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RESEARCH REPORT 228

Contract NASw-1101

September 1965

prepared for

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ABSTRACT

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The diocotron (or slipping stream) instability of low density ($\omega_p \ll \omega_c$) electron beams in crossed electric and magnetic fields is considered for a cylindrical geometry with a radial electric field and an azimuthal magnetic field. In the analysis the electrons are assumed to have no thermal energy, collisional effects are neglected, the quasi-static approximation is made, and perturbations of the electric field along the magnetic field are ignored. For a simple density distribution the important normal modes of the electron beam correspond to two discrete eigenvalues. A condition for the stability of these modes is derived. This condition shows that, within the approximations of the analysis, the electron beam can be stabilized against diocotron modes of all wavelengths by proper selection of dimensions. The application of this theory to a proposed toroidally-shaped space radiation shield is discussed. Assuming the long straight cylinder geometry to be a reasonable approximation to a torus with a large radius ratio, it is shown that the crossed field electron beam surrounding a space radiation shield will, within the same approximations, be stable against diocotron instabilities when the beam is sufficiently thick.

author

I. INTRODUCTION

This note considers the diocotron (or slipping stream) instabilities which might occur in a geometry applicable to a proposed toroidally-shaped space radiation shield. The shield is described in detail by Levy and Janes[1] and its important features are shown in Fig. 1. The geometry of the shield is the same as that of a toroidal magnetron as analyzed by Buneman,[2] except that for the space shield no outer conducting (or reactive) surface is involved. Following Buneman, it will be assumed that the torus can be approximated by a long straight cylinder which can support waves moving parallel to its axis. The wavelengths of these waves, however, must be shorter than approximately 2π times the major radius R of the torus. This approximation neglects the asymmetry of the magnetic field intensity about the minor axis of the torus and the curvature of the electron flow direction. However, for reasonable radius ratios, it is anticipated that the results from an exact toroidal geometry formulation will not differ from those given by this analysis in any fundamental way.

Instead of considering only a geometry applicable to the space shield, the more general geometry of Fig. 2 will be analyzed. In the analysis it will be assumed that the electron density is sufficiently low relative to the magnetic field intensity so that $\omega_p \ll \omega_c$, the symbols referring, respectively, to the plasma and cyclotron frequencies. In accord with this assumption, perturbations in the component of the electric field along the magnetic field will be ignored. It is further assumed that the electrons have no thermal energy and that their drift speeds are much less than the speed of light (the quasi-static approximation).

The diocotron (or slipping stream) instability has been known for some time and forms the basis of the small-signal theory of the crossed field microwave magnetron. In most treatments of the problem[3,4,5,6] a planar model of the electron beam is used. This approximation is generally adequate, especially when the wavelengths of interest are short compared to the characteristic dimensions of the beam. However, the long wavelength behavior of the diocotron instability is substantially influenced by the particular geometry under consideration. This is especially true when, as in the space radiation shield, the plasma frequency ω_p is much less than the cyclotron frequency ω_c. In this case[5,7] growth occurs only for wavelengths which are long compared to the beam thickness. Since there is an absolute maximum wavelength for perturbations in the space shield geometry it might be possible to exclude unstable modes altogether if the beam were made sufficiently thick. This note gives a quantitative evaluation of this effect.

