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COMMUNICATION THEORY APPLIED TO ANTENNA SYSTEMS

BY
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I. INTRODUCTION

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The properties of an antenna system are determined not only by the antenna but also by the electronics associated with the antenna. For this reason, there is an important area in the field of antenna research where the antenna is treated as a signal processing device rather than a simple transducer. Many of the published papers that analyze this type of antenna system use a terminology which is closer to that of communication theory than antenna theory. To understand such papers the reader must be knowledgeable in both fields. This is not the usual case and the purpose of this report is to define some of the fundamental concepts of communication theory and show how they can be applied to antenna systems. The report is divided into two parts: chapters II and III review some results of Fourier (harmonic) analysis and linear circuit theory; chapters IV and V develop analogous results for antenna systems. The emphasis throughout the report has been placed on the physical interpretation of concepts and not on the mathematics used to derive or define the concepts.

author

II. REVIEW OF HARMONIC ANALYSIS

A function $g(\xi)$ is related to its complex spectrum $G(\eta)$ by the Fourier transform pair:

$$g(\xi) = \int_{-\infty}^{\infty} G(\eta)\, e^{j2\pi\xi\eta}\, d\eta \tag{1}$$

$$G(\eta) = \int_{-\infty}^{\infty} g(\xi)\, e^{-j2\pi\xi\eta}\, d\xi \tag{2}$$

In order for the function to have a complex spectrum the integral of equation (2) must converge. This condition will be satisfied if $g(\xi)$ is absolutely integrable, that is,

$$\int_{-\infty}^{\infty} |g(\xi)|\, d\xi$$

is finite.

It is convenient to use the notation

$$g(\xi) \longleftrightarrow G(\eta)$$

to indicate that the functions $g(\xi)$ and $G(\eta)$ are related by the Fourier transforms of equations (1) and (2). The double-headed arrow signifies a reversible transformation between the two functions.

The significance of the complex spectrum is apparent when equation (1) is written in the form

$$g(\xi) = \int_{-\infty}^{\infty} \left[G(\eta)\, d\eta\right] e^{j2\pi\xi\eta} \tag{3}$$

The integrand is a complex sinusoid of frequency η and infinitesimal amplitude $G(\eta)d\eta$. Equation (3) states that the function $g(\xi)$ is synthesized by an infinite number of sinusoids of all frequencies η in the continuous infinite range $(-\infty, \infty)$. The amplitude density spectrum $|G(\eta)|$ does not give the actual amplitudes of the sinusoids because all amplitudes are of infinitesimal magnitude; it is rather a function which shows the <u>relative</u> magnitudes of the infinity of complex sinusoids used to synthesize the function $g(\xi)$.

The physical significances of the transform variables ξ and η depend on the problem. Most often the variables are the time t (seconds) and the temporal frequency f (cycles/second) but other physical variables occur. It is important to note that the concept of a spectrum is not restricted to sinusoids of temporal frequency. The sinusoidal function of η, $G(\eta)\, e^{j2\pi\xi\eta}$, is still called the Fourier component of $g(\xi)$ with frequency η even though η may have units other than cycles per second.

If a time function $g(t)$ represents a voltage waveform, its complex spectrum $G(f)$ will have the units of volts per unit frequency (volt-sec). The function

$$\Phi(f) = |G(f)|^2$$

has the units of watt-seconds per unit frequency and is called the energy density spectrum of $g(t)$. If $g(t)$ is applied to a 1-ohm load of pure resistance, the total energy consumed by the resistance will be given by the integral of $\Phi(f)$ over all frequencies. Using equations (1) and (2) it can be shown[(1)] that the Fourier transform of the energy density spectrum is

$$\int_{-\infty}^{\infty} g(t)\, g(t+\tau)\, dt$$

The integral states that the function $g(t)$ is multiplied by itself delayed τ seconds and the product integrated over all time. The result is a function only of the displacement τ and is indicated by

$$R(\tau) = \int_{-\infty}^{\infty} g(t)\, g(t+\tau)\, dt \tag{4}$$

$R(\tau)$ is known as the autocorrelation function of $g(t)$.

Figure 1a illustrates the relationships between $g(t)$ and its various transforms. The single-headed arrows are used to indicate the irreversible transformations between $g(t)$ and its autocorrelation function and between the complex spectrum and the energy density spectrum.

If $g(t)$ is a random signal, the integral of equation (2) will not converge and the signal has no complex spectrum. The autocorrelation function for the random

(1) Statistical Theory of Communication, Y. W. Lee, Wiley pp. 36-37.

signal is defined as

$$\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} g(t)\, g(t + \tau)\, dt$$

i.e. the autocorrelation function is the time average of the random function multiplied by itself delayed. If the signal results from a stationary random process (one whose statistics are independent of time), the autocorrelation function will depend only on the delay τ and is indicated by

$$R(\tau) = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} g(t)\, g(t + \tau)\, dt$$

The Fourier transform of the autocorrelation function has the units of watts per unit frequency and is defined as the power density spectrum of $g(t)$. Figure 1b

$$\begin{tikzcd} g(t) \arrow[r, leftrightarrow] \arrow[d] & G(f) \arrow[d] \\ R(\tau) \arrow[r, leftrightarrow] & \Phi(f) \end{tikzcd}$$

(a) $g(t)$ is nonrandom signal with finite energy

$$\begin{tikzcd} g(t) \arrow[d] & \\ R(\tau) \arrow[r, leftrightarrow] & \Phi(f) \end{tikzcd}$$

(b) $g(t)$ is stationary random signal

Figure 1 – Relationships Between a Signal and its Transforms

illustrates the relationships between the random function and its various transforms. Notice that the only way to determine the power density spectrum of a stationary random signal is by means of the autocorrelation function. This is one reason for the importance of autocorrelation functions in analyses treating random signals. Figure 2 shows the various transforms for the pulse signal

$$
\begin{aligned}
g(t) &= A \quad -\frac{T}{2} \leq \tau \leq \frac{T}{2} \\
&= 0 \quad \text{elsewhere}
\end{aligned}
$$

The complex spectrum is calculated from equation (2) as

$$
G(f) = \int_{-\infty}^{\infty} g(t)\, e^{-j2\pi ft}\, dt = A \int_{-T/2}^{T/2} e^{-j2\pi ft}\, dt
$$

$$
= \frac{A}{(-j2\pi f)} \left[e^{-j\pi fT} - e^{j\pi fT}\right] = \frac{A}{\pi f} \sin \pi fT
$$

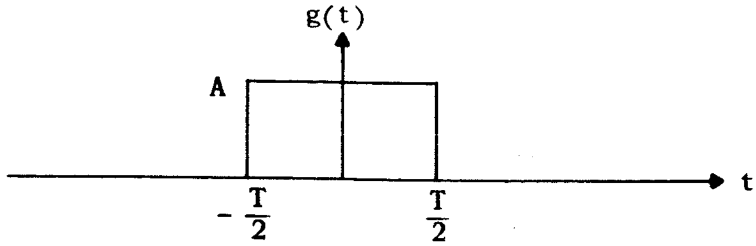
$$
G(f) = AT \frac{\sin \pi fT}{\pi fT}
$$

