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JUNCTURE STRESS FIELDS IN MULTICELLULAR SHELL STRUCTURES

Final Report

Nine Volumes

Vol. VI Analysis of Multicellular Propellant Pressure Vessels by the Stiffness Method

by

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FOREWORD

This report is the result of a study on the numerical analysis of stresses and deformations of multicellular propellant pressure vessels by the stiffness method. Work on this study was performed by staff members of Lockheed Missiles and Space Company in cooperation with the George C. Marshall Space Flight Center of the National Aeronautics and Space Administration under Contract NAS 8-11480. Contract technical representative was H. Coldwater.

This volume is the sixth of a nine-volume final report of studies conducted by the department of Solid Mechanics, Aerospace Sciences Laboratory, Lockheed Missiles & Space Company. Project Manager was K. J. Forsberg; E. Y. W. Tsui was Technical Director for the work.

The nine volumes of the final report have the following titles:

Vol. I	Numerical Methods of Solving Large Matrices
Vol. II	Stresses and Deformations of Fixed-Edge Segmental Cylindrical Shells
Vol. III	Stresses and Deformations of Fixed-Edge Segmental Conical Shells
Vol. IV	Stresses and Deformations of Fixed-Edge Segmental Spherical Shells
Vol. V	Influence Coefficients of Segmental Shells
Vol. VI	Analysis of Multicellular Propellant Pressure Vessels by the Stiffness Method
Vol. VII	Buckling Analysis of Segmental Orthotropic Cylinders under Uniform Stress Distribution
Vol. VIII	Buckling Analysis of Segmental Orthotropic Cylinders under Non-uniform Stress Distribution
Vol. IX	Summary of Results and Recommendations

SUMMARY

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This volume presents a technique to set up and solve the boundary-value problem of structures composed of shell elements. The problem is formulated from the standpoint of transformations of coordinate systems, the compatibility and equilibrium requirements at the junctures and the solution of a large set of algebraic equations. The technique, known as the direct matrix stiffness or displacement method is introduced to the multicellular shell structure to predict the stresses and deformations due to applied loads and thermal gradients. It is shown that the coupling and ordering of equations plays an important role in forming a desirable overall banded matrix. An example which demonstrates the practicability of the method is given. A digital program designed for the juncture matching has been developed. Although this program is not sophisticated, it can be used for a complete juncture problem.

Author

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NOTATION

D	flexural rigidity of shell = $E\hat{h}^3/12(1 - \nu^2)$
E	modulus of elasticity
F	nondimensionalized force vector
F_i	boundary force at station i
F^f	boundary forces of fixed-edge shell due to applied forces or thermal gradients
G	shear modulus
$\hat{h}$	thickness of shell
i, j	dummy subscripts
k	stiffness matrix
k_{ij}	stiffness influence coefficients
$\overline{M}_{(\ )}, \overline{N}_{(\ )}$	moments and stress resultants
$\overline{Q}_{(\ )}$	transverse shears
u, v, w	displacement components
ζ, η, z	orthogonal coordinates along boundaries of a shell
δ	nondimensionalized displacement vector
δ_i	boundary deformations (displacements or rotations) at Station 1
λ	dimensionalizing matrix for forces $\hat{F} = \lambda F$
μ	dimensionalizing matrix for displacements $\hat{\delta} = \mu\delta$
ν	Poisson's ratio
Φ	transformation matrix of coordinates
$(\hat{\ })$	dimensional quantity

Section 1

INTRODUCTION

As a result of an investigation of juncture stress fields peculiar to the multicellular pressure vessels (Fig. 1), a theory for the prediction of the membrane and bending stresses and the corresponding deformations for such shell structures was formulated.*

