NASA TECHNICAL MEMORANDUM

NASA TM X-53367

December 10, 1965

N66-15829
(ACCESSION NUMBER)

(THRU)

32
(PAGES)

1
(CODE)

TMX 53367
(NASA CR OR TMX OR AD NUMBER)

32
(CATEGORY)

FACILITY FORM 602

NASA TM X- 53367

ELASTIC STABILITY OF A SLENDER BAR WITH FREE-FREE ENDS UNDER DYNAMIC LOADS

By Frank C. Liu
AERO-ASTRODYNAMICS LABORATORY

GPO PRICE $ ______

CFSTI PRICE(S) $ ______

Hard copy (HC) $2.00

Microfiche (MF) .50

ff 653 July 65

NASA

George C. Marshall
Space Flight Center,
Huntsville, Alabama

TECHNICAL MEMORANDUM X-53367

ELASTIC STABILITY OF A SLENDER BAR WITH FREE-FREE ENDS UNDER DYNAMIC LOADS

By

Frank C. Liu

George C. Marshall Space Flight Center
Huntsville, Alabama

ABSTRACT

15829

Problems of elastic stability of a space rocket under dynamic, non-conservative loads are presented. The spacecraft is treated as a slender, uniform cylindrical bar with free-free ends. Analysis of the frequency of the transverse vibration coupled with the longitudinal vibration under the action of time-varying thrust force based on Galerkin's method leads to Mathieu's equation. The elastic stability boundaries of the frequency ratio, longitudinal to transverse, versus the thrust are obtained. The dynamic buckling thrust of a uniform bar with various boundary conditions is presented. The aero-dynamic force induced from the oscillation of the rocket in supersonic speed on the critical thrust is investigated. A numerical illustration is given to show the relationship of the critical thrust versus flight speed at various flight altitudes.

Author

NASA - GEORCE C. MARSHALL SPACE FLIGHT CENTER

TECHNICAL MEMORANDUM X-53367

ELASTIC STABILITY OF A SLENDER BAR WITH FREE-FREE ENDS UNDER DYNAMIC LOADS

By

Frank C. Liu

AERO-ASTRODYNAMICS LABORATORY

TABLE OF CONTENTS

DEFINITION OF SYMBOLS

Symbol	Definition
A	Effective load carrying cross-sectional area
A_n	$= a \sqrt{2(1 - \cos \theta_n \tau_o)/(\theta_n \tau_o)^2}$
a	Constant, defined by equation (11)
c	$\sqrt{EA/m}$
c_∞	Velocity of sound in air
E	Modulus of elasticity
I	Bending moment of inertia
K	$= P_o/(\pi^2 EI/L^2)$
L	Length of bar
M	Mach number $= U/c_\infty$
m	Mass per unit length of bar
N	Axial force
P, P_o	Thrust force function and its maximum value
P_∞	Atmospheric pressure
q	Aerodynamic load per unit length
R	Radius of bar
r_n	$= \theta_n/\Omega_1$
t	Time variable
t_o	Thrust buildup time
U	Velocity of flight

DEFINITION OF SYMBOLS (Concluded)

Symbol	
u	Longitudinal displacement of bar
v	Normalized mode of free transverse vibration
w	Transverse displacement
x	Coordinate along bar
α	$= P_o/mL^2\Omega_1^2 = \dfrac{\pi^2}{(\beta_1 L)^4} K$
β	$= \pi R\gamma P_\infty M/mL\Omega_1^2$
γ	$= 1.4$, polytropy index of air
ζ_a	$= \frac{1}{2}\pi\gamma RLp_\infty/mLc_\infty\Omega_1$, aerodynamic damping factor
ζ_s	Structural damping factor
θ_n	$= n\pi c/L$, longitudinal frequency
μ_n	Parameter of Mathieu equation defined by equation (23)
τ_o	$= t_o\Omega_1$
Ω_i	Natural frequency of transverse vibration.

TECHNICAL MEMORANDUM X-53367

ELASTIC STABILITY OF A SLENDER BAR WITH FREE-FREE ENDS UNDER DYNAMIC LOADS

SUMMARY

Problems of elastic stability of a space rocket under dynamic, non-conservative loads are presented. The spacecraft is treated as a slender, uniform cylindrical bar with free-free ends. Analysis of the frequency of the transverse vibration coupled with the longitudinal vibration under the action of time-varying thrust force based on Galerkin's method leads to Mathieu's equation. The elastic stability boundaries of the frequency ratio, longitudinal to transverse, versus the thrust are obtained. The dynamic buckling thrust of a uniform bar with various boundary conditions is presented. The aero-dynamic force induced from the oscillation of the rocket in supersonic speed on the critical thrust is investigated. A numerical illustration is given to show the relationship of the critical thrust versus flight speed at various flight altitudes.

INTRODUCTION

The means of finding the static buckling load of a slender column is the well known Euler's method. If the external forces have a potential, they are called conservative forces, in which case this method is applicable. If the forces are nonconservative, this method is not applicable [1]. The theory of elastic stability of nonconservative problems is based on the investigation of the small oscillation of the system about its equilibrium position. Thorough discussion and many practical examples of this so-called dynamic method may be found in V. V. Bolotin's books listed as References 1 and 2.

The thrust force and the aerodynamic load acting on a space vehicle have fixed directions relative to the vehicle and vibrate together with it when oscil-lation is taking place. Hence, these forces are nonconservative. Determination of the elastic stability of the vehicle, while disregarding the local buckling and panel flutter, is the object of this study. The space vehicle may be treated simply as a uniform, slender cylindrical bar with both ends free. A thrust force is applied at the tail end along the body axis; this force is built up from zero to its maximum magnitude in a short time interval and then is kept constant. Two dynamic stability problems are presented here.

Dynamic Coupling of Longitudinal and Transverse Vibrations

The velocity of the longitudinal stress waves traveling in a solid bar is equal to the speed of sound in the bar. In general, the frequency of the stress waves traveling back and forth in a bar is much greater than its natural frequency of transverse vibration; thus, the coupling is negligible. However, a rocket usually has great mass density per length and a small effective load-carrying cross-sectional area. Its longitudinal frequency may be of the same order of magnitude as its transverse vibration frequency. In this case the elastic coupling becomes significant.

Elastic Stability of a Space Vehicle under the Combined Action of a Thrust and Aerodynamic Load

For a space rocket of large cross-sectional area, the aerodynamic force and damping effect caused by its transverse oscillation in supersonic air flow may have some significance to its dynamic stability. Such information may be helpful to structural design engineers.