The energy density spectrum is given by

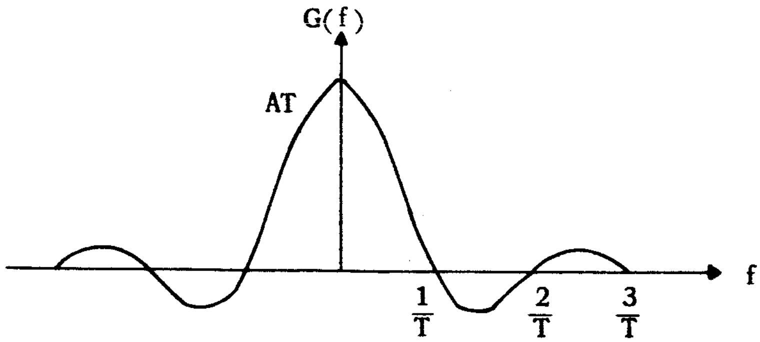
$$
\Phi(f) = |G(f)|^2 = (AT)^2 \left|\frac{\sin \pi fT}{\pi fT}\right|^2
$$

The determination of the autocorrelation function is simplified by a graphical interpretation of the definition (4). At $\tau = 0$, the autocorrelation function is simply the area under the function squared

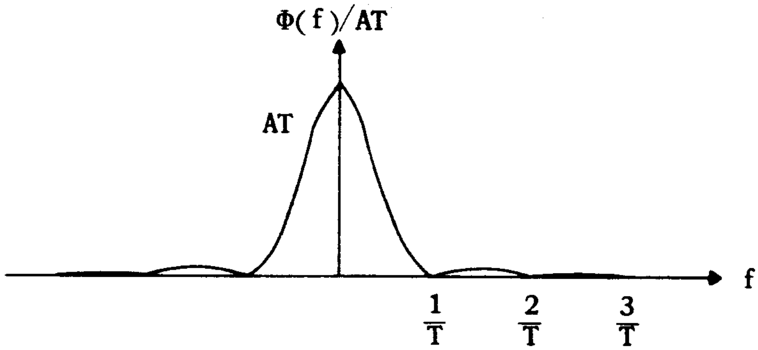
$$
R(0) = \int_{-\infty}^{\infty} g^2(t)\, dt = E^2\, T
$$



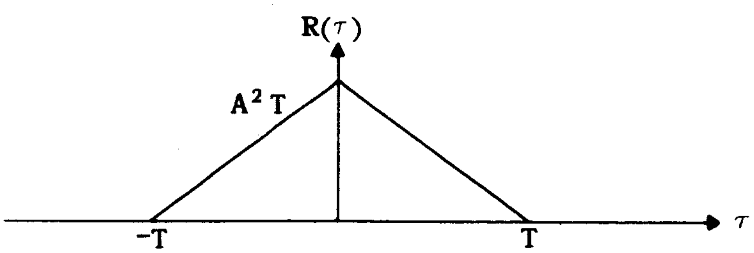
(a) signal



(b) complex spectrum



(c) energy density spectrum



(d) autocorrelation function

Figure 2 – Transforms of a Pulse Signal

As $|\tau|$ increases, the area of overlap between $g(t)$ and $g(t+\tau)$ will decrease linearly. Beyond $|\tau| = T$ there is no overlap and the autocorrelation function is equal to zero. The function therefore has the triangular shape shown in Figure 2d.

III. REVIEW OF LINEAR CIRCUIT ANALYSIS

Consider the circuit shown in Figure 3. The input and output of the system are functions only of time and we indicate that $x(t)$ produces the output $y(t)$ by the notation $x(t) \Rightarrow y(t)$. The function $h(t)$ is the response of the circuit to a unit-impulse excitation, that is, $\delta(t) \Rightarrow h(t)$, where the unit impulse is a mathematical function defined by the following properties:

a) $\delta(t) = 0 \quad t \neq 0$

b) $\delta(0) \to \infty$

c) $$\int_{t_1}^{t_2} \delta(t)\, dt \begin{array}{l} = 1 \text{ if range of integration includes } t = 0 \\ = 0 \text{ otherwise} \end{array}$$

The circuit is assumed to satisfy the following conditions:

1. Linearity

$$a x_1(t) + b x_2(t) \Rightarrow a y_1(t) + b y_2(t)$$

Adding two inputs results in adding their outputs. Also if an input is multiplied by a real constant, its output is multiplied by the same constant

2. Time-Invariance

$$x(t + t_0) \Rightarrow y(t + t_0)$$

The input-output relation of the circuit is invariant to a translation of the time axis. Satisfaction of these two conditions leads directly to the instantaneous input-output relation of the system

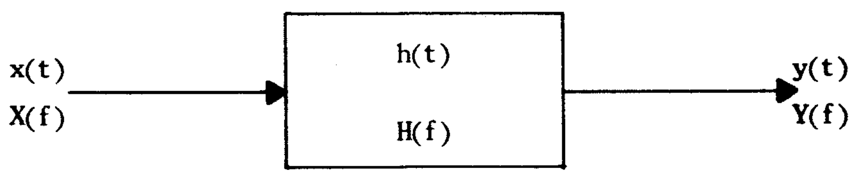
$$y(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t - \tau)\, d\tau \tag{5}$$

Equation (5) shows that the operation of the linear circuit is characterized by the unit-impulse response function. The integral is known as the superposition theorem and holds whether $x(t)$ and $y(t)$ be periodic, transient, or random.

If $x(t)$ is a nonrandom signal, it will possess a Fourier transform and equation (5) can be transformed to yield

$$Y(f) = X(f)\, H(f) \tag{6}$$

Equation (6) relates the input and output spectrums by means of $H(f)$, the transform of the inpulse-response function. $H(f)$ is known as the system transfer function.



$h(t)$ = unit-impulse response function

$H(f)$ = System transfer function

Figure 3 – Linear Circuit Functions

Suppose that the circuit of Figure 3 is to be used as a filter, that is, we want the system to pass the input signal $x(t)$ undistorted and block all other signals. The output signal will be an exact replica of the input if the spectrums are identical; according to (6) this condition will be met if the system transfer function is constant over the bandwidth occupied by the input signal. If $H(f)$ is zero outside this bandwidth, it will prevent unwanted signals from appearing at the output. Theoretically, it is possible to synthesize a circuit with the desired transfer junction because any practical time signal $x(t)$ will have a spectrum that is band limited in frequency.

IV. APERTURE THEORY

According to the scalar theory of diffraction, if all sources of an electromagnetic field are enclosed by a surface, then knowledge of the field (or its spatial derivative) on the surface is sufficient to determine the field at any point in the source-free region. In many practical problems the surface may be taken as an infinite plane and the field on the plane will be zero except over a finite area. This area through which power flow is confined is termed an aperture even though no physical hole exists. For convenience, we can assume that the infinite plane is the xy-plane as shown in Figure 4.

