Taub-NUT Space as a Counterexample to Almost Anything

by

C. W. Misner

TECHNICAL REPORT NO. 529

November 1965

UNIVERSITY OF MARYLAND
DEPARTMENT OF PHYSICS AND ASTRONOMY
COLLEGE PARK, MARYLAND

A contributed lecture given at the American Mathematical Society's 1965 Summer Seminar on Relativity and Astrophysics at Cornell University, August 1965.

Taub-NUT Space as a Counterexample to Almost Anything*

by

C.W. Misner
University of Maryland
Department of Physics and Astronomy
College Park, Maryland

* Supported in part by NASA Grant NsG 436

Taub-NUT Space as a Counterexample to Almost Anything

C.W. Misner

This lecture will discuss some of the peculiar properties of the metric

$$ds^2 = (t^2 + \ell^2)(d\theta^2 + \sin^2\theta d\phi^2)$$

$$+ U(t)(2\ell)^2(d\psi + \cos\theta d\phi)^2 \qquad (1)$$

$$+ 2(2\ell)(d\psi + \cos\theta d\phi)dt$$

where

$$U(t) = -1 + 2\,\frac{mt + \ell^2}{t^2 + \ell^2} \qquad (2)$$

This metric satisfied the empty-space Einstein equations

$$R_{\mu\nu} = 0 \qquad (3)$$

and has been discovered by both of the prime exact-solution-finding methods mentioned by Kerr in his lecture this morning. Taub (1951) discovered it in a systematic development of a class of metrics with high symmetry. Later it was rediscovered by Newman, Unti, and Tamburino (1963) studying a class of algebraically special metrics. Actually, Taub gave the metric in a coordinate system covering only the region where $U(t) > 0$ ("Taub space") in which the $t = \text{const}$ hypersurfaces are space-like, while Newman, Unti, and Tamburino gave the region where $U(t) < 0$ ("NUT space") in which the ψ-lines ($t\theta\phi$ constant) are time-like.

This Taub-NUT space has many unusual properties, some of which are also known in other metrics. I will give a short list and then discuss a few:

1. Although the Taub region $U > 0$ can be interpreted as a cosmological solution with homogeneous but non-isotropic space sections, it evolves into NUT space which seems to have no reasonable interpretation.

2. The NUT region contains closed time-like lines.

3. The NUT region does not contain any decent space-like hypersurfaces.

4. Although the curvature tensor vanishes as one approaches infinity in space-like directions, asymptotically rectangular coordinates ($g_{\mu\nu} \to \eta_{\mu\nu}$) do not exist.

5. Taub space allows, besides Eq. (1) another, inequivalent, maximal analytic extension.

6. Taub-NUT space is non-singular in a meaningful mathematical sense

but is not geodesically complete.

7. There are closed geodesics (circles) on which one cannot extend the solution of the geodesic equation to infinite values of the path parameter.

The first four points are discussed in my (1963) paper on NUT space and will not be considered further here. Points 6. and 7. are based on a study of the geodesics which Taub and I (1966) hope to finish writing soon. An example of inextendable closed geodesics (7) for a simpler metric is given in the paragraph containing Eq. (1) in my 1963 paper. The question of the non-uniqueness of analytic continuation for metrics (point 5) is also based on the paper by Taub and Misner (1966) and will occupy us for the remainder of this talk.

Before discussing analytic continuations we must discuss analyticity. A function of several real variables $f(xy \dots z)$ is analytic at $x_o y_o \dots z_o$ if it has a power series expansion about that point with a non-zero radius of convergence. A function $f(P)$ on a manifold M is analytic if it can represent as an analytic function $f(x^1, x^2, \dots x^4)$ of the coordinates in the various coordinate patches defining the analytic structure of the manifold (see Auslander and MacKenzie 1963 for the definition of manifolds and of differentiable structure). Tensors are analytic if their components are analytic functions of the coordinates in each coordinate patch. A symmetric tensor $g_{\mu\nu}$ is a metric only if it has the proper signature (-+++); in particular $g = \det g_{\mu\nu}$ cannot vanish. The metric of Eq. (1) fails to satisfy the signature requirement of $\theta = 0,\pi$ since in these $t\psi\theta\phi$ coordinates one has

$$(-g)^{\frac{1}{2}} = 2\ell(t^2 + \ell^2)\sin\theta \tag{4}$$

We will therefore try to interpret Eq. (1) as defining the metric on a coordinate patch where $t\psi\theta\phi$ have the ranges

$$\begin{array}{ll} -\infty < t < +\infty & 0 < \theta < \pi \\ 0 < \psi < 4\pi & 0 < \phi < 2\pi \end{array} \tag{5}$$

Analyticity of the metric in this range is obvious, as is the choice of limits on t and θ. It remains to be seen whether there exists any larger manifold containing this coordinate patch on which the metric can be analytic. We will study some simpler examples before returning to this question.

