ARIZONA STATE UNIVERSITY

Tempe, Arizona

College of Engineering Sciences

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"INVESTIGATION OF DYNAMIC BEHAVIOR OF THIN SPHERICAL SHELLS"

Principal Investigator: James P. Avery

A. INTRODUCTION:

Although the effective date of the grant was June 1, 1965, the formal research effort started with the academic year, that is, September 1, 1965. Consequently, this report covers a period of three months.

The research effort of the first three months has been devoted largely to the following tasks: 1) study of relevant literature, 2) formulation of a mathematical description of the vibration problem and of a set of governing integral equations 3) outlining the approach to be employed in obtaining a solution, and 4) developing a solution of the fundamental (Green's function) problem which provides the basis for the solution of the general boundary value problem.

The progress in each of these tasks is discussed in the following sections of this report.

B. LITERATURE SURVEY:

The literature that appears to be relevant to this study falls into three categories: (1) that which deals directly with the spherical shell vibration problem, (2) the literature on plate and shell theory pertinent to the fundamental Green's function problem, and (3) the literature on mathematical methods applicable to the solution of either the fundamental problem or the governing equations of the general problem.

The basic literature (together with recent contributions) on spherical shell vibration has been reviewed with the objective of gaining insight into the physical features of the problem, as well as of achieving a familiarity with alternate approaches to shell vibration problems.

In the area of plate and shell theory, basic works of such authors as Flugge, Timoshenko, Love, E. Reisner, have been studied to establish and guide the solution of the governing differential equations of the fundamental problem. Related studies of the plate on an elastic foundation have also been reviewed.

To develop a solution to the governing differential equations for the fundamental problem, basic works on differential equations and mathematical analysis have been reviewed. Also literature on numerical analysis and matrix methods have been studied to provide guidance in the solution of the governing integral equations of the general boundary value problem.

C. MATHEMATICAL FORMULATION OF THE PROBLEM:

A formulation (employing a Green's function approach) of the general spherical shell vibration problem has been developed as part of the first three months research effort. This formulation is outlined in the following paragraphs. (A more complete description is presented in Appendix I.)

The vibration problem (with damping not included) is first replaced by an equivalent static problem. The inertial loading of the dynamic problem is replaced by a distributed static load proportional to the displacement. Additionally, an artifice of an elastic foundation is introduced in the static problem such that the foundation reaction is proportional to the displacement vector but in opposite sense. Thus if the applied load is proportional to displacement, then also the net load on the shell (applied load less foundation reaction) adheres to this proportionality. Symbolically:

(a) For the vibration problem

$$\vec{q} = \mu\,\omega^2\,\vec{u}$$

where,

$\vec{q}$ = inertial shell force per unit surface area

μ = shell mass per unit surface area

ω = natural angular frequency

$\vec{u}$ = displacement vector of the middle surface

(b) For the shell on the elastic foundation

$$\vec{q} = \frac{1}{\lambda}\,\vec{u} - k\vec{u} = \left(\frac{1}{\lambda} - k\right)\vec{u}$$

where

$\vec{q}$ = net load on shell (per unit surface area)

λ = proportionality factor between applied load and displacement

k = foundation modulus

Consequently if $(\frac{1}{\lambda} - k)$ is made equal to $\mu\,\omega^2$, the static problem is equivalent to the vibration problem.

The equivalent static problem is next formulated in terms of fundamental influence functions. In brief, if the displacement and stress fields are known for a unit load (and unit couple) applied to a point on the complete sphere (on an elastic foundation), then through superposition the required relationships may be written satisfying boundary conditions (specified along a given contour) as well as the condition that the applied load be proportional to displacement.

For condensed notation, let:

A(m,n)	be the displacement vector at point m (on the sphere) due to a unit load vector applied at point n on the sphere.
B(s,n)	be the boundary condition "residual" vector* (four dimensional) at point s on the contour, C, due to a unit load vector at point n.
C(s,t)	be the boundary condition residual vector at point s on the contour C due to corrective load system unit vector**(four dimensional) at point t on the contour, C.
D(m,t)	be the displacement vector at point m on the sphere due to the corrective load unit vector at point t on the contour, C.
$\vec{q}$ (n)	be the applied load (intensity) vector at point n
$\vec{u}$ (m)	be the displacement vector at point m
$\vec{L}$ (t)	be the contour corrective load system at point t on the contour, C.

