/|\underset{\sim}{k}|$ at the speed $\omega/|\underset{\sim}{k}|$. When $\underset{\sim}{k}$ is chosen equal to zero, this loading reduces to the coherent sinusoidal loading $e^{i\omega t}$. Thus

$$H(0,\omega) = H(\omega) \tag{8}$$

where $H(\omega)$ is the conventional transfer function used in Eq.(1).

Similarly, the power spectral density $\Phi_f(\underset{\sim}{k},\omega)$ is defined as the multiple Fourier transform

$$\Phi_f(\underset{\sim}{k},\omega) = \int_{-\infty}^{\infty} \mathcal{C}_f(\underset{\sim}{\rho},\tau)\, e^{-i(\underset{\sim}{k}\circ\underset{\sim}{\rho}+\omega\tau)}\, d^3\underset{\sim}{\rho}\, d\tau \tag{9}$$

of the correlation function $\mathcal{C}_f$ of the load field in space and in time. Taking the inverse transform in $\underset{\sim}{k}$ of both members of Eq.(9) and making $\underset{\sim}{\rho} = 0$ yields the relation

$$\frac{1}{(2\pi)^3}\int_{-\infty}^{\infty} \Phi_f(\underset{\sim}{k},\omega)\ d^3\underset{\sim}{k} = \int_{-\infty}^{\infty} \mathcal{C}_f(0,\tau)\ e^{-i\omega\tau}\ d\tau = \Phi_f(\omega) \qquad (10)$$

where $\Phi_f(\omega)$ is the conventional power spectral density used in Eq.(1). Since the correlation function $\mathcal{C}_f(\underset{\sim}{\rho},\tau)$ is an even function of each of the components of $\underset{\sim}{\rho}$, its transform $\Phi_f(\underset{\sim}{k},\omega)$ is similarly an even function of each of the wave numbers k_i. We may thus write the following relations:

$$\int_{-\infty}^{\infty} k_i\ \Phi_f(\underset{\sim}{k},\omega)\ d^3\underset{\sim}{k} = 0\ , \qquad \int_{-\infty}^{\infty} k_i\ k_j\ \Phi_f(\underset{\sim}{k},\omega)\ d^3\underset{\sim}{k} = 0 \quad (i \neq j) \qquad (11)$$

We shall now expand the function $|H(\underset{\sim}{k},\omega)|^2$ in a Taylor series in the variables k_i. We write

$$|H(\underset{\sim}{k},\omega)|^2 = |H(0,\omega)|^2 + \sum_{i=1}^{3}\left[\frac{\partial|H(\underset{\sim}{k},\omega)|^2}{\partial k_i}\right] k_i + \frac{1}{2}\sum_{i=1}^{3}\sum_{j=1}^{3}\left[\frac{\partial^2|H(\underset{\sim}{k},\omega)|^2}{\partial k_i\,\partial k_j}\right]_{\underset{\sim}{k}=0} k_i\ k_j + \ldots \qquad (12)$$

Substituting into Eq.(2) and taking into account the relations (11), we have

$$\Phi_q(\omega) = |H(0,\omega)|^2\ \frac{1}{(2\pi)^3}\int_{-\infty}^{\infty} \Phi_f(\underset{\sim}{k},\omega)\ d^3\underset{\sim}{k} + \frac{1}{2}\sum_{i=1}^{3}\left[\frac{\partial^2|H(\underset{\sim}{k},\omega)|^2}{\partial k_i^2}\right]_{\underset{\sim}{k}=0} \frac{1}{(2\pi)^3}\int_{-\infty}^{\infty} k_i^2\ \Phi_f(\underset{\sim}{k},\omega)\ d^3\underset{\sim}{k} + \ldots \qquad (13)$$

Recalling Eqs.(8) and (10), we observe that the first term in the right-hand member of Eq.(13) represents the power spectral density $\Phi_{q_o}(\omega)$ obtained from Eq.(1).