ANALYSIS

Coupling of Longitudinal and Transverse Vibrations

The well known equations of transverse and longitudinal vibrations of a uniform bar and its boundary conditions of free-free ends are as follows [1, 3]:

$$EI \frac{\partial^4 w}{\partial x^4} + m \frac{\partial^2 w}{\partial t^2} = \frac{\partial}{\partial x}\left(N \frac{\partial w}{\partial x}\right) + q(x, t) \tag{1}$$

$$N(x, t) = EA \frac{\partial u}{\partial x} \tag{2}$$

$$EA \frac{\partial^2 u}{\partial x^2} = m \frac{\partial^2 u}{\partial t^2} \tag{3}$$

$$\left(\frac{\partial^2 w}{\partial x^2}\right)_{x=0} = \left(\frac{\partial^2 w}{\partial x^2}\right)_{x=L} = \left(\frac{\partial^3 w}{\partial x^3}\right)_{x=0} = \left(\frac{\partial^3 w}{\partial x^3}\right)_{x=L} = 0 \tag{4}$$

$$EA\left(\frac{\partial u}{\partial x}\right)_{x=0} = -P(t) \qquad \left(\frac{\partial u}{\partial x}\right)_{x=L} = 0 \tag{5}$$

The coupling of longitudinal vibration with the transverse vibration as given by the first term on the right-hand side of equation (1) can be obtained easily with the aid of Figure 1 in which N(x, t) denotes the tensile force along the axis of the bar.

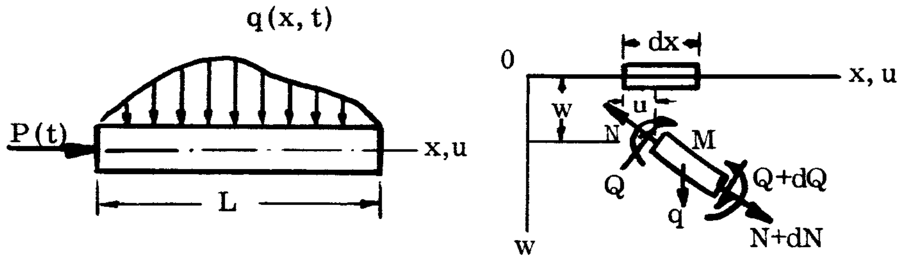

FIGURE 1. COORDINATE SYSTEM

1. Solution of u(x, t). - To eliminate the nonhomogeneous boundary condition of u given by equation (5), we introduce a new variable $\bar{u}$ and write the solution of equation (3) in the form

$$u(x, t) = \bar{u}(x, t) - \frac{P(t)}{EA}(x - x^2/2L). \tag{6}$$

From equations (5) and (6),

$$\left(\frac{\partial \bar{u}}{\partial x}\right)_{x=0} = \left(\frac{\partial \bar{u}}{\partial x}\right)_{x=L} = 0 . \tag{7}$$

Substituting from equation (6) into equation (3) yields

$$\frac{\partial^2 \bar{u}}{\partial t^2} - c^2 \frac{\partial^2 \bar{u}}{\partial x^2} = \frac{P(t)}{mL} + \frac{1}{EA}(x - x^2/2L)\frac{d^2P(t)}{dt^2} . \qquad c^2 = EA/m . \tag{8}$$

Now, if we let

$$\bar{u}(x, t) = \sum_{n=0}^{\infty} \tilde{u}_n(t) \cos \frac{n\pi x}{L} ,$$

which satisfies the boundary conditions given by equation (7), the approximate solution of the unknown function $\tilde{u}_n(t)$ may be determined using Galerkin's method. This involves substituting the assumed solution into equation (8) and then multiplying both sides of the equation by cos $n\pi x/L$ and integrating from $x = 0$ to $x = L$. This results in

$$\frac{d^2\tilde{u}_n}{dt^2} + \theta_n{}^2\tilde{u}_n = -\frac{2L}{n^2\pi^2}\frac{d^2P(t)}{dt^2} , \quad n = 1, 2, \ldots \,.\, \theta_n = n\pi c/L \tag{9}$$

$$\frac{d^2\tilde{u}_o}{dt^2} = \frac{P(t)}{mL} \,. \tag{10}$$

The last equation represents the rigid body motion of the vehicle.

Equation (9) indicates that the solution of $\tilde{u}(t)$ depends on the second derivative of the thrust function P(t). It usually takes from one half to one second for a rocket engine to build up to its full power; then the thrust remains fairly constant until engine cutoff. The thrust buildup curve may be represented by a second order curve with t_o as the buildup time and P_o as its maximum thrust. Hence, let us assume

$$P(t) = P_o[(1 + \tfrac{1}{2}a)(t/t_o) - \tfrac{1}{2}a(t/t_o)^2], \quad 0 < a \leqq 2, \quad 0 \leq t \leq t_o \tag{11}$$

$$= P, \quad t \geq t_o ,$$

from which we obtain

$$\frac{d^2P(t)}{dt^2} = -P_o(a/t_o^2) \quad [S(t) - S(t - t_o)] , \tag{12}$$

where S(t) is a unit step function. The thrust function and its second derivative is represented by Figure 2.

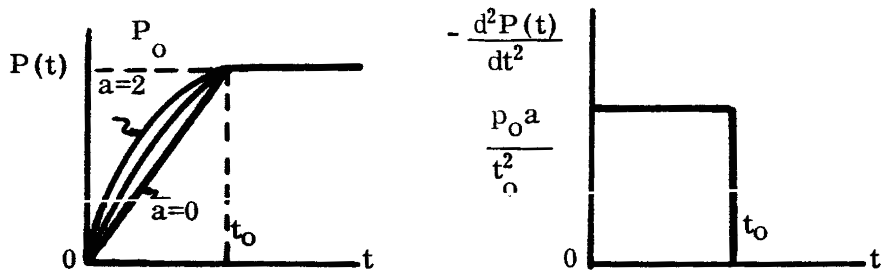

FIGURE 2. THRUST FUNCTION AND ITS SECOND DERIVATIVE

The solution of equation (9) may be obtained easily by using the Laplace transform,

$$\tilde{u}_n(t) = \frac{2LP_oA_n}{n^2\pi^2EA} \cos(\theta_n t + \phi_n), \qquad t \geqq t_o, \tag{13}$$

where

$$A_n = (a/\tau_o^2)\sqrt{2(1 - \cos r_n\tau_o)/r_n^4}, \qquad \phi_n = \tan^{-1}\frac{\sin\theta_n t_o}{1 - \cos\theta_n t_o},$$

$$r_n = \theta_n/\Omega_1, \qquad \tau_o = \Omega_1 t_o .$$

In study of stability problems, the initial conditions can be disregarded; hence, we may drop out ϕ_n from equation (13) and use P_o to replace P(t) in dealing with the transverse vibration of the bar. Thus, we have

$$u(x, t) = \sum_{n=1}^{\infty} \frac{2LP_oA_n}{n^2\pi^2EA} \cos\theta_n t \cos\frac{n\pi x}{L} - (x - x^2/2L)P_o/EA + \frac{1}{2}(P_o/mL)t^2 \tag{14}$$

and

$$N(x, t) = - \sum_{n=1}^{\infty} \frac{2P_o A_n}{n\pi} \cos \theta_n t \sin \frac{n\pi x}{L} - (1 - x/L) P_o \, . \tag{15}$$

2. Solution of Transverse Vibration w(x, t). -The solution of free vibration (N = 0 and q = 0) of a free-free beam is [4]

$$w(x, t) = \sum_{i-1}^{\infty} C_i v_i(x) \, e^{j\Omega_i t} , \tag{16}$$

where C_i is a constant, $j = \sqrt{-1}$, and $v_i(x)$ is the ith normalized vibration mode,

$$v_i(x) = \frac{1}{\sqrt{L}} \left[\mathrm{Ch}\beta_i x + \cos \beta_i x - \gamma_i (\mathrm{Sh}\beta_i x + \sin \beta_i x) \right] . \tag{17}$$

