

NON-RESONANT NUCLEAR REACTIONS AT STELLAR TEMPERATURES*

The purpose of this note is to describe a systematic and accurate procedure for calculating the rates of non-resonant nuclear reactions at stellar temperatures and to indicate the approximations that are involved, and the corrections that are necessary, when using the currently fashionable formulae (Caughlan and Fowler 1962; Parker, Bahcall, and Fowler 1964).

The number of nuclear reactions, P , occurring per unit of time per unit of volume between nuclei of type one and type two is

$$P = n_1 n_2 (1 + \delta_{12})^{-1} \langle \sigma v \rangle \quad (1)$$

where n_1 and n_2 are the number densities of nuclei of type one and two and $\langle \sigma v \rangle$ is the interaction cross section times the relative velocity averaged over a Maxwell-Boltzmann distribution. In analyzing non-resonant reactions at stellar temperatures, it is conventional and convenient to represent the cross section (in the center of mass frame) by (Burbidge, Burbidge, Fowler, and Hoyle 1957; Cameron 1963; Fowler and Vogl 1964)

$$\sigma(E) = E^{-1} S(E) \exp - (b E^{-1/2}) \quad (2a)$$

where the exponential factor represents the Gamow penetration factor, and

$$b = 2 \pi \alpha Z_1 Z_2 (m A c^2/2)^{1/2} \quad (2b)$$

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Here, m_A is the reduced mass of the colliding nuclei. The reaction rate P is therefore proportional to an integral of the form

$$I = \int_0^{\infty} dE S(E) \exp - (E/kT + bE^{-1/2}) \quad (3)$$

The usual evaluation (Caughlan and Fowler 1962; Fowler and Vogl 1964) of I is accomplished by representing S in Eq. (3) by a two-term power series about E equals zero and replacing the exponential by a Gaussian centered at an energy E_0 chosen such that the exponent, $E/kT + b/\sqrt{E}$, is an extremum. Instead of making approximations with I in the form given by Eq. (3), we shall first rewrite I in a form that permits a power series expansion in a parameter, τ^{-1} , that is small for all stellar problems encountered so far.

In the usual notation (Parker, Bahcall, and Fowler 1964; Fowler and Vogl 1964),

$$P = S_{\text{eff}} \times 7.20 \times 10^{-19} n(1) n(2) f_{1,2} \tau^2 e^{-\tau} (1 + \delta_{1,2})^{-1} \\ (A Z_1 Z_2)^{-1} \text{ reactions cm}^{-3} \text{ sec}^{-1} \quad (4a)$$

where $f_{1,2}$ is an electron screening factor and

$$\tau = 3 E_0 / kT \\ \approx 42.48 (Z_1^2 Z_2^2 A / T_6)^{1/3} \quad (4b)$$

with

$$E_0 = (b k T / 2)^{2/3} \\ = \left[(\pi \alpha Z_1 Z_2 k T)^2 (m_n A c^2 / 2) \right]^{1/3} \quad (4c)$$

Here, S_{eff} is the correction factor which arises from the variation of S with energy and the departure of the exponential in Eq. (3) from a Gaussian shape. More explicitly, one can show that

$$S_{\text{eff}} = \left[(\tau/\pi)^{1/2} (2 E_0)^{-1} e^{+\tau} \right] \int_0^{\infty} \exp - (E/kT + bE^{-1/2}) S(E) dE, \quad (5)$$

which has, of course, the form indicated in Eq. (3). The bracketed factors have been chosen for simplicity in the final answer.

The quantity τ , defined by Eq. (4b), is large (typically 15 to 40) for all non-resonant nuclear reactions of interest (p-p chain, CNO bi-cycle, etc.). We therefore express S_{eff} as a power series in τ^{-1} . This may be accomplished by the substitution (motivated by Salpeter's treatment of the p-p reaction, Salpeter 1952):

$$E = E_0 (1 + \tau^{-1/2} u)^2 \quad (6)$$

Inserting the above definition of u in Eq. (5), one finds

$$S_{\text{eff}} = (\pi \tau)^{-1/2} \int_{-\tau^{1/2}}^{\infty} du (\exp - u^2) (\exp + \left[(2u^3/3)(\tau^{1/2} + u)^{-1} \right] (\tau^{1/2} + u) S(E_0 (1 + \tau^{-1/2} u)^2) du \quad (7)$$

Since $e^{-\tau} \ll \tau^{-1}$, S_{eff} can be readily evaluated as a power series in τ^{-1} by replacing the lower limit of the integral in Eq. (7) by infinity and expanding $\exp \left[2u^3/3 (\tau^{1/2} + u)^{-1} \right]$ and $S(E_0 (1 + \tau^{-1/2} u)^2)$ as a Taylor series in u about u equals zero. The successive terms can be grouped in descending integral powers of τ (since e^{-u^2} is an even function of u).

One finds in this way that

$$S_{\text{eff}} = S(E_0) \left[1 + \tau^{-1} \left(\frac{5}{12} + \frac{5}{2} \frac{S'(E_0) E_0}{S(E_0)} + \frac{S''(E_0) E_0^2}{S(E_0)} \right) + O(\tau^{-2}) \right] \quad (8)$$

where

$$S'(E_0) = \left. \frac{dS(x)}{dx} \right|_{x=E_0}$$

and

$$S''(E_0) = \left. \frac{d^2 S(x)}{dx^2} \right|_{x=E_0}$$

Note that up to order τ^{-2} , only first and second derivatives of S enter Eq. (8).

In order to relate the above expression for S_{eff} to the usual formula (Caughlan and Fowler 1962; Parker, Bahcall, and Fowler 1964), one must express the relevant quantities in terms of their values at E equals zero (not E_0). If one does so, there is no a priori assurance that, to order τ^{-1} , only first and second derivatives of S are important. However, this is a plausible assumption and is necessary in order to obtain a not too complicated formula. We find (neglecting all derivatives higher than the second derivative):

$$S_{\text{eff}} \approx S(0) \left[1 + \frac{S'(0) E_0 + S''(0) E_0^2/2}{S(0)} + \frac{1}{\tau} \left(\frac{5}{12} + \frac{(\frac{35}{12}) E_0 S'(0) + (\frac{89}{24}) S''(0) E_0^2}{S(0)} \right) \right] \quad (9a)$$

or

$$s_{\text{eff}} = s(0) \left[1 + \frac{5}{12\tau} + \frac{s'(0)}{s(0)} \left(E_0 + \frac{35}{36} kT \right) + \frac{s''(0) E_0}{s(0)} \right. \\ \left. \left(E_0/2 + \frac{89}{72} kT \right) \right] \quad (9b)$$

If the last term in Eq. (9b) is set equal to zero, then the above expression reduces to the usually quoted formula.

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