SEVENTH PROGRESS REPORT

on

STRESSES AND DEFORMATIONS IN THIN SHELLS OF REVOLUTION

to

National Aeronautics and Space Administration

under

NASA Research Grant NsG-274 (S 1 and 2)

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Office of Research Services

December 30, 1965

Seventh Progress Report on

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Introduction

The objective of this investigation continues to be the development of methods of analysis for thin shells of revolution subjected to axi-symmetrical loading of high intensity.

Progress during the report period

During the past six months, the following activity associated with the project may be noted:

1. Paper on "Computer Analysis of Axisymmetrically Loaded Shells of Revolution", by E. P. Popov and Z. A. Lu has been presented at the IASS Symposium on Shell Structures, September 3, 1965, in Budapest. Proceedings of the Conference are in press (see item 1 of the previous report).

2. Computer programms for determining frequencies associated with axisymmetrical vibrations as well as the response of circular plates and shells to time-dependent force or acceleration input for a number of boundary conditions has been completed. The capabilities of the developed solution are illustrated in the attached note on "Dynamic Response of Shells of Revolution Based on Finite Element Approach", by E. P. Popov and H. Y. Chow. A more detailed report is in preparation.

3. The developments achieved on the project and technical assistance were given to James Chisholm, a graduate student, in connection with his M.S. research on pressurized torroidal shells.

4. Extensive work was done postulating elastic-ideally plastic and elastic-isotropic hardening materials. Computer programs for the analysis of circular plates have been achieved. Based on this work, a synopsis of a paper has been submitted to the 5th U. S. National Congress of Applied Mechanics. This synopsis titled "A Bending Analysis of Elastic-Plastic Circular Plates" by E. P. Popov, M. Khojasteh-Bakht, and S. Yaghmai is enclosed. A more complete description and extension of the developed procedures is in progress. The selected approach appears to be suitable in general for inelastic response of rotational shells.

5. The procedures for developing methods of analysis for predicting large-deflection response of circular plates and rotational shells remain under consideration.

Budget

A budgetary statement on this project will be sent separately after the December expenses are reported.

Dynamic Response of Shells of Revolution Based on Finite Element Approach

by

E. P. Popov* and H. Y. Chow**

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Synopsis

The displacement method of analysis using matrix procedures is widely used in static and dynamic structural problems (Refs. 1, 2). For shells of revolution, finite elements in the form of conical frustra joined at nodal circles have been found to be very effective in the solution of problems (Refs. 3, 4). An outline of a solution*** based on these concepts for axisymmetrical response to dynamic loads is given in this discussion. Two examples are included. The data for one of the examples are taken from a recent paper**** by S. Klein and R. J. Sylvester. Excellent agreement between the two solutions is found which serves to corroborate the results found independently. The other example shows a possible advantage of the method described here in some problems.

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*** A more complete report by the authors to NASA is in preparation.

**** "The linear elastic dynamic analysis of shells of revolution by the matrix displacement method" Conference on Matrix Methods in Structural Mechanics, Oct. 1965, Wright-Patterson AFB.

Equations of Motion

Any shell of revolution can be approximated with a sufficient degree of accuracy for practical purposes by a finite number of elements consisting of plate, conical, or cylindrical rings. All of such elements are joined at nodal circles. The matrix formulation of the general dynamic response of such a substitute structure can be stated as follows:

$$[M] \left\{\ddot{q}(t)\right\} + [K] \left\{q(t)\right\} = \left\{f(t)\right\} \tag{1}$$

where the column matrices $\left\{f(t)\right\}$ and $\left\{q(t)\right\}$ are, respectively, the generalized forces and generalized displacements of the structure. The mass matrix $[M]$ and the structural stiffness matrix $[K]$ are symmetrical and furthermore $[K]$ is a positive definite. Therefore, Eq. 1 can be uncoupled (Ref. 5) and solved using the normal mode superposition technique.

A direct numerical integration method was used in this investigation to solve the uncoupled second order differential equations. For each mode a different interval of time for each integral is used. This procedure retains the necessary accuracy for the higher frequency modes and avoids the unnecessary, time consuming computations for the lower frequencies.

The mass matrix $[M]$ and the stiffness matrix $[K]$ for a whole structure are determined from the assemblage of basic solutions for the elements. Since this procedure is well known, only the formulation used to establish element stiffness and mass matrices is discussed here.