Setting

$$F_i(\omega) = \frac{1}{2}\left[\frac{\partial^2 |H(\underset{\sim}{k},\omega)|^2}{\partial k_i^2}\right]_{\underset{\sim}{k}=0} \qquad (14)$$

and

$$G_i(\omega) = \frac{1}{(2\pi)^3}\int_{-\infty}^{\infty} k_i^2 \; \Phi_f(\underset{\sim}{k},\omega) \; d^3\underset{\sim}{k} \qquad (15)$$

and neglecting higher-order terms we may therefore write the corrections (4) and (5) in the forms

$$\Delta\ \Phi_q(\omega) = \Phi_q(\omega) - \Phi_{q_o}(\omega) = \sum_{i=1}^{3} F_i(\omega)\ G_i(\omega) \qquad (16)$$

$$\Delta\ \overline{q}^2 = \overline{q}^2 - \overline{q}_o^2 = \frac{1}{2\pi}\int_{-\infty}^{\infty} \sum_{i=1}^{3} F_i(\omega)\ G_i(\omega)\ d\omega \qquad (17)$$

The result obtained shows that the dynamic characteristics of the structure and the statistical characteristics of the load field may be handled separately in the computation of the correction $\Delta\ \Phi_q(\omega)$; the former affect only the functions $F_i(\omega)$ and the latter the functions $G_i(\omega)$. It may be further noted that, in the case of an isotropic load field, the three functions $G_i(\omega)$ are equal. Denoting their common value by $G(\omega)$ we may write for that case

$$\Delta\ \Phi_f(\omega) = [\sum_{i=1}^{3} F_i(\omega)]\ G(\omega) \qquad (18)$$

Since the functions $G_i(\omega)$ are non-negative, the correction $\Delta\overline{q}^2$ will be negative and the approximation (1) conservative if the functions $F_i(\omega)$ are negative for all values of ω. If the functions $F_i(\omega)$ are positive for all values of ω, the approximation (1) is clearly non-conservative. If the functions $F_i(\omega)$ have positive and negative values, the integration in (17) must be carried out to determine whether the approximation is conservative or not.

Before we turn our attention to the computation of the functions $F_i(\omega)$ we shall show that the expressions (16) and (17) obtained for the corrections are of the order of L^{-2}, where L is the scale of the load field and that the terms neglected in writing these expressions are of the order of L^{-4} and higher. Introducing the dimensionless wave numbers $\kappa_i = L\, k_i$ and the corresponding vector $\underset{\sim}{\kappa}$, we define the function

$$\Phi'_f(\underset{\sim}{\kappa}, \omega) = (1/L)^3 \; \Phi_f(\underset{\sim}{\kappa}/L, \omega)$$

and referring to Eq.(10), verify that the integral

$$\frac{1}{(2\pi)^3} \int \Phi'_f(\underset{\sim}{\kappa}, \omega) \; d^3\underset{\sim}{\kappa} = \Phi_f(\omega)$$

is independent of the scale L. Substituting for Φ_f in terms of Φ'_f in (15), we write

$$G_i(\omega) = \frac{1}{L^2} \frac{1}{(2\pi)^3} \int_{-\infty}^{\infty} \kappa_i^2 \; \Phi_f(\underset{\sim}{\kappa}, \omega) \; d^3\underset{\sim}{\kappa}$$

which shows that the functions $G_i(\omega)$, and thus the corrections $\Delta\,\Phi_q(\omega)$ and $\Delta\,\overline{q}^2$, are for a given structure of the order of L^{-2}. A similar analysis would show that the first of the non-zero terms neglected in (13) is of the order of L^{-4}.

3. Computation of the Functions $F_i(\omega)$

The computation of the functions $F_i(\omega)$ from the transfer function $H(\underset{\sim}{k}, \omega)$ itself would be quite cumbersome since it would involve the determination of the response of the system to trains of sinusoidal waves of all possible wave-lengths moving in all directions and at every possible velocity. We shall show in this section that they may instead be computed from the response of the system to a few simple coherent sinusoidal loadings.