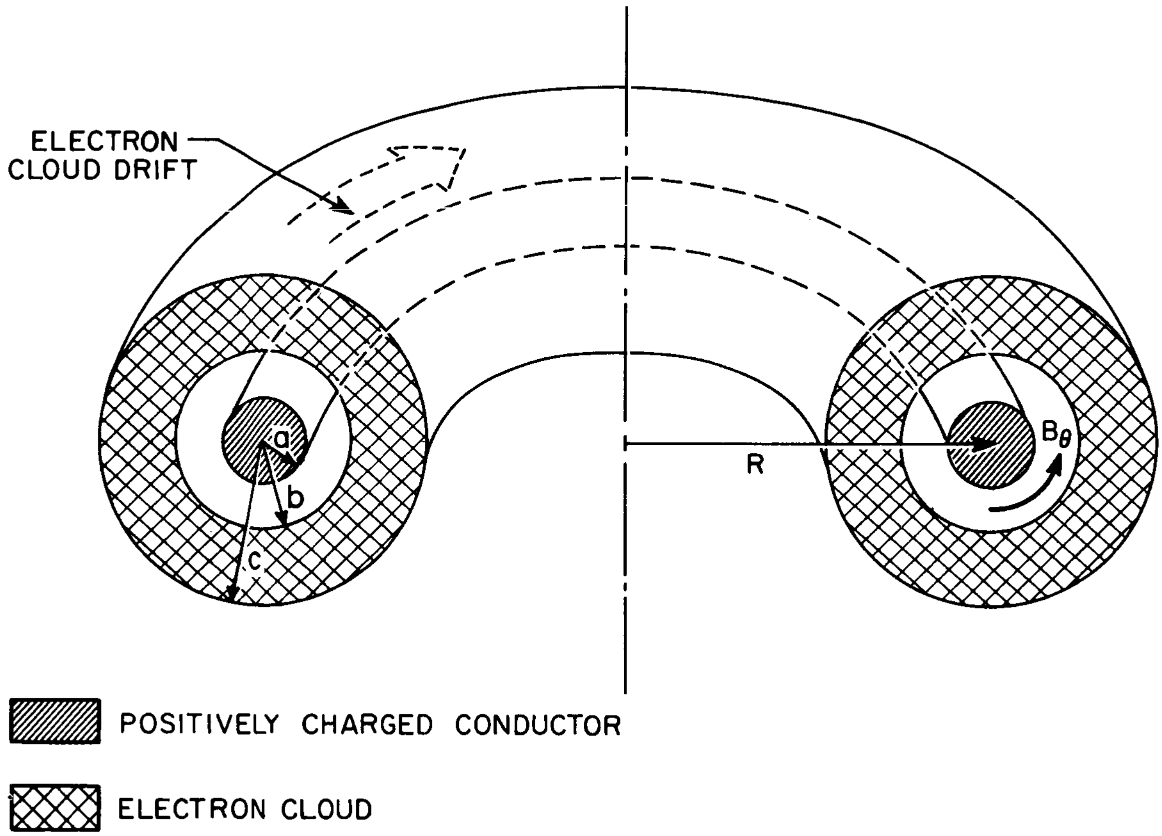

Fig. 1 Idealization of space radiation shield. The space vehicle (the inner conductor of radius a) is electrostatically shielded from high energy protons by virtue of the fact that it is charged to a very high positive potential.

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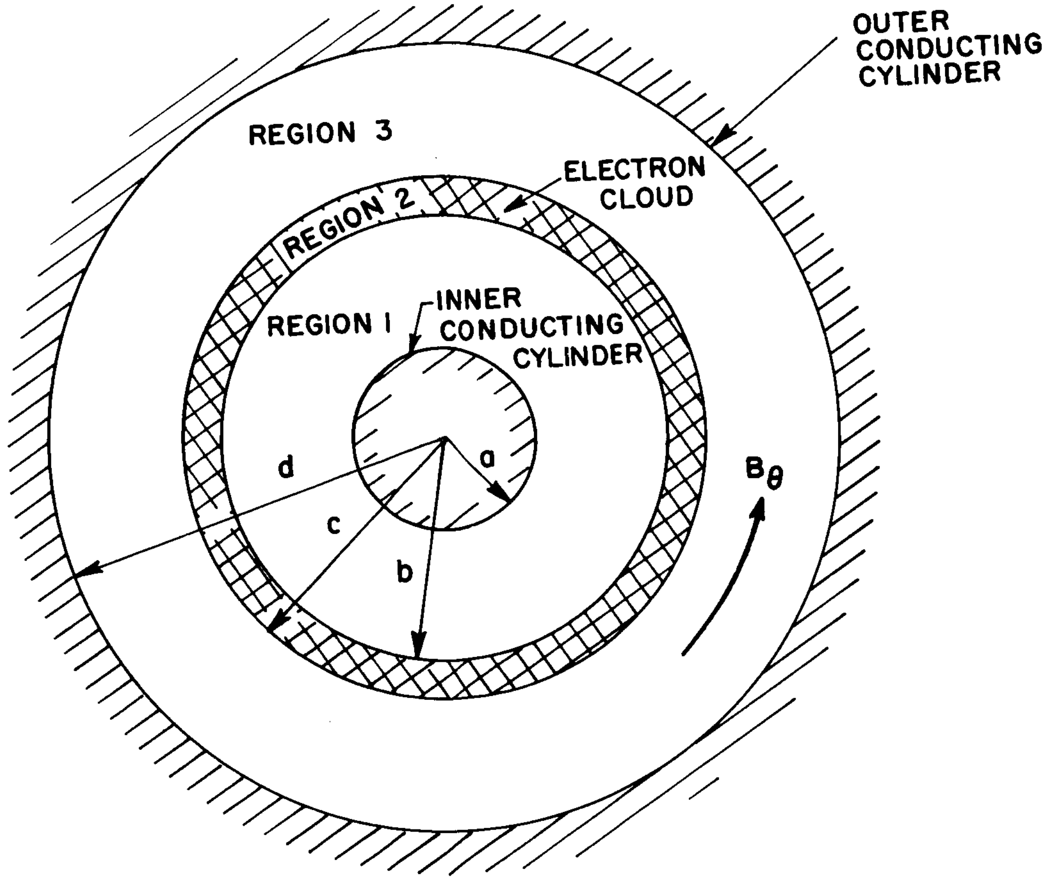

Fig. 2 Cross section of the geometry considered in the text.