Then employing superposition, the specified boundary conditions along C are satisfied if the boundary condition "residuals" vanish; that is,

$$\iint_S B(s,n)\,\vec{q}(n)\,d\sigma + \int_C C(s,t)\,\vec{L}(t)\,ds = 0 \qquad (1)$$

The applied load-displacement relationship is also obtained from superposition,

$$\iint_S A(m,n)\,\vec{q}(n)\,d\sigma + \int_C D(m,t)\,\vec{L}(t)\,ds = \vec{u}(m) \qquad (2)$$

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* The boundary condition residuals refer to deviations from four specified boundary conditions along the contour, C.

** The corrective load vector includes four components of a load system (three force components and one force-couple) applied to the sphere at the contour, C.

Were the function $\vec{L}(t)$ eliminated between equations (1) and (2) (by means of inverse operators), the resulting equation could be symbolized by:

$$\iint_S G(m,n)\,\vec{q}(n)\,d\sigma = \vec{u}(m) \tag{3}$$

Then for $\vec{q}(n) = \frac{1}{\lambda}\vec{u}(n)$, equation (3) becomes

$$\iint_S G(m,n)\,\vec{u}(n)\,d\sigma = \lambda\,\vec{u}(m) \tag{4}$$

which defines the eigenvalue problem for the general static equivalent problem, and hence, for the original vibration problem.

D. PROBLEM SOLUTION:

The solution of the general boundary value problem defined in the preceding section (the static equivalent to the vibration problem) can be conveniently divided into two general efforts. First, the influence functions (or Green's functions) that appear in the governing integral equations (1) and (2) must be obtained. Secondly, a general method must be developed for the solution of the integral equations.

Considering first the Green's functions, the research effort to date has consisted of the following activities: (1) definition of the auxiliary problems (of which the Green's functions are solutions) (2) development of the governing differential equations and (3) development of a series solution of these equations under the appropriate boundary conditions. (These equations and their solutions are discussed further in section E.)

The auxiliary problems are concerned with displacements and stress resultants arising from a unit force or unit force couple applied to the complete sphere on the elastic foundation. Specifically, three separate problems require solution and are defined by the application to the sphere of:

(1) A unit normal load

(2) A unit tangential load

(3) A unit couple (with tangential axis).

In each of these problems the displacements and stress resultants are sought; however, as the stress resultants are expressible as functions of derivatives of displacements, a solution in terms of displacements is sufficient.

Moreover, since a rotation of the middle surface is a directional derivative of normal displacement, then from Betti's reciprocal theorem, a solution to problem (3) (for the unit couple) is provided directly by solutions to problems (1) and (2). Finally, if the normal load problem (1) is first solved, the solution to problem (2) is in part obtained again by means of the reciprocal theorem; that is, the normal displacements for the unit tangential load would be obtained.

In summary then, the auxiliary problems reduce to (A) solving for displacements resulting from a unit normal load and (B) solving for the tangential components of displacement resulting from a unit tangential load.

Turning next to the solution of the integral equations (once they are obtained), the research effort to date has been limited to the formulation of a suitable numerical approach. In essence, the method consists of solving a finite difference approximation to the integral equations, with the integral operators replaced by influence matrices and the unknown functions by n-dimensional vectors (corresponding to n points on the surface). The eigenvectors and eigenvalues would then be found by the Vianello-Stadola iterative method. The steps in the numerical solution are described in more detail in Appendix II.

E. THE GREEN'S FUNCTIONS:

The research effort to date in the area of development of the Green's functions, has been limited to obtaining equations and developing a solution to the first fundamental problem (that of the unit normal load applied to the spherical shell on an elastic foundation). The equations were derived and compared with those found in several references.* These equations were then modified by a variable substitution to eliminate trigonometric coefficient functions.