Let $E(x/\lambda, y/\lambda)$ indicate the transverse electric field over the aperture and $A(S_1\ S_2)$ be the two-dimensional Fourier transform of the aperture field. The Fourier transform pair may be written in the following form:

$$E\left(\frac{x}{\lambda}, \frac{y}{\lambda}\right) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} A(S_1\ S_2)\, e^{-j2\pi(S_1 x/\lambda + S_2 y/\lambda)}\, dS_1\, dS_2 \tag{7}$$

$$A(S_1\ S_2) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} E\left(\frac{x}{\lambda}, \frac{y}{\lambda}\right) e^{j2\pi(S_1 x/\lambda + S_2 y/\lambda)}\, d\left(\frac{x}{\lambda}\right) d\left(\frac{y}{\lambda}\right) \tag{8}$$

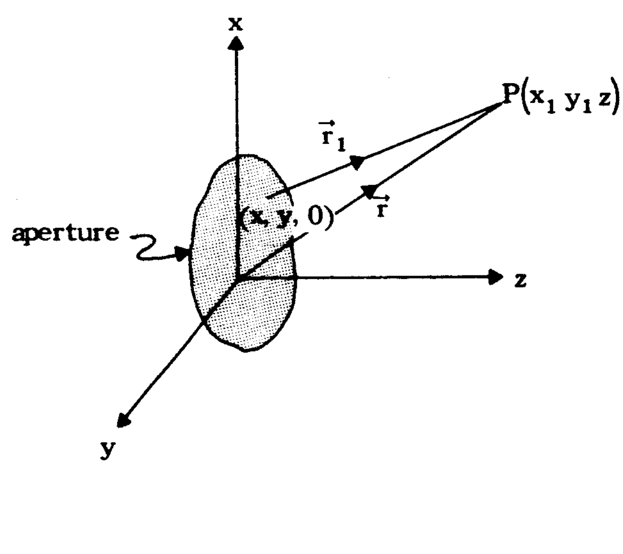


Figure 4 – Coordinate System for Aperture Antennas

where

$$S_1 = \sin\theta \cos\phi$$

$$S_2 = \sin\theta \sin\phi$$

$$\lambda = \text{wavelength}$$

The integrand of equation (7) is the mathematical representation of a plane wave of amplitude $A(S_1\ S_2)dS_1\,dS_2$ traveling in the S_1, S_2 direction. Equation (7), then, synthesizes the aperture field distribution by superimposing an infinite number of plane waves traveling in all directions. The function $A(S_1\ S_2)$ which gives the relative amplitudes of the plane waves is known as the angular spectrum.

The field at any point P in the source-free region of space is given by

$$U_P = \frac{1}{4\pi} \int_{-\infty}^{\infty}\!\!\int E\left(\frac{x}{\lambda}, \frac{y}{\lambda}\right) \frac{e^{-jkr_1}}{r_1} \left[\left(jk + \frac{1}{r_1}\right)\vec{a}_z \cdot \vec{r}_1 + jk\vec{a}_z \cdot \vec{S}\right] d\left(\frac{x}{\lambda}\right) d\left(\frac{y}{\lambda}\right) \tag{9}$$

where $\vec{r}_1$ is a unit vector in the direction from the aperture point (x, y) to the field point (x, y, z), $\vec{S}$ is a unit vector in the direction of the ray at (x, y), and $k = 2\pi/\lambda$. At large distances from the aperture, equation (9) can be approximated by

$$U_P \approx \frac{j}{\lambda} \frac{e^{-jkr}}{r} \int_{-\infty}^{\infty}\!\!\int E\left(\frac{x}{\lambda}, \frac{y}{\lambda}\right) e^{j2\pi(S_1 x/\lambda + S_2 y/\lambda)}\, d\left(\frac{x}{\lambda}\right) d\left(\frac{y}{\lambda}\right) = \frac{j}{\lambda} \frac{e^{-jkr}}{r} A(S_1\ S_2)$$

where r is the distance from the origin to the field point P. The function $A(S_1\ S_2)$, therefore, gives the dependence of the field on the angular variables and may be identified as the conventional angular radiation pattern. Since the angular spectrum, by definition, is the Fourier transform of the aperture field, the far-field radiation pattern is also the Fourier transform of the aperture field.

Rather than using two-dimensional Fourier transforms, it will be convenient to confine our attention to one-dimensional apertures with an aperture field distribution $E(x/\lambda)$ and far-field radiation pattern $A(\sin\theta)$.

Analogous to the energy-density spectrum of time signals we have the function

$$\Phi(\sin\theta) = |A(\sin\theta)|^2$$

which, is the far-field power pattern of the antenna. The Fourier transform of the power pattern is equal to the aperture autocorrelation function

$$R\left(\frac{S}{\lambda}\right) = \int_{-\infty}^{\infty} E\left(\frac{x}{\lambda}\right) E\left(\frac{x+S}{\lambda}\right) d\left(\frac{x}{\lambda}\right) \tag{10}$$

Equation (10) states that the autocorrelation function is found by multiplying the field at the point x/λ in the aperture by the field at another point separated by S/λ and then integrating the product over all points of the aperture. The result is a function only of the separation between the field points. Figure 5 summarizes the relationships between the aperture field distribution and its various transforms; Figure 6 illustrates the transforms for a uniformly illuminated aperture.

$$\begin{tikzcd} E\left(\frac{x}{\lambda}\right) \arrow[r, leftrightarrow] \arrow[d] & A(\sin\theta) \arrow[d] \\ R\left(\frac{s}{\lambda}\right) \arrow[r, leftrightarrow] & \Phi(\sin\theta) \end{tikzcd}$$

Figure 5 – Relationships Between an Aperture Field and Its Transforms

An important result of the Fourier transform relation between an aperture field and the far-field pattern is the following:

If

$$E\left(\frac{x}{\lambda}\right) \longleftrightarrow A(u)$$

where

$$u = \sin\theta$$

then

$$E\left(\frac{x}{\lambda}\right) \exp\left(-j\,2\pi u_0 \frac{x}{\lambda}\right) \longrightarrow A\left(u - u_0\right)$$

The result may be proved from the definition of the Fourier integral

$$A(u) = \int_{-\infty}^{\infty} E\left(\frac{x}{\lambda}\right) e^{j\,2\pi u x/\lambda}\, d\left(\frac{x}{\lambda}\right)$$

$$A\left(u - u_0\right) = \int_{-\infty}^{\infty} E\left(\frac{x}{\lambda}\right) e^{j\,2\pi (u-u_0) x/\lambda}\, d\left(\frac{x}{\lambda}\right)$$

$$= \int_{-\infty}^{\infty} \left[E\left(\frac{x}{\lambda}\right) e^{-j\,2\pi u_0 x/\lambda}\right] e^{j\,2\pi u x/\lambda}\, d\left(\frac{x}{\lambda}\right)$$

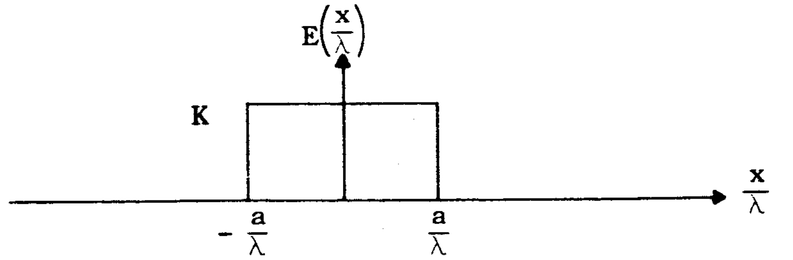
$$\therefore\ A\left(u - u_0\right) \longrightarrow E\left(\frac{x}{\lambda}\right) \exp\left(-j\,2\pi u_0 \frac{x}{\lambda}\right) \qquad (11)$$

The physical interpretation of equation (11) is that a linear phase shift $\exp\left(-j\,2\pi u_0\, x/\lambda\right)$ applied to an aperture distribution will shift the radiation pattern to the direction u_0. Note that the shift is a function of $u = \sin\theta$, not of θ itself.