The Element Stiffness and Mass Matrices

The homogeneous solution for a basic finite element can be expressed in matrix form as follows:

$$\left\{d_i(s,t)\right\} = \begin{Bmatrix} \chi(s,t) \\ V(s,t) \\ W(s,t) \end{Bmatrix} = \left[X_{ij}(s)\right] \left\{A_j(t)\right\} \tag{2}$$

and

$$\left\{S_i(s,t)\right\} = \begin{Bmatrix} M_s(s,t) \\ N_s(s,t) \\ Q_s(s,t) \end{Bmatrix} = \left[Y_{ij}(s)\right] \left\{A_j(t)\right\} \tag{3}$$

and

$i = 1,2,3$ and $j = 1,2....,6$

where $\left\{d_i(s,t)\right\}$ are displacement-variables which are comprised of rotational $\chi(s,t)$ meridianal $V(s,t)$, and normal $W(s,t)$ displacements; $\left\{S_i(s,t)\right\}$ are force-variables which consist of meridianal moments $M_s(s,t)$, meridianal stress-resultants $N_s(s,t)$, and shearing stress-resultants $Q_s(s,t)$.

$[X_{ij}(s)]$ and $[Y_{ij}(s)]$ are solution matrices of a homogeneous problem for an element. $\left\{A_j(t)\right\}$ is a column matrix which can be determined from the boundary values at each end of the shell segment.

Closed form solutions of Eq. 1 were developed (Refs. 3,4 and authors report to NASA in preparation) for circular annular rings, conical frustra, and cylindrical segments. In this formulation, if such elements represent portions of the actual structure, no limitation on the size of elements needs to be placed.

Using closed form solutions of Eq. 1, the element stiffness matrices $[k]$ were developed and programmed for the above type of elements. The basic relation for determining $[k]$ can be deduced by considering strain energy U of an element, and can be shown to be

$$[k] = [T]^T [C] [B^{-1}] [T] \tag{4}$$

where $[T]$ is a coordinate transformation matrix relating shell element coordinates to the global coordinates, and matrices $[C]$ and $[B]$ are matrices $[Y]$ and $[X]$, respectively, upon substitution into them of the boundary values of s. Since this relationship is but a slightly different form was previously reported (Refs. 3,4), no further comments will be made here.

To determine the mass matrix $[m]$ for an element, the fundamental displacement-variable vector $\left\{d_i\right\}$, Eq. 2, must be re-cast in terms of its six generalized global nodal coordinates, i.e.,

$$\left\{d_i(s,t)\right\} = [X_{ij}(s)] \; [B_{jk}^{-1}] \; [T_{km}] \; \left\{q_m(t)\right\} \tag{5}$$

here $i = 1,2,3$ and $j, k, m, = 1,2,3....6$.

The general expression for the kinetic energy T(t) (Refs. 6,7) of a shell element can be written as

$$T(t) = \frac{1}{2} \int_s [m \rho_A^2 \dot{\chi}^2 (s,t) + m \dot{V}^2 (s,t) + m \dot{W}^2 (s,t)] 2\pi r(s) ds \quad (6)$$

where m is mass per unit of surface area, and ρ_A is the radius of gyration of the section of a shell segment.

Upon substituting the displacement variables involved in Eq. 5 into Eq. 6, one obtains

$$T(t) = \frac{1}{2} < \dot{q}(t) > [T]^T [B^{-1}]^T [\int_s 2\pi [E(s)] r(s) ds] [B^{-1}] [T] \left\{\dot{q}(t)\right\} \quad (7)$$

By comparing this complex matrix expression with the usual one for kinetic energy, definition of the mass matrix [m] is obtained:

$$[m] = [T]^T [B^{-1}]^T [\int_s 2\pi [E(s)] r(s) ds] [B^{-1}] [T] \quad (8)$$

This element mass matrix [m] was determined and programmed for annular rings using an exact displacement field. For conical frustra the mass matrix [m] for an element (Ref. 6) was developed on the basis of an assumed polynomial function to represent the displacement field. For the above reason, the range of applicability of the developed program as it relates to the size of elements is different for the two cases.

Examples and Conclusions

As the first example consider an elastic circular plate clamped along the edge subjected to a ring load as shown in Fig. 1. The ring load P is applied as a step function in time. To determine the dynamic response of this plate by the developed method, only two elements* need to be used, since both the [m] and [k] matrices are programmed using the exact displacement field. Alternatively, an arbitrary number of elements may be used and 20 elements were selected to obtain a solution for comparative purposes. The results of the two solutions are plotted in Figs. 4a and 4b. Differences between the two solutions are negligible. The solution based on the use of 20 elements actually is a little less accurate due to the unavoidable accumulation of numerical errors.

The second example is for the dynamic response of a shallow spherical cap shown in Fig. 2. The data are from the Klein and Sylvester example. The results of an output for a 14 element solution are shown in Figs. 5a, 5b and 5c. These results are seen to be in excellent agreement with the Klein and Sylvester solution and this provides a good check on the two independently developed programs. In the solution of this problem no advantage can be gained by taking large finite elements.