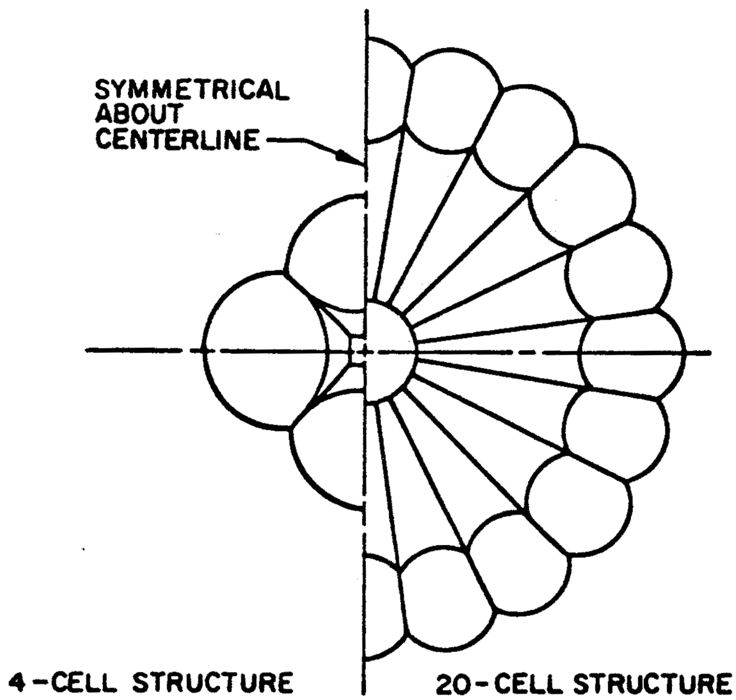

Fig. 1 Multicellular Shell Structure

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* "Investigation of Juncture Stress Fields in Multicellular Shell Structures," by E. Y. W. Tsui, F. A. Brogan, J. M. Massard, P. Stern, and C. E. Stuhlman, Technical Report M-03-63-1, Lockheed Missiles & Space Company, Sunnyvale, Calif., Feb 1964 - NASA CR-61050.

Due to the fact that analytic solutions are still lacking, it was decided to solve the problem numerically by means of a finite-difference technique. To ensure the feasibility of such a numerical solution, a direct method of solving large matrices with a high-speed digital computer was also developed.

According to the previous work, if the stiffness or displacement method is used, the total forces and hence the corresponding stresses along the juncture of the shell segments (Fig. 2) may be expressed concisely in the following matrix form

$$F = k\delta + F^f \tag{1.1}$$

where k is the stiffness matrix, δ are the deformations, and F^f are the fixed-end forces due to applied loads or thermal gradients. In view of this situation, it is logical to solve the problem systematically by the established general procedure of analysis already described. This procedure may be stated briefly as follows:

1. Determination of the fixed-end forces, F^f, along the boundary as well as stresses and deformations in the interior of shell segments due to loads
2. Determination of the influence coefficients, k_{ij}, along the boundaries of shell segments, i.e., the induced forces at points i due to unit deformations ($\delta = 1$) at points j
3. Determination of the actual deformations, δ, along the shell boundaries; this requires the satisfaction of both compatibility and equilibrium conditions at the junctures of the structure

Once all the work involved in these three steps is completed, the total stresses and deformations in the specific discrete interior locations may be obtained.

