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SUMMARY

When the Bessel function representing the induced electric field in a plasma under a Stix coil is expanded in terms of the natural plasma modes, the coefficients of the expansion may be approximated by assuming that the eigenfunctions are orthogonal. However, the eigenfunctions are not orthogonal. When the orthogonality assumption is not used, there results an infinite system of equations involving an infinite set of unknowns. Kantorovich developed a technique that can be adapted to this problem. The technique produces both a deficient and an excessive value of the coefficient. Because the correct value lies between these two limits, the greatest error resulting from the orthogonality assumption can be obtained. In the cases tested it was found that the errors involved in determining the coefficients using the orthogonality assumption were less than 0.1 percent for the fundamental mode.

INTRODUCTION

When the efficiency of adding energy to a plasma wave by means of a Stix coil (ref. 1) was calculated, a Bessel function in the analysis was expanded into an infinite series of eigenfunctions (refs. 1 and 2) as follows:

$$J_1(\nu R) = \sum_{m=1}^{\infty} a_m J_1(\nu_m R)$$

(Symbols are defined in appendix A.) The coefficients of the infinite series were then determined by the usual Fourier series technique, which requires that the eigenfunctions be orthogonal. This straightforward approach yielded an easily evaluated, closed-form expression for the coefficients:

$$a_m = \frac{\int_0^{R_1} RJ_1(\nu R)J_1(\nu_m R)dR}{\int_0^{R_1} RJ_1^2(\nu_m R)dR}$$

The eigenfunctions, however, are not orthogonal; consequently, the cross product terms are not identically zero,

$$\int_0^{R_1} RJ_1(\nu_m R)J_1(\nu_n R)dR \neq 0$$

and the preceding expression for the a_m's is incorrect.

There exist techniques for solving problems involving an infinite set of nonorthogonal eigenfunctions, such as the method of Green's functions (ref. 3) and Schmidt's orthogonalization process (ref. 4). These require expansion of the original nonorthogonal functions into polynomials which form an orthogonal set. However, these techniques result in expressions which are difficult to evaluate and compare with the coefficients in the original series, that is, with the a_m's in

$$\sum_{m=1}^{\infty} a_m J_1(\nu_m R)$$

However, by proceeding in a manner similar to that used with the Fourier-Bessel expansion, a direct comparison can be made. In this latter technique, the cross product terms do not vanish, and, instead of the simple expression for a_m presented previously, there results an infinite set of simultaneous equations,

$$\int_0^{R_1} RJ_1(\nu R)J_1(\nu_n R)dR = \sum_{m=1}^{\infty} a_m \int_0^{R_1} RJ_1(\nu_m R)J_1(\nu_n R)dR$$

for which it is not possible to obtain an exact solution. However, by an iteration method described in reference 5, it is possible to calculate upper and lower limits of the coefficients. If the iteration process is extended, these two limits can be brought as close together as desired. In this manner, a direct comparison can be made between the orthogonal and nonorthogonal coefficients. This report calculates these upper and lower values of the coefficients for several plasma conditions near ion cyclotron resonance. In addition, these limits on the coefficients are compared with the values of the coefficients obtained from the orthogonal expansion used in references 1 and 2.

THEORY

The physical problem in reference 1 concerned the coupling of radiofrequency power into a plasma by generating ion cyclotron waves. A cylindrical plasma of infinite length, confined by a steady longitudinal magnetic field and surrounded by a vacuum, is capable of exhibiting many natural modes of oscillation. Among these are oscillations identified as hydromagnetic or ion cyclotron waves (ref. 6). The radial variation of the electric field in the plasma for the mth mode of oscillation is proportional to a cylindrical Bessel function of the first order and first kind, $J_1(\nu_m R)$. The eigenvalue ν_m, which is the radial wave number of the mth natural mode of oscillation, is determined from the plasma dispersion relation and the boundary condition at the vacuum-plasma interface (ref. 6). The axial wave number κ_m of the mth natural mode is simultaneously found.

If a disturbance in the plasma column is driven by a current sheet (Stix coil) periodic in both time and space, the radial variation of the electric field in the plasma is proportional to the $J_1(\nu R)$ where ν is the radial wave number associated with the driving electric field and is determined from the dispersion relation and the axial wavelength of the current sheet.

In reference 1 the forced electric field consequently was written as

$$E \propto J_1(\nu R) = \sum_{m=1}^{\infty} a_m J_1(\nu_m R) \tag{1}$$

the electric field of the mth natural mode being proportional to

$$a_m J_1(\nu_m R)$$

It is necessary to consider this expansion because the efficiency of power transfer was expressed for each natural mode separately. If a_m can be evaluated for nonorthogonal $J_1(\nu_m R)$ and compared with the coefficient obtained when the orthogonal assumption is made, the maximum error of the coefficients can be determined. Because the electric fields of the natural modes are proportional to the a_m's, the relative errors of the a_m's are also the relative errors of the field modes. To calculate the a_m's, multiply equation (1) by $RJ_1(\nu_n R)$ and integrate with respect to R over the range $0 \leq R \leq R_1$, where R_1 is the radius of the plasma. Then,

$$\int_0^{R_1} RJ_1(\nu_n R)J_1(\nu R)dR = \sum_{\substack{m=1 \\ n\neq m}}^{\infty} a_m \int_0^{R_1} RJ_1(\nu_n R)J_1(\nu_m R)dR + a_n \int_0^{R_1} RJ_1^2(\nu_n R)dR \tag{2}$$

If the $J_1(\nu_n R)$'s are assumed orthogonal, all integrals of the

$$\int_0^{R_1} RJ_1(\nu_n R)J_1(\nu_m R)dR \qquad n \neq m$$

form vanish. However, these integrals can be expressed as

$$\int_0^{R_1} RJ_1(\nu_n R)J_1(\nu_m R)dR = \frac{R_1 J_1(\nu_m R_1)J_1(\nu_n R_1)}{\nu_m^2 - \nu_n^2}\left[\nu_n \frac{J_o(\nu_n R_1)}{J_1(\nu_n R_1)} - \nu_m \frac{J_o(\nu_m R_1)}{J_1(\nu_m R_1)}\right]$$