The developed program of course also can be successfully applied to deep shells as well as to shell-like enclosures. For example, the dynamic

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* A disk of 10 in. radius, and an annular ring bounded by 10 in. and 20 in. radii.

response of the sphere shown in Fig. 3 was readily found using a solution based on 50 elements. (Results not reported here).

The dynamic response of linear elastic shells of revolution of arbitrary meridian shape and thickness variation can be determined using finite element approach. The accuracy appears to be excellent, and once a program is developed a solution is achieved very rapidly. Occasionally, solutions based on exact displacement fields for an element mass and stiffness matrices may prove advantageous.

Acknowledgements

The authors wish to acknowledge with gratitude the financial support of NASA which made possible the development of the reported programs.

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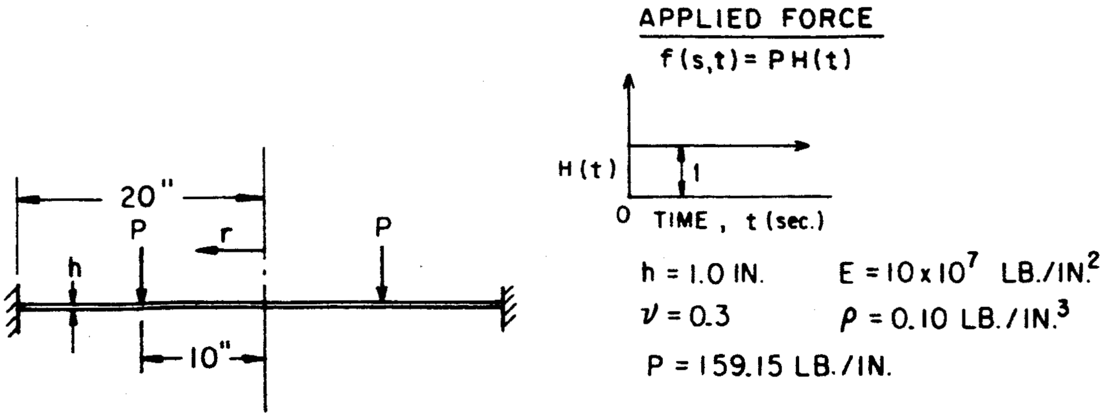