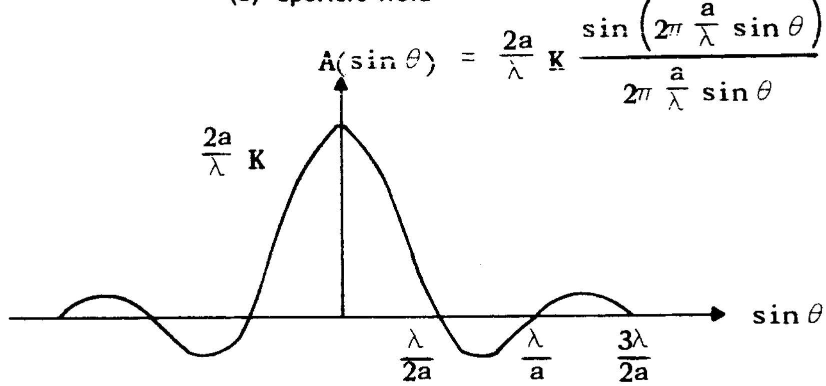
V. IMAGING PROPERTIES OF ANTENNAS

The receiving antenna shown in Figure 7 may be treated as an optical image-forming system whose operation is described by the equation

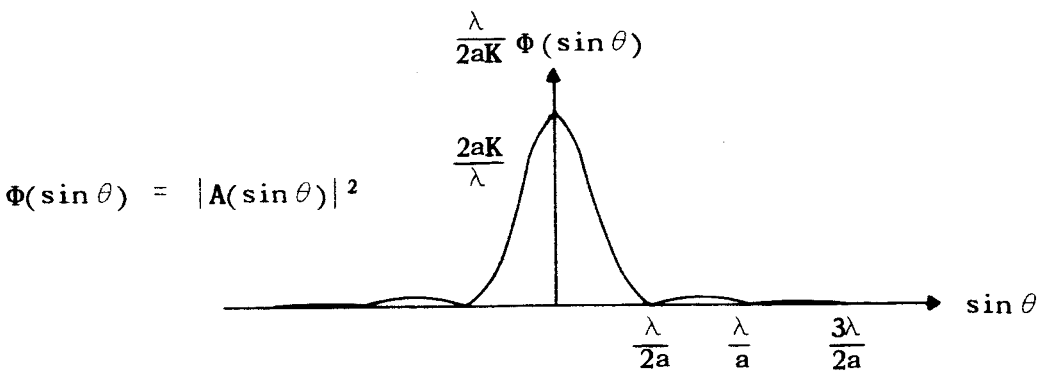
$$I(u') = \int_{-1}^{1} G(u' - u)\; O(u)\, du \qquad (12)$$



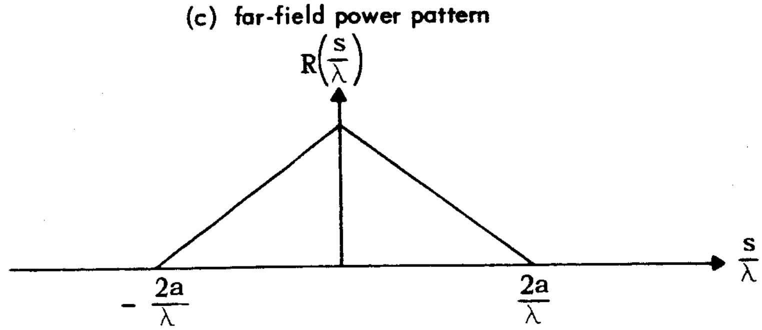
(a) aperture field



(b) far-field radiation pattern



(c) far-field power pattern



(d) operature autocorrelation function

Figure 6 – Transforms of a Uniformly-Illuminated Aperture

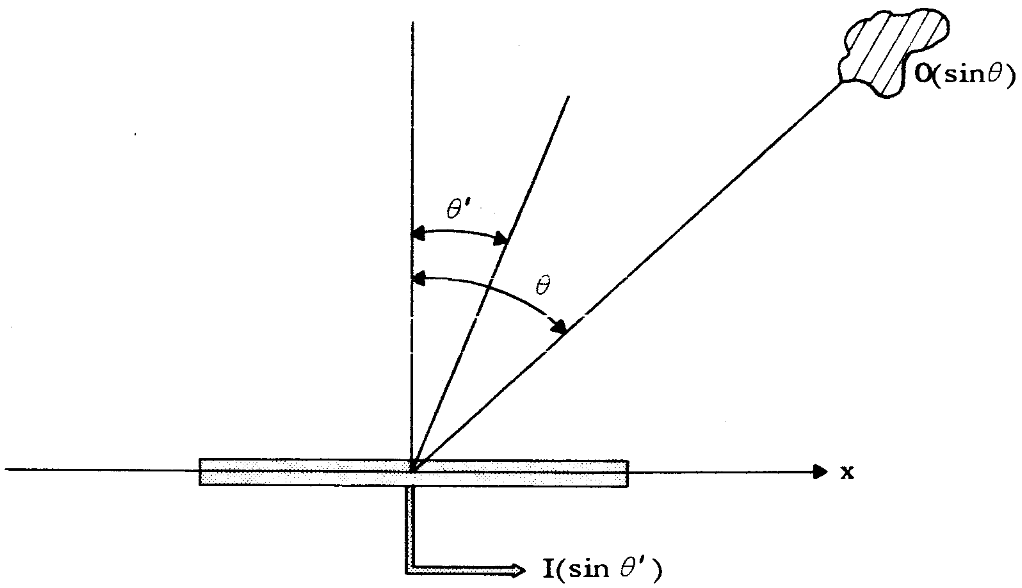


Figure 7 – Scanning Antenna System

where $u = \sin\theta$, θ is the observation angle, $u' = \sin\theta'$, and θ' is the scan angle. The function G(u) is known in optics as the spread function and describes the distribution of light in the image plane due to a point source in the object plane. The equation yields the image $I(u')$ of an object (or source) in terms of the source distribution function O(u) and the spread function. Sin θ is used as the variable because the scanning is assumed to be performed electronically by producing a linear phaseshift across the aperture.(2) The physical interpretation of the quantities $I(u')$, O(u), and G(u) depends on the coherency of the source; the limiting cases of complete coherence (sources that emit signals differing only by a constant amplitude and phase factor) and complete incoherence (sources that emit statistically independent signals) will be treated.