The symbols in the two equations above are defined as follows: β_i is the eigenvalue of the frequency equation

$$\cos \beta L \; \mathrm{Ch} \, \beta L = 1; \tag{18}$$

Ω_i is the natural circular frequency of free vibration

$$\Omega_i^{\,2} = \beta_i^{\,4} \, EI/m = (\beta_i L)^4 \, (\pi^2 EI/L^2)/\pi^2 m L^2 \; ; \tag{19}$$

and γ_i is a constant

$$\gamma_i = \frac{\mathrm{Ch}\beta_i L - \cos \beta_i L}{\mathrm{Sh}\beta_i L - \sin \beta_i L} \quad .$$

The values of β_i and γ_i are

$$\beta_i L = 4.73000, \; 7.8532, \; 10.996, \; \ldots, \cong \tfrac{1}{2}(2i + 1)\pi$$

$$\gamma_i = .98250, \; 1.00078, \; .99997, \; \ldots, \cong 1 \, .$$

To solve equation (1) with $q(x, t) = 0$, we assume the solution in the form

$$w(x, t) = \sum_{i=1}^{\infty} v_i(x) f_i(t) , \tag{20}$$

where $f_i(t)$ is an unknown function to be determined. After substituting $w(x,t)$ from equation (20) and $N(x, t)$ from equation (15) into equation (1), multiplying the equation by $v_k(x)$, and then integrating with respect to x from 0 to L, we obtain a system of infinite simultaneous equations:

$$\frac{d^2 f_k}{dt^2} + \Omega_k^{\,2} \left(1 - \alpha a_{kk} - 2 \sum_{n=1}^{\infty} \mu_{n_{kk}} \cos \theta_n t\right) f_k$$

$$- \Omega_k^{\,2} \sum_{\substack{i=1 \\ i \neq k}}^{\infty} \left(a_{ki} + 2 \sum_{n=1}^{\infty} \mu_{n_{ki}} \cos \theta_n t\right) f_i = 0, \quad k = 1, 2, \ldots \tag{21}$$

where

$$\alpha = P_o / m L^2 \Omega_1^{\,2} = \frac{\pi^2}{(\beta_1 L)^4} K = .01972K \qquad K = P_o / (\pi^2 EI/L^2)$$

$$a_{ki} = (\Omega_1/\Omega_k)^2 L \int_o^L v_k [v_i' - (L - x) v_i''] \, dx$$

$$= (\Omega_1/\Omega_k)^2 L [L v_k(0) v_i'(0) + \int_o^L (L - x) v_k' v_i' dx]$$

$$\mu_{n_{ki}} = -\alpha A_n (\Omega_1/\Omega_k)^2 L \int_o^L v_k \left(\frac{L}{n\pi} v_i'' \sin \frac{n\pi x}{L} + v_i' \cos \frac{n\pi x}{L}\right) dx$$

$$= - \alpha A_n (\Omega_1/\Omega_k)^2 \frac{L^2}{n\pi} \int_o^L v_k' v_i' \sin \frac{n\pi x}{L} \, dx \, .$$

The prime in the above equations denotes derivative with respect to x. Using the integration formulas given by Reference 5 we obtain

$$a_{ki} = \left(\frac{\Omega_1}{\Omega_k}\right)^2 4\beta_i L \left[\frac{\beta_k}{\beta_k{}^4 - \beta_i{}^4}\left\{(\gamma_i\beta_k{}^3 - \gamma_k\beta_i{}^3) + [1 - (-1)^{i+k}]\frac{4\beta_i{}^3\beta_k{}^3}{(\beta_k{}^4 - \beta_i{}^4)L}\right\} - \gamma_i\right] \tag{22}$$

$k \neq i$

$$a_{ii} = (\Omega_1/\Omega_i)^2\gamma_i\beta_i L\left[\tfrac{1}{2}(\gamma_i\beta_i L + 6) - 4\right] . \tag{23}$$

The approximate formula for $\mu_{n_{ii}}$ may be written in the form

$$\mu_{n_{ii}} \cong \alpha(\beta_i L)^2 [1 - (-1)^n] A_n \left[\frac{1}{4\beta_i{}^2 L^2 + n^2\pi^2} + \frac{1}{n^2\pi^2} + 2\,\frac{2\beta_i{}^2 L^2 + n^2\pi^2}{4\beta_i{}^4 L^4 + n^4\pi^4}\right] . \tag{24}$$

The above formulas give

$a_{11} = 6.151$ $\qquad$ $a_{12} = -9.211$

$a_{21} = .4783$ $\qquad$ $a_{22} = 3.027$

$\mu_1 = \mu_{1_{11}} = 7.15\ A_1 = .1423 K A_1$.

Because μ_n decreases with $1/n^2$, let us consider only the first mode of both longitudinal and transverse vibrations. Then from equation (21) we obtain

$$\frac{d^2 f_1}{dt^2} + \Omega_1{}^2 (1 - \alpha a_{11} - 2\mu_1 \cos\theta_1 t)\, f_1 = 0 . \tag{25}$$

Equation (25) may be rewritten in the form of the well known Mathieu equation,

$$\frac{d^2 f_1}{dt^2} + \overline{\Omega}_1^2 (1 - 2\overline{\mu}_1 \cos \theta_1 t)\, f_1 = 0 \ , \tag{26}$$

where

$$\overline{\Omega}_1^2 = \Omega_1^2 (1 - \alpha a_{11}) \qquad\qquad \overline{\mu} = \mu/(1 - \alpha a_{11}) \ .$$

Applying the stability boundary for the Mathieu equation given by Reference 2 (pp. 24-28), we have the critical frequency ratio, longitudinal to transverse:

$$r_1{*} = \frac{2}{n} \sqrt{1 - \alpha a_{11} \pm \mu_n} \tag{27a}$$

$$r_1{*} = \frac{1}{n} \sqrt{1 - \alpha a_{11} + \mu_n^{\,2}/\, 3(1 - \alpha a_{11})} \tag{27b}$$

$$r_1{*} = \frac{1}{n} \sqrt{1 - \alpha a_{11} - 2\mu_n^{\,2}/(1 - \alpha a_{11})} \ . \tag{27c}$$

The above equations for $n = 1$ and $\mu_1 = .1423KA_1$ may be solved graphically, if we write these equations in the forms

$$A_1 = \pm\,(1 - \alpha a_{11} - \tfrac{1}{4} r_1^2)/.1423K \tag{28a}$$

$$A_1^2 = (1 - \alpha a_{11} - r_1^2)(1 - \alpha a_{11})/.00675K^2 \tag{28b}$$

$$A_1^2 = (1 - \alpha a_{11} - r_1^2)(1 - \alpha a_{11})/.0405K^2 \ . \tag{28c}$$

Also, from equation (13)

$$A_1 = a\sqrt{2(1 - \cos r_1 \tau_o)}/(r_1 \tau_o)^2 \ . \tag{28}$$

The procedure is as follows:

(1) Plot A_1 vs r_1 from equations (28) and (28a), respectively, by using K as a parameter and note that $\alpha a_{11} = .1213K$; a and τ_0 are fixed.