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* Ex: Flugge, Stresses in Shell.

The fundamental set of solutions for the resulting differential equations were developed in the form of powers series (with a logarithmic coefficient function appearing in one solution).

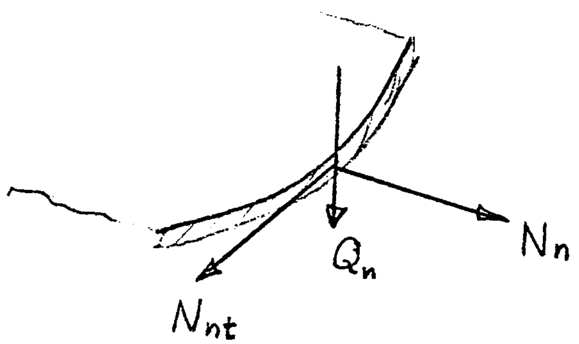
The recursion formulae associated with the series solutions have been programmed for computer evaluation. The required boundary conditions are enforced in the computer solution.

The details of the reformulation and the series solution of the equations are presented in Appendix III.

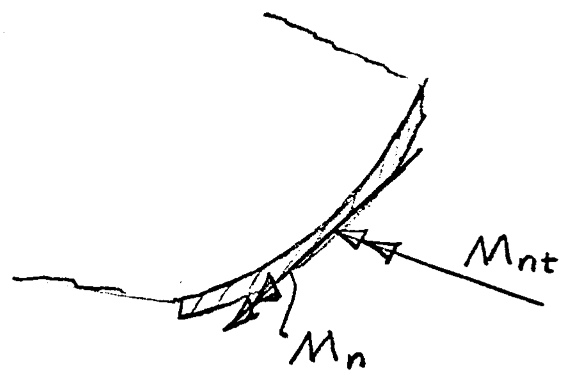
APPENDIX I   MATHEMATICAL FORMULATION

The natural frequencies and mode shapes are sought for a segment of a spherical shell subject to specified boundary conditions. As developed in section (C) of the report text, a substitute static equivalent problem is instead considered, in which a complete spherical shell on an elastic foundation is loaded over that portion of the spherical surface enclosed by a contour C (corresponding to the edge of the segment) and loaded by line loads and moments, along the contour C. The conditions to be met in order to make the two problems equivalent are (1) the static distributed surface force must be proportional to displacement (2) the desired boundary conditions must be enforced along the contour C.

To simplify the present discussion, we consider the case of a free boundary (along contour C); that is, the stress resultants (per unit length) along the edge must be zero. According to Kirchhoff's formulation, N_n, N_{nt}, M_n and V_n must each vanish at the boundary, where referring to the sketches,



Forces/unit length



Force couples/unit length

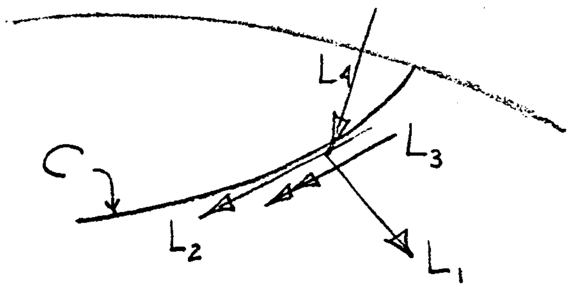
N_n & N_{nt} are membrane stress resultants

M_n is the bending moment

$V_n = (Q_n + \frac{\partial M_{nt}}{\partial s})$, a static equivalent normal edge reaction.

If we denote the four quantities N_n, N_{nt}, M_n, V_n by R_1, R_2, R_3, R_4 and refer to them as boundary condition "residuals", we note the boundary conditions are met if, $R_i = 0$ along the contour C.

Next consider possible "line loads" applied to the complete sphere along the contour, which loads are to assist in meeting the boundary conditions.



Referring to the above sketch, let:

L_2, L_4, L_1 be rectangular components of force (per unit length) which are respectively, tangential to the curve C, normal to the surface, and in the third orthogonal direction

L_3 be a force couple (per unit length) with axis tangential to the curve C.