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II. BASIC FORMULATION

The geometry to be considered (Fig. 2) consists of two concentric, perfectly conducting cylinders of radii a and d aligned along the z axis. An azimuthal magnetic field whose intensity varies inversely with the distance r from the axis, acts in the space between the two conductors. This magnetic field will be written as

$$B_{\Theta}^{(o)}(r) = B^{(o)} r_o/r. \tag{2.1}$$

It is consistent with the assumption that the electron drift speed is small compared to the speed of light to ignore the change in $B_{\Theta}^{(o)}$ induced by the current in the electron beam. In the equilibrium (unperturbed) state the space between the electrodes is filled with electrons having a density distribution $n^{(o)}(r)$ which will be left arbitrary for the moment. The electron density distribution $n^{(o)}(r)$ determines an equilibrium radial electric field $E_r^{(o)}(r)$ through Gauss' law:

$$\frac{1}{r}\frac{d}{dr}\left[r\ E_r^{(o)}(r)\right] = -\frac{en^{(o)}(r)}{\epsilon_o}. \tag{2.2}$$

The radial electric field on the inner electrode is related to the electric charge (per unit axial length) at that surface by

$$E_r^{(o)}(a) = \frac{Q}{2\pi a\epsilon_o}. \tag{2.3}$$

In the unperturbed state the electron cloud moves in the axial direction with a velocity $v_z^{(o)}(r) = E_r^{(o)}(r)/B_{\Theta}^{(o)}(r)$. Now that part of the radial electric field which is due to the charge on the inner conductor is proportional to Q/r. Since the magnetic field $B_{\Theta}^{(o)}$ is also proportional to $1/r$, a particular value of Q merely determines the magnitude of a uniform velocity of translation for the entire electron cloud. It therefore follows that the charge on the inner conductor can be made to vanish by transferring to a coordinate system moving with an appropriate uniform velocity in the axial direction. Now such a transformation can have no effect on the stability or otherwise of the electron beam. All that happens is that the real part of the frequency of the normal modes (waves) is

Doppler shifted; the complex part of the frequency which determines stability is unaffected. Therefore, without loss of generality, the value of Q may be taken to be zero. From Eq. (2.3) it is seen that this choice also makes the radial component of the electric field at the inner conductor vanish. The real part of the frequency appropriate to a non-zero value of Q can be found by Doppler-shifting the frequencies deduced from this analysis. The preceding remarks show that the charge on the inner conductor and hence the potential applied between the two conductors have no effect on the stability of the electron beam. Thus, in this geometry the electron beam cannot be stabilized merely by applying a large potential between the conducting walls.

It is anticipated that the frequencies of interest in this study will be of the order of ω_p^2/ω_c. This frequency is much less than ω_p and very much less than ω_c. Therefore, it seems reasonable to ignore perturbations of the component of the electric field in the direction of the magnetic field and hence to consider only two dimensional perturbations. These observations justify taking for the electronic equations of motion:

$$E_r = v_z B_\Theta^{(o)}(r) \; ; \quad E_z = - v_r B_\Theta^{(o)}(r) \tag{2.4}$$

where the subscripts r, Θ and z of B, E, and v indicate, respectively, radial, azimuthal and axial field components. Applying the quasi-static approximation the electric field is simply related to a potential ϕ by the equations:

$$E_r = - \frac{\partial \phi}{\partial r} \; ; \quad E_z = - \frac{\partial \phi}{\partial z} . \tag{2.5}$$

The condition of conservation of electrons requires that

$$\frac{\partial n}{\partial t} + \underset{\sim}{\nabla} \cdot n \underset{\sim}{v} = 0. \tag{2.6}$$

Using Eqs. (2.4) and (2.5) the equation of conservation of electrons, Eq. (2.6), can be written as

$$\frac{\partial n}{\partial t} + \frac{1}{B_\Theta^{(o)}(r)} \frac{\partial \left(\phi, \; n/\left[B_\Theta^{(o)}(r)\right]^2\right)}{\partial (z, r)} = 0. \tag{2.7}$$