(2) The intersections of these two curves give the upper and lower bounds of r_1 of the instability region for the given K. It is shown as the upper branch in Figure 3.

(3) Plot A_1^2 vs r_1 from equation (28b) and the square of equation (28) for given K, a, and τ_0.

(4) The intersection of these two curves gives the upper bound of the instability region of the lower branch shown in Figure 3.

(5) Similarly, equations (28c) and (28) will yield the lower bound of the instability region of the lower branch.

It is interesting to note that the thrust buildup time $t_0 (= \tau_0/\Omega_1)$ plays an important role in determining dynamic elastic stability of a space rocket.

Dynamic Buckling Thrust and Frequency of Vibration

Let us return to equation (25) and disregard the cos $\theta_1 t$ term in the equation; in other words, we will neglect the effect of the longitudinal stress wave; thus, we obtain

$$\frac{d^2 f_1}{dt^2} + \Omega_1^2 (1 - \alpha a_{11}) f_1 = 0 . \tag{29}$$

Therefore, the fundamental frequency of transverse vibration of a uniform beam under the influence of a thrust force is

$$\Omega = \sqrt{1 - \alpha a_{11}}\, \Omega_1 . \tag{30}$$

The critical dynamic thrust P_o* is the force that reduces the frequency of vibration to zero. It follows immediately that

$$\alpha * = 1/a_{11} \quad \text{or} \quad K* = P_o*/(\pi^2 EI/L^2) = (\beta_1 L)^4/\pi^2 a_{11} . \tag{31}$$

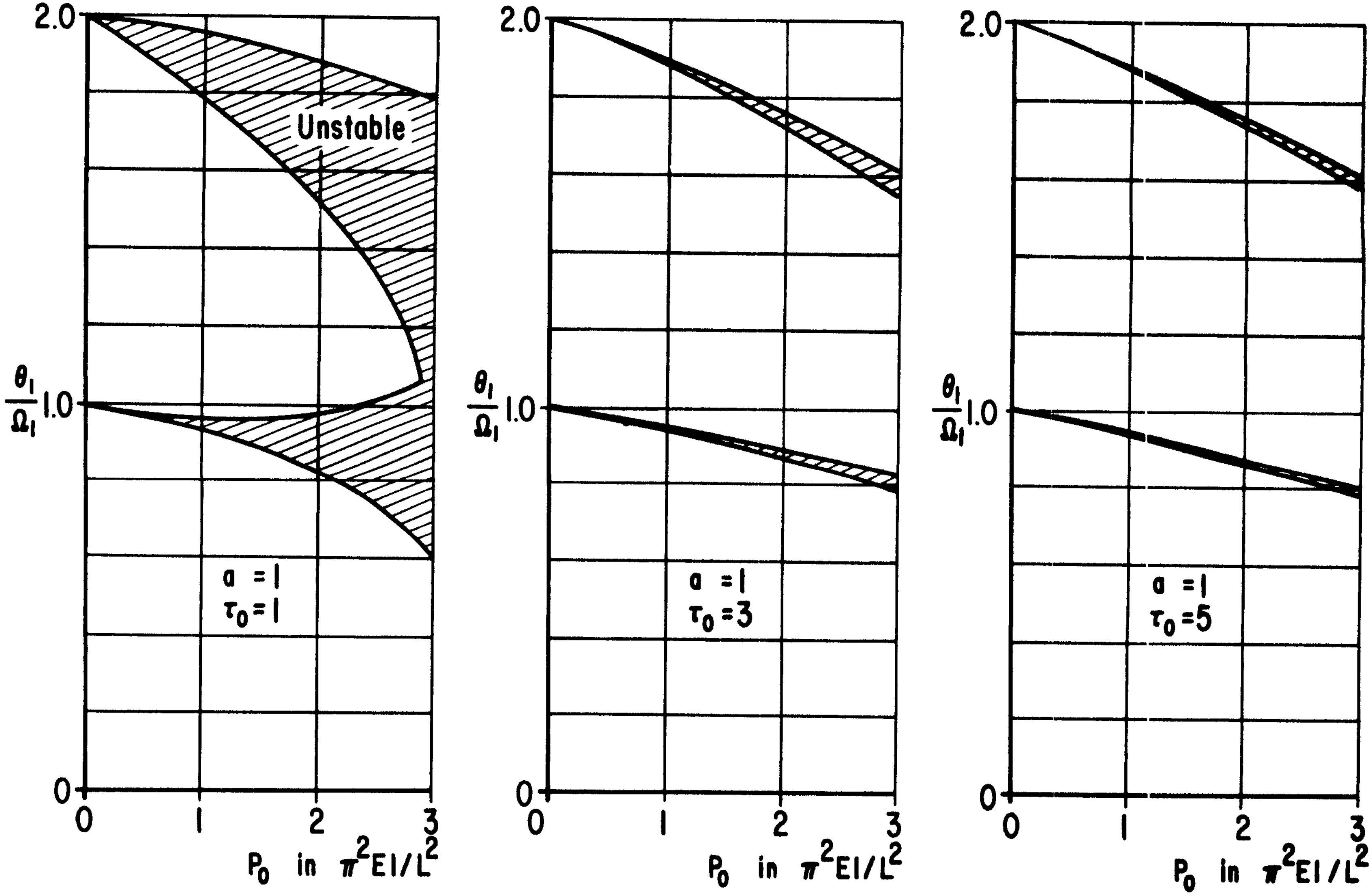

FIGURE 3 THE ELASTIC STABILITY B OUNDARIES

For a free-free uniform beam, $K^* = 8.23$, and for beams with other end conditions, K^* can be obtained easily with the aid of References 4 and 5. Formulas for various end conditions are given in Tables I and II.

The Static Approach. -The critical thrust of a bar based on the static approach is actually the buckling inertia load caused by acceleration. It is assumed that the thrust force is increased gradually such that no transverse vibration is induced. In some cases, this approach fails to yield the critical inertia load, because under the given end conditions an equilibrium configuration of the beam cannot be attained without having transverse vibration. The critical values of thrust which causes inertia buckling of a clamped-free beam and a supported-supported beam, obtained from Reference 6, are .794 and 1.88 times $\pi^2 EI/L^2$, respectively.

Elastic Stability Under Aerodynamic Load

It is a well known phenomenon that, as air flow passing around an elastic, slender body, energy is emitted from the body into the surrounding medium when the velocity of propagation of transverse wave of the body is greater than the velocity of the flow; and conversely, the energy is absorbed by the body from the medium. In the latter case the energy absorbed by the body has a tendency to increase its magnitude of vibration. This is commonly called self-excited motion or flutter. Our main concern here is not the panel flutter of the skin of the shell structure; we are interested in estimating how the high speed aerodynamic flow affects the critical thrust. To this end, we shall avoid the complicated three-dimensional problem of an elastic shell in a potential flow such as that treated in Reference 1 (pp. 218 - 230). Our analysis will be made on the following assumptions:

(1) The deformation of the cross section of the hollow cylindrical bar is negligible; therefore, we may take the radial deformation of the rocket shell as the product of the transverse deflection of the free-free, uniform bar and $\cos\theta$ with θ measured from the plane of transverse motion around the cross section.

$$w_r(x,\theta,t) = w(x,t)\cos\theta \,.$$

(2) The bar is sufficiently long in comparison with its diameter so that the end effect may be ignored.