As a very simple example consider the metric

$$ds^2 = dt^2 + d\phi^2 \tag{6}$$

on a coordinate patch $-\infty < t < \infty$, $0 < \phi < 2\pi$ (Fig. 1). We are accustomed to interpret this metric as a cylinder $R \times S'$. Here R means the real line $-\infty < t < \infty$, and S' is the 1-sphere or circle (base of the cylinder). I shall not attempt to explain how one looks at a metric on a single coordinate patch, such as Eq. (1) or Eq. (6), and tries to guess what the full manifold containing this patch should be. But we shall now consider how to verify a guess. I guess that the metric

$$ds^2 = \frac{dx^2 + dy^2}{x^2 + y^2} \tag{7}$$

which is clearly analytic in the region $0 < x^2 + y^2 < +\infty$ might be related to the metric of Eq. (6). The coordinate transformation

$$\begin{aligned} x &= e^t \cos\theta \\ y &= e^t \sin\theta \end{aligned} \tag{8}$$

shows indeed that these metrics are equivalent on the region $\theta \neq 0$ (which is the largest region to which this correspondence between xy and $t\theta$ coordinates can be extended in a one-to-one way) but Eq. (7) is to be preferred since it is analytic also on this positive x-axis of the xy plane. One may begin to worry about the "singularity" of $x^2 + y^2 \to 0$ in Eq. (7); since it corresponds to $t \to -\infty$ in Eq. (6) it clearly is <u>not</u> a singularity. A coordinate independent way of stating this is to note that every geodesic which approaches the boundary of the manifold ($x^2 + y^2 \to 0$ or ∞) has infinite length.

Some slightly more complicated metrics to serve as further examples are

$$ds_I^2 = dt^2 + d\theta^2 + \sin^2\theta d\phi^2 \tag{9}$$

$$ds_{II}^2 = \frac{1}{3}(2 + \cos\theta)dt^2 + d\theta^2 + \sin^2\theta d\phi^2 \tag{10}$$

$$ds_{III}^2 = dt^2 + e^{2t}d\theta^2 + \sin^2\theta d\phi^2 \tag{11}$$

The first metric here obviously represents a higher dimensional cylinder $R \times S^2$ which is just a linear (R) stack of ordinary spheres (S^2), while the other two metrics represent some deformations of this first hyper-cylinder. What is not so obvious to the unpractised eye is that while the first two metrics are analytic, the third is not even continuous over the whole cylinder $R \times S^2$. Analyticity of the first metric (by which we now mean the existence of an analytic extension of it to cover all of $R \times S^2$) is demonstrated by writing

$$ds_I^2 = \frac{1}{r^2}(dx^2 + dy^2 + dz^2) \tag{12}$$

with

$$r^2 \equiv x^2 + y^2 + z^2 \tag{13}$$

which is obviously analytic for $0 < r^2 < +\infty$, and which corresponds to Eq. (9) by the transformation

$$\begin{aligned} x &= e^t \sin\theta\cos\phi \\ y &= e^t \sin\theta\sin\phi \\ z &= e^t \cos\theta \end{aligned} \tag{14}$$

Use the same transformation on ds_{II}^2 and write

$$ds_{II}^{2} = ds_{I}^{2} + \frac{1}{3}(\cos\theta - 1)dt^{2} \tag{15}$$

Then since $\cos\theta = z/r$ is analytic for $r^2 > 0$ and

$$dt = \frac{1}{r^2}(xdy + ydy + zdz)$$

is an analytic differential form in this region, the xyz components of ds_{II}^{2} will also be analytic functions. The condition of proper signature is verified from

$$\begin{aligned} g_{II} &= \frac{1}{3}(2 + \cos\theta)\sin^2\theta\left[\frac{\partial(t\theta\phi)}{\partial(xyz)}\right]^2 \\ &= (2r + z)/3r^7 > 0 \end{aligned} \tag{16}$$