In order to linearize Eq. (2.7) it will be assumed that the potential ϕ and the electron density distribution n each have zero order (unperturbed) and first order (perturbed) components and are of the form

$$\left.\begin{aligned} \phi &= \phi^{(o)}(r) + \phi^{(1)}(r) \exp\left[i(kz - \omega t)\right] \\ n &= n^{(o)}(r) + n^{(1)}(r) \exp\left[i(kz - \omega t)\right] \end{aligned}\right\} \tag{2.8}$$

where as usual the physical quantities are the real parts of the complex quantities appearing in Eq. (2.8). This choice represents a wave moving in the z direction (along the direction of electron flow) which will be growing in time (unstable) if the imaginary part of ω is positive. On linearization with the assumed forms of ϕ and n, Eq. (2.7) yields:

$$\left[\omega - kv_z^{(o)}(r)\right] n^{(1)}(r) = \frac{k\phi^{(1)}(r)}{r^2 B_\Theta^{(o)}(r)} \frac{d}{dr} \left[r^2 n^{(o)}(r)\right] . \tag{2.9}$$

Substituting Eq. (2.9) in Poisson's equation yields the following differential equation for the perturbation potential:

$$\left[\omega - kv_z^{(o)}\right] \left\{\frac{1}{r} \frac{d}{dr} \left[r \frac{d\,\phi(r)}{dr}\right] - k^2\phi(r)\right\} = \frac{ek\phi(r)}{\epsilon_o r^2 B_\Theta^{(o)}(r)} \frac{d}{dr} \left[r^2 n^{(o)}(r)\right]$$

(2.10)

where here and henceforth the superscript (1) on the perturbation potential has been omitted for convenience.

Equation (2.10) is similar to the equation governing the Kelvin-Helmholtz instability which has been treated extensively in the aerodynamic literature.[8, 9, 10, 11, 12] The analogy between the aerodynamic and electromagnetic cases has been discussed by Levy[7] and will not be considered further here. In order to solve the analogous aerodynamic problem, it is customary to set the term on the right of Eq. (2.10) to zero by a particular choice of the shear which is analogous to $n^{(o)}(r)$ here. Similarly, in this case, the zero order electron density distribution will be assumed to be

$$n^{(o)}(r) = \begin{cases} O , & a \le r < b \\ Nb^2/r^2 , & b \le r \le c \\ O , & c < r \le d \end{cases} \tag{2.11}$$

where $N = n^{(o)}(b)$ is a constant. This choice makes $\frac{d}{dr} \left[r^2 n^{(o)}(r)\right]$ zero in each of the three regions. It is interesting to note that this choice also makes ω_p^2/ω_c^2 constant throughout the electron beam. In the interior of each of the three regions, the electron density distribution of Eq. (2.11) reduces Eq. (2.10) to the much simpler form:

$$\frac{1}{r}\frac{d}{dr}\left[r\frac{d\phi(r)}{dr}\right] - k^2\phi(r) = 0 \qquad (2.12)$$

whose solutions are the modified Bessel functions of the first and second kind of order zero. In addition, from Eq. (2.9), the perturbed electron density $n^{(1)}(r)$ vanishes in the interior of each of the three regions. Thus, the perturbation is greatly simplified and involves no perturbation charge density at all in the interior of the electron cloud, but merely an accumulation of charge at each of the two free surfaces.

This observation leads to a consideration of the conditions to be applied to the perturbation potential across the free surfaces at $r = b$ and $r = c$. First, clearly the perturbed potential $\phi(r)$ must be continuous across the surface. In order to obtain the change in $\frac{d\phi(r)}{dr}$ across the surface $r = b$ it is convenient to integrate Eq. (2.10) for a short distance from $r = b - \delta$ to $r = b + \delta$ and then let $\delta \rightarrow 0$. In performing the integration it is useful to note that the bracket containing ω is virtually constant over this small range and can therefore be taken out of the integration. Since $\phi(r)$ is continuous, the integral of $\phi(r)$ over a vanishingly small range vanishes. Using these facts, integrating Eq. (2.10) and taking the limit $\delta \rightarrow 0$, the following jump condition is obtained:

$$\left[\omega - kv_z^{(o)}(b)\right]\left\{\left.\frac{d\phi(r)}{dr}\right|_{b_+} - \left.\frac{d\phi(r)}{dr}\right|_{b_-}\right\} = k\phi(b)\,\frac{\omega_p^2}{\omega_c} \qquad (2.13)$$

where ω_p and ω_c (which vary through the beam) are evaluated at $r = b$. A similar analysis at $r = c$ results in the jump condition

$$\left[\omega - kv_z^{(o)}(c)\right]\left\{\left.\frac{d\phi(r)}{dr}\right|_{c_+} - \left.\frac{d\phi(r)}{dr}\right|_{c_-}\right\} = -\,k\phi(c)\,\frac{b}{c}\,\frac{\omega_p^2}{\omega_c}\,. \qquad (2.14)$$

The specification of the problem is completed by noting that the appropriate boundary conditions for the perturbed potential on the conducting electrodes at $r = a$ and $r = d$ are $\phi(a) = \phi(d) = 0$.