Coherent Source—For this case O(u) is the object field strength distribution as seen from the phase center of the antenna, G(u) is equal to the far-field amplitude pattern A(u), and $I(u')$ is the output voltage of the antenna as a function of the scan variable. It is convenient to extend the integration limits of

---

(2)The image equation for a mechanically-scanned antenna is

$$I(\theta') = \int_{-1}^{1} G(\theta' - \theta)\, O(\theta)\, d\theta$$

instead of (12). For this type antenna there is no simple relation between the spread function and the aperture distribution.

equation (12) to infinity so that

$$I(u') = \int_{-\infty}^{\infty} A(u' - u)\, O(u)\, du \tag{13}$$

which has the same form as the response of a linear circuit to an input time function. $A(u)$ is analogous to the impulse response function of the circuit; the spatial signals $O(u)$ and $I(u)$ correspond to the input and output time signals respectively. The antenna system, therefore, may be represented by the spatial circuit shown in Figure 8. Analogous to the linearity and time-invariance conditions of temporal circuits we assume:

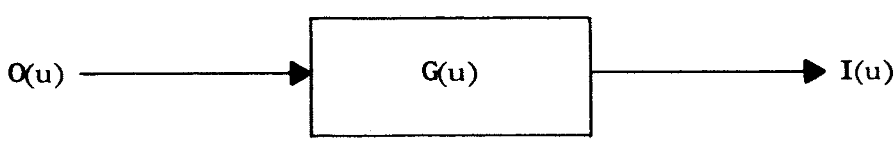
A. The response of the antenna to several sources is the superposition of the responses to each of the sources separately.

B. The response to a point source is independent of the direction of the source. This means that shifting the antenna pattern does not change its amplitude or shape.

To see why $A(u)$ is used as the impulse response function consider the response of the antenna to a point source in the direction u_0. As the antenna scans past the source, its output will trace out the far-field pattern of the antenna centered in the direction u_0 as shown in Figure 9. Mathematically, this method of measuring the radiation pattern may be expressed

$$I(u') = \int_{-\infty}^{\infty} A(u' - u)\, \delta\left(u - u_0\right) du = A\left(u' - u_0\right)$$

which is equivalent to stating that the impulse response function is the radiation pattern.



$$I(u') = \int_{-\infty}^{\infty} G(u' - u)\, O(u)\, du$$

Figure 8 – Spatial Circuit Representation of an Antenna System

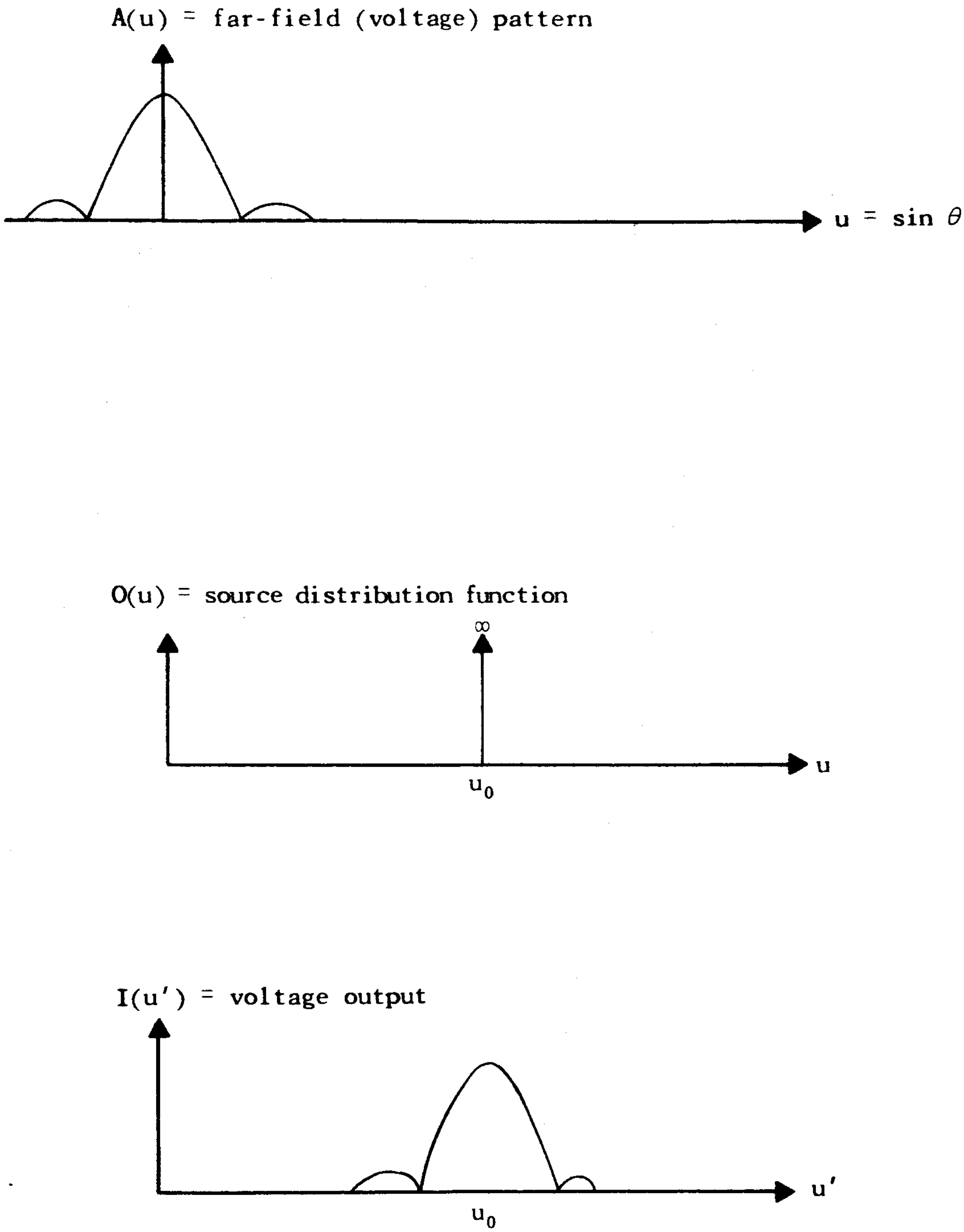


Figure 9 – For the Derivation of the Impulse Response Function

The Fourier transform of equation (13) may be written as

$$\mathcal{S}(\gamma) = \mathcal{A}(\gamma)\,\mathcal{O}(\gamma) \tag{14}$$

where the script letters represent the transformed functions and γ is the transform variable. Equation (14) describes the performance of the antenna system in the γ-domain and is equivalent to the frequency-domain representation of time signals in circuit theory. By analogy, the transform variable γ is known as the spatial frequency.

The function $\mathcal{A}(\gamma)$ may be identified as the transfer function of the antenna system and is defined by

$$\mathcal{A}(\gamma) = \int_{-\infty}^{\infty} A(u)\, e^{j2\pi u\gamma}\, du$$

It has been shown, however, that the transform of the far-field pattern is equal to the aperture field distribution

$$E\left(\frac{x}{\lambda}\right) = c\int_{-\infty}^{\infty} A(u)\; e^{j2\pi u x/\lambda}\; du$$

where c is a constant. Therefore, the system transfer function is the same as the aperture field, except for a constant, since both are related to the far-field pattern by a Fourier transform. The spatial frequency γ must also be the same as the aperture variable x/λ.