Since, in general, the eigenfunctions are nonorthogonal, the right side cannot be expected to vanish for all n and m combinations. If the eigenfunctions were orthogonal, either $J_1(\nu_n R_1)$ would be zero or the term $\nu_n R_1\left[J_o(\nu_n R_1)/J_1(\nu_n R_1)\right]$ would be constant for all n. All cross products would then vanish. However, $J_1(\nu_n R_1)$ does not vanish, and $\nu_n R_1\left[J_o(\nu_n R_1)/J_1(\nu_n R_1)\right]$ is actually a slowly varying function of $\nu_n R_1$ which approaches a finite limit. This latter condition is responsible for the small error calculated herein. The nth equation of the set represented by equation (2) becomes, after integration,

$$C_n = \sum_{m=1}^{\infty} (1 - \delta_{nm})a_m A_{nm} + a_n A_{nn} \tag{3}$$

where

$$C_n \equiv \frac{J_1(\nu R_1)J_1(\nu_n R_1)}{\nu^2 - \nu_n^2}\left[\nu_n R_1 \frac{J_o(\nu_n R_1)}{J_1(\nu_n R_1)} - \nu R_1 \frac{J_o(\nu R_1)}{J_1(\nu R_1)}\right] \tag{4}$$

$$A_{nm} = \frac{J_1(\nu_n R_1) J_1(\nu_m R_1)}{\nu_m^2 - \nu_n^2} \left[\nu_n R_1 \frac{J_o(\nu_n R_1)}{J_1(\nu_n R_1)} - \nu_m R_1 \frac{J_o(\nu_m R_1)}{J_1(\nu_m R_1)} \right]_{m \neq n} \tag{5}$$

$$A_{nn} \equiv \frac{R_1^2}{2} \frac{J_1^2(\nu_n R_1)}{(\nu_n R_1)^2} \left\{ \left[\nu_n R_1 \frac{J_o(\nu_n R_1)}{J_1(\nu_n R_1)} \right]^2 - 2\nu_n R_1 \frac{J_o(\nu_n R_1)}{J_1(\nu_n R_1)} + (\nu_n R_1)^2 \right\} \tag{6}$$

and

$$\delta_{nm} = \begin{cases} 1 & n = m \\ 0 & n \neq m \end{cases}$$

Solving for a_n in equation (3) yields

$$a_n = \sum_{m=1}^{\infty} (\delta_{nm} - 1) \frac{A_{nm}}{A_{nn}} a_m + \frac{C_n}{A_{nn}}$$

In Kantorovich's notation (ref. 5), define

$$C_{nm} \equiv (\delta_{nm} - 1) \frac{A_{nm}}{A_{nn}}$$

and

$$b_n \equiv \frac{C_n}{A_{nn}}$$

Hence,

$$a_n = \sum_{m=1}^{\infty} C_{nm} a_m + b_n \tag{7}$$

If orthogonality is assumed, the sum term vanishes and the solution for a_n is just b_n, the Fourier-Bessel coefficient. The error is obtained by comparing the values of a_n determined from equation (7) with the b_n.

In order to evaluate a_n, it is first desirable to determine the signs of the C_{nm}

terms. Because all the subscripted ν's that appear in A_{nm} and A_{nn} (eqs. (5) and (6)) are wave numbers of the natural modes, they satisfy the boundary condition (ref. 6) for undamped waves

$$\nu_n R_1 \frac{J_o(\nu_n R_1)}{J_1(\nu_n R_1)} = -k_n R_1 \frac{K_o(k_n R_1)}{K_1(k_n R_1)} \tag{8}$$

where $k_n = (\kappa_n^2 - \Omega^2)^{1/2}$, $\kappa_n^2 > \Omega^2$, and $K_o(k_n R_1)$ and $K_1(k_n R_1)$ are modified Bessel functions of the second kind of order zero and one, respectively. Substituting the right side of equation (8) in equation (6) results in the following expression for A_{nn}:

$$A_{nn} = \frac{R_1^2}{2} \frac{J_1^2(\nu_n R_1)}{(\nu_n R_1)^2} \left\{ \left[k_n R_1 \frac{K_o(k_n R_1)}{K_1(k_n R_1)} \right]^2 + 2k_n R_1 \frac{K_o(k_n R_1)}{K_1(k_n R_1)} + (\nu_n R)^2 \right\}$$

Because the modified Bessel functions are always positive, A_{nn} is always positive and the sign of C_{nm} will be opposite that of A_{nm}. To determine the signs of the A_{nm}'s, consider equation (5). Because the term

$$\frac{J_o(\nu_n R_1)}{J_1(\nu_n R_1)} = \frac{1}{(\nu_n R_1)} \left[\nu_n R_1 \frac{J_o(\nu_n R_1)}{J_1(\nu_n R_1)} \right] = -\frac{1}{\nu_n R_1} \left[k_n R_1 \frac{K_o(k_n R_1)}{K_1(k_n R_1)} \right]$$

is negative, the derivative

$$\frac{d\left[\nu_n R_1 \frac{J_o(\nu_n R_1)}{J_1(\nu_n R_1)} \right]}{d(\nu_n R_1)} = 2 \frac{J_o(\nu_n R_1)}{J_1(\nu_n R_1)} - (\nu_n R_1) - (\nu_n R_1) \frac{J_o^2(\nu_n R_1)}{J_1^2(\nu_n R_1)}$$

is also negative. Therefore, $\nu_n R_1 \left[J_o(\nu_n R_1)/J_1(\nu_n R_1) \right]$ is a monotonically decreasing function. Consequently, when $m > n$, that is, $\nu_m > \nu_n$,

$$\nu_n \frac{J_o(\nu_n R_1)}{J_1(\nu_n R_1)} - \nu_m \frac{J_o(\nu_m R_1)}{J_1(\nu_m R_1)} > 0 \tag{9a}$$

and

$$\nu_m^2 - \nu_n^2 > 0 \tag{9b}$$

Because the reverse inequality holds for both expressions when $m < n$, the ratio of the left side of these inequalities is always positive. The sign of A_{nm} (eq. (5)) then depends only on the sign of $J_1(\nu_m R)J_1(\nu_n R)$. Because both ν_m and ν_n must satisfy the boundary condition expressed in equation (8) and because both $K_o(k_n R_1)$ and $K_1(k_n R_1)$ are greater than zero, $J_o(\nu_n R_1)$ and $J_1(\nu_n R_1)$ must have opposite signs. It is shown in figure 1 that the nonparity of J_1 and J_o occurs only in the bracketed regions between the zeros of the Bessel functions.