Noting first that

$$|H(\underset{\sim}{k}, \omega)|^2 = H(\underset{\sim}{k}, \omega) \; H^*(\underset{\sim}{k}, \omega)$$

where H^* denotes the complex conjugate of H, and substituting in Eq.(14), we write

$$F_i(\omega) = \left[\frac{\partial H}{\partial k_i} \frac{\partial H^*}{\partial k_i} + \frac{1}{2} H \frac{\partial^2 H^*}{\partial k_i^2} + \frac{1}{2} H^* \frac{\partial^2 H}{\partial k_i^2} \right]_{\underset{\sim}{k}=0} \qquad (19)$$

But from Eq.(6) we have

$$\left[\frac{\partial H(\underset{\sim}{k},\omega)}{\partial k_i}\right]_{\underset{\sim}{k}=0} = -\, i \int_{-\infty}^{\infty} h(\underset{\sim}{s},\sigma)\, s_i\, e^{-i\omega\sigma}\, d^3\underset{\sim}{s}\, d\sigma \tag{20}$$

and

$$\left[\frac{\partial^2 H(\underset{\sim}{k},\omega)}{\partial k_i^2}\right]_{\underset{\sim}{k}=0} = -\int_{-\infty}^{\infty} h(\underset{\sim}{s},\sigma)\, s_i^2\, e^{-i\omega\sigma}\, d^3\underset{\sim}{s}\, d\sigma \tag{21}$$

Consider now the two particular deterministic sinusoidal loadings

$$f_{i1}(\underset{\sim}{r},t) = r_i\, e^{i\omega t}, \qquad f_{i2}(\underset{\sim}{r},t) = r_i^2\, e^{i\omega t} \tag{22}$$

which, at any given isntant, are represented respectively by a linear and a parabolic distribution of loads in the r_i coordinate. Recalling[2] that the response of the system to a load field $f(\underset{\sim}{r},t)$ is

$$q(\underset{\sim}{r},t) = \int_{-\infty}^{\infty} h(\underset{\sim}{s},\sigma)\, f(\underset{\sim}{r}-\underset{\sim}{s},\, t-\sigma)\, d^3\underset{\sim}{s}\, d\sigma$$

and choosing the reference point at the origin of the system of axes $(\underset{\sim}{r}=0)$, we express the responses of the system to the loadings (22) respectively as

$$H_{i1}(\omega)\, e^{i\omega t} = \int_{-\infty}^{\infty} h(\underset{\sim}{s},\sigma)\, (-s_i)\, e^{-i\omega(t-\sigma)}\, d^3\underset{\sim}{s}\, d\sigma \tag{23}$$

and

$$H_{i2}(\omega)\, e^{i\omega t} = \int_{-\infty}^{\infty} h(\underset{\sim}{s},\sigma)\, s_i^2\, e^{-i\omega(t-\sigma)}\, d^3\underset{\sim}{s}\, d\sigma \tag{24}$$

where H_{i1} and H_{i2} represent complex amplitudes. Comparing (20) and (21) with (23) and (24) respectively, we easily verify that

$$\left[\frac{\partial H(\underset{\sim}{k},\omega)}{\partial k_i}\right]_{\underset{\sim}{k}=0} = i\, H_{i1}(\omega), \qquad \left[\frac{\partial^2 H(\underset{\sim}{k},\omega)}{\partial^2 k_i}\right]_{\underset{\sim}{k}=0} = -H_{i2}(\omega) \tag{25}$$

Substituting from (25) into (19), we have

$$F_i(\omega) = H_{i1}(\omega)\, H^*_{i1}(\omega) - \frac{1}{2}[H(\omega)\, H^*_{i2}(\omega) + H^*(\omega)\, H_{i2}(\omega)] \tag{26}$$

Introducing the phase angles $\theta(\omega)$ and $\theta_{i2}(\omega)$ of the complex amplitudes $H(\omega)$ and $H_{i2}(\omega)$, we may express the functions $F_i(\omega)$ in the alternate form

$$F_i(\omega) = |H_{i1}(\omega)|^2 - |H(\omega)|\ |H_{i2}(\omega)|\ \cos\,(\theta_{i2}-\theta) \tag{27}$$