But studying ds_{III}^{2} in xyz coordinates leads to no interesting extension for we have

$$ds_{III}^{2} = ds_{I}^{2} + (e^{2t} - 1)d\theta^{2} \tag{17}$$

and from the transformation (14) one sees that

$$d\theta = \frac{z}{r^2}\frac{xdx + ydy}{(x^2 + y^2)^{\frac{1}{2}}} - \frac{(x^2 + y^2)^{\frac{1}{2}}}{r^2}dz \tag{18}$$

is <u>not</u> an analytic differential form on the z-axis where $x/(x^2 + y^2)^{\frac{1}{2}}$ for instance is not a continuous function. We could then ask whether some coordinate transformation different from Eq. (14) might not lead to a form of $ds_{III}^{\,2}$ which allowed some analytic extension beyond the region to $-\infty < t < +\infty$, $0 < \theta < \pi$, $0 < \phi < 2\pi$ where Eq. (11) deomonstrates analyticity. But a computation of the curvature tensor (see Appendix A of my 1963 paper for rapid computational techniques which here gave $R_{\mu\nu\alpha\beta}$ starting from Eq. (11) in twelve minutes work) shows that

$$R_{ij}R^{ij} - \tfrac{1}{2}R^2 = 2(e^t \sin\theta)^{-2} \tag{19}$$

The lines of infinite curvature at $\theta = 0$ and $\theta = \pi$ are therefore natural boundaries to this space; since the metric puts these lines arbitrarily close to regular points of the space we say that this space is intrinsically singular.

An analytic extension of the line element of Eq. (1) is obtained by the coordinate transformation

$$\begin{aligned} w &= e^{\frac{1}{2}t}\cos\tfrac{1}{2}\theta\cos\tfrac{1}{2}(\phi + \psi) \\ x &= e^{\frac{1}{2}t}\sin\tfrac{1}{2}\theta\cos\tfrac{1}{2}(\phi - \psi) \\ y &= e^{\frac{1}{2}t}\sin\tfrac{1}{2}\theta\sin\tfrac{1}{2}(\phi - \psi) \\ z &= e^{\frac{1}{2}t}\cos\tfrac{1}{2}\theta\sin\tfrac{1}{2}(\phi + \psi) \end{aligned} \tag{20a}$$

This transformation can also be written

$$q = e^{\frac{1}{2}t}e^{\frac{1}{2}k\phi}e^{\frac{1}{2}i\theta}e^{\frac{1}{2}k\varphi} \tag{20b}$$

where q is the quaternion

$$q = w + ix + jy + kz$$

and ijk are the imaginary quaternion units which obey $k^2 = -1$, $ij = -ji = k$, etc. This transformation shows that $\psi\theta\phi$ are the "Euler angle" coordinates on S^3 which are familiar in discussions of the rotation group SO(3). (See, for instance, Corben and Stehle 1960, Appendix IV.)

In order to write the Taub-NUT metric in $wxyz$ coordinates it is convenient first to define several differential forms

$$\begin{aligned}
\sigma_x &= 2|q|^{-2}(x dw - w dx - z dy + y dz)\\
\sigma_y &= 2|q|^{-2}(y dw + z dx - w dy - x dz)\\
\sigma_z &= 2|q|^{-2}(z dw - y dx + x dy - w dz)\\
dt &= 2|q|^{-2}(w dw + x dx + y dy + z dz)
\end{aligned} \tag{21}$$

where

$$|q|^2 \equiv w^2 + x^2 + y^2 + z^2 = e^t \tag{22}$$

These differential forms are obviously analytic on the region $0 < |q|^2 < \infty$, and from the following equation we see that the components of the transformed metric are also

$$ds^2 = (t^2 + \ell^2)(\sigma_x{}^2 + \sigma_y{}^2) + U(t)(2\ell)^2\sigma_z{}^2 + 2(2\ell)\sigma_z dt \tag{23}$$

This equation can also be written

$$ds^2 = g_{\mu\nu}\omega^\mu\omega^\nu \tag{24}$$

where

$$\begin{aligned} \omega^0 &= dt \\ \omega^1 &= \sigma_x(t^2 + \ell^2)^{\frac{1}{2}} \\ \omega^2 &= \sigma_y(t^2 + \ell^2)^{\frac{1}{2}} \\ \omega^3 &= \sigma_z(2\ell) \end{aligned} \tag{25}$$

and

$$g_{\mu\nu} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & U & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{26}$$

In this form it is easy to verify the signature requirement on the metric is satisfied, for we need only verify the signature of the $g_{\mu\nu}$ matrix in Eq. (26) and check that the basis differential froms of Eq. (25) are linearly independent. This linear independence is clear when one notes that the coefficients in Eqs. (21) form non-zero orthogonal vectors in the standard flat Euclidean metric on wxyz space. The signature of $g_{\mu\nu}$ in Eq. (26) is obvious when $U(t) = 0$, and cannot change since

$$g = \det g_{\mu\nu} = -1 \neq 0 \tag{27}$$

It can be shown that the metric of Eq. (23) on the domain $0 < |q|^2 < +\infty$ of wxyz space is maximal in the sense that it cannot be identified with a coordinate patch on an even larger connected manifold. Taub and I (1966) have shown this by studying its geodesics and verifying that every geodesic arc which approaches the boundaries ($|q|^2 \to 0$ or $|q|^2 \to \infty$) is infinitely long as measured by any affine path parameter for the geodesic equation. In contrast, geodesics which start from points outside, but on the boundary of, a coordinate patch can outer it with finite, even arbitrarily small, changes in the path parameter.

