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MAXIMUM LIFT-TO-DRAG RATIOS OF SLENDER, FLAT-TOP, HYPERSONIC BODIES PART 1

BY

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PART 1

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SUMMARY

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An investigation of the lift-to-drag ratio attainable by a slender, flat-top, homothetic body flying at hypersonic speeds is presented under the assumptions that the pressure distribution is Newtonian and the skin-friction coefficient is constant.

It is shown that a value of the thickness ratio exists such that the lift-to-drag ratio is a maximum; this particular value is such that the friction drag is one-third

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of the total drag. The subsequent optimization of the longitudinal and transversal contours is reduced to the extremization of products of powers of integrals related to the lift, the pressure drag, and the skin-friction drag. For the longitudinal contour, the variational approach shows that a conical solution is the best. For the transversal contour, a triangular configuration satisfies the Euler equation for every cross-sectional elongation ratio μ; it satisfies the Legendre condition for $0 \leq \mu \leq 0.65$ but violates it in the neighborhood of the plane of symmetry for $0.65 \leq \mu \leq 1$. The lift-to-drag ratio of the optimum conical body of triangular cross section increases as the elongation ratio of the cross section decreases; for a friction coefficient $C_f = 10^{-3}$, the highest attainable lift-to-drag ratio is $E = 5.29$.

1. INTRODUCTION

In a previous report (Ref. 1), an investigation of the lift-to-drag ratio attainable by a slender, flat-top, homothetic body at hypersonic speeds was presented under the assumption that the pressure distribution is Newtonian and the skin-friction coefficient is constant. Direct methods were employed, and the analysis was confined to the class of bodies whose longitudinal contour is a power law and whose transversal contour is semielliptical or triangular. For these special bodies, the lift-to-drag ratio depends on three parameters: the thickness ratio, the exponent of the power law, and the elongation ratio of the cross section. Therefore, by means of the ordinary theory of maxima and minima, the combination of parameters maximizing the lift-to-drag ratio can be found.

In this report, the limitations set forth in Ref. 1 are removed and the indirect methods of the calculus of variations are employed in order to determine the optimum longitudinal and transversal contours. The hypotheses employed are as follows: (a) a plane of symmetry exists between the left-hand and right-hand sides of the body; (b) the upper surface of the body is flat (reference plane); (c) the base plane is perpendicular to both the plane of symmetry and the reference plane; (d) the body is slender in the longitudinal sense, that is, the square of the slope of any meridian contour is small with respect to one; (e) the body is homothetic, in the sense that each cross section is geometrically similar to the base cross section and has the same orientation; (f) the free-stream velocity is perpendicular to the base plane and, therefore, is parallel to the line of intersection of the plane of symmetry and the reference plane; (g) the pressure coefficient is twice the cosine squared of the angle formed by the free-stream velocity and the normal to each surface element; (h) the skin-friction coefficient is constant; and (i) the contribution of the tangential forces to the lift is negligible with respect to the contribution of the normal forces.

2. DRAG AND LIFT

We consider the class of flat-top bodies and define two coordinate systems (Fig. 1): a Cartesian coordinate system Oxyz and a cylindrical coordinate system $Oxr\theta$. For the Cartesian coordinate system, the origin O is the apex of the body; the x-axis is the intersection of the plane of symmetry and the reference plane, positive toward the base; the z-axis is contained in the plane of symmetry, perpendicular to the x-axis, and positive downward; and the y-axis is such that the xyz-system is right-handed. For the cylindrical coordinate system, r is the distance of any point from the x-axis, and θ measures the angular position of this point with respect to the xy-plane.

Next, we focus our attention on those bodies $r(x, \theta)$ such that any transversal contour is geometrically similar to that of the base and has the same orientation. The geometry of these homothetic bodies is given by (Ref. 1)

$$r = \ell(\tau/\mu)\, A(\xi)\, B(\theta) \tag{1}$$

where ℓ denotes the length and where

$$\tau = \frac{r(\ell, \pi/2)}{\ell} \quad , \quad \mu = \frac{r(\ell, \pi/2)}{r(\ell, 0)} \tag{2}$$

are the thickness ratio and the elongation ratio, respectively; also, $\xi = x/\ell$ is a nondimensional abscissa, $A(\xi)$ a function describing the longitudinal contour such that

$$A(0) = 0, \qquad A(1) = 1 \tag{3}$$

and $B(\theta)$ a function describing the transversal contour such that

$$B(0) = 1, \qquad B(\pi/2) = \mu \tag{4}$$

With this understanding and in the light of the hypotheses of the introduction, the drag and the lift can be rewritten as (Ref. 1)

$$\begin{aligned} D/q\,\ell^2 &= \tau^4 I_1 J_1 + C_f\, \tau I_2 J_2 \\ L/q\,\ell^2 &= \tau^3 I_3 J_3 \end{aligned} \tag{5}$$