Equation (14) indicates that the output of the antenna will be an exact replica of the source distribution function if $A(\gamma)$ is a constant over the spatial frequency bandwidth of the source. Figure 10 shows the required transfer function for a source whose spectrum occupies a finite bandwidth. The required aperture distribution is found from $\mathcal{A}(\gamma)$ by replacing γ with the aperture variable x/λ.

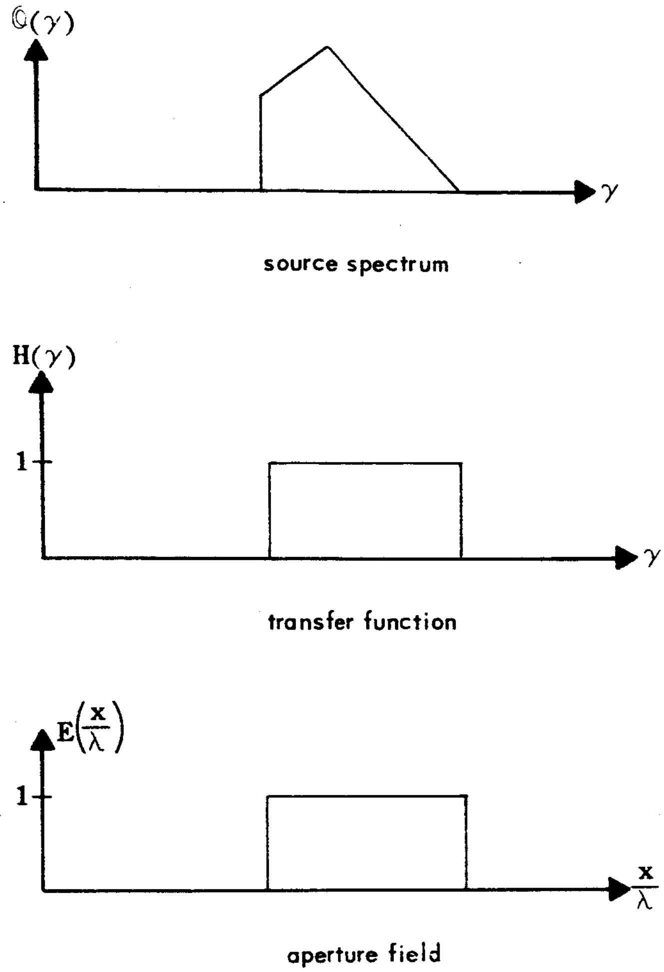


Figure 10 – Spatial Filter Characteristics of an Antenna

The figure shows that spatial frequency bandwidth of an antenna is equivalent to the aperture size.

Unlike time signals whose spectrums are band limited, spatial signals usually have spectrums covering essentially all spatial frequencies. The large bandwidth is caused by the small angular dimension of the source as seen by the antenna and is one of the most important differences between temporal and spatial signals. The most common source, a point source, has a spectrum that is constant over the entire band of spatial frequencies $(-\infty, \infty)$. Since the aperture is finite in extent (covering spatial frequencies from zero to some cutoff value), the output of the spatial filter with a point source input will be a distorted version of the source distribution function. Stated in another way, the source distribution function is "smoothed out" due to the low-pass spatial filter characteristic of the antenna.

It is interesting to see how the concept of a matched filter applies to an antenna. In circuit theory, a linear filter that maximizes the output signal-to-noise power ratio is termed a matched filter because its characteristics are determined by the signal to be detected. For white gaussian noise at the input, the transfer function of the filter should be proportional to the complex conjugate of the spectrum of the input signal. The requirement for an antenna to be a spatial matched filter, therefore, is

$$\mathcal{A}(\gamma) = K\mathcal{O}^*(\gamma)$$

or

$$E\left(\frac{x}{\lambda}\right) = K\mathcal{O}^*\left(\frac{x}{\lambda}\right)$$

i.e. the antenna aperture distribution should be the complex conjugate of the field produced by the source across the aperture. For a point source in the direction u_0

$$\mathcal{O}\left(\frac{x}{\lambda}\right) = \exp\left(j\,2\pi\frac{x}{\lambda}u_0\right)$$

and the required aperture distribution is

$$E\left(\frac{x}{\lambda}\right) = K\,\exp\left(-j\,2\pi\frac{x}{\lambda}u_0\right) \tag{15}$$

An antenna with such an aperture distribution will track the point target in such a manner as to maximize the power received from the target relative to the background noise power. The physical interpretation of equation (15) is that the antenna should be electronically scanned toward the target.

Incoherent Sources—For this case $O(u)$ is the intensity (power) distribution of the source, $G(u)$ is the far-field power pattern $\Phi(u)$, and $I(u')$ is the output power of the antenna. The image equation becomes

$$I(u') = \int_{-\infty}^{\infty} \Phi(u' - u)\, O(u)\, du \tag{16}$$

and the antenna may be represented as a spatial circuit with $\Phi(u)$ as the impulse response function. Equation (16) may be Fourier transformed to yield

$$\mathcal{I}(\gamma) = \mathbb{P}(\gamma)\, \mathbb{O}(\gamma)$$

where the system transfer function, defined by

$$\mathbb{P}(\gamma) = \int_{-\infty}^{\infty} \Phi(u)\, e^{j 2\pi u \gamma}\, du \ ,$$

is the same as the aperture autocorrelation function, equation (10), since both functions are the Fourier transform of the far-field power pattern. The spatial frequency γ may be identified as the aperture variable S/λ .

By way of summary, there are two basic relations for an antenna that enable us to apply the concepts of communication theory to an antenna system:

1. The image equation, (13), expresses an input-output relation which is similar to the input-output relation of a linear circuit. We are able, therefore, to treat the antenna as a black box and use the techniques of circuit analysis to analyze antenna systems.

2. The transform relations, equations (7) and (8), between the aperture distribution and the far-field pattern enable us to Fourier transform the image equation and relate the spatial frequency spectrums of the input and output signals. This furnishes an alternate method of describing antenna systems similar to the frequency domain representation of linear circuits.

The above theory will be applied to two types of antenna systems in order to demonstrate the effectiveness of this technique for solving antenna problems.