At this point the problem has been completely specified. The characteristic equation for ω can be derived by writing out the eigenfunctions which satisfy Eq. (2.12) and applying the boundary and jump conditions. The resulting characteristic equation will be a polynomial in ω which will be of the same degree as the number of surfaces where $n^{(o)}(r)$ is discontinuous. For the present case with only two such surfaces the characteristic equation is simply a quadratic in ω. Since the coefficients of the quadratic are real, either both roots will be real or the roots will occur in complex conjugate pairs. In the latter case, one of the roots corresponds to a growing (unstable) wave and the other to an evanescent (damped) wave. Thus, stability can be claimed only when all the roots of the characteristic equation are real, in which case each wave can propagate at constant amplitude.

The method described above cannot, as it stands, be used to make any firm statement about stability. This is because such a statement can be made only when the complete set of normal modes is obtained. In the above treatment only those modes corresponding to discontinuities in $n^{(o)}(r)$ have been considered. That this set of modes is not complete is seen by observing that no initial condition involving a perturbation in the charge density in the interior of the beam can be described by them. The remaining modes correspond to a continuous spectrum of eigenvalues and come essentially from the vanishing at some point of the $[\omega - kv_z^{(o)}(r)]$ term. An analogous problem has been extensively treated by Case[13, 14] and Dikii[15] in connection with the problem of aerodynamic shear flow. These analyses, which ignore dissipative mechanisms (viscosity) indicate that all of the normal modes corresponding to the continuous spectrum of eigenvalues decay like various powers of the time for long times and therefore are not important in a determination of stability. Since dissipative mechanisms are not considered in this paper (i.e. collisions are neglected and the electrons are assumed to be cold) formally the continuous spectrum of eigenvalues can be ignored in the stability analysis. Stability is therefore determined solely by the normal modes corresponding to the discrete eigenvalues. Inclusion of dissipative mechanisms could cause the continuous spectrum of eigenvalues to become important in the stability analysis; however, such considerations are beyond the scope of the present treatment.

III. ESTABLISHMENT OF THE STABILITY CONDITION

Since no further difficulty of a theoretical nature remains, the eigenfunctions, the dispersion relation, and the condition that both roots of the latter be real will now be obtained. The first step is to determine the zero order potential and electric field that are implied by the distribution of charge specified in Eq. (2.11). Since a coordinate system in which Q = 0 is being employed, the zero order electric field in region 1 ($a \leq r < b$) vanishes identically. Taking the inner conductor ($r = a$) to be at zero potential, the zero order potential in region 1 also vanishes identically. The zero order distributions in the two remaining regions are found to be:

region 2; $b \leq r \leq c$

$$E_r^{(o)}(r) = -\frac{Neb^2}{r\epsilon_o} \ell n \frac{r}{b}$$

$$\phi^{(o)}(r) = \frac{Neb^2}{2\epsilon_o} \left[\ell n \frac{r}{b}\right]^2 \tag{3.1}$$

region 3; $c < r \leq d$

$$E_r^{(o)}(r) = -\frac{Neb^2}{r\epsilon_o} \ell n \frac{c}{b}$$

$$\phi^{(o)}(r) = \frac{Neb^2}{2\epsilon_o} \ell n \frac{c}{b} \; \ell n \frac{r^2}{bc} . \tag{3.2}$$

The solutions of Eq. (2.12) are linear combinations of the modified Bessel functions $I_o(kr)$ and $K_o(kr)$. The perturbation potential in region 2 ($b \leq r \leq c$) will be assumed to be

$$\phi(r) = \beta I_o(kr) + \gamma K_o(kr) \tag{3.3}$$

where β and γ are arbitrary constants. The perturbation potential in region 1 ($a \leq r < b$) must vanish at $r = a$ and be continuous with Eq. (3.3) at $r = b$. From these conditions the perturbation potential in region 1 is found to be:

$$\phi(r) = \frac{\beta I_o(kb) + \gamma K_o(kb)}{K_o(ka)I_o(kb) - K_o(kb)\, I_o(ka)} \left[K_o(ka)I_o(kr) - I_o(ka)K_o(kr)\right]. \tag{3.4}$$

Similarly, the perturbation potential in region 3 ($c < r \leq d$) must vanish at $r = d$ and be continuous with Eq. (3.3) at $r = c$ and it is found to be:

$$\phi(r) = \frac{\beta\, I_o(kc) + \gamma\, K_o(kc)}{K_o(kd)I_o(kc) - K_o(kc)\, I_o(kd)} \left[K_o(kd)I_o(kr) - I_o(kd)\, K_o(kr)\right]. \tag{3.5}$$