Because the absolute magnitude of J_o/J_1 varies from zero to infinity within these regions, there will be a ν_m which falls within each of the bracketed regions. The value denoted by ν_1 will be in the first region, ν_2 in the second, etc. However,

$$J_1(\nu_n R_1)\begin{cases} > 0 & \text{n odd} \\ < 0 & \text{n even} \end{cases}$$

This allows A_{nm} to be expressed as

$$A_{nm} = (-1)^{n+m}\left|\frac{R_1 J_1(\nu_m R_1)J_1(\nu_n R_1)}{\nu_m^2 - \nu_n^2}\left[\nu_n \frac{J_o(\nu_n R_1)}{J_1(\nu_n R_1)} - \nu_m \frac{J_o(\nu_m R_1)}{J_1(\nu_m R_1)}\right]\right| \tag{10}$$

Equation (10) explicitly exhibits the sign of A_{nm}. Thus, C_{nm} can be written as

$$C_{nm} = (-1)^{n+m+1}\left|C_{nm}\right| \tag{11}$$

Particular care must be exercised when the signs of the coefficients vary. Rearrange equation (7) into the following form:

$$a_n = \sum_{m=1}^{N}{}' C_{nm}a_m + \sum_{m=1}^{N}{}'' C_{nm}a_m + b_n + \sum_{m=N+1}^{\infty} C_{nm}a_m \tag{12}$$

where N is a finite number, $\sum'$ indicates a sum of terms in which $C_{nm} > 0$ (i.e., $n + m$ is odd), $\sum''$ indicates a sum of terms in which $C_{nm} < 0$ (i.e., $n + m$ is even), and the last term is a remainder term which will be approximated on page 8.

An expression for a deficient solution can be synthesized from equation (12) by replacing the terms on the right side with expressions that are smaller. Similarly, an excessive solution may be obtained by substituting larger terms. The remainder term

$$\sum_{m=N+1}^{\infty} C_{nm} a_m$$

is modified by selecting a number $\mathscr{K}$ which is larger than every a_m for every $m > N+1$. The numerical technique for this selection is given in appendix B. The remainder term can be written as

$$\sum_{m=N+1}^{\infty} C_{nm} a_m < \mathscr{K} R_N^{(n)}$$

where

$$R_N^{(n)} \equiv \sum_{m=N+1}^{\infty} \left| C_{nm} \right| \tag{13}$$

If $\tilde{a}_n$ represents a deficient solution and $\bar{a}_n$ an excessive one, write

$$\tilde{a}_n = \sum_{m=1}^{N}{}' C_{nm} \tilde{a}_m + \sum_{m=1}^{N}{}'' C_{nm} \bar{a}_m + b_n - \mathscr{K} R_N^{(n)} \qquad (n = 1, 2, \ldots, N) \tag{14}$$

and

$$\bar{a}_n = \sum_{m=1}^{N}{}' C_{nm} \bar{a}_m + \sum_{m=1}^{N}{}'' C_{nm} \tilde{a}_m + b_n + \mathscr{K} R_N^{(n)} \qquad (n = 1, 2, \ldots, N) \tag{15}$$

Equations (14) and (15) represent 2N coupled equations in 2N unknowns. This set of equations may be simplified somewhat by considering $\bar{a}_n - \tilde{a}_n$ and $\bar{a}_n + \tilde{a}_n$ as new variables. Hence, when equation (14) is subtracted from equation (15), there results

$$(\bar{a}_n - \tilde{a}_n) = \sum_{m=1}^{N}{}' C_{nm} (\bar{a}_m - \tilde{a}_m) + \sum_{m=1}^{N}{}'' (-C_{nm})(\bar{a}_m - \tilde{a}_m) + 2\mathscr{K} R_N^{(n)}$$

However, because C_{nm} in the $\sum''$ sum is always negative,

$$(\bar{a}_n - \tilde{a}_n) = \sum_{m=1}^{N} \left|C_{nm}\right|(\bar{a}_m - \tilde{a}_m) + 2\mathscr{R}R_N^{(n)} \qquad (n = 1, 2, \ldots, N) \tag{16}$$

Similarly, when equations (14) and (15) are added,

$$(\bar{a}_n + \tilde{a}_n) = \sum_{m=1}^{N}{}' C_{nm}(\bar{a}_m + \tilde{a}_m) + \sum_{m=1}^{N}{}'' C_{nm}(\bar{a}_m + \tilde{a}_m) + 2b_n$$

$$= \sum_{m=1}^{N} C_{nm}(\bar{a}_m + \tilde{a}_m) + 2b_n \qquad (n = 1, 2, \ldots, N) \tag{17}$$

Equations (16) and (17) each consist of N equations and N unknowns, which may be solved by any one of many techniques. After values for $\bar{a}_n - \tilde{a}_n$ and $\bar{a}_n + \tilde{a}_n$ are obtained, $\bar{a}_n$ and $\tilde{a}_n$ are readily found. All that is required, consequently, is to increase N until $\bar{a}_n$ and $\tilde{a}_n$ are nearly equal within desired limits.