(3) The linear approximation theory may be applied to determine the aerodynamic force exerted on the oscillating bar at supersonic speed. Let us denote U as the speed of the rocket or the velocity of air flow relative to the rocket, and by using the coordinates shown in Figure 4, we may write the linearized aerodynamic pressure in the form:

TABLE I. CHARACTERISTIC FUNCTION AND EIGENVALUES OF VIBRATION OF A UNIFORM BEAM

B. C.	Characteristic Functions, $\sqrt{L}v(x)$	γ	Charact. Equation	Eigenvalues, $\beta_n L$ and γ_n			
				n = 1	n = 2	n = 3	n > 3
C-C	$\cosh\beta x - \cos\beta x - \gamma(\sinh\beta x - \sin\beta x)$	$\frac{\cosh\beta L - \cos\beta L}{\sinh\beta L - \sin\beta L}$	$\cos\beta L \cosh\beta L = 1$	4.7300 .9825	7.8532 1.0008	10.9956 .99997	$\cong \frac{\pi}{2}(2n+1)$ $\cong 1$
F-F	$\cosh\beta x + \cos\beta x - \gamma(\sinh\beta x + \sin\beta x)$	$\frac{\cosh\beta L - \cos\beta L}{\sinh\beta L - \sin\beta L}$	$\cos\beta L \cosh\beta L = 1$	same as C-C			
C-F	$\cosh\beta x - \cos\beta x - \gamma(\sinh\beta x - \sin\beta x)$	$\frac{\cosh\beta L + \cos\beta L}{\sinh\beta L + \sin\beta L}$	$\cos\beta L \cosh\beta L = -1$	1.8751 .7341	4.6941 1.0185	7.8547 .9992	$\cong \frac{\pi}{2}(2n-1)$ $\cong 1$
C-S	$\cosh\beta x - \cos\beta x - \gamma(\sinh\beta x - \sin\beta x)$	$\frac{\cosh\beta L - \cos\beta L}{\sinh\beta L - \sin\beta L}$	$\tan\beta L = \tanh\beta L$	3.9266 1.0008	7.0686 1.00000	10.210 1.0000	$\cong \frac{\pi}{4}(4n+1)$ $\cong 1$
F-S	$\cosh\beta x + \cos\beta x - \gamma(\sinh\beta x + \sin\beta x)$	$\frac{\cosh\beta L - \cos\beta L}{\sinh\beta L - \sin\beta L}$	$\tan\beta L = \tanh\beta L$	same as C-S			
S-S	$\sqrt{2}\sin\beta x$		$\sin\beta L = 0$	π	2π	3π	$n\pi$

TABLE II. CRITICAL THRUST OF A UNIFORM BEAM

	$a = Lv(0)v'(0) + \int_0^L (L-x)v'^2dx$	γ_1	$\beta_1 L$	$\pi^2/(\beta_1 L)^4$	a_{11}	$K* = (\beta_1 L)^4/\pi^2 a_{11}$
clamped-free	$\frac{1}{2}(\gamma^2\beta^2L^2 - 2\gamma\beta L + 4)$	.7341	1.8751	.7984	1.5709	.7974
free-clamped	$\frac{1}{2}(\gamma^2\beta^2L^2 - 2\gamma\beta L - 4)$					.517
clamped-clamped	$\frac{1}{2}(\gamma^2\beta^2L^2 - 2\gamma\beta L)$	.9825	4.7300	.01972	6.151	8.23
free-free						
clamped-supported	$\frac{1}{2}(\gamma^2\beta^2L^2 - 2\gamma\beta L + 1)$	1.00078	3.9297	.04152	4.2913	5.61
supported-clamped	$\frac{1}{2}(\gamma^2\beta^2L^2 - 1)$				7.221	3.33
free-supported	$\frac{1}{2}(\gamma^2\beta^2L^2 - 2\gamma\beta L - 3)$				2.2913	10.36
supported-free	$\frac{1}{2}(\gamma^2\beta^2L^2 + 3)$				9.221	2.61
supported-supported	$\frac{1}{2}\pi^2n^2$			$1/\pi^2$	$\frac{1}{2}\pi^2$	2.0

$$p = \frac{\gamma p_\infty}{c_\infty}\left(\frac{\partial w_r}{\partial t} - U\frac{\partial w_r}{\partial x}\right) . \tag{32}$$

Then the dynamic load per unit length, $q(x, t)$, in equation (1) is

$$q = -2\int_{-\pi/2}^{\pi/2} p \cos\theta \, R d\theta = -\frac{\pi R\gamma p_\infty}{c_\infty}\left(\frac{\partial w_r}{\partial t} - U\frac{\partial w_r}{\partial x}\right) . \tag{33}$$

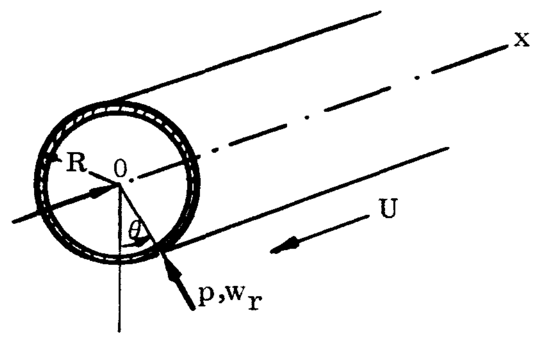

FIGURE 4. COORDINATE SYSTEM

Stability Criteria. -Now, we proceed to solve equation (1) in which $q(x, t)$ is substituted from equation (33) and $N(x, t)$ is substituted from equation (15) by neglecting the longitudinal stres wave. Similarly, applying Galerkin's method, we obtain, by retaining only the first two terms,

$$\frac{d^2f_1}{dt^2} + 2(\zeta_a + \zeta_s)\,\Omega_1\frac{df_1}{dt} + \Omega_1^2(1 - \alpha a_{11} - \beta b_{11})f_1 - \Omega_1^2(\alpha a_{12} + \beta b_{12})f_2 = 0$$

$$\frac{d^2f_2}{dt^2} + 2(\zeta_a + \zeta_s)\,\Omega_1\frac{df_2}{dt} + \Omega_2^2(1 - \alpha a_{22} - \beta b_{22})f_2 - \Omega_2^2(\alpha a_{21} + \beta b_{21})f_1 = 0 , \tag{34}$$

where Ω_1 and Ω_2 are the first and second natural frequencies of free vibration, ζ_a and ζ_s are the aerodynamic and structural damping factors, respectively,

$$\zeta_a = \frac{\pi\gamma R L p_\infty}{2mLc_\infty \Omega_1}, \quad \text{and} \quad \beta = \frac{\pi R\gamma p_\infty M}{mL\Omega_1^2} \qquad (M = \text{Mach No.} = U/c_\infty). \tag{35}$$

$$b_{ki} = (\Omega_1/\Omega_k)^2 L \int_0^L v_k v_i' \, dx = 4\,[1 - (-1)^{i+k}](\Omega_1/\Omega_k)^2 \beta_i^4/(\beta_k^4 - \beta_i^4) \,. \tag{36}$$

Note that the integration formula given by Reference 5 has been used to obtain the last expression, from which we calculate

$b_{11} = 0$ $\quad$ $b_{22} = 0$ $\quad$ $b_{12} = -9.212$ $\quad$ $b_{21} = 0.1596$

Here, we point out that the rigid body modes have no effect on the elastic stability of a free-free bar. The proof is very simple. Let us add the rigid body motion f_R and rotation f_θ to equation (20) to give

$$w(x, t) = f_R(t) + x f_\theta(t) + \sum_{i=1}^{\infty} v_i(x) f_i(t) \tag{20a}$$

Substituting the above equation, with

$$N(x, t) = - P_o(1 - x/L)$$

and $q(x, t)$, given by equation (33), into equation (1) and then multiplying the resultant equation by $v_k(x)$, we integrate it from $x = 0$ to $x = L$. Using the integrals given by Reference 5,

$$\int_0^L x v_k(x)\, dx = 0 \qquad \text{and} \qquad \int_0^L v_k(x)\, dx = 0 \ ,$$

we note immediately that the equations obtained are identical with equation (34).