Applying the jump conditions given by Eqs. (2.13) and (2.14) at $r = b$ and $r = c$, respectively, results in the conditions

$$\begin{aligned} &\frac{\omega}{kb}\left[\beta\, I_o(ka) + \gamma K_o(ka)\right] \\ &\quad = \left[\beta\, I_o(kb) + \gamma K_o(kb)\right] \left[I_o(ka)\, K_o(kb) - I_o(kb)\, K_o(ka)\right] \end{aligned} \tag{3.6}$$

and

$$\begin{aligned} &\frac{1}{kc}\left[\beta\, I_o(kd) + \gamma K_o(kd)\right] \left[\omega + kb\, \ell n\, \frac{c}{b}\right] \\ &\quad = -\frac{b}{c}\left[\beta I_o(kc) + \gamma K_o(kc)\right]\left[I_o(kc)\, K_o(kd) - I_o(kd)\, K_o(kc)\right]. \end{aligned} \tag{3.7}$$

In these equations and henceforward, the unit of frequency has been taken to be ω_p^2/ω_c which, as before, is evaluated at $r = b$. The dispersion relation is now obtained by writing down the condition for consistency of these two linear homogeneous equations in β and γ. Defining

$$a_{ab} = I_o(ka)\, K_o(kb) - I_o(kb)\, K_o(ka) \tag{3.8}$$

etc.

the dispersion relation may be written as

$$
\alpha_{ad} \quad \omega^2 + kb \left[\alpha_{ac}\,\alpha_{cd} - \alpha_{ab}\,\alpha_{bd} + \alpha_{ad}\,\ell n\,\frac{c}{b}\right]\omega
$$

$$
- k^2 b^2 \alpha_{ab} \left[\alpha_{bd}\,\ell n\,\frac{c}{b} + \alpha_{bc}\,\alpha_{cd}\right] = 0. \tag{3.9}
$$

The condition for stability is that the roots ω be real and consequently that the discriminant of the quadratic in ω, Eq. (3.9), be positive. After some reduction, this condition can be written as:

$$
\left[\alpha_{ab}\,\alpha_{bd} + \alpha_{ac}\,\alpha_{cd} + \alpha_{ad}\,\ell n\,\frac{c}{b}\right]^2 - \left[2\,\alpha_{ab}\,\alpha_{cd}\right]^2 \geq 0. \tag{3.10}
$$

To the extent that the approximations of the analysis are valid, this is both a necessary and a sufficient condition for stability.

IV. DEDUCTIONS FROM THE STABILITY CONDITION

In this section the general stability condition of inequality (3.10) is examined. First a simple geometrical condition for stability of diocotron modes of all wavelengths is obtained. Next the implications of this geometrical condition are considered. Finally the limiting case which approximates the space radiation shield is examined.

The geometrical implications of the general stability condition of inequality (3.10) will now be considered. In the short wavelength limit ($k \to \infty$), it can be shown that inequality (3.10) is always satisfied. In the long wavelength limit ($k \to 0$), inequality (3.10) is satisfied if and only if

$$4 \ln \frac{b}{a} \ln \frac{d}{c} \leq \left[\ln \frac{c}{b} \right]^2 . \tag{4.1}$$

Numerical computations have shown that the function on the left of inequality (3.10) is a monotonically increasing function of k. The longest wavelengths are the most likely to be unstable, that is, in a given geometry. Therefore, inequality (4.1) is a necessary and sufficient condition for stability of diocotron modes of all wavelengths. Physically this geometrical inequality is satisfied and diocotron stability is guaranteed for electron beams which are sufficiently thick relative to the distance between the two conducting electrodes.

Some simple deductions can be made directly from inequality (4.1). First it can be seen that thin electron beams ($c \approx b$) always have diocotron modes of some wavelength which are unstable. Secondly, if either the inner or outer conductor is removed (i.e. $a \to 0$ or $d \to \infty$), there is always a diocotron mode of some (sufficiently long) wavelength which is unstable. Thus, for the space radiation shield case ($d \to \infty$), there can be no geometrical arrangement of the electron beam and the inner conductor which will make diocotron modes of all wavelengths stable. This statement requires the system to be infinitely long - a point of crucial importance which will be discussed later.

The two special geometries in which the electron beam extends up to either conducting wall are of particular interest. In these cases either $a = b$, or $c = d$, and the left side of inequality (3.10) becomes a perfect square. In addition, the left side of inequality (4.1) vanishes for these cases. Hence, stability is guaranteed whenever the beam is in contact with either conductor. This result is in agreement with the observation of Knauer[16] that the diocotron instability is essentially an interaction between waves on two surfaces; when one wave is removed by being brought into contact with a perfect conductor, instability can no longer result. Actually, for stability of diocotron modes of all wavelengths it is not necessary for the conductor to come into contact with the electron beam; it need only come close enough to satisfy inequality (4.1).