We note that $H(\omega)$ is obtained from the response of the system to the uniform coherent sinusoidal loading $e^{i\omega t}$ and that its determination is part of the simplified analysis based on Eq.(1). Thus, in the most general case, the computation of the functions $F_i(\omega)$ and of the correction $\Delta\ \phi_q(\omega)$ depends on the determination of the response of the system to the six simple deterministic loadings defined in (22). Note that in many applications one will be concerned with the spacewise variation of the load field in one dimension only; the computation of the correction $\Delta \phi_q(\omega)$ will then require the determination of the response of the system to only two loadings.

4. Example

As an example of application of the proposed method we shall consider the rigid beam of length 2b shown in Fig.1. The beam is subjected to a random loading f(x,t), of scale L large compared to b, and the response q(t) of interest is the acceleration of the right-hand tip of the beam (x=b). The beam is supported by two identical springs located at distance a from its center, and we assume for simplicity that the system is undamped. This assumption precludes the determination of the mean square of the response, but it facilitates the discussion of the correction $\Delta \phi_q(\omega)$.

We assume that the simplified analysis based on Eq.(1), i.e., on the assumption of a coherent loading, random in time only, has been carried out and that the corresponding approximate expression $\phi_{q_o}(\omega)$ for the power spectral density has been obtained. This computation requires the determination of the conventional transfer function $H(\omega)$

representing the complex amplitude of the response of the system to the uniform sinusoidal loading of Fig. 2a. Denoting by M the mass of the beam, by ω_o its natural frequency in the translational mode, and by μ its centroidal radius of gyration, we have

$$H(\omega) = \frac{2b\omega^2}{M(\omega^2-\omega_o^2)} \tag{28}$$

The computation of the correction $\Delta\,\phi_q(\omega)$ necessitated by the spacewise variation of the load field requires the determination of the response of the system to the two additional loadings shown in Figs. 2b and 2c. The corresponding complex amplitudes are

$$H_1(\omega) = \frac{(2/3)\,b^3\omega^2}{M(\mu^2\omega^2-a^2\omega_o^2)} \tag{29}$$

and

$$H_2(\omega) = \frac{(2/3)b\,\omega^2}{M(\omega^2-\omega_o^2)} \tag{30}$$

Substituting from Eqs. (28), (29), and (30) into Eq. (27), assuming that the beam is a slender rod ($\mu^2 = b^2/3$), and observing that for a rigid beam the responses to the loadings (a) and (c) are in phase ($\theta_2 = \theta$), we have

$$F(\omega) = \frac{4b^2\omega^4}{M^2} \left[\frac{b^2}{(b^2\omega^2-3a^2\omega_o^2)^2} - \frac{1}{3(\omega^2-\omega_o^2)^2}\right] \tag{31}$$

or, setting $(\omega/\omega_o)^2 = u$ and $(a/b)^2 = v$,

$$F(\omega) = \frac{4b^2u^2}{M^2} \left[\frac{1}{(u-3v)^2} - \frac{1}{3(u-1)^2}\right] \tag{32}$$

The regions of the uv plane for which $F(\omega)$ is positive and those for which it is negative are shown in Fig.3.

Let us first consider the case where the springs are attached at the tips of the beam ($a = b$), which is represented in Fig. 3 by the horizontal line $v = 1$. At low frequencies, the beam responds less readily in rotation than in translation; we have $F(\omega) < 0$ and, from Eq.(18), $\Delta\,\phi_q(\omega) < 0$, which indicates that the correct value of $\phi_q(\omega)$ is smaller than the approximate value $\phi_{q_o}(\omega)$ obtained from Eq.(1). This condition persists as ω and u increase and as we pass through resonance in translation ($u = 1$). However, for higher frequencies the situation is reversed; the beam responds more readily in rotation and we have $F(\omega) > 0$, $\Delta\phi_q(\omega) > 0$. If we now choose a value of <u>a</u> smaller than the radius of gyration $b/\sqrt{3}$ of the beam, corresponding to the dashed horizontal line in Fig.3, we find that at low frequencies $F(\omega)$ and $\Delta\phi_q(\omega)$ are positive and the correct value of $\phi_q(\omega)$ is larger than the approximate value $\phi_{q_o}(\omega)$ obtained from Eq.(1). The situation persists as ω increases and as we pass through resonance in rotation ($v = u/3$), but reverses itself as we approach and pass through resonance in translation ($u = 1$). For higher frequencies, the beam responds again more readily in rotation and $F(\omega)$ becomes positive.