The solution of equations (16) and (17) depends on calculating values for the C_{nm}'s, which in turn depends on calculating many natural modes. However, the higher modes are very difficult to determine by normal techniques. These techniques generally break down because the solutions are near singular points. In addition, evaluating the C_{nm}'s requires calculating differences of nearly equal quantities (cf., eq. (5)). This results in a considerable loss in computing accuracy. These difficulties may be circumvented, however, by expanding the Bessel functions in a manner that utilizes certain mathematical characteristics of the natural modes. To accomplish this, notice that the wave numbers for the natural modes approach the zeros of $J_o(\nu_n R)$ as n increases without limit (appendix C). It then becomes convenient to expand both $J_o(\nu_n R_1)$ and $J_1(\nu_n R_1)$ about the nth zero of the zero-order Bessel function of the first kind. If j_{on} represents the nth zero and t_n the expansion variable, it can be shown (see appendix D) that

$$J_o(\nu_n R_1) \approx \left(-t_n + \frac{t_n^2}{2j_{on}}\right) J_1(j_{on}) \tag{18}$$

and

$$J_1(\nu_n R_1) \approx \left(1 - \frac{t_n}{j_{on}} + \frac{t_n^2}{j_{on}^2} - \frac{t_n^2}{2^2}\right) J_1(j_{on}) \tag{19}$$

where terms higher than the second power in t_n have been neglected. The series from which these are derived are convergent. The expression $\nu_n R_1 \left[J_o(\nu_n R_1)/J_1(\nu_n R_1)\right]$ that appears in C_{nm} (see eqs. (5) and (6)) consequently may be written as

$$\nu_n R_1 \frac{J_o(\nu_n R_1)}{J_1(\nu_n R_1)} = \frac{-4j_{on}^2 t_n\left(j_{on} + \frac{t_n}{2}\right)}{4j_{on}^2 - 4j_{on}t_n + 4t_n^2 - j_{on}^2 t_n^2} \tag{20}$$

because

$$\nu_n R_1 = j_{on} + t_n \tag{21}$$

The variable t_n which appears in equations (18), (19), and (20) can be determined by utilizing the dispersion relation (ref. 6)

$$\nu^2 = \frac{\Omega^4\alpha^2 - \Omega^2\left[2\alpha + (\kappa^2 - \Omega^2)\right](\kappa^2 - \Omega^2) + (\kappa^2 - \Omega^2)^2}{\Omega^2\left[\alpha + (\kappa^2 - \Omega^2)\right] - (\kappa^2 - \Omega^2)} \tag{22}$$

and the boundary condition

$$\nu R_1 \frac{J_o(\nu R_1)}{J_1(\nu R_1)} = -(\kappa^2 - \Omega^2)^{1/2} R_1 \frac{K_o\left[(\kappa^2 - \Omega^2)^{1/2} R_1\right]}{K_1\left[(\kappa^2 - \Omega^2)^{1/2} R_1\right]} \tag{23}$$

If equations (18) and (19) are substituted into equation (23) and use is made of the fact that all k_n occur near the singularity of equation (22) (see appendix C), algebriac expressions can be derived (see appendix E) for the values of t_n. Thus, all the quantities necessary for evaluating $\bar{a}_n$ and $\tilde{a}_n$ have been determined.

The values of the coefficients which result from incorporating the orthogonality relation are the values of b_n. Consequently, the error can readily be evaluated by comparing b_n with the excessive and deficient values of a_n.

DISCUSSION OF RESULTS

The evaluation of error due to the orthogonality assumption for the nth mode of the electric field requires a numerical calculation of $\tilde{a}_n$, $\bar{a}_n$, and b_n for a specific case.

Since calculation of the b_n for one specific physical model had already been made (ref. 2) (with the assumption of orthogonality of the Bessel functions), this model was used herein. Only conditions that fall in regions of interest were checked. Of course, any region where a wave number is complex cannot be checked because the final equations presented herein would not be applicable. The results of reference 2 are in terms of an efficiency which is proportional to the nth mode of the electric field. Consequently, the greatest percentage error calculated in the coefficients herein would also be the greatest error in efficiency.

With these conditions in mind, corrections described in this report were determined for the selection of points indicated in figure 2 (ref. 2). The values of $\bar{a}_n$, $\tilde{a}_n$, and b_n are presented in table I. The accuracy was sufficient to allow an error calculation to within at least 0.1 percent. The results presented in table I indicate that the orthogonality assumption of references 1 and 2 was well founded. The errors in determining the coefficients are insignificant for the fundamental mode, being less than 0.1 percent for the conditions tested.

The only condition where there is a considerable difference is in the second and third modes near the first maximum in the efficiency. However, the percentage of the total field energy involved in the second and third modes at this condition is quite small when compared with the field energy of the fundamental. Thus, the quantity of primary interest in the present case, the efficiency of energy addition, is negligibly affected by the errors resulting from the orthogonality assumption. If the field amplitude of a specific higher mode were required, for diagnostic purposes, for example, the orthogonality assumption would prove unsatisfactory for some conditions.

CONCLUDING REMARKS

Nonorthogonal eigenfunctions are not uncommon in physical problems: they arise especially when wave propagation is involved. When an expansion in terms of these eigenfunctions is desired, the most expeditious approach could well be to use the orthogonality assumption and develop a Fourier series. The resulting simplification in the analysis may easily justify the procedure. The accuracy of the results might be evaluated by comparing the Fourier-Bessel coefficients with those determined by the numerical methods used in this report.

Indeed there are other examples in the literature where the orthogonality approximation was used to obtain a Fourier expansion (ref. 7). This approach may be justifiable when only a slight modification of the boundary conditions will give orthogonal eigenfunctions. The eigenfunctions for the vacuum boundary case might then be termed almost-orthogonal functions.

Lewis Research Center,
National Aeronautics and Space Administration,
Cleveland, Ohio, November 30, 1965.