In Eqs. (5), the positive quantities I_1, I_2, I_3 are defined as

$$I_1 = \int_0^1 A\dot{A}^3 \, d\xi \,, \quad I_2 = \int_0^1 A \, d\xi \,, \quad I_3 = \int_0^1 A\dot{A}^2 \, d\xi \tag{6}$$

where $\dot{A} = dA/d\xi$. Also, the positive quantities J_1, J_2, J_3 are defined as

$$J_1 = (4/\mu^4)K_1 \,, \quad J_2 = (2/\mu)K_2 \,, \quad J_3 = (4/\mu^3)K_3 \tag{7}$$

where

$$K_1 = \int_0^{\pi/2} \left[B^6/(B^2 + \dot{B}^2) \right] d\theta$$

$$K_2 = \int_0^{\pi/2} \left[2/\pi + (B^2 + \dot{B}^2)^{1/2} \right] d\theta \tag{8}$$

$$K_3 = \int_0^{\pi/2} \left[B^4/(B^2 + \dot{B}^2) \right] (B \sin\theta - \dot{B} \cos\theta) \, d\theta$$

and $\dot{B} = dB/d\theta$.

3. LIFT-TO-DRAG RATIO

From the previous formulas, it appears that -- if the length ℓ, the thickness ratio τ, the longitudinal contour $A(\xi)$, and the transversal contour $B(\theta)$ are given -- the drag and the lift can be evaluated from Eqs. (5) through (8). Once these quantities are known, one can determine the aerodynamic efficiency or lift-to-drag ratio

$$E = L/D \tag{9}$$

which, in the light of Eqs. (5), can be written as

$$E = \tau^2 I_3 J_3 / (\tau^3 I_1 J_1 + C_f I_2 J_2) \tag{10}$$

4. OPTIMUM THICKNESS RATIO

We now assume that the longitudinal contour $A(\xi)$ and the transversal contour $B(\theta)$ are arbitrarily prescribed, and study the effect of the thickness ratio τ on the lift-to-drag ratio (10). Clearly, the lift-to-drag ratio is an extremum when the thickness ratio satisfies the relationship

$$E_\tau = 0 \tag{11}$$

whose explicit form

$$\tau/\sqrt[3]{C_f} = \sqrt[3]{2(I_2/I_1)(J_2/J_1)} = \mu\sqrt[3]{(I_2/I_1)(K_2/K_1)} \tag{12}$$

means that the friction drag is one-third of the total drag. The associated lift-to-drag ratio is given by

$$E\sqrt[3]{C_f} = \sqrt[3]{(4/27)\,(I_3^3/I_1^2\,I_2)(J_3^3/J_1^2J_2)} = (2/3)\sqrt[3]{(I_3^3/I_1^2I_2)\,(K_3^3/K_1^2K_2)} \tag{13}$$

and is a maximum owing to the fact that

$$E_{\tau\tau} = -(2/3\;C_f)\,(I_3/I_2)\,(J_3/J_2) = -\,(4/3\mu^2C_f)\,(I_3/I_2)\,(K_3/K_2) \le 0 \tag{14}$$

5. OPTIMUM LONGITUDINAL CONTOUR

Next, we consider bodies optimized with respect to the thickness ratio τ, assume that the transversal contour $B(\theta)$ is arbitrarily prescribed, and study the effect of the longitudinal contour $A(\xi)$ on the lift-to-drag ratio (13). Since the lift-to-drag ratio depends on the longitudinal contour through the expression

$$I = I_3{}^3/I_1{}^2\, I_2 \tag{15}$$

we formulate the following problem: "In the class of functions $A(\xi)$ which satisfy the end conditions (3), find that particular function which extremizes the functional (15), where the integrals I_1, I_2, I_3 are defined by Eqs. (6)."