If we introduce a reference polar axis for the complete sphere, surface points may be located by the polar coördinates ϕ and θ, the latitudinal and meridinal angles respectively, (ϕ measured from the pole).

Let two three-dimensional vectors q_α, u_α (greek indices for three dimensions) be defined such that:

q_1, q_2, q_3 are components of the applied surface force (per unit surface area) in normal direction, tangential to the meridian circle, and tangential to the latitude circle, respectively.

u_1, u_2, u_3 are the displacement components of the shell middle surface (in the normal, and two tangential directions, respectively).

We define next the Green's functions:

$A_{\alpha\beta}(\phi,\theta,\bar{\phi}\bar{\theta})$ as the displacement u_α (at point ϕ,θ) due to an applied force vector p_β (at point $\bar{\phi},\bar{\theta}$) whose components are each unity.

$B_{i\beta}(s,\bar{\phi},\bar{\theta})$ as the boundary condition residuals R_i (at point $\underline{s}$ on the contour C) due the same applied force vector p_β(at $\bar{\phi},\bar{\theta}$)

$C_{ij}(s,t)$ as the boundary condition residuals R_i(at point $\underline{s}$ on contour C) due to an applied line load system K_j (at point $\underline{t}$ on contour C) whose components are each unity.

$D_{\alpha j}(\phi,\theta,t)$ as the displacement u_α (at point ϕ,θ) due to the same line load system K_j (at point $\underline{t}$).

Then for the applied force, q_β acting over the surface element $\underline{d\sigma}$, we may express the resulting contribution to the boundary condition residuals:

$$dR_i = \sum_{\beta=1}^{3} q_\beta \, d\sigma \; B_{i\beta}$$

or employing the summation convention (for repeated indices):

$$dR_i(t) = q_\beta \, B_{i\beta} \, d\sigma \quad \text{(a function of } \underline{t}\text{)}.$$

Similarly, for an applied force system, L_i acting over the arc length ds (of the contour C), the contribution to residuals is:

$$dR_i = L_j \, C_{ij} \, ds$$

Thus through superposition, the requirement that residuals vanish along contour C is met if,

$$\iint_S B_{i\beta} \, (t,\bar{\phi},\bar{\theta}) \; q_\beta \, (\bar{\phi},\bar{\theta}) \, d\sigma + \int_C C_{ij} \, (t,s) \, L_j(s) \, ds = 0 \tag{1a}$$

In a similar fashion, the contributions to displacement, from the applied surface load q_α (acting on $d\sigma$) and the line load L_i (acting over ds) are respectively:

$$du_\alpha = q_\beta A_{\alpha\beta} \quad ; \quad du_\alpha = L_j D_{\alpha j}$$

and hence, the displacement at surface point (ϕ,θ) is:

$$\int_S\int A_{\alpha\beta}(\phi,\theta,\bar{\phi},\bar{\theta})\, q_\beta(\bar{\phi},\bar{\theta})\, d\sigma + \int_C D_{\alpha j}(\phi,\theta,s)\, L_j(s)\, ds = u_\alpha(\phi,\theta) \qquad (2a)$$

The formulation of the problem is completed with the further requirement that

$$q_\alpha(\phi,\theta) = \frac{1}{\lambda} u_\alpha(\phi,\theta) \qquad (3b)$$

where λ is the eigenvalue of the problem.

APPENDIX II METHOD OF SOLUTION

If the integral equations (1a) and (2a), of Appendix I, are replaced by finite difference approximations to these equations, the integral operators become rectangular matrices and the functions u_α, q_α and L_i become column matrices (or "vectors").

Let the surface S (enclosed by the contour C) be subdivided into N elements, the n^{th} element denoted by $\Delta\sigma_n$.

Also let the contour C be subdivided into M segments, with Δs_m denoting the m^{th} segment.