Let us now turn to another question of analytic continuation, its uniqueness. We may and do now choose to consider Eq. (23) on the region $0 < |q|^2 < \infty$ as giving one maximal analytic extension of Taub space, which is the region

$$e^{t_1} < |q|^2 < e^{t_2} \tag{28}$$

where t_1 and t_2 are the two zeros of $U(t)$. Since analytic continuation of functions on the real line is a unique process, we might expect this also to be true for analytic metric manifolds. To exhibit non-uniqueness we first introduce a new coordinate system on Taub space by the transformation

$$\psi = \Psi - \int_m^t (\ell U)^{-1} dt \tag{29}$$

in which we retain the old $t\theta\phi$ coordinates. The metric then becomes

$$\begin{aligned} ds^2 &= (t^2 + \ell^2)(d\theta^2 + \sin^2\theta d\phi^2) \\ &\quad + U(t)(2\ell)^2(d\psi + \cos\theta d\phi)^2 \\ &\quad - 2(2\ell)(d\psi + \cos\theta d\phi)dt \end{aligned} \tag{30}$$

which differs from Eq. (1) only by one minus sign and the capitalization of Ψ. Although the transformation (29) is only regular in the region $t_1 < t < t_2$, limited by logarithmic singularities at the zeros of $U(t)$, the metric of Eq. (30) can clearly be continued to all values of t and its analyticity verified by a transformation analogous to Eq. (20)

$$Q = W + iX + jY + kZ = e^{\frac{1}{2}t}e^{\frac{1}{2}k\phi}e^{\frac{1}{2}i\theta}e^{\frac{1}{2}k\Psi} \tag{31}$$

Rather than continue with the details of this example, we consider a simpler case based on the 2-dimensional metric

$$ds^2 = 2d\psi dt + td\psi^2 \tag{32}$$

which can be interpreted (by assigning ψ a period of, say, 2π) as a metric of signature (-,+) on the cylinder $S^1 \times R$. The most significant difference between this example and Taub-NUT space is that $S^1 \times R$ is _not_ simply connected and can be covered by the plane $R \times R$ (assign ψ the range $-\infty$ to $+\infty$) while $S^3 \times R$ _is_ simply connected and no range for ψ larger than 0 to 4π is possible there. An inequivalent analytic extension of the region $t > 0$ in the metric (32) is obtained by first making the coordinate transformation

$$\psi = \Psi - 2\ell nt \tag{33}$$

on the $t > 0$ region. A simple computation using

$$d\psi = d\Psi - 2t^{-1}dt \tag{34}$$

in Eq. (32) gives

$$ds^2 = -2d\Psi dt + td\Psi^2 \tag{35}$$

The coordinate transformation (33) shows the equivalence of the metrics (32) and (35) on the regions $t > 0$. To show their inequivalence when each is considered to define a metric manifold on which t varies from $-\infty$ to $+\infty$ we look at the curve defined on the ψt cylinder by

$$\begin{aligned} \psi &= 0 \\ t &= -\lambda \end{aligned} \tag{36}$$

with $-\infty < \lambda < +\infty$. Since (λ) and $t(\lambda)$ are each analytic functions of λ here this is an analytic curve. Its image on the $t > 0$ part of the Ψt cylinder is also an analytic curve

$$\begin{aligned} \Psi &= 2\ln(-\lambda) \\ t &= -\lambda \end{aligned} \tag{37}$$

where $\lambda < 0$. However the tangent vector to this curve is

$$\begin{aligned} v^{\Psi} &= \frac{d\Psi}{d\lambda} = \frac{2}{\lambda} \\ v^{t} &= dt/d\lambda = -1 \end{aligned} \tag{38}$$

and becomes infinite (i.e. $v^{\Psi} \to -\infty$) as $\lambda \to 0$. Since the analytic structure of the Ψt cylinder is defined by taking $d\Psi$ to be an analytic differential, an analytic extension of this curve segment $(-\infty < \lambda < 0)$ would have to make $d\Psi/d\lambda$ an analytic function at $\lambda = 0$, which is impossible. Thus this curve cannot be extended in the Ψt manifold, while it can be in the ψt manifold, demonstrating the inequivalence of these two manifolds.