The functional (15) is a product of powers of integrals whose end points are fixed and is governed by the theory set forth in Ref. 2. In this reference, it is shown that the previous problem is equivalent to that of extremizing the integral

$$\tilde{I} = \int_0^1 F(A,\ \dot{A},\ \lambda_1,\ \lambda_2)\, d\xi \tag{16}$$

where the fundamental function is defined as

$$F = A(\dot{A}^2 - \lambda_1 \dot{A}^3 - \lambda_2) \tag{17}$$

and the undetermined, constant Lagrange multipliers are given by

$$\lambda_1 = 2I_3/3I_1 \quad , \quad \lambda_2 = I_3/3I_2 \tag{18}$$

Since the fundamental function does not contain the independent variable explicitly, standard methods of the calculus of variations show that the Euler equation

$$dF_{\dot{A}}/d\xi - F_A = 0 \tag{19}$$

admits the following first integral (see, for instance, Chapter 1 of Ref. 3):

$$F - \dot{A}F_{\dot{A}} = C \tag{20}$$

whose explicit form is

$$A(2\lambda_1 \dot{A}^3 - \dot{A}^2 - \lambda_2) = C \tag{21}$$

Upon integrating Eq. (21) over the range 0, 1 and accounting for the definitions (6), we obtain the relationship

$$2\lambda_1 I_1 - I_3 - \lambda_2 I_2 = C \tag{22}$$

which is consistent with Eqs. (18) providing the integration constant has the value

$$C = 0 \tag{23}$$

Consequently, the differential equation of the extremal arc (21) becomes

$$2\lambda_1 \dot{A}^3 - \dot{A}^2 - \lambda_2 = 0 \tag{24}$$

and implies that

$$\dot{A} = C_1 \tag{25}$$

where C_1 is a constant. Upon integrating this differential equation, we obtain the relationship

$$A = C_1 \xi + C_2 \tag{26}$$

where, because of the end conditions (3), the constants take the values

$$C_1 = 1 \quad , \quad C_2 = 0 \tag{27}$$

In conclusion, the optimum longitudinal contour is described by

$$A = \xi \tag{28}$$

and, therefore, is conical. For this cone, the integrals (6) take the values

$$I_1 = I_2 = I_3 = 1/2 \tag{29}$$

and the Lagrange multipliers (18) are given by

$$\lambda_1 = 2/3 \quad , \quad \lambda_2 = 1/3 \tag{30}$$

Finally, the optimum thickness ratio (12) and the lift-to-drag ratio (13) become

$$\tau/\sqrt[3]{C_f} = \mu\sqrt[3]{K_2/K_1} \quad , \quad E\sqrt[3]{C_f} = (2/3)\sqrt[3]{K_3^3/K_1^2K_2} \tag{31}$$

Incidentally, the solution obtained maximizes the lift-to-drag ratio, owing to the

fact that

$$F_{\dot{A}\dot{A}} = 2A(1-3\lambda_1\dot{A}) = -2\xi \le 0 \tag{32}$$

6. OPTIMUM TRANSVERSAL CONTOUR

Finally, we consider configurations optimized with respect to the thickness ratio τ and the longitudinal contour $A(\xi)$, and study the effect of the transversal contour $B(\theta)$ on the lift-to-drag ratio (31-2). Since the lift-to-drag ratio depends on the transversal contour through the expression

$$K = K_3{}^3/K_1{}^2 K_2 \tag{33}$$

we formulate the following problem: "In the class of functions $B(\theta)$ which satisfy the end conditions (4), find that particular function which extremizes the functional (33), where the integrals K_1 , K_2, K_3 are defined by Eqs. (8)."

For each given elongation ratio μ, the functional (33) is a product of powers of integrals whose end points are fixed and is governed by the theory set forth in Ref. 2. Therefore, the previous problem is equivalent to that of extremizing the integral

$$\widetilde{K} = \int_0^{\pi/2} F(\theta, B, \dot{B}, \lambda_1 , \lambda_2) d\theta \tag{34}$$

where the fundamental function is defined as

$$F = \left[B^4/(B^2 + \dot{B}^2)\right](B \sin\theta - \dot{B}\cos\theta) - \lambda_1\left[B^6/(B^2 + \dot{B}^2)\right] - \lambda_2\left[2/\pi + (B^2 + \dot{B}^2)^{1/2}\right] \tag{35}$$