The integrals of equation (1a) may then be approximated by mechanical quadratures; for example,

$$\int_S\int B_{i\beta}\, q_\beta\, d\sigma \doteq \sum_{n=1}^{N} B_{i\beta}(m,n)\, q_\beta(n)\, \Delta\sigma_n$$

where $B_{i\beta}$ (m,n) is $B_{i\beta}$ evaluated at the central points of the m^{th} segment on contour C and the n^{th} surface element.

$q_\beta(n)$ is the central value of q_β for surface element $\Delta\sigma_n$.

Next let the symbol [B] denote a 4M by 3N rectangular matrix whose element in row $[4(m-1) + i]$ and column $[3(n-1) + \beta]$ is $B_{i\beta}(m,n)\,\Delta\sigma_n$. Also let $\vec{q}$ be a column matrix whose element in row $[3(n-1) + \beta]$ is $q_\beta(n)$. then

$$\sum_{n=1}^{N} B_{i\beta}(m,n)\; q(n)\; \Delta\sigma_n \;=\; [B]\,\vec{q}$$

We define other rectangular and column matrices correspondingly to represent the other integral operators and functions appearing in equations (1a) and (2a):

Matrix	Element	Row	Column
[C]	$C_{ij}(m,p)\Delta s_p$	$4(m-1) + i$	$4(p-1) + j$
[A]	$A_{\alpha\beta}(r,n)\Delta\sigma_n$	$3(r-1) + \alpha$	$3(n-1) + \beta$
[D]	$D_{\alpha j}(r,p)\Delta s_p$	$3(r-1) + \alpha$	$4(p-1) + j$
$\vec{L}$	$L_j(p)$	$4(p-1) + j$	
$\vec{u}$	$u_\alpha(r)$	$3(r-1) + \alpha$	

The mechanical quadrature approximation to equations (1a) and (2a) is then:

$$[B]\,\vec{q} \;+\; [C]\,\vec{L} \;=\; 0 \tag{1b}$$

$$[A]\,\vec{q} \;+\; [D]\,\vec{L} \;=\; u \tag{2b}$$

If $[C^{-1}]$ denotes the inverse of matrix [C], then from equation (1b):

$$\vec{L} \;=\; -[C^{-1}]\;[B]\;\vec{q} \tag{3b}$$

This may be substituted into equation (2b) to obtain:

$$[A]\vec{q} - [D][C^{-1}][B]\vec{q} = \vec{u}$$

or

$$\left\{[A] - [D][C^{-1}][B]\right\}\vec{q} = \vec{u} \tag{4b}$$

Finally, if $\vec{q} = \frac{1}{\lambda}\vec{u}$, we may write:

$$[G]\vec{u} = \lambda\vec{u} \tag{5b}$$

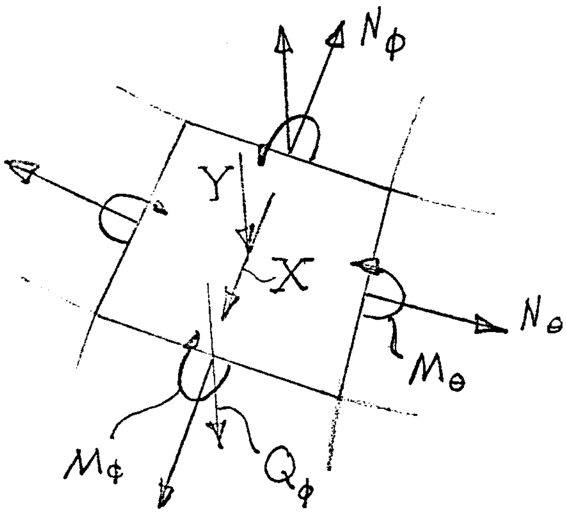
where

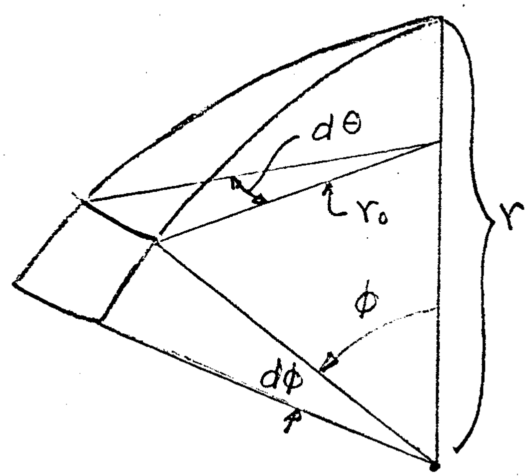
$$[G] = \left\{[A] - [D][C^{-1}][B]\right\}$$