FIG. 1 A CIRCULAR PLATE UNDER DYNAMIC LOADING

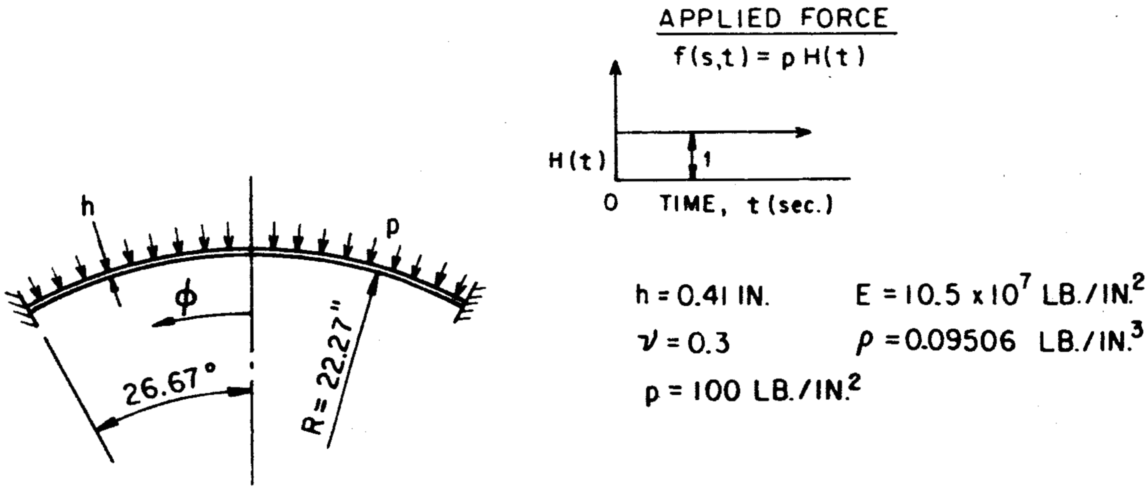

FIG. 2 A SPHERICAL CAP UNDER DYNAMIC LOADING

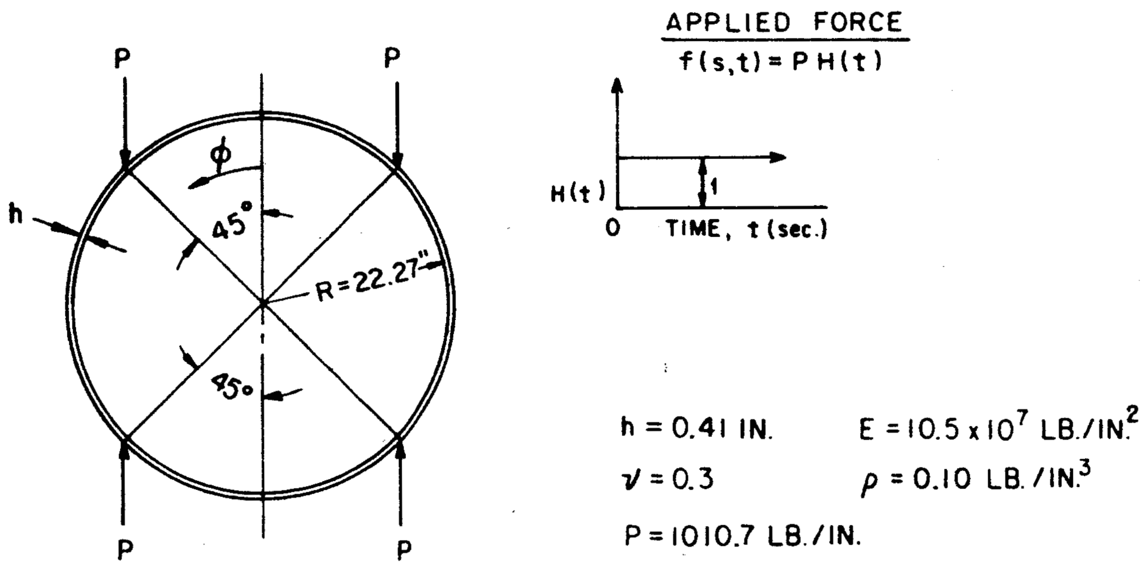

FIG. 3 A SPHERE UNDER DYNAMIC LOADING

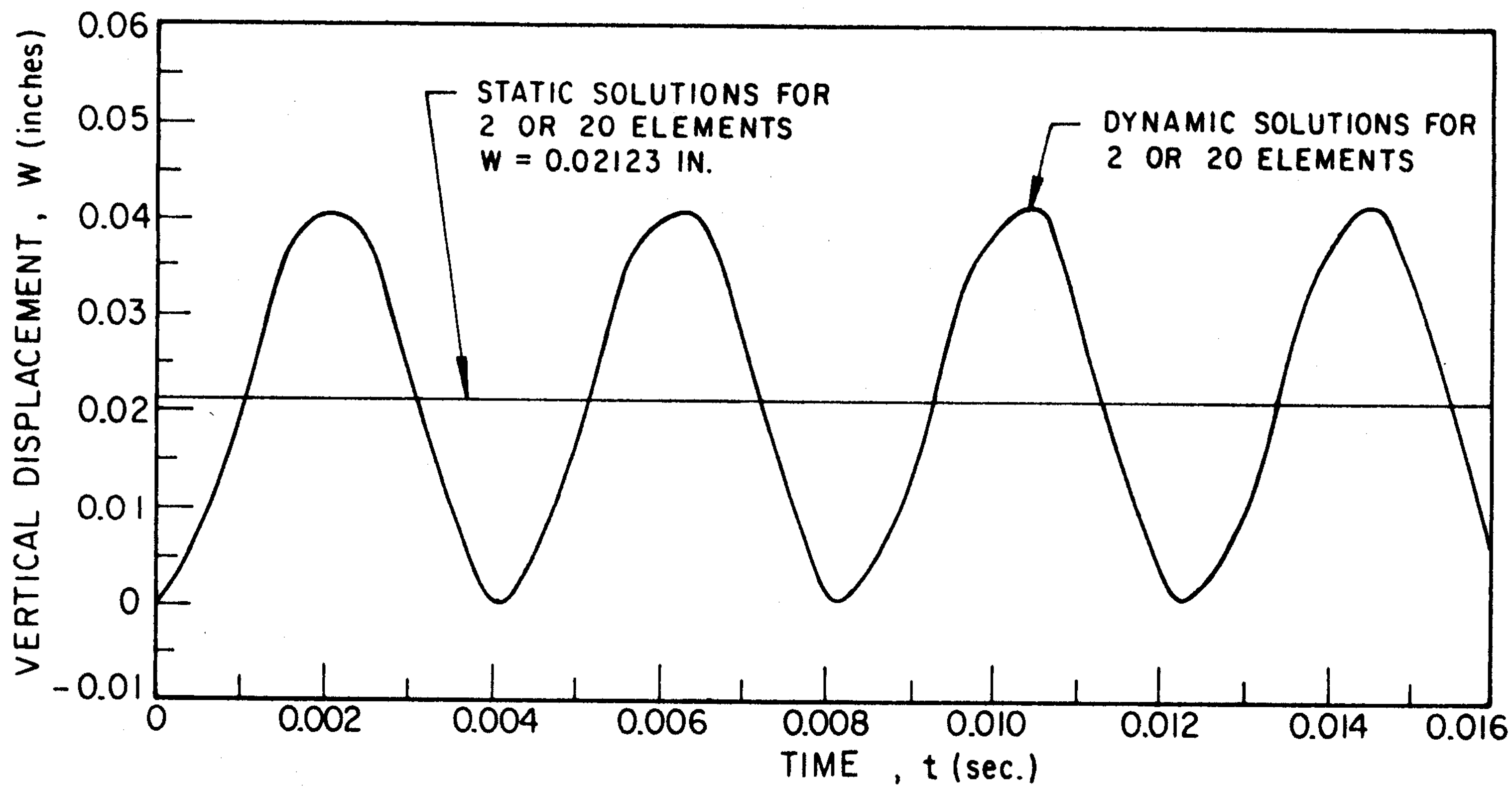

FIG. 4a VERTICAL DISPLACEMENT RESPONSE AT r = 10 IN.
FOR THE CIRCULAR PLATE

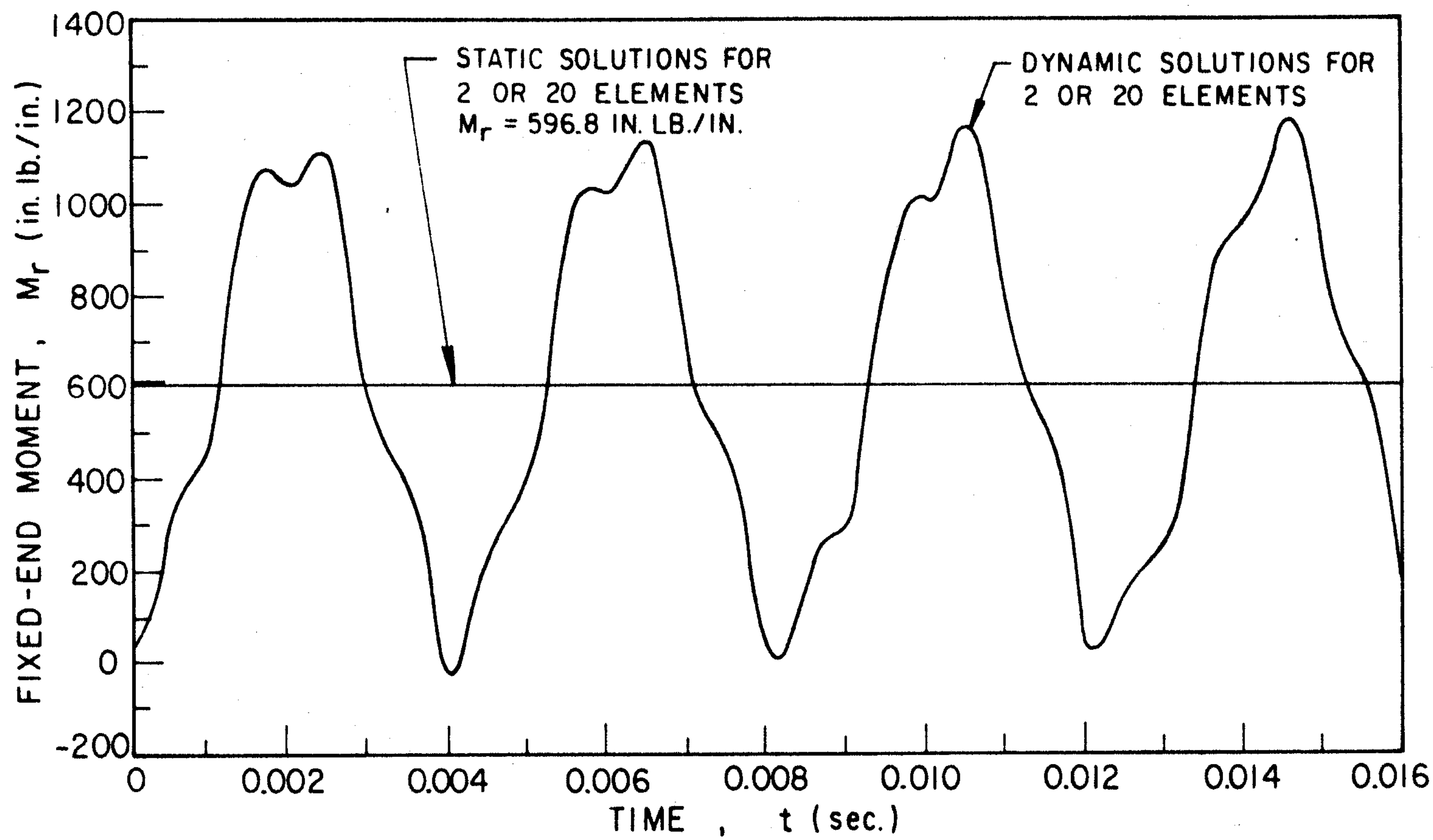

FIG. 4b FIXED-END MOMENT RESPONSE AT r = 20 IN.
FOR THE CIRCULAR PLATE

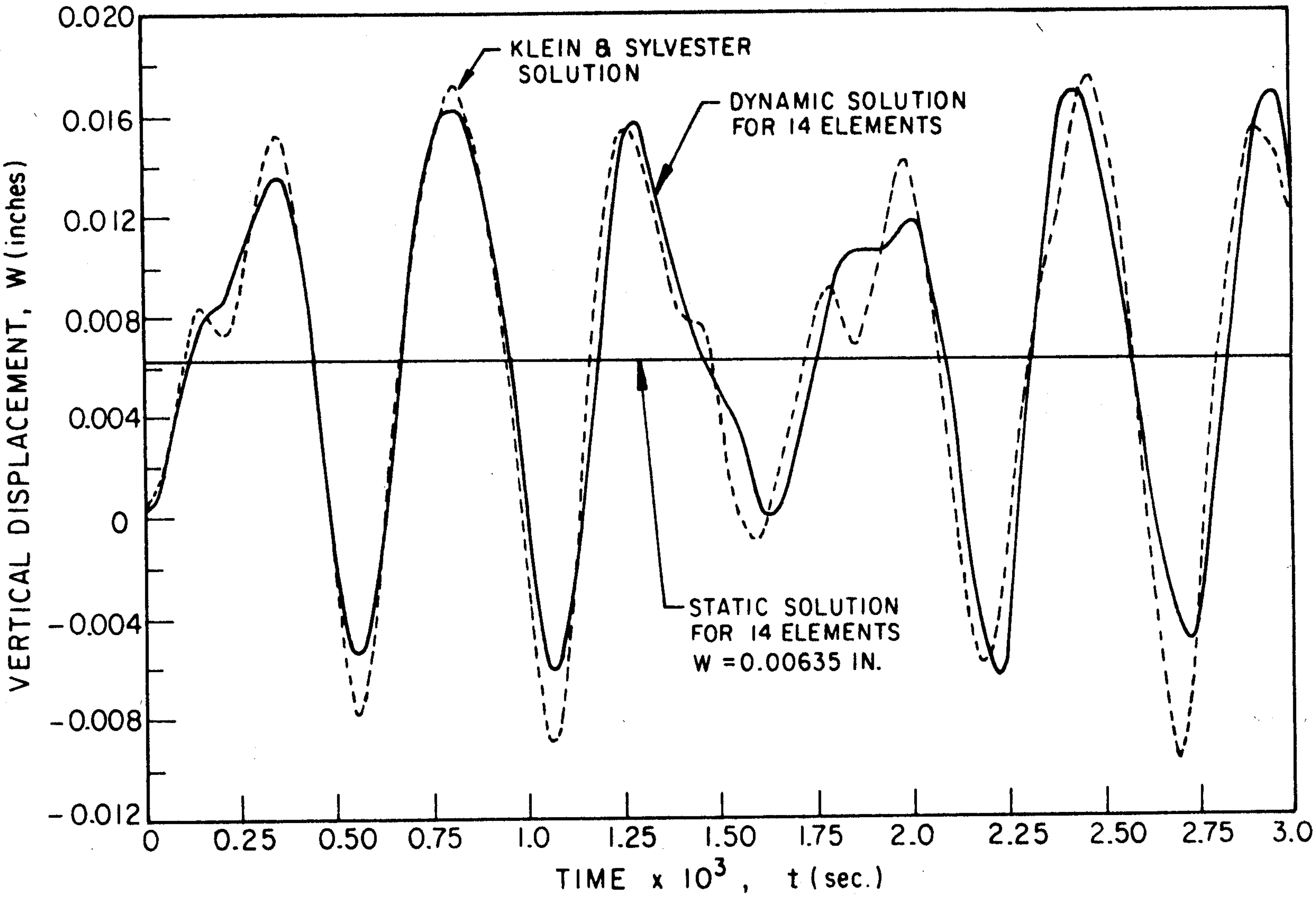

FIG. 5a VERTICAL DISPLACEMENT RESPONSE AT $\phi = 0°$ FOR THE SPHERICAL CAP

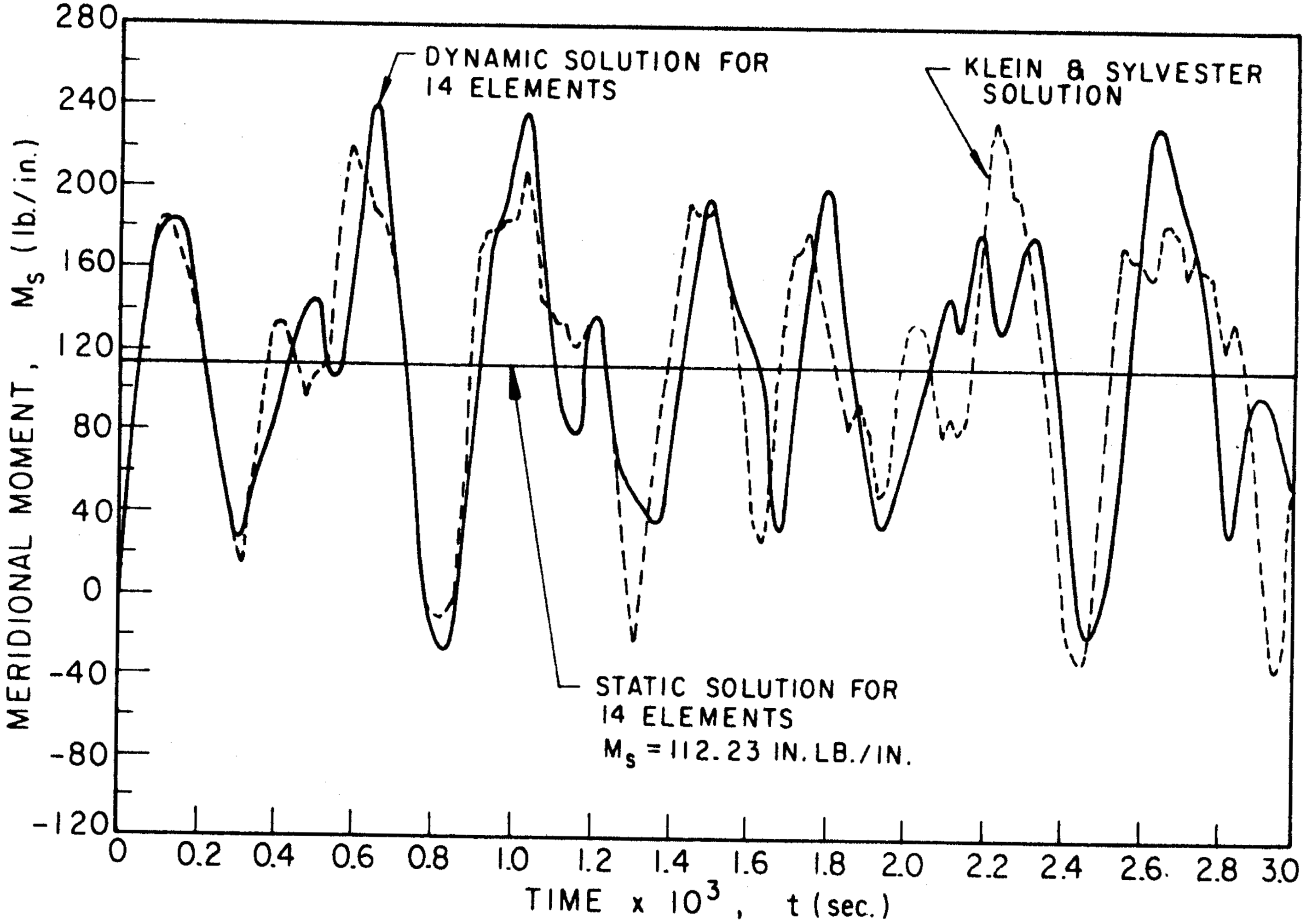