APPENDIX A

SYMBOLS

A_{nm}	defined by eq. (5)
A_{nn}	defined by eq. (6)
$\tilde{a}$	deficient coefficient
$\bar{a}$	excessive coefficient
a_m	coefficient of mth term in Bessel expansion of $J_1(\nu R)$
b_n	coefficient obtained by orthogonality assumption
C_n	defined by eq. (4)
C_{nm}	$(\delta_{nm} - 1)A_{nm}/A_{nn}$
j_{on}	nth zero of zero-order Bessel function of first kind J_o
k_n	$(\kappa_n^2 - \Omega^2)^{1/2}$
R	nondimensional radius (ref. 1)
R_1	nondimensional plasma radius (ref. 1)
α	reciprocal of square of nondimensional Alfvén velocity for dense plasmas (ref. 1)
$\mathscr{K}$	number larger than every a_m (evaluated in appendix B)
δ_{nm}	Kronecker delta
κ	nondimensional axial wave number (ref. 1)
ν	nondimensional radial wave number (ref. 1)
Ω	nondimensional frequency (ref. 1)

Subscripts:

m	pertaining to mth natural mode of plasma oscillation
n	pertaining to nth natural mode of plasma oscillation

APPENDIX B

EVALUATION OF $\mathscr{K}$

In order to pick a number, $\mathscr{K}$, which is greater than every a_m for all $m \geq N + 1$, a set of numbers $\mathscr{K}_n$ must be examined. According to reference 5 (p. 26), if

$$\sum_{m=1}^{\infty} \left|C_{nm}\right| < 1$$

there exists a set of numbers $\mathscr{K}_n$, given by

$$\mathscr{K}_n \geq \frac{b_n}{1 - \sum_{m=1}^{\infty} \left|C_{nm}\right|}$$

such that the largest will satisfy the condition desired of $\mathscr{K}$, that is, $\mathscr{K} > a_m$ for $m \geq N + 1$. To guarantee that $\mathscr{K}$ was indeed larger than any $\mathscr{K}_n$, the relation

$$\mathscr{K} = 1.1 \left(\max\left|\mathscr{K}_n\right|\right)$$

was used to allow for any computational errors.

Because $\mathscr{K}_n$ consists of an infinite set of elements, it is not particularly straightforward to find their maximum. However, if $\left|\mathscr{K}_n\right|$ can be shown to be a monotonically decreasing function of n, its maximum is just the first element of the series. Difficulty arises, though, in evaluating $\sum_{m=1}^{\infty} \left|C_{nm}\right|$ because the elements of the sum contain complicated collections of Bessel functions. As an alternative, the Bessel functions are approximated by expanding them about the zeros of J_o. In appendix C, it is shown that the arguments $\nu_n R_1$ will be near the zeros of J_o and that the arguments $k_n R_1$ will be near $\left(\kappa_o^2 - \Omega^2\right)^{1/2} R_1$, where κ_o is given by equation (C1). In addition, for higher order solutions the arguments will approach these limits more closely. This is demonstrated in the expansion of the Bessel function, carried out in appendixes D and E.

In order to verify that $\left|\mathscr{K}_n\right|$ is a monotonically decreasing function, approximate expressions for C_n, A_{nm}, and A_{nn} must be derived. It follows from equation (D7) that for $t_n << 1$

$$\nu_n R_1 \frac{J_o(\nu_n R_1)}{J_1(\nu_n R_1)} \approx -j_{on} t_n \tag{B1}$$

and from equation (E4) that

$$t_n \approx \frac{D_1}{n}$$

where $D_1 < j_{ol}$. Because

$$\nu_n R_1 \approx j_{on} \approx (n - 1/4)\pi \tag{B2}$$

(appendix E) equation (B1) may be written as

$$\nu_n R_1 \frac{J_o(\nu_n R_1)}{J_1(\nu_n R_1)} \approx -\left(1 - \frac{1}{4n}\right)\pi D_1 \tag{B3}$$

Because the zeros of J_o occur near the maximum of J_1,

$$J_1(\nu_n R_1) \approx \sqrt{\frac{2}{\pi \nu_n R_1}} \tag{B4}$$

Equations (B2), (B3), and (B4) are the approximations that reduce C_n, A_{nm}, and A_{nn} to algebraic expressions. Substituting these approximations and the condition $n >> 1$ into C_n (eq. (4)) results in

$$C_n \approx \frac{\sqrt{2}\, R_1^2 D_1 J_1(\nu R_1)}{\pi^2} \frac{\left[\left(n - \frac{1}{4}\right)\frac{1}{n} + F\right]}{\left(n - \frac{1}{4}\right)^{5/2}} \tag{B5}$$

where F is a parameter independent of n and given by

$$F = \frac{\nu R}{\pi D_1} \frac{J_o(\nu R_1)}{J_1(\nu R_1)}$$

Equation (B5) reduces to

$$C_n \approx \frac{\sqrt{2}\, R_1^2 D_1 J_1(\nu R_1)}{\pi^2} \frac{(1 + F)}{\left(n - \frac{1}{4}\right)^{5/2}} \propto \frac{1}{n^{5/2}} \tag{B6}$$

Similarly, A_{nn} for $n >> 1$ becomes

$$A_{nn} \approx \frac{R_1^2}{\pi^2 n} \propto \frac{1}{n}$$

Consequently,

$$b_n = \frac{C_n}{A_{nn}} \propto \frac{1}{n^{3/2}}$$

The term b_n consequently has for $n >> 1$ an absolute value that is monotonically decreasing. Finally, A_{nm} for large argument may be written as

$$A_{nm} \propto \frac{1}{(m + n)\left(m - \frac{1}{4}\right)^{1/2} mn^{3/2}}$$

The expression for $|C_{nm}|$ is, consequently,

$$|C_{nm}| \propto \frac{1}{(m + n)\left(m - \frac{1}{4}\right)^{1/2} mn^{1/2}}$$

which is a monotonically decreasing function of n. Because each term of the series

$$\sum_{m=1}^{\infty} |C_{nm}|$$

is a decreasing function of n, the sum over m also is a monotonically decreasing function of n. Therefore, $|\mathscr{K}_n|$ monotonically decreases for large n. To determine the variation of $\mathscr{K}_n$ for small n, numerical calculations were made of $|\mathscr{K}_n|$ for all n less than 100, where the sum in $\mathscr{K}_n$ included 300 terms. In the range $1 < n < 100$, $|\mathscr{K}_n|$ was also continuously decreasing. Consequently, the maximum value of this set occurs at a condition of $n = 1$.