If, for some choice of the geometrical parameters a, b, c, and d, inequality (4.1) is not satisfied, there is always some critical value of k, k_{crit}, which makes inequality (3.10) an equality. Values of k larger than k_{crit} lead to stable diocotron modes whereas values of k smaller than k_{crit} lead to growing or unstable diocotron modes. Table I lists, by way of example, the value of k_{crit} for a few choices of the parameters a, b, c, and d. From these numerical results it can be seen that as the electron beam becomes thicker relative to the distance between the two conducting electrodes, the critical value of k, k_{crit}, becomes smaller. Finally, as the beam becomes thick enough to satisfy inequality (4.1), the value of k_{crit} goes to zero and stability is guaranteed for diocotron modes of all wavelengths.

As previously noted, in the space radiation shield case ($d \rightarrow \infty$) inequality (4.1) is not satisfied. Therefore, the diocotron modes with wave numbers between zero and some k_{crit} will be unstable. However, in the toroidally-shaped space radiation shield the electron beam is re-entrant and a periodic boundary condition must be applied to k. Thus, k can take on only integral multiples of the minimum value of k, k_{min}. In a torus, the maximum wavelength of waves propagating along the minor axis of the torus is approximately 2π times the major radius R of the torus. Thus, the minimum value of k is given approximately by

$$k_{min} \simeq \frac{2\pi}{2\pi R} = 1/R \tag{4.2}$$

If the space radiation shield is designed so that k_{min} is larger than k_{crit}, then all of the diocotron modes possible for the re-entrant electron beam would be stable.

In examining the space radiation shield case it is convenient to consider a critical value of c instead of k_{crit}. As previously noted with respect to Table I, for constant values of the parameters a and b, the value of k_{crit} decreases as c (a measure of the thickness of the beam) increases. The symbol c^* will be used to denote that value of c which makes k_{crit} exactly equal to k_{min}. Thus, values of c greater than c^* correspond to stable electron beams and vice versa. In the space radiation shield case the stability condition of inequality (3.10) must be modified by letting the radius d of the outer electrode approach infinity. Letting $d \rightarrow \infty$, the stability condition of inequality (3.10) becomes

TABLE I

a	b	c	d	k_{crit}
0	2	3	4	1.28
0	2	6	10	0.37
0	3	6	10	0.46
1	2	3	4	1.20
1	2	3	10	1.26
1	2	6	10	0.11
1	3	4	10	1.30
1	3	6	10	0.40
1	3	8	10	0.03
1	2	3	∞	1.25
1	2	6	∞	0.20
1	3	6	∞	0.41
0	2	3	∞	1.32
0	2	6	∞	0.38
0	3	6	∞	0.47

Table I. Value of k_{crit} above which stability exists and below which instability exists for given values of a, b, c, and d. The units of a, b, c, and d are arbitrary and the units of k_{crit} are the reciprocal of these same arbitrary units.

$$\left[a_{ab}\, K_o(kb) + a_{ac}\, K_o(kc) + K_o(ka)\, \ell n\, \frac{c}{b}\right]^2 - \left[2\, a_{ab}\, K_o(kc)\right]^2 \geq 0. \tag{4.3}$$

The critical value c^* is found by making inequality (4.3) an equality and solving the resulting equation numerically for the value of c which satisfies it.

Fig. 3 is a plot derived from the stability condition of inequality (4.3). Essentially, it shows how thick the electron beam must be to assure stability. In Fig. 3 the abscissa (1-a/b) measures how near the inner edge of the electron beam is to the outer edge of the torus. When (1-a/b) = 0, the electron beam touches the torus and (in the sense of this section) stability is guaranteed. Since k is approximately k_{min}, the parameter ka is approximately equal to a/R (the radius ratio of the torus) which can be considered a fixed quantity in the stability analysis. Similarly kc^* is approximately equal to c^*/R, a measure of the thickness of the electron beam relative to its maximum thickness R. The region $kc^* > 1$ is not important in the present analysis since this region is excluded for toroidal geometry.

From Fig. 3 it can be seen that as b approaches a, the critical value c^* decreases. Physically this means that as the beam is brought closer to the torus, the required thickness of the electron beam becomes smaller. Also, for a given value of (1-a/b), the ratio of the required thickness of the electron beam to the major radius R of the torus becomes smaller for decreasing values of the parameter ka (the minor to major radius ratio of the torus). Thus, in general, for a reasonable electron beam thickness, the space radiation shield should be designed with a moderately large major to minor radius ratio and the inner edge of the electron beam as close as possible to the torus. A typical reading of Fig. 3 shows that for (1-a/b) = 0.1 (a/b = 0.9), and ka = 0.20 (a major to minor radius ratio for the torus of 5:1) the beam thickness should be at least 45% of the major radius for stability.