FIG. 5c MERIDIONAL MOMENT RESPONSE AT ϕ = 26.67° FOR THE SPHERICAL CAP

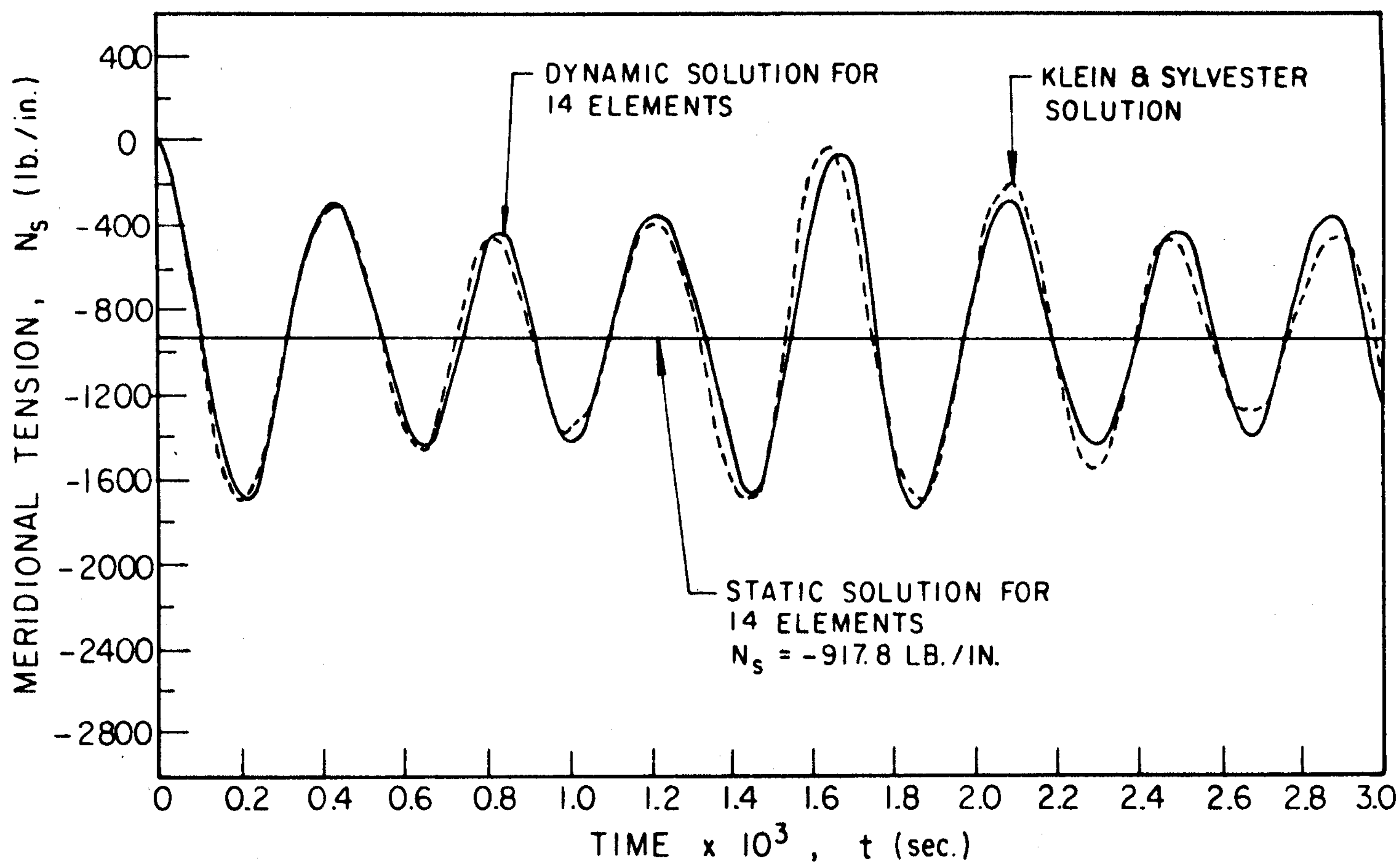

FIG. 5b MERIDIONAL TENSION RESPONSE AT $\phi = 26.67°$ FOR THE SPHERICAL CAP

A BENDING ANALYSIS OF ELASTIC-PLASTIC CIRCULAR PLATES*

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A general bending analysis of circular plates with small-deflections and axial symmetry is developed in the paper. It is postulated that the Kirchhoffean assumption neglecting shearing effects is applicable, and that the von Mises yield condition with the associated flow rule