Substituting $f_1 = B_1 e^{\Omega t}$ and $f_2 = B_2 e^{\Omega t}$ into equation (34) and equating the determinant of the coefficients of B_1 and B_2 to zero yields the characteristic equation

$$a_o\Omega^4 + a_1\Omega^3 + a_2\Omega^2 + a_3\Omega + a_4 = 0 \ , \tag{37}$$

where

$$a_o = 1$$

$$a_1 = 2\epsilon\Omega_1 \qquad \epsilon = 2(\zeta_a + \zeta_s)$$

$$a_2 = \Omega_1^2[(1 + \overline{r} + \epsilon^2) - \alpha(a_{11} + \overline{r}\, a_{22})] \qquad \overline{r} = (\Omega_2/\Omega_1)^2 = (\beta_2/\beta_1)^4 = 7.5985$$

$$a_3 = \Omega_1^3[1 + \overline{r} - \alpha(a_{11} + \overline{r}\, a_{22})]\epsilon$$

$$a_4 = \Omega_1^4\overline{r}\ [1 - \alpha(a_{11} + a_{22}) + \alpha^2(a_{11}a_{22} - a_{12}a_{21}) - (a_{12}b_{21} + a_{21}b_{12})\alpha\beta$$
$$- b_{12}b_{21}\beta^2] \ .$$

We recall that the Routh-Hurwitz stability criteria are as follows:

$$\text{(a)} \quad a_1 > 0 \ ; \quad \text{or} \quad \epsilon > 0 \tag{38}$$

$$\text{(b)} \begin{vmatrix} a_1 & a_3 \\ a_o & a_2 \end{vmatrix} > 0 \ ; \quad \text{or} \quad \alpha^* < \frac{\frac{1}{2}\epsilon^2 + 1 + \overline{r}}{a_{11} + \overline{r}\, a_{22}} \cong .294 \tag{39}$$

$$\text{(c)} \begin{vmatrix} a_1 & a_3 & 0 \\ a_o & a_2 & a_4 \\ 0 & a_1 & a_3 \end{vmatrix} > 0 \qquad \text{or} \qquad A\alpha^2 - B\alpha + C > 0, \tag{40}$$

where

$$A = (a_{11} + \overline{r}\, a_{22})/4\overline{r} - (a_{11}a_{22} - a_{12}a_{21}) = 4.1512$$

$$B = \frac{1}{2\overline{r}}\ (1 + \overline{r} + \epsilon^2)(a_{11} + \overline{r}a_{22}) - [a_{11} + a_{22} - (a_{12}b_{21} + a_{21}b_{12})\beta\,]$$

$$= 1.9263(8.5985 + \epsilon^2) - (9.194 - 5.8763\beta)$$

$$C = (1 + \bar{r})(1 + \bar{r} + 2\epsilon^2)/4r - (1 - b_{12}b_{21}\beta^2) = .2829(8.5985 + \epsilon^2)$$

$$- (1 + 14.7028\beta^2)$$

(d) $d_4 > 0$.

Condition (a) is satisfied if the damping factor is positive; and (d) is satisfied for all values of α; (b) requires that α^* be smaller than .294 or $K^* < 14.9$; and (c) can be used to determine the critical thrust for a given β.

Frequency of Transverse Vibration. -Let us set the damping factor equal to zero ($a_1 = a_3 = 0$) in equation (37) from which we may readily solve for the frequency of vibration by using the thrust P_o or the aerodynamic factor β as a parameter. A one-term approximation without taking into account aerodynamic effect is given by equation (30). Note that the coalescence point of these curves, by definition, is the limit of stability of motion.

NUMERICAL EXAMPLE

Let us consider a large rocket with the following structural data:

$R = 4.572$ m (15 ft),

$L = 106.68$ m (350 ft),

Weight $= mgL = 2.72 \times 10^6$ kg (6×10^6 lb),

$EI = 1.4632 \times 10^{10}$ m^2-kg (50×10^{12} in^2-lb),

from which we calculate

$\Omega_1 = 4.66$ rad/sec and $\pi^2 EI/L^2 = 12.7 \times 10^6$ kg (28×10^6 lb).

The values of ζ_a and β given by equation (35) are given in Table III.

By using equation (40) the critical thrust versus the flight speed of the rocket is plotted in Figure 5 with flight altitudes at 3048 m (10,000 ft), 6096 m (20,000 ft), and 9164 m (30,000 ft), respectively. These curves indicate that structural damping has little effect on the critical thrust.

TABLE III. VALUES OF ζ_a AND β

Altitude	ζ_a	β
3048 m (10,000 ft)	.0192	.0236 M
6096 m (20,000 ft)	.0137	.0158 M
9164 m (30,000 ft)	.0090	.0102 M
15,240 m (50,000 ft)	.0036	.00394 M
30,480 m (100,000 ft)	.00033	.00036 M
42,672 m (140,000 ft)	.000034	.000068 M

The variation of vibration frequency, which is represented by equation (37) with $a_1 = a_3 = 0$, is illustrated by Figures 6 and 7. In Figure 6 the frequency ratio, frequency with thrust to frequency without thrust, is plotted against P_o with β as a parameter, while in Figure 7 it is plotted against β with P_o as a parameter. Note that the one-term approximation given by equation (30) is also shown in Figure 6. For a given rocket the value of β is proportional to Mach number at a constant altitude. This is also shown in the figures.

CONCLUSIONS

(1) It has been proven that the rigid body modes do not affect the elastic stability of a uniform bar with free-free ends.

(2) From a structural design point of view, the critical thrust is more than four times greater when the rocket is treated as free-free with an axial thrust than when the rocket is treated as supported-supported with a vertical, conservative force.

(3) Figure 3 indicates that the region of elastic instability decreases with the thrust buildup time of the rocket.

(4) The critical thrust decreases rapidly at high supersonic speed and low flight altitude.

(5) Structural damping does not have significant effect on the critical thrust.

(6) One must keep in mind that the aerodynamic load given by equation (33) is a very crude approximation, but that it leads to the simplest formulation to serve the purpose of our study. Certainly, one can find many other approaches based on more realistic assumptions. Discussion of these approaches is beyond the scope of this report.