The geodesics for the metric (32) are easily obtained. One finds that any geodesic segment from a finite point $(\psi_o t_o)$ to a boundary $(t = \pm\infty)$ is infinitely long in the sense of the affine path parameter, so this space is maximal. However not all geodesics can be extended to parameter values both $\lambda \to +\infty$ and $\lambda \to -\infty$. For instance

$$\begin{aligned} t &= 0 \\ \psi &= -2\ln\lambda \end{aligned} \tag{39}$$

is a (null) geodesic, where λ has a maximum range $0 < \lambda < \infty$. Consequently this space is not complete.

A further curious property of the example of Eq. (32) which was discovered in a discussion with Bonner is that it is flat. This one may verify by starting from the flat metric

$$ds^2 = d\xi^2 - d\eta^2 = d(\xi + \eta)d(\xi - \eta) \tag{40}$$

and making the transformation

$$\begin{aligned} \xi + \eta &= 2te^{\frac{1}{2}\psi} \\ \xi - \eta &= -2te^{-\frac{1}{2}\psi} \end{aligned} \tag{41}$$

under which the half plane $\eta > \xi$ covers the $t\psi$ cylinder infinitely many times.

REFERENCES

L. Auslander and R.E. MacKenzie (1963)
Introduction to Differentiable Manifolds, (McGraw-Hill, New York)

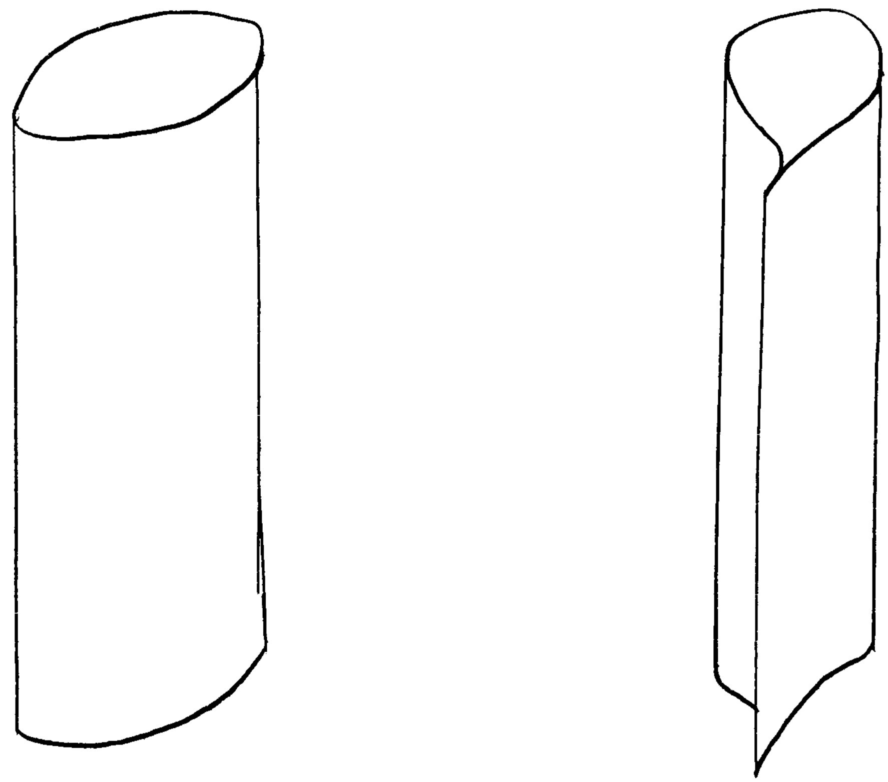
H.C. Corben and P. Stehle (1960)
Classical Mechanics (2nd edition) (Wiley & Sons, New York)

C.W. Misner (1963)
J. Math. Phys. 4, 924 - 937

E. Newman, L. Tamburino and T. Unti (1963)
J. Math. Phys. 4, 915 - 923

A. Taub (1951)
Ann. Math. 53, 472

A. Taub and C.W. Misner (1966)
(to be submitted to JETP)



(a) (b)

<u>Fig. 1</u> Both the cylinders shown here are flat and can be represented away from the seam ($\phi = 0$ or 2π) by the metric of Eq. (6). Only (a) is a smooth (analytic) manifold, while (b) with its sharp edged seam does not inherit a useful class of differentiable functions from the Euclidean 3-space in which it is embedded.