and the undetermined, constant Lagrange multipliers are given by

$$\lambda_1 = 2K_3/3K_1 \quad , \quad \lambda_2 = K_3/3K_2 \tag{36}$$

The extremal solution is described by the Euler equation

$$dF_{\dot{B}}/d\theta - F_B = 0 \tag{37}$$

which, in explicit form, is given by

$$\ddot{B} = \frac{\lambda_1 P_1(B, \dot{B}) + \lambda_2 P_2(B, \dot{B}) + P_3(\theta, B, \dot{B})}{\lambda_1 Q_1(B, \dot{B}) + \lambda_2 Q_2(B, \dot{B}) + Q_3(\theta, B, \dot{B})} \tag{38}$$

where

$$P_1 = 2B^4(2B^4 + 7B^2\dot{B}^2 + 9\dot{B}^4)$$

$$P_2 = (B^2 + 2\dot{B}^2)(B^2 + \dot{B}^2)^{3/2} \tag{39}$$

$$P_3 = -2B^2\left[(B^4 + 5B^2\dot{B}^2 + 8\dot{B}^4)B\sin\theta + (B^4 + B^2\dot{B}^2 - 4\dot{B}^4)\dot{B}\cos\theta\right]$$

and

$$
\begin{aligned}
Q_1 &= 2B^5(3\dot{B}^2 - B^2) \\
Q_2 &= B(B^2 + \dot{B}^2)^{3/2} \\
Q_3 &= 2B^3\left[(B^2 - 3\dot{B}^2)\, B \sin\theta - (3B^2 - \dot{B}^2)\, \dot{B} \cos\theta\right]
\end{aligned} \tag{40}
$$

There is no method known to these authors for obtaining the general integral of this differential equation and, as a consequence, numerical integration is necessary. Prior to undertaking this task, these authors have investigated the possibility that the family of triangular contours described by

$$B = \mu/(\sin\theta + \mu\cos\theta) \tag{41}$$

might be a particular solution(*). That this is the case can be shown with the following reasoning. First of all, the triangular contours (41) satisfy the end conditions (4). Next, the evaluation of the integrals (8) yields the relationships

$$K_1 = \mu^3/(1 + \mu^2)\,, \quad K_2 = 1 + \sqrt{1 + \mu^2}\,, \quad K_3 = \mu^2/(1 + \mu^2) \tag{42}$$

(*) The excellent aerodynamic qualities of the bodies of triangular cross section are suggested by the analysis of Ref. 1.

so that the Lagrange multipliers (36) are given by

$$\lambda_1 = 2/3\mu \ , \quad \lambda_2 = \mu^2/3(1+\mu^2)(1+\sqrt{1+\mu^2}) \tag{43}$$

Then, by direct substitution into Eq. (38), it can be verified that the assumed optimum contour (41) and the associated multipliers (43) reduce this Euler equation to an identity regardless of the cross-sectional elongation ratio μ. Consequently, the thickness ratio and the lift-to-drag ratio become (Figs. 2 and 3)

$$\tau/\sqrt[3]{C_f} = \sqrt[3]{(1+\mu^2)(1+\sqrt{1+\mu^2})}$$

$$E\sqrt[3]{C_f} = 2/3\sqrt[3]{(1+\mu^2)(1+\sqrt{1+\mu^2})} \tag{44}$$

Incidentally, the solution obtained maximizes the lift-to-drag ratio providing the Legendre condition

$$F_{\dot{B}\dot{B}} = -B(B^2+\dot{B}^2)^{-3}\left[\lambda_1 Q_1(B,\dot{B}) + \lambda_2 Q_2(B,\dot{B}) + Q_3(\theta,B,\dot{B})\right] \le 0 \tag{45}$$

is satisfied at every point of the extremal arc. After Eqs. (40), (41), and (43) are accounted for, it can be verified that Ineq. (45) is satisfied everywhere if the cross-sectional elongation ratio is in the range $0 \leq \mu \leq 0.6511$. For larger values of the elongation ratio, Ineq. (45) is violated in the neighborhood of the final point; hence, a triangular cross section cannot be optimal in the range $0.6511 \leq \mu \leq 1$. A further investigation is needed and is to be presented in a forthcoming report (Part 2).

7. DISCUSSION AND CONCLUSIONS

In the previous sections, the optimization of the lift-to-drag ratio of a slender, flat-top, homothetic body flying at hypersonic speeds is presented under the assumptions that the pressure distribution is Newtonian and the skin-friction coefficient is constant.