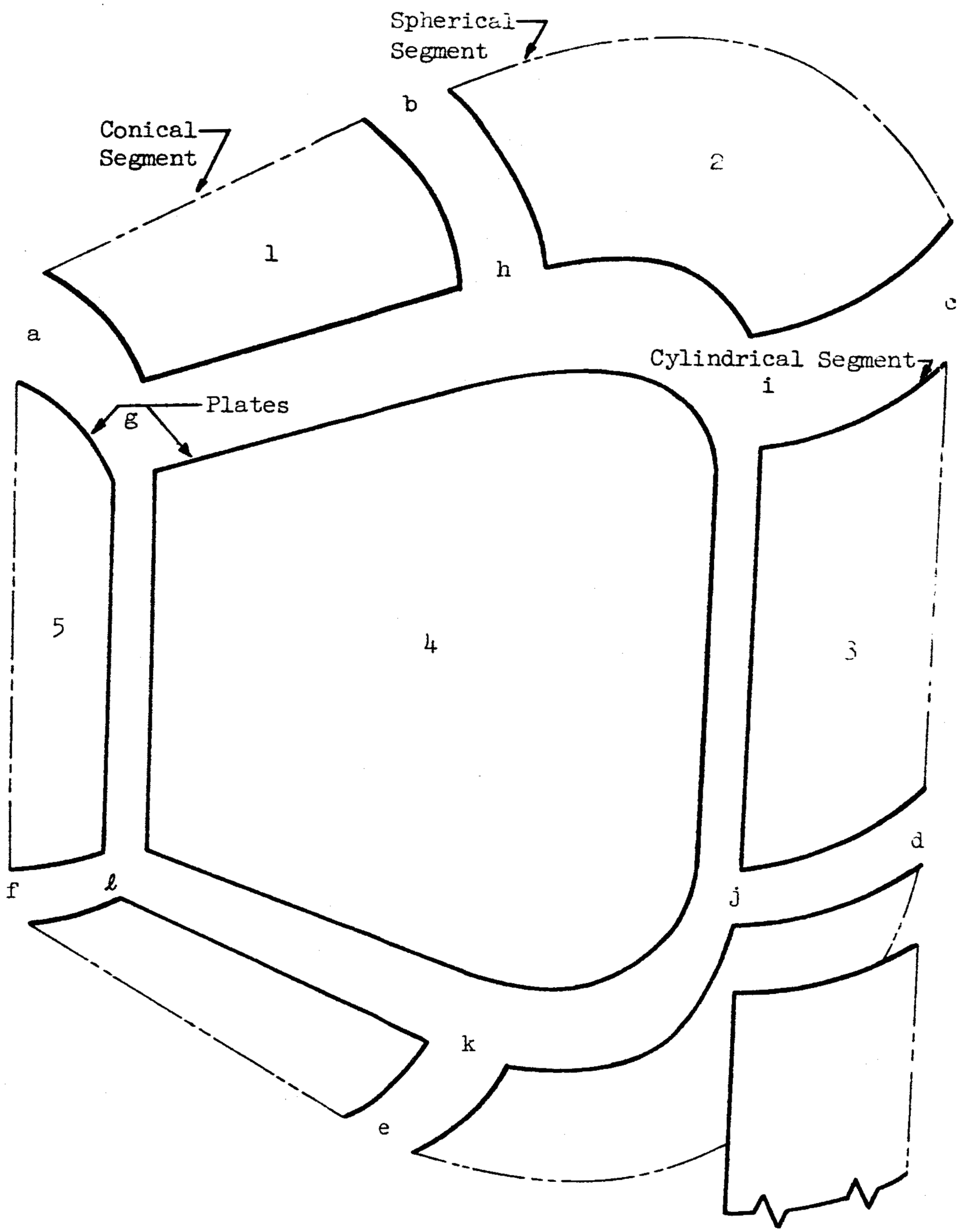


Fig. 2 Basic Shell Elements of Multicellular Structure

This volume presents the work involved in Step 3 and includes a technique to set up an overall stiffness matrix for shell structures. This matrix is based on the stiffness matrices of the individual shell elements and compatibility and equilibrium requirements along their juncture lines. The stiffness matrix method is applied to a multicellular shell structure and a simple numerical example is given to demonstrate the technique. Both overall relaxation and successive approximation methods of determining the actual deformations along the shell boundaries are given.

Section 2

STIFFNESS METHOD FOR SHELL STRUCTURES

2.1 General

Previous volumes contain information on the determination of fixed-edge forces and stiffness influence coefficients of shell elements which form the multicellular shell structure shown in Fig. 2. With this knowledge it is possible to determine by the "stiffness" method the boundary deformations and forces required to join the various elements together to form a continuous structure. Stresses and displacements at any point in the structure are completely determined by the loading and boundary deformation of each element. Therefore, the problem is to determine the deformation at the juncture of the shell elements for the complete structure when it is subjected to a prescribed loading. The stiffness method obtains its name from the fact that the juncture problem is reduced to the solution of a set of simultaneous equations

$$\overline{K}\,\delta^{R} = \overline{F} \tag{2.1}$$

in which the unknowns are the redundant juncture deformations δ^R, the vector $\overline{F}$ is composed of known fixed-edge forces and the matrix $\overline{K}$ is the "overall" stiffness matrix. This set of equations is based on the element constitutive equation

$$F = k\delta + F^{f} \tag{2.2}$$

by requiring compatibility of the deformations δ and equilibrium of the forces F between the shell elements along juncture lines. In principle the problem is straightforward; however, in carrying out the analysis, problems such as the coordinate system, ordering of the equations, and the solution of a large system of simultaneous equations become evident.

2.2 Coordinate System

The constitutive boundary relations for the individual shell elements were found by using the most convenient orthogonal coordinate system for the given geometry, thus appropriate shell coordinates were used for the local coordinate system for the cylinder, cone, and sphere. All forces F and deformations δ were nondimensionalized in order to cover a wide range of parameters. Therefore, as a first step in obtaining the overall stiffness matrix it is necessary to dimensionalize the constitutive boundary equations as noted in Vol. V or to nondimensionalize these quantities by the same constants.