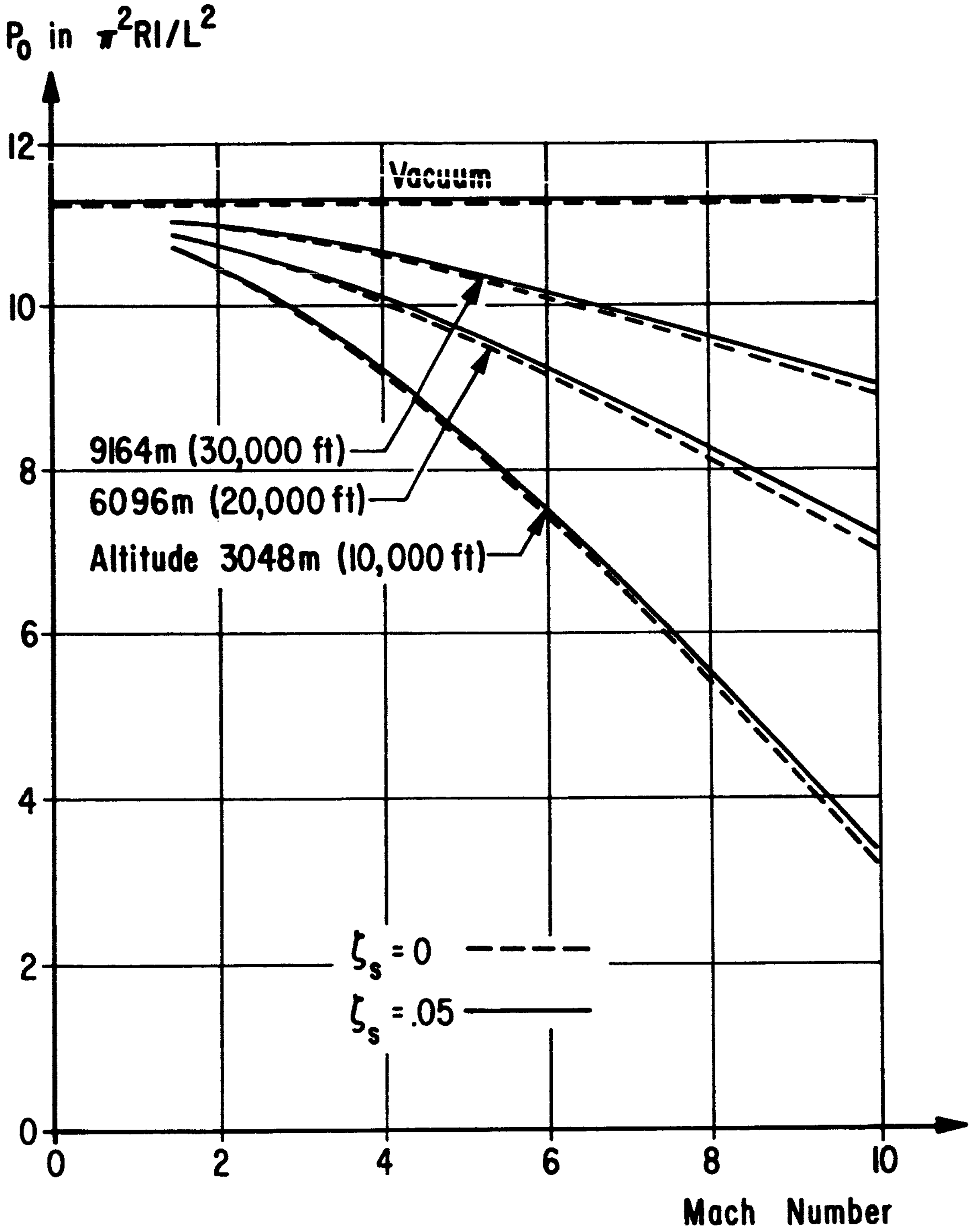

FIGURE 5. CRITICAL THRUST VS FLIGHT SPEED

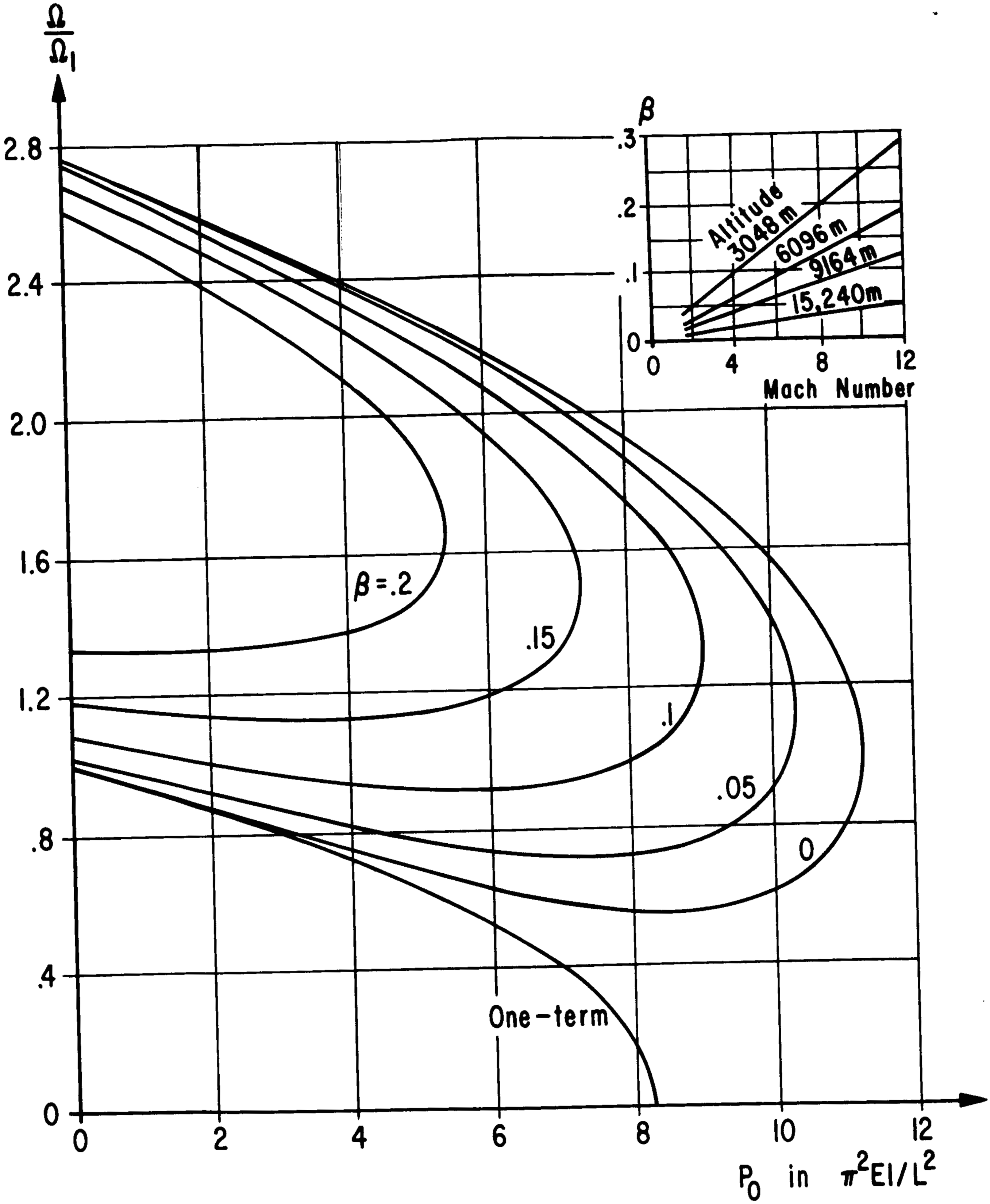

FIGURE 6. VARIATION OF FREQUENCY VS THRUST

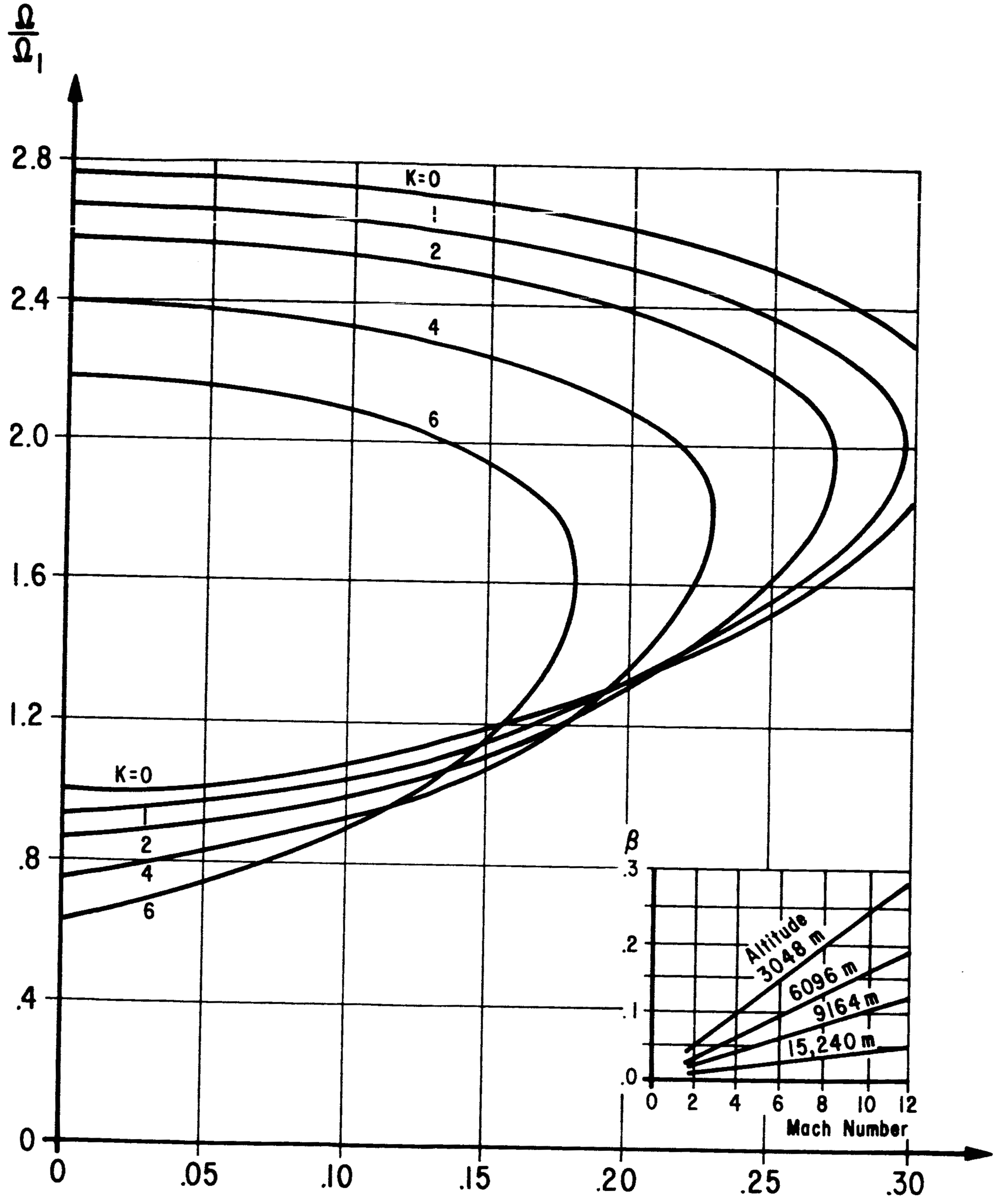

FIGURE 7. VARIATION OF FREQUENCY VS β

REFERENCES

1. Bolotin, V. V.: Nonconservative Problems of the Theory of Elastic Stability. (trans.), The MacMillan Co., N. Y., 1963.

2. Bolotin, V. V.: The Dynamic Stability of Elastic Systems. (trans.), Holden-Day, Inc., San Francisco, 1964.

3. Hoff, N. J.: The Dynamics of the Buckling of Elastic Columns. J. Appl. Mech., Mar. 1951, pp. 68-74.

4. Young, D.; and Flegar, R. P.: Tables of Characteristic Functions Representing Normal Modes of Vibration of a Beam. The University of Texas Publication, July 1, 1949.

5. Felgar, R. P.: Formulas for Integrals Containing Characteristic Functions of a Vibrating Beam. Bur. Eng. Res., Univ. of Texas, Austin, Texas, 1950.

6. Timoshenko, S.: Theory of Elastic Stability. McGraw-Hill, N. Y., 1936, p. 122.

BIBLIOGRAPHY

1. Beal, T. R.: Dynamic Stability of a Flexible Missile under Constant and Pulsating Thrusts. AIAA J., Mar. 1965, pp. 486-494.