Analysis of a Simple Interferometer

The simple interferometer shown in Figure 11 consists of a pair of identical antennas separated by a distance L and connected to a receiver by cables of equal length. The system is used to measure the source distribution function (which is related to the brightness temperature distribution) of an incoherent object such as the sun or a radio star. If $E(x/\lambda)$ denotes the aperture field of one antenna located by itself at the origin and $A_0(u)$ its far-field amplitude pattern, then

$$E\left(\frac{x}{\lambda}\right) \longleftrightarrow A_0(u)$$

and

$$E\left(\frac{x \pm \frac{L}{2}}{\lambda}\right) \longleftrightarrow A_0(u)\, e^{\pm jkuL/2}$$

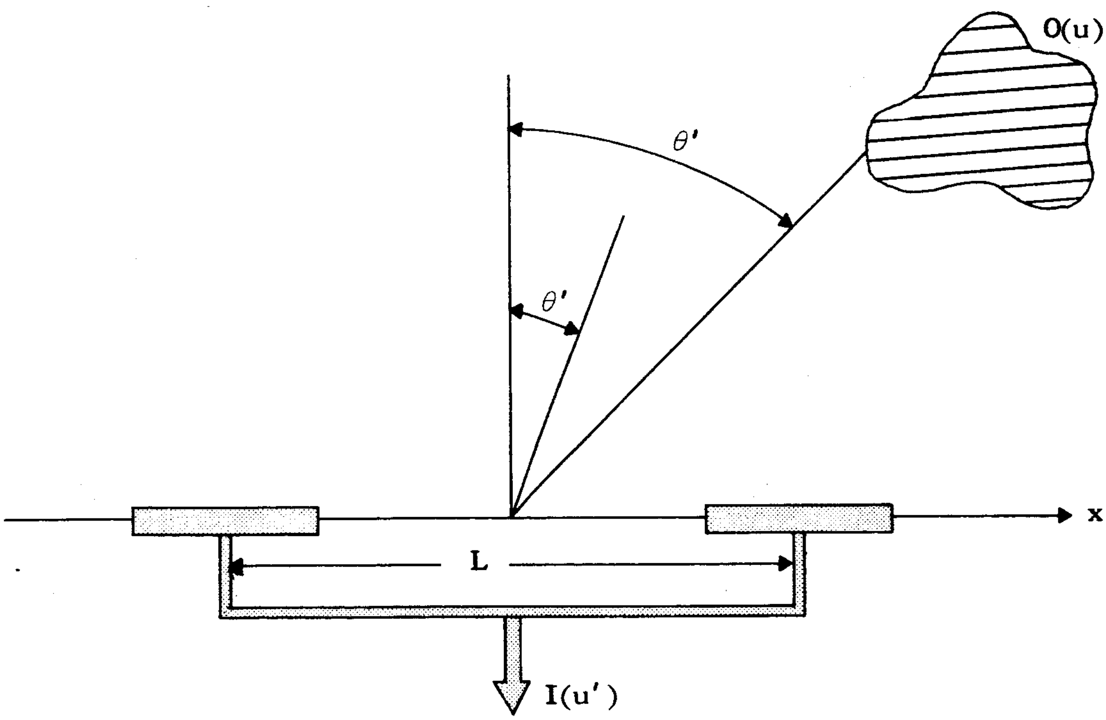


Figure 11 – Simple Interferometer

The aperture field of the two-element interferometer is given by

$$E\left(\frac{x+\frac{L}{2}}{\lambda}\right)+E\left(\frac{x-\frac{L}{2}}{\lambda}\right)$$

with a far-field amplitude pattern

$$A(u) = A_0(u)\left[e^{jkuL/2}+e^{-jkuL/2}\right] = 2A_0(u)\cos\left(k\frac{L}{2}u\right)$$

and a far-field power pattern

$$\Phi(u) = |A(u)|^2 = 2A_0^2(u)\left[1+\cos(kLu)\right]$$

Using equation (16) the output power of the interferometer is

$$I(u') = \int_{-\infty}^{\infty} \Phi_0(u'-u)\left\{1+\cos\left[kL(u'-u)\right]\right\}O(u)\,du \tag{17}$$

where $\Phi_0(u) = 1/2|A_0(u)|^2$ denotes the power fed to the receiver by a single antenna when the object is a point source. Expanding equation (17)

$$I(u') = \int_{-\infty}^{\infty} \Phi_0(u'-u)\,O(u)\,du + \int_{-\infty}^{\infty} \Phi_0(u'-u)\cos\left[kL(u'-u)\right]O(u)\,du$$

$$= \int_{-\infty}^{\infty} \Phi_0(u'-u)\,O(u)\,du + \cos(kLu')\int_{-\infty}^{\infty} \Phi_0(u'-u)\cos(kLu)\,O(u)\,du$$

$$+ \sin(kLu')\int_{-\infty}^{\infty} \Phi_0(u'-u)\sin(kLu)\,O(u)\,du$$

We assume that the source is centered at $u = u_0$ and its width is narrow enough that $A_0(u' - u)$ is effectively constant over the range of integration. Then

$$I(u') = \Phi_0\left(u' - u_0\right) \int_{-\infty}^{\infty} O(u)\,du + \Phi_0\left(u' - u_0\right) \cos(kLu') \int_{-\infty}^{\infty} O(u) \cos(kLu)\,du$$

$$+ \Phi_0\,(u' - u)\sin(kLu') \int_{-\infty}^{\infty} O(u)\sin(kLu)\,du$$

$$I(u') = \Phi_0\left(u' - u_0\right) \int_{-\infty}^{\infty} O(u)\,du + \Phi_0\left(u' - u_0\right) \cos(kLu')\,V\cos\alpha$$

$$+ \Phi_0\left(u' - u_0\right) \sin(kLu')\,V\sin\alpha$$

$$I(u') = \Phi_0\left(u' - u_0\right) \int_{-\infty}^{\infty} O(u)\,du + \Phi_0\left(u' - u_0\right) V\cos(kLu' - \alpha) \tag{18}$$

where

$$V\cos\alpha = \int_{-\infty}^{\infty} O(u)\cos(kLu)\,du \tag{19a}$$

$$V\sin\alpha = \int_{-\infty}^{\infty} O(u)\sin(kLu)\,du \tag{19b}$$

Equation (18) shows that the output of the interferometer consists of a constant term equal to the power available from a single antenna and a sinusoidally-oscillating term whose amplitude and phase depend on the functions V and α. The output is plotted in Figure 12 for an arbitrary element pattern $A_0(u)$.