There remains the verification that

$$\sum_{m=1}^{\infty} |C_{nm}| < 1$$

where the inequality must hold for all n. Expressions for A_{nn} and A_{nm} must be written that are valid in the region of small n. These are

$$A_{nn} \approx \frac{R_1^2 J_1^2(j_{on})}{2}\left(1 - \frac{4t_n^2}{j_{on}^2}\right)$$

$$\approx \frac{R_1^2}{\pi^2\left(n - \frac{1}{4}\right)}\left[1 - \frac{4t_n^2}{\left(n - \frac{1}{4}\right)^2 \pi^2}\right] \tag{B7}$$

and

$$A_{nm} \approx \frac{R_1^2}{2\pi^2 nm\left(n - \frac{1}{4}\right)^{1/2}\left(m - \frac{1}{4}\right)^{1/2}\left(m + n - \frac{1}{2}\right)} \tag{B8}$$

From equations (B7) and (B8) the sum may be written as

$$\sum_{m=1}^{\infty} |C_{nm}| \approx \frac{\left(n - \frac{1}{4}\right)^{1/2}}{2n\left[1 - \frac{4t_n^2}{\left(n - \frac{1}{4}\right)^2 \pi^2}\right]} \sum_{m=1}^{\infty} \frac{1}{m\left(m - \frac{1}{4}\right)^{1/2}\left(m + n - \frac{1}{2}\right)} \tag{B9}$$

Because $n \geq 1$, equation (B9) reduces tc

$$\sum_{m=1}^{\infty} \left|C_{nm}\right| < \frac{1}{2n^{1/2}\left[1 - \frac{4t_n^2}{\left(n - \frac{1}{4}\right)^2 \pi^2}\right]} \sum_{m=1}^{\infty} \frac{1}{\left(m - \frac{1}{4}\right)^{1/2} m^2}$$

In order to estimate the magnitude of $\sum_{m=1}^{\infty} \left|C_{nm}\right|$, t_n will have to be evaluated. The most adverse value of t_n occurs when $n = 1$. Because t_1 never exceeds 0.2 for the cases calculated, the maximum value of $\left(1 - \frac{4^3 t_1^2}{3^2 \pi^2}\right)^{-1}$ will always be less than 1.03, and consequently,

$$\sum_{m=1}^{\infty} \left|C_{nm}\right| < \frac{1.03}{2n^{1/2}} \left(0.156 + \sum_{m=1}^{\infty} \frac{1}{m^2}\right)$$

However, from reference 8

$$\sum_{m=1}^{\infty} \frac{1}{m^2} = 1.644$$

Therefore,

$$\sum_{m=1}^{\infty} \left|C_{nm}\right| < \frac{0.93}{n^{1/2}}$$

which is less than 1.

APPENDIX C

CHARACTERISTICS OF NATURAL MODES

The natural modes are determined by the simultaneous solution of the dispersion relation (ref. 6)

$$\nu^2 = \frac{\Omega^4\alpha^2 - \Omega^2\left[2\alpha + (\kappa^2 - \Omega^2)\right](\kappa^2 - \Omega^2) + (\kappa^2 - \Omega^2)^2}{\Omega^2\left[\alpha + (\kappa^2 - \Omega^2)\right] - (\kappa^2 - \Omega^2)} \tag{22}$$

and the boundary condition

$$\nu R_1 \frac{J_o(\nu R_1)}{J_1(\nu R_1)} = -(\kappa^2 - \Omega^2)^{1/2} R_1 \frac{K_o\left[(\kappa^2 - \Omega^2)^{1/2} R_1\right]}{K_1\left[(\kappa^2 - \Omega^2)^{1/2} R_1\right]} \tag{23}$$

These two equations are presented in figure 3 (fig. 2 of ref. 2) as a log log plot.

Because ν^2 obtained from equation (22) is a monotonically decreasing function with respect to κ^2 and because the regions of possible solutions of ν_n (fig. 1) are located at larger values of ν as n increases, κ_n will decrease and approach the finite limit of

$$\kappa_o^2 = \frac{\Omega^2\alpha}{1 - \Omega^2} + \Omega^2 \tag{C1}$$

The right side of equation (23) consequently approaches a constant as ν_n increases. Because ν_n increases without limit and because the amplitude of J_1 is continuously decreasing, $\nu_n R_1$ must approach the zeros of J_o in order for equation (23) to be satisfied. These characteristics of the natural modes (ν_n and κ_n) suggest an expansion of J_o, J_1, and ν_n about the zeros of J_o and an expansion of κ_n about the value of κ_o given by equation (C1).

APPENDIX D

EXPANSION OF J_0 AND J_1 ABOUT ZEROS OF J_0

It is convenient (see appendix C) to expand $J_o(\nu_n R_1)$ and $J_1(\nu_n R_1)$ about j_{on}, which is the nth zero of the zero-order unmodified Bessel function of the first kind. Hence, because

$$J_o(\nu_n R_1) = J_o(j_{on} + t_n)$$

$$J_1(\nu_n R_1) = J_1(j_{on} + t_n)$$

and $|t_n| < |j_{on}|$, the following addition formula (ref. 9, p. 143) may be utilized:

$$J_o(j_{on} + t_n) = \sum_{m=-\infty}^{\infty} J_{-m}(j_{on})J_m(t_n)$$

$$= J_o(t_n)J_o(j_{on}) + 2\sum_{m=1}^{\infty} (-1)^m J_m(t_n)J_m(j_{on}) \tag{D1}$$