It is shown that a value of the thickness ratio exists which maximizes the lift-to-drag ratio; this particular value is such that the friction drag is one-third of the total drag. The subsequent optimization of the longitudinal and transversal contours is reduced to the extremization of products of powers of integrals related to the lift, the pressure drag, and the skin-friction drag. For the longitudinal contour, the variational approach shows that a conical solution is the best. For the transversal contour, a triangular cross section satisfies the Euler equation for every cross-sectional elongation ratio μ; it satisfies the Legendre condition for $0 \leq \mu \leq 0.6511$ but violates it in the neighborhood of the plane of symmetry for $0.6511 \leq \mu \leq 1$.

It is of interest to check the lift-to-drag ratios attainable with conical bodies of triangular cross section against those attainable with conical bodies having different cross-sectional contours, specifically:

$$
\begin{aligned}
&\text{(a)} \qquad B = 1 - (1-u)\sin\theta \\
&\text{(b)} \qquad B = u/\sqrt{\sin^2\theta + u^2\cos^2\theta} \\
&\text{(c)} \qquad B\sin\theta = u\,(1 - B\cos\theta)^m \\
&\text{(d)} \qquad B\sin\theta = u\left[1 - (B\cos\theta)^m\right]
\end{aligned}
\tag{46}
$$

where (a) denotes a sinusoidal contour, (b) denotes an elliptical contour, and (c) and (d) are power law contours. Numerical analyses performed at Rice University with an IBM 7040 Digital Computer show that, for each given elongation ratio in the range $0 \leq u \leq 1$, the body of triangular cross section is aerodynamically superior to the bodies (a) and (b). It is also superior to the body (c) regardless of the exponent m. An analogous remark holds for the body (d) as long as the elongation ratio does not approach the value $u = 1$. For values of u in the neighborhood of 1, an exponent m can be found such that the lift-to-drag ratio of the body of triangular cross section and that of the body (d) differ by only the fourth significant figure. Thus, even though the body of triangular cross section does not meet all the requirements of the calculus of variations in the range $0.6511 \leq \mu \leq 1$, it exhibits excellent aerodynamic characteristics by comparison with the bodies (a) through (d). For this reason, while the present investigation is to be completed in a forthcoming report, it is probable that the lift-to-drag ratio of the true variational solution will not differ substantially from that of the body of triangular cross section.

In closing, the following comments are pertinent:

(a) The lift-to-drag ratio of the body of triangular cross section increases as the elongation ratio decreases and achieves its highest value $E\sqrt[3]{C_f} = 0.529$ at $\mu = 0$, corresponding to a thickness ratio in $\tau/\sqrt[3]{C_f} = 0.126$. This limiting result, obviously to be interpreted with a grain of salt, means that wing-like configurations rather than body-like configurations are aerodynamically desirable at hypersonic speeds. It is clear that a practical vehicle can only be constructed by replacing the mathematical solution $\mu = 0$ with a neighboring value, for instance, $\mu = 0.2$. As Fig. 3 shows, the loss in the lift-to-drag ratio is small.

(b) The conical bodies of triangular cross section exhibit sharp corners at $\theta = 0$ and $\theta = \pi/2$. Hence, their main drawback is the severe heat transfer occurring at the lines of intersection between the surfaces composing the vehicle. Consequently, the present sharp-edge configurations must be replaced by faired configurations in which the transistion from one surface to another occurs with a finite curvature. If this is done, lift-to-drag ratios smaller than those predicted here are to be expected.