The eigenvalue problem symbolized by equation (5b) may be solved conveniently by the Vianello-Stadola iterative method. Upon obtaining each eigenvalue and mode shape, the matrix [G] is then purified of that mode shape characteristic such that higher order modes and eigenvalues will emerge.

APPENDIX III SOLUTION OF FIRST FUNDAMENTAL PROBLEM

The first fundamental problem is that associated with a spherical shell on an elastic foundation subjected to an applied unit normal load. If the unit normal load is located at the pole ($\phi = 0$) then the problem possesses polar symmetry. Employing the notation and sign conventions used by Timoshenko in "Theory of Plates and Shells", the stress resultants for the polar symmetric problem are as they appear in the sketch below:





Equilibrium of the element leads to:

$$
\begin{aligned}
(N_\phi r_0)^\circ - N_\theta r \cos\phi - (Q_\phi r_0) &= -r_0 r Y \\
(N_\phi r_0) + N_\theta r \sin\phi + (Q_\phi r_0)^\circ &= -r_0 r Z \\
Q_\phi r_0 r = (M_\phi r_0)^\circ - M_\theta r \cos\phi &
\end{aligned}
\tag{1c}
$$

where $(\ )^\circ$ denotes $\frac{d(\ )}{d\phi}$.

For the symmetric problem, stress-strain and strain-displacement relationships may be combined to obtain expressions for the stress resultants in terms of displacements:

$$
\begin{aligned}
N_\phi &= \frac{K}{r}\left[(\bar{v}^\circ - \bar{w}) + \nu(\bar{v}\cot\phi - \bar{w})\right] \\
N_\theta &= \frac{K}{r}\left[(\bar{v}\cot\phi - \bar{w}) + \nu(\bar{v}^\circ - \bar{w})\right] \\
M_\phi &= -\frac{D}{r^2}\left[(\bar{v} + \bar{w}^\circ)^\circ + \nu(\bar{v} + \bar{w}^\circ)\cot\phi\right] \\
M_\theta &= -\frac{D}{r^2}\left[(\bar{v} + \bar{w}^\circ)\cot\phi + \nu(\bar{v} + \bar{w}^\circ)^\circ\right]
\end{aligned}
\tag{2c}
$$

where $D = \dfrac{Eh^3}{12(1-\nu^2)}$ $\quad K = \dfrac{Eh}{(1-\nu^2)}$ $\quad \nu$ = poisson's ratio and $\bar{w}$, $\bar{v}$ are displacement components.

The elastic foundation reaction loads are:

$$
Y = -k\bar{v} \quad , \qquad Z = -k\bar{w} \qquad (\underline{k} \text{ is the foundation modulus.}) \tag{3c}
$$

Defining dimensionless variables and constants:

$$
w = \frac{\bar{w}}{r} \qquad v = \frac{\bar{v}}{r} \qquad \alpha = \frac{D}{Kr^2} \qquad \beta = \frac{kr^2}{K} \quad ,
$$

the substitution of expressions (2c) and (3c) into the equilibrium equations (1c) leads to two equations in v and w:

$$L(v) + \frac{\alpha}{1+\alpha} L(w^\circ) - \frac{1+\nu}{1+\alpha} w^\circ \sin^2 \phi - \frac{\beta}{1+\alpha} v \sin^2 \phi = 0 \tag{4c}$$

$$J(w^\circ) + J(v) - \frac{1+\nu}{\alpha}\left[v^\circ \sin^3 \phi + v \sin^2 \phi \cos \phi - 2w \sin^3 \phi\right] - \frac{\beta}{\alpha} w \sin^3 \phi = 0 \tag{5c}$$