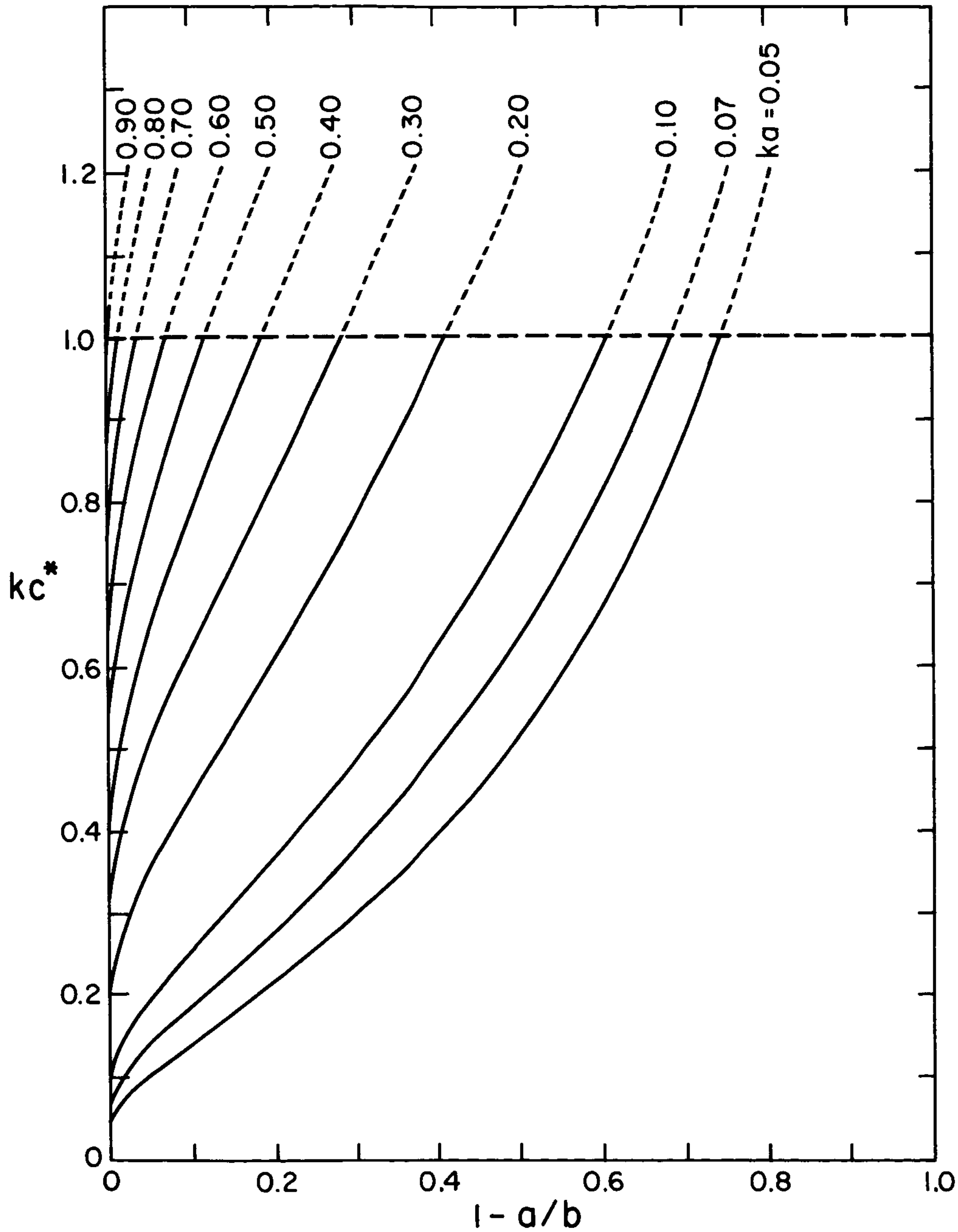

Fig. 3 Stability results for the space radiation shield case ($d \rightarrow \infty$) computed from inequality (4.3). Regions of stability are above and to the left of each curve of constant ka and regions of instability are below and to the right of each curve.

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CONCLUSIONS

Diocotron instabilities in a cylindrical geometry with a radial electric field and an azimuthal magnetic field have been studied. To the extent that the approximations of the analysis are valid, it has been demonstrated that proper selection of dimensions can ensure stability against diocotron instabilities of all wavelengths in this geometry. The application of this theory to a proposed space radiation shield has been discussed. By further assuming that a torus can be adequately approximated by a cylinder, it has been found that a space radiation shield of reasonable physical proportions can be made stable against diocotron instabilities.

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