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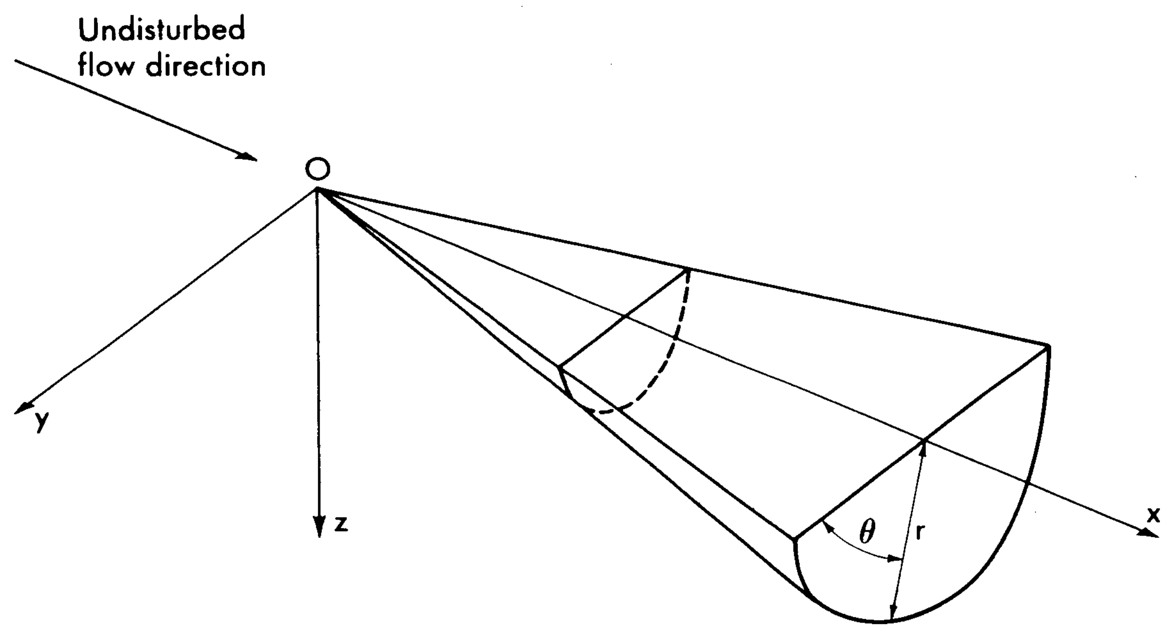
LIST OF CAPTIONS

Fig. 1 Coordinate system.

Fig. 2 Thickness ratio of conical bodies of triangular cross section.

Fig. 3 Lift-to-drag ratio of conical bodies of triangular cross section.



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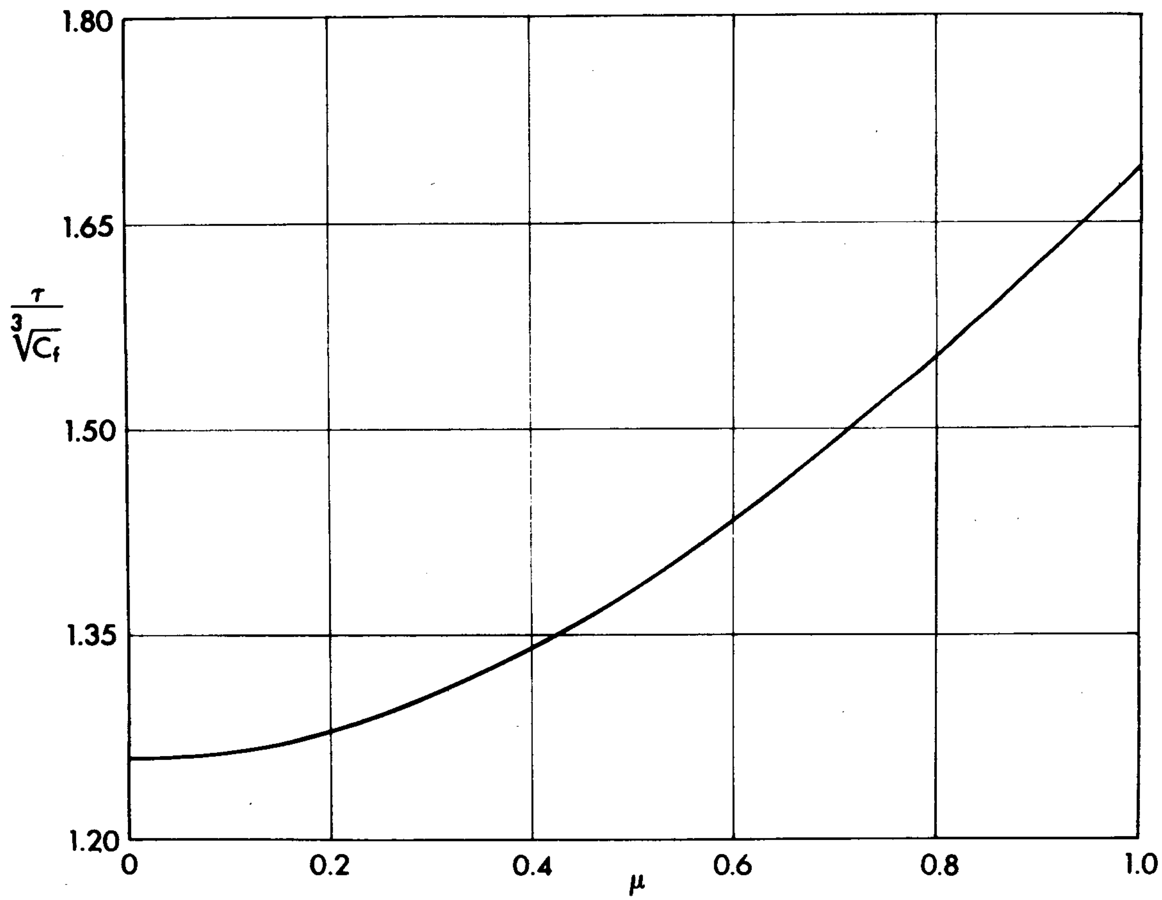