2. Seide, P.: Effect of Constant Longitudinal Acceleration on the Transverse Vibration of Uniform Beams. TDR-169(3560-30) TN-6, Aerospace Corp., Oct. 21, 1963.

APPROVAL

ELASTIC STABILITY OF A SLENDER BAR WITH FREE-FREE ENDS UNDER DYNAMIC LOADS

By

Frank C. Liu

The information in this report has been reviewed for security classification. Review of any information concerning Department of Defense or Atomic Energy Commission programs has been made by the MSFC Security Classification Officer. This report, in its entirety, has been determined to be unclassified.

This report has also been reviewed and approved for technical accuracy.

Geissler

DR. E. D. GEISSLER
Director, Aero-Astrodynamics Laboratory

DISTRIBUTION

R-AERO
Dr. Geissler
Mr. Horn
Mr. Dahm
Mr. Reed
Mr. Lavender
Mr. Lindberg
Mr. Rheinfurth
Mr. Ryan
Mr. Stone
Mr. Baker
Mr. W. W. Vaughan
Mrs. Chandler
Dr. McDonough
Mr. J. L. Milner
Mr. J. A. Lovingood
Mr. Kiefling
Dr. Sperling
Mr. P. Hays
Mr. G. Wilhold
Mr. D. H. Townsend
Mr. F. W. Swift
Mr. Cummings
Mr. Verderaime
Dr. Liu (30)

MS-IP

MS-IL (8)

MS-T (5)

MS-H

I-RM-M

CC-P

EXTERNAL

Scientific and Technical
Information Facility (25)
Attn: NASA Representative (S AK/RKT)
P. O. Box 33
College Park, Maryland 20740