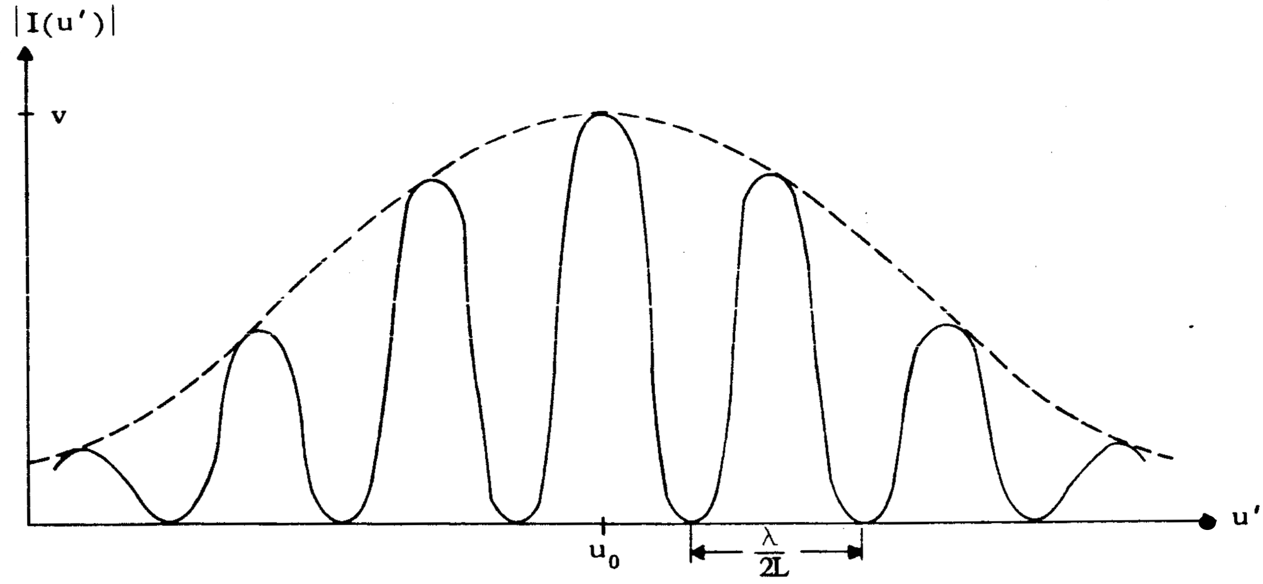


Figure 12 – Sinusoidally-Oscillating Output of Interferometer

Writing equation (19) in complex form

$$V\, e^{j\alpha} = \int_{-\infty}^{\infty} O(u)\, e^{j2\pi uL/x}\, du$$

and comparing this with the transform of the source distribution function

$$\mathbb{O}(\gamma) = \int_{-\infty}^{\infty} O(u)\, e^{j2\pi u\gamma}\, du$$

shows that the output of the interferometer is a measurement of one Fourier component of O(u) at the spatial frequency $\gamma = L/x$. Measurements taken at all antenna spacings would yield the complete spectrum $O(\gamma)$ from which the true source distribution function could, in principle, be found by an inverse Fourier transformation.

Analysis of a Nonlinear Antenna System

Consider next the array shown in Figure 13. The far-field pattern of antenna A is given by

$$G_A(u) = A(u) \exp\left(-j2\pi u \frac{a}{\lambda}\right)$$

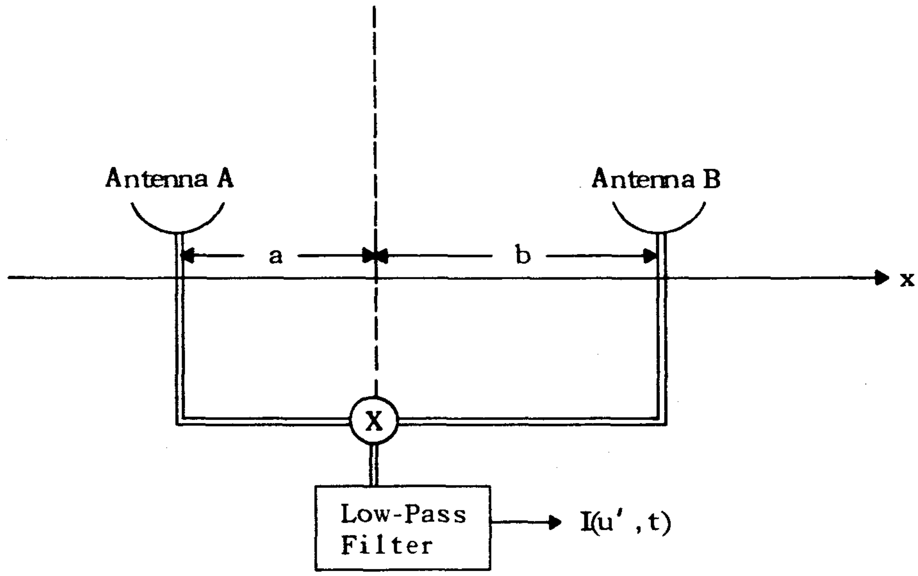


Figure 13 – A Nonlinear Antenna System

where $A(u)$ is the pattern when the antenna is located at the phase center of the array, $x = 0$. Similarly, the pattern of antenna B is given by

$$G_B(u) = B(u) \exp\left(j2\pi u \frac{b}{\lambda}\right)$$

The output of antenna A is

$$I_A(u') = \int G_A(u' - u)\, O(u)\, du$$

$$= \int A(u' - u) \exp\left[-j 2\pi \frac{a}{\lambda} (u' - u)\right] O(u)\, du$$

$$I_A(u', t) = \mathrm{Re}\left\{I_a(u')\, e^{j\omega t}\right\}$$

$$I_A(u', t) = \int A(u' - u)\, O(u) \cos\left[\omega t - \frac{2\pi a}{\lambda} (u' - u)\right] du$$

Similarly

$$I_B(u', t) = \int B(u' - u)\, O(u) \cos\left[\omega t + \frac{2\pi b}{\lambda} (u' - u)\right] du$$

The output of the multiplier is

$$Q(u', t) = I_A(u', t)\, I_B(u', t)$$

$$= \int A(u' - u)\, O(u) \cos\left[\omega t - \frac{2\pi a}{\lambda} (u' - u)\right] du$$

$$\cdot \int B(u' - u)\, O(u) \cos\left[\omega t + \frac{2\pi b}{\lambda} (u' - u)\right] du$$

$$Q(u', t) = \iint A(u' - u)\, B(v' - v)\, O(u)\, O(v)$$

$$\cdot \cos\left[\omega t - \frac{2\pi a}{\lambda}(u' - u)\right] \cos\left[\omega t + \frac{2\pi b}{\lambda}(v' - v)\right] du\, dv$$

The output of the low-pass filter is given by

$$I(u', t) = \frac{1}{2}\iint A(u' - u)\, O(u)\, B(v' - v)\, O(v) \cos\left[\frac{2\pi a}{\lambda}(u' - u) + \frac{2\pi b}{\lambda}(v' - v)\right] du\, dv$$

If antenna B is located at the phase center of the array, then $b = 0$ and

$$I(u', t) = \frac{1}{2}\iint A(u' - u)\, O(u)\, B(v' - v)\, O(v) \cos\left[\frac{2\pi a}{\lambda}(u' - u)\right] du\, dv$$

$$= \frac{1}{2}\int A(u' - u)\, O(u) \cos \frac{2\pi a}{\lambda}(u' - u)\, du \int B(u' - u)\, O(u)\, du \qquad (20)$$

Assume next that antenna B is omnidirectional so that the second integral of (20) is a constant. Then

$$I(u', t) = K\int A(u' - u)\, O(u) \cos\left[\frac{2\pi a}{\lambda}(u' - u)\right] du$$

If antenna A is a uniformly-illuminated line source of length 2a,

$$A(u) = \frac{1}{\pi u} \sin\left(2\pi u \frac{a}{\lambda}\right)$$

and

$$I(u', t) = K \int \frac{\sin\left[2\pi \frac{a}{\lambda} (u' - u)\right]}{\pi(u' - u)} \cos\left[\frac{2\pi a}{\lambda} (u' - u)\right] O(u) \, du$$

or

$$I(u', t) = K \int \frac{\sin\left[4\pi \frac{a}{\lambda} (u' - u)\right]}{2\pi(u' - u)} O(u) \, du \tag{21}$$

Equation (21) is the expression for the output of a uniformly-illuminated line source of length 4a. The imaging properties of the nonlinear antenna system, therefore, are the same as those of a linear array with twice the overall aperture size.

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