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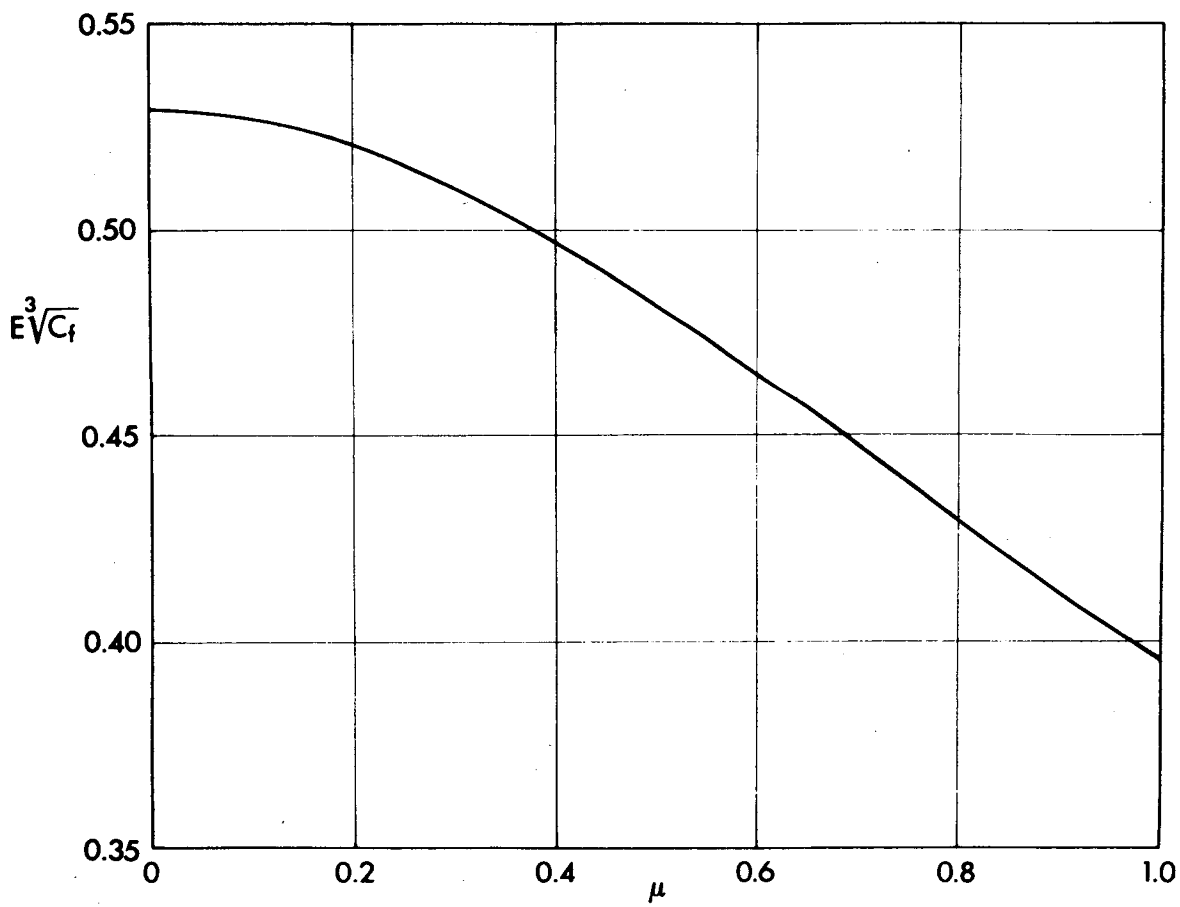

Fig. 3 Lift-to-drag ratio of conical bodies of triangular cross section.