where the operator L( ) and J( ) are defined by:

$$L(\ ) \equiv (\ )^{\circ\circ} \sin^2\phi + (\ )^\circ \sin \phi \cos \phi - (\ )(\cos^2\phi + \nu \sin^2\phi$$

$$J(\ ) \equiv (\ )^{\circ\circ\circ}\sin^3\phi + 2(\ )^{\circ\circ}\sin^2\phi \cos \phi - (\ )^\circ(\nu \sin^3\phi + \sin \phi) + (\ )\left[(1-\nu)\right.$$

$$\left.\sin^2\phi \cos \phi + \cos \phi\right]$$

The higher order derivatives of v appearing in equation (5c) may be eliminated by subtracting $\sin \phi \frac{d}{d\phi}$ (equation 4c) and adding $\cos \phi$ (equation 4c). Additionally, the trigonometric coefficient functions may then be eliminated in both equations through the introduction of new variables:

$$x = \sin \phi \qquad \text{and} \qquad y = \frac{v}{\cos \phi}$$

The resulting two equations are:

$$G(y) + \frac{\alpha}{1+\alpha} G(w') - (1+\nu) x^2 w' - \left[(1+\nu) + \frac{\beta}{1+\alpha}\right] x^2 y = 0 \tag{I}$$

$$H(w) - \left[\frac{1+\alpha}{\alpha}(1+\nu) - \beta\right]\left[(x^3 - x^5)y' + (x^2 - 2x^4)y\right] = 0 \tag{II}$$

where:

$$(\ )' \equiv \frac{d}{dx}$$

$$G(\ ) \equiv (x^2 - x^4)(\ )'' + (x - 4x^3)(\ )' - (\ )$$

$$H(\ ) \equiv (x^3 - 2x^5 + x^7)(\ )^{IV} + (2x^2 - 10x^4 + 8x^6)(\ )''' -$$

$$(x + 8x^3 - 12x^5)(\ )'' + (\ )' + \frac{1+\alpha}{\alpha}\left[2(1+\nu) + \beta\right] x^3(\ )$$

The boundary conditions to be met at the pole, $\phi = 0$, are:

$$w'(0) = 0, \qquad y(0) = 0 \qquad \text{and} \qquad \lim_{x \to 0} (Q_\phi 2\pi x) = 1$$

A sufficiently large value of the elastic foundation constant $\underline{\beta}$ is assumed to insure rapid decay of displacements and stress resultants with increasing ϕ (in the physical problem) such that a negligible error is introduced if stress resultants are set equal to zero for an appropriately large value of x (say x_1). Thus at $x = x_1$, the boundary conditions become:

$$N_\phi(x_1) = 0, \qquad M_\phi(x_1) = 0\ , \qquad Q_\phi(x_1) = 0$$

It is possible to obtain formally series solutions (in x) for w and y of the form:

$$w = \sum_{n=0}^{\infty} a_n x^{r+n} \qquad\qquad y = \sum_{n=1}^{\infty} b_n x^{r+n}$$

The indicial equations of the differential equations I and II are then respectively:

$$r\ (r-2) = 0$$

$$r^2 (r-2)^2 = 0$$

Although the roots $(r = 0,2)$ differ by an integer, independent solutions corresponding to each root are obtainable, as the second order recurrence relationship for equation II is identically satisfied for arbitrary a_2:

$$0 \cdot a_2 = 0$$

Consequently, the solutions take the form of ordinary power series in $\underline{x}$:

$$w = \sum_{n=0}^{\infty} a_n x^n \qquad y = \sum_{n=1}^{\infty} b_n x^n \tag{6c}$$

Thus three linearly independent solutions are obtained by letting the arbitrary constants a_0, a_2 and b_1 each independently be non-zero (while the remaining two are zero). Moreover, the solutions so obtained satisfy identically the boundary conditions:

$$w'(0) = 0 \qquad , \qquad y(0) = 0$$

Consequently, the four remaining boundary conditions require for their satisfaction a fourth linearly independent solution. The repeated roots of the indicial equation (for the differential equation II) suggest a solution of the form

$$w = \hat{w}(\ln x) + \sum c_n x^n \qquad y = \hat{y}(\ln x) + \sum d_n x^n \tag{7c}$$

where $\hat{w}$ and $\hat{y}$ are a combination of series solutions already obtained.