Now, because $J_o(j_{on}) = 0$, write from the recurrence relations (ref. 9, p. 45) that

$$\left.\begin{aligned}
J_2(j_{on}) &= \frac{2}{j_{on}} J_1(j_{on}) \\
J_3(j_{on}) &= \left(\frac{2^3}{j_{on}^2} - 1\right) J_1(j_{on}) \\
J_4(j_{on}) &= \left(\frac{2^4 \cdot 3}{j_{on}^3} - \frac{2^2}{j_{on}}\right) J_1(j_{on}) \\
J_5(j_{on}) &= \left(\frac{2^7 \cdot 3}{j_{on}^4} - \frac{2^3 \cdot 5}{j_{on}^2} + 1\right) J_1(j_{on})
\end{aligned}\right\} \tag{D2}$$

In addition, for t small

$$J_m(t_n) = \frac{t_n^m}{2^m m!} \tag{D3}$$

Substituting (D2) and (D3) into (D1) and dropping all the terms of the third power or higher of t_n result in

$$J_o(\nu_n R_1) \approx \left(-t_n + \frac{t_n^2}{2j_{on}}\right) J_1(j_{on}) \tag{D4}$$

Because the series appearing in equation (D1) is convergent, the series from which equation (D4) was obtained is also convergent. The addition formula for $J_1(j_{on} + t_n)$ may be written as

$$J_1(j_{on} + t_n) = \sum_{m=-\infty}^{\infty} J_{1-m}(j_{on}) J_m(t_n)$$

Rearranging the terms results in

$$J_1(j_{on} + t_n) = \sum_{m=1}^{\infty} (-1)^{m-1} \left[J_{m-1}(t_n) - J_{m+1}(t_n)\right] J_m(j_{on}) \tag{D5}$$

Substituting equations (D2) and (D3) into equation (D5) results in

$$J_1(\nu_n R_1) \approx \left(1 - \frac{t_n}{j_{on}} + \frac{t_n^2}{j_{on}^2} - \frac{t_n^2}{2^2}\right) J_1(j_{on}) \tag{D6}$$

Because the series appearing in equation (D5) is convergent, the series from which equation (D6) was obtained is also convergent. Consequently, the expression $\nu_n R_1 \left[J_o(\nu_n R_1)/J_1(\nu_n R_1)\right]$ that appears in the expression for A_{nm} (eq. (5)) may be written as

$$\nu_n R_1 \frac{J_o(\nu_n R_1)}{J_1(\nu_n R_1)} = \frac{-4j_{on}^2 t_n \left(j_{on} + \frac{t_n}{2}\right)}{4j_{on}^2 - 4j_{on}t_n + 4t_n^2 - j_{on}^2 t_n^2} \tag{D7}$$

APPENDIX E

EVALUATION OF t_n

The term t_n may be found by first substituting the expansion formula of appendix D into the two expressions used to determine ν_n and κ_n (eqs. (22) and (23)). In addition, similar expansion formulas for the κ_n variable are required before equation (23) reduces to a purely algebraic expression. Once this is done, the algebriac equations can be solved for t_n.

A characteristic of the solutions of κ_n that is useful to evaluate higher modes is the property that κ_n is a very slowly decreasing function which has a finite lower limit. Because κ_n lies close to a fixed value, it is possible to expand the right side of equation (23) by Taylor's expansion. If the expansion is carried out around

$$\kappa_o^2 = \frac{\Omega^2\alpha}{1 - \Omega^2} + \Omega^2 \tag{E1}$$

(see appendix C), the result is

$$\frac{-4j_{on}^2 t_n\left(j_{on} + \frac{t_n}{2}\right)}{4j_{on}^2 - 4j_{on}t_n + 4t_n^2 - j_{on}^2 t_n} = -\left\{B + \left[2B - (\kappa_o^2 - \Omega^2)R_1^2 + B^2\right]\frac{\kappa_o\,\Delta_n}{\kappa_o^2 - \Omega^2}\right.$$

$$\left. + \left[-(3 + 2B)(\kappa_o^2 - \Omega^2)R_1^2 + B(2 + 5B + 2B^2)\right]\frac{\kappa_o^2\,\Delta_n^2}{(\kappa_o^2 - \Omega^2)^2}\right\} \tag{E2}$$

where

$$B = (\kappa_o^2 - \Omega^2)^{1/2} R_1 \frac{K_0\left[(\kappa_o^2 - \Omega^2)^{1/2} R_1\right]}{K_1\left[(\kappa_o^2 - \Omega^2)^{1/2} R_1\right]}$$

and $\Delta_n = \kappa_n - \kappa_o$. Substituting $\nu_n R_1 = j_{on} + t_n$ and $\kappa_n = \kappa_o + \Delta_n$ into equation (22) gives

$$\frac{(j_{on} + t_n)^2}{R_1^2} = \frac{\Omega^6 \alpha^2}{(1 - \Omega^2)^2 (2\kappa_o \Delta_n + \Delta_n^2)} - (2\kappa_o \Delta_n + \Delta_n^2) \tag{E3}$$

The nth positive zero of $J_o(j_{on})$ can readily be determined from the relation (ref. 9, p. 505)

$$j_{on} \simeq \left(n - \frac{1}{4}\right)\pi + \frac{1}{8\left(n - \frac{1}{4}\right)\pi} - \frac{31}{384\left[\left(n - \frac{1}{4}\right)\pi\right]^3} + \frac{3779}{15360} \frac{1}{\left[\left(n - \frac{1}{4}\right)\pi\right]^5} - \frac{6277237}{3440640} \frac{1}{\left[\left(n - \frac{1}{4}\right)\pi\right]^7} + \ldots$$

Equations (E2) and (E3) are two nonlinear algebraic equations in the two unknowns Δ_n and t_n and consequently can be solved.

A somewhat simpler expression for t_n will suffice if the first few natural modes are known. For these higher modes, an empirical relation of the form

$$t_n = \sum_{m=1}^{p} \frac{D_m}{n^m} \tag{E4}$$

can be written. The first p values of t_n are obtained from

$$\nu_n R_1 = j_{on} + t_n$$

where the ν_n's were obtained from reference 2. Using these values of t_n in equation (E4) yields p simultaneous equations for the D_m's. The resulting equation for t_n can then be used to calculate t_n for n's > p. These first few modes were obtained in reference 2 by solving equations (22) and (23) directly by a modified Newton-Raphson technique. However, this method (ref. 2) breaks down when used for the higher modes. The results from the empirical relation (E4) and from equations (E2) and (E3) indicate that both approaches were compatible with the original equations, that is, equations (22) and (23).