holds. The solutions are obtained for elastic-perfectly plastic solid as well as for isotropic hardening material. In both instances incremental constitutive laws are used which have caused a considerable mathematical difficulty in the solution of boundary value problems.

The numerical solutions are achieved by dividing a plate into small circular annulii and a number of very thin layers along its depth.

In linear theory of plasticity the infinitesimal increments of strain and stress tensors are related linearly. In the proposed solutions the infinitesimal increments are replaced by small finite increments. Within an increment the change of material properties is accounted for. At any stage of loading, relations analogous to those for anisotropic elastic media between the stress and strain increment tensors are used. For each increment of loading the problem is reduced to a solution of non-homogeneous anisotropic linear elastic problem. The problem is formulated in matrix algebra using the stiffness method.

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* This research was supported by NASA under NsG-274 grant.

Examples of solutions for simply supported and fixed-ended plates are given for elastic-perfectly plastic and hardening materials. The questions of solution convergence and the effect of loading paths are discussed. Comparison is made with some existing solutions for plates based on the deformation (total strain) theory of plasticity. Extension of the proposed approach to other axisymmetrical problems is indicated.

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To be submitted for publication to ASCE Engineering Mechanics Division Journal as "Anal. of Elast.-Perf. Plastic Cir. Plates" and "Bending of Circ. Plates of Hardening Mat'l".