It may be verified that a fourth solution consistent with the boundary conditions on w and y can be found formally, if

$$\hat{w} = \sum \hat{a}_n x^n \qquad , \qquad \hat{y} = \sum \hat{b}\, x^n$$

where:

$$\hat{a}_1 = 0 \qquad \hat{b}_1 = -\frac{2\alpha}{1+\alpha}\, a_2 \tag{8c}$$

The coefficients c_0, c_2, d_1 are arbitrary and for convenience are set equal to zero.

Four recurrence formulae are obtained upon substitution of (6c) and (7c) (under conditions 8c) into equations I and II:

$$H_1 b_n = H_2 b_{n-2} + H_3 a_{n+1} + H_4 a_{n-1}$$

$$J_1 a_n = J_2 a_{n-2} + J_3 a_{n-4} + J_4 b_{n-3} + J_5 b_{n-5}$$

$$H_1 d_n = H_2 d_{n-2} + H_3 c_{n+1} + H_4 c_{n-1} + K_1 \hat{a}_{n+1} + K_2 \hat{a}_{n-1} + K_3 \hat{b}_n + K_4 \hat{b}_{n-2}$$

$$J_1 c_n = J_2 c_{n-2} + J_3 c_{n-4} + J_4 d_{n-3} + J_5 d_{n-5} + L_1 \hat{a}_n + L_2 \hat{a}_{n-2} + L_3 \hat{a}_{n-4} + L_4 \hat{b}_{n-3} + L_5 \hat{b}_{n-5}$$

(9c)

where:

$$H_1 = (n-1)(n+1)$$

$$H_2 = (n-2)(n+1) + \left[(1+\nu) + \frac{\beta}{1+\alpha}\right]$$

$$H_3 = -\frac{\alpha}{1+\alpha}(n-1)(n+1)^2$$

$$H_4 = \frac{\alpha}{1+\alpha}(n-1)\left[(n+1)(n+2) + (1+\nu)\right]$$

$$J_1 = n^2(n-2)^2$$

$$J_2 = 2(n-2)^3(n-3)$$

$$J_3 = -(n-2)(n-3)(n-4)(n-5) - \frac{1+\alpha}{\alpha}\left[2(1+\nu) + \beta\right]$$

$$J_4 = \left[\frac{1+\alpha}{\alpha}(1+\nu) - \beta\right](n-2)$$

$$J_5 = -\left[\frac{1+\alpha}{\alpha}(1+\nu) - \beta\right](n-3)$$

$$K_1 = -\frac{\alpha}{1+\alpha}(n+1)(3n-1)$$

$$K_2 = \frac{\alpha}{1+\alpha}\left[(n-1)(3n-1)-2\right] + (1+\nu)$$

$$K_3 = -2n$$

$$K_4 = (2n-1)$$

$$L_1 = -4n(n-1)(n-2)$$

$$L_2 = 2(n-2)^2(4n-11)$$

$$L_3 = -2\left\{(n-4)\left[(n-5)(2n-3) + 4\right] - 1\right\}$$

$$L_4 = \left[\frac{1+\alpha}{\alpha}(1+\nu) - \beta\right]$$

$$L_5 = -\left[\frac{1+\alpha}{\alpha}(1+\nu) - \beta\right]$$

The arbitrary constants a_0, a_2, b_1 and $\hat{a}_2$ may be found by applying the remaining four boundary conditions:

$$\lim_{x \to 0}(Q_\phi \, 2\pi x) = 1 \quad , \quad Q_\phi(x_1) = 0 \quad , \quad N_\phi(x_1) = 0 \quad , \quad M_\phi(x_1) = 0$$