REFERENCES

1. Stix, Thomas H.: Generation and Thermalization of Plasma Waves. Phys. Fluids, vol. 1, no. 4, July-Aug. 1958, pp. 308-317.
2. Woollett, Richard R.: Energy Addition to an Atomic Hydrogen Plasma at Off-Resonant Conditions. NASA TN D-2982, 1965.
3. Morse, Philip M.; and Feshbach, Herman: Methods of Theoretical Physics. McGraw-Hill Book Co., Inc., 1953.
4. Dicke, Robert H.; and Wittke, James P.: Introduction to Quantum Mechanics. Addison-Wesley Pub. Co., Inc., 1960.
5. Kantorovich, L. V.; and Krylov, V. I.: Approximate Methods of Higher Analysis. Third ed., John Wiley and Sons, Inc., 1964.
6. Stix, Thomas H.: Oscillations of a Cylindrical Plasma. Phys. Rev., vol. 106, no. 6, June 15, 1957, pp. 1146-1150.
7. Wait, James R.: The Mode Theory of VLF Ionospheric Propagation for Finite Ground Conductivity. Proc. IRE, vol. 45, no. 6, June 1957, pp. 760-767.
8. Jolley, L. B. W.: Summation of Series. Dover Publications, 1961, p. 240.
9. Watson, G. N.: A Treatise on the Theory of Bessel Functions. Cambridge University Press, 1962.

TABLE I. - DEFICIENT, EXCESSIVE, AND APPROXIMATE COEFFICIENTS

Ion density, ions/cm^3	Magnetic field, G	Mode, n	Deficient coefficient, $\tilde{a}_n$	Excessive coefficient, $\bar{a}_n$	Orthogonal coefficient, b_n
10^{12}	4.2015×10^{3}	1	1.00064	1.00071	1.00068
		2	3.44680×10^{-4}	3.81019×10^{-4}	-1.73485×10^{-3}
		3	-4.88554	-4.58876	6.31477×10^{-4}
	4.24585×10^{3}	1	5.90566×10^{-1}	5.90595×10^{-1}	5.90722×10^{-1}
		2	-1.75595	-1.75580	-1.76466
		3	9.00958×10^{-2}	9.01084×10^{-2}	9.05804×10^{-2}
	4.26×10^{3}	1	4.91661×10^{-1}	4.91683×10^{-1}	4.91782×10^{-1}
		2	-1.51788	-1.51771	-1.52439
		3	7.82748×10^{-2}	7.82843×10^{-2}	7.86396×10^{-2}
	4.31396×10^{3}	1	1.75898×10^{-1}	1.75904×10^{-1}	1.75931×10^{-1}
		2	-5.71984×10^{-2}	-5.71955×10^{-2}	-5.73651×10^{-2}
		3	2.97105	2.97130	2.98030
10^{14}	4.61766×10^{3}	1	1.66201×10^{-1}	1.66203×10^{-1}	1.66208×10^{-1}
		2	-5.59803×10^{-2}	-5.59779×10^{-2}	-5.60098×10^{-2}
		3	2.92708	2.92737	2.92893

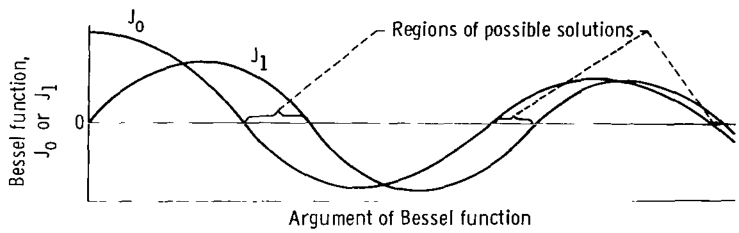

Figure 1. - Schematic of Bessel functions, J_0 and J_1.

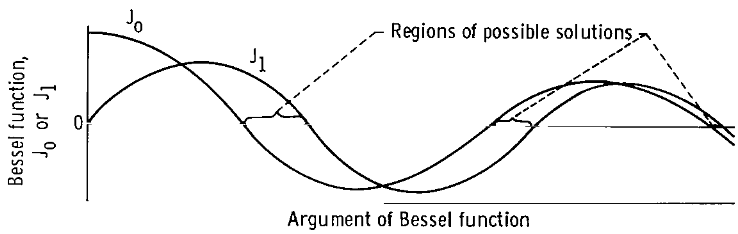

Figure 1. - Schematic of Bessel functions, J_0 and J_1.

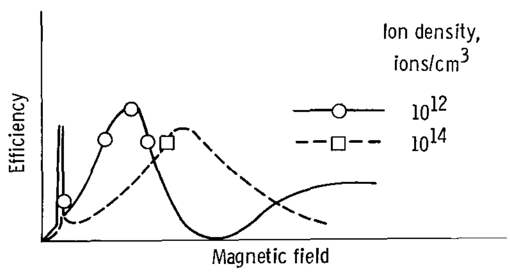

Figure 2. - Data points of reference 2 selected for comparison.

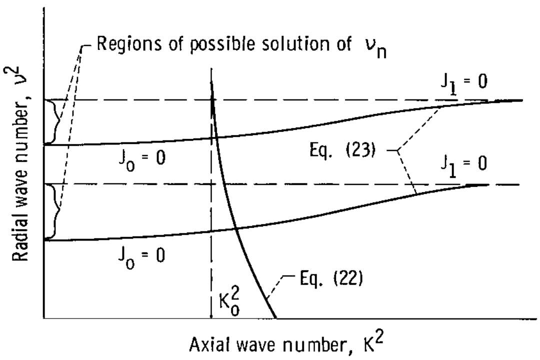

Figure 3. - Graphic solution for natural modes of plasma oscillation.

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