5. Conclusion

It has been shown that the effect of spacewise variations in a random load field on the response of a multidimensional linear system may be determined by computing the response of the system to a few simple deterministic loadings, provided that the scale of the field is large. This approach makes it possible to determine whether a simplified analysis based on the assumption of no spacewise variations in the load field is conservative or not. It also yields the correction $\Delta\phi_q$ which should be added to the value of the power spectral density obtained from the simplified analysis. While the effect of this correction on the value of the mean square $\overline{q^2}$ of the system response has been emphasized in our presentation, it should be noted that the corrected value of the power spectral density $\phi_q(\omega)$ may be used to determine many other statistical characteristics of the response which depend directly upon ϕ_q, such as the expected number of crossings of a given level per unit time, the expected number of maxima or minima of the response per unit time[3], the probability of exceedance of a given level during a given time interval, etc. In view of the present renewal of interest in large transport airplanes, it is believed that the approach presented here will find many useful applications.

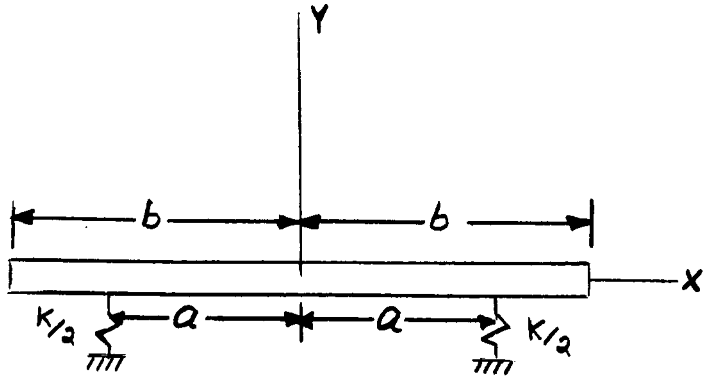

FIG. 1

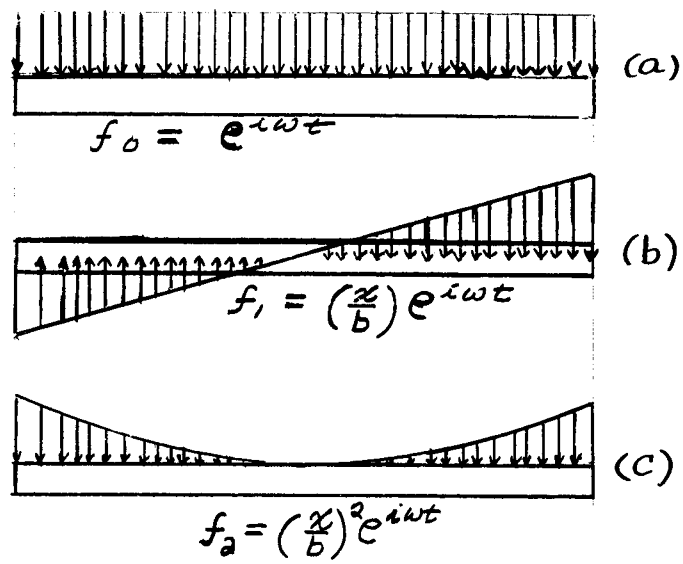

FIG. 2

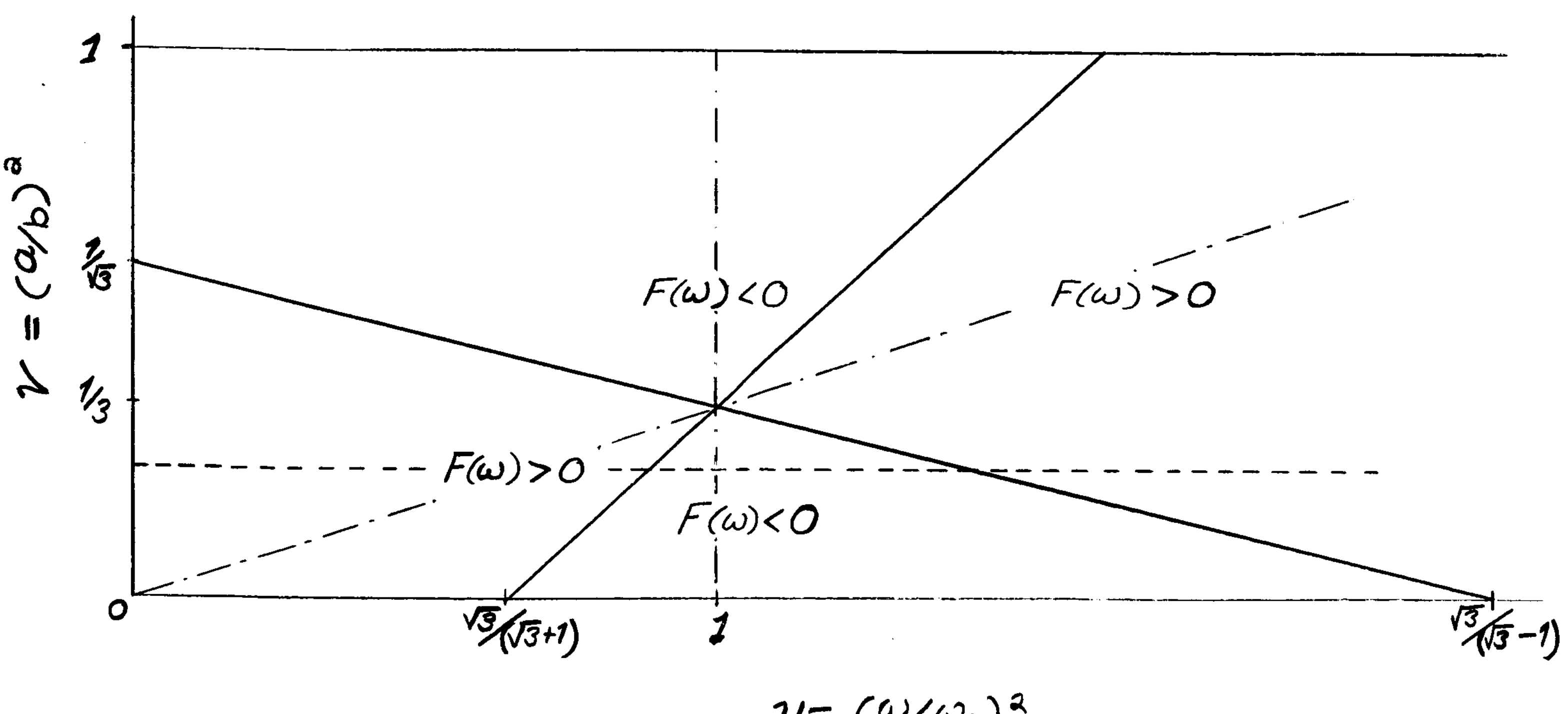

FIG. 3

References

1. H.S. Riebner, "Spectral Theory of Buffeting and Gust Response: Unification and Extension", Journal of the Aeronautical Sciences, vol. 23, 1956, p. 1075

2. F.P. Beer, "On the Response of Linear Systems to Time-Dependent, Multidimensional Loadings", Journal of Applied Mechanics, Transactions of the ASME, vol. 28, Series E, No. 1, 1961, p. 50

3. J.S. Bendat, "Principles and Applications of Random Noise Theory", Wiley, New York, 1958