For a general structure it is sometimes useful to consider a rectangular global coordinate system for all forces and deformations. A shell structure on the other hand has a common tangent line $\overline{ab}$ as seen in Fig. 3 along which the rotation ω_η and the displacement component u_η are parallel at the juncture of two or more elements. For this case it is advantageous to use the local coordinate system of element 1 or 2 for reference of direction for force and deformation components. Along the juncture of the two elements shown in Fig. 3

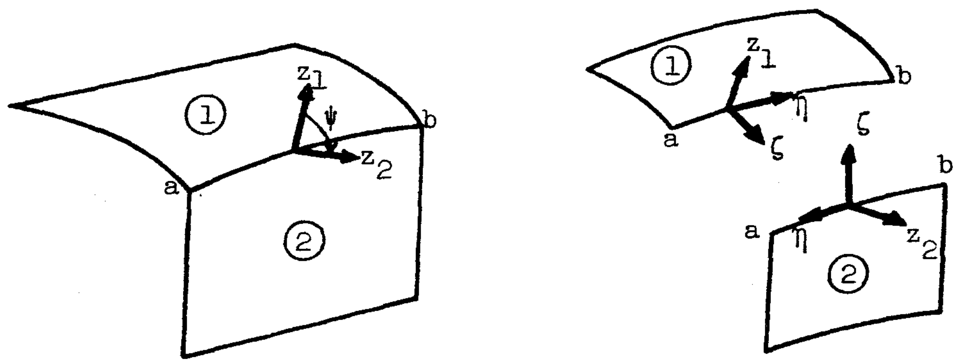


Fig. 3 Juncture Line of Two Shell Elements

the local coordinate η is parallel for both elements. Thus to transform the boundary coordinate of Element 2 to conform to the coordinate system of Element 1 involves a 180° rotation about the z_2 axis and a rotation by the amount $-\psi$ about the η axis. This rotation of coordinates is given by

$$\begin{Bmatrix} \eta \\ \zeta \\ z \end{Bmatrix}_1 = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -\cos\psi & -\sin\psi \\ 0 & -\sin\psi & \cos\psi \end{bmatrix} \begin{Bmatrix} \eta \\ \zeta \\ z \end{Bmatrix}_2 \tag{2.3}$$

In certain cases it can occur that the η coordinates of Elements 1 and 2 are in the same direction as shown in Fig. 4. Then the transformation of the

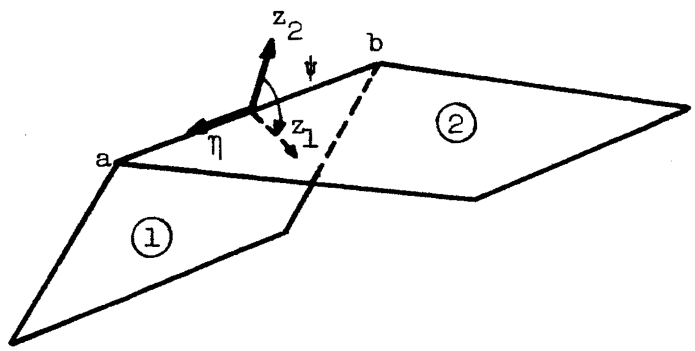


Fig. 4 Orientation of Coordinates Along the Juncture of Two Shell Elements

local coordinates of Element 2 to those of Element 1 is found by a rotation of $-\psi$ about the η axis and is given by

$$\begin{Bmatrix} \eta \\ \zeta \\ z \end{Bmatrix}_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\psi & \sin\psi \\ 0 & -\sin\psi & \cos\psi \end{bmatrix} \begin{Bmatrix} \eta \\ \zeta \\ z \end{Bmatrix}_2 \tag{2.4}$$

Since the components of boundary forces and deformations are directed along the boundary coordinates they can be transformed to different orientations by the rotations defined by Eqs. (2.3) or (2.4). Instead of using a number of different rotation matrices, it is possible to use only one given by

$$\Phi = \begin{bmatrix} 1 & & & \\ & \cos\psi & \sin\psi & \\ & -\sin\psi & \cos\psi & \\ & & & 1 \end{bmatrix} \tag{2.5}$$

in conjunction with a diagonal matrix for changing signs of the main diagonal. For instance, the direction of the forces F of Element 2 can be transformed to those of Element 1 by

$$\begin{Bmatrix} \bar{N}_\eta \\ \bar{N}_\zeta \\ \bar{Q} \\ \bar{M} \end{Bmatrix}_1 = \Phi \lambda \begin{Bmatrix} \bar{N}_\eta \\ \bar{N}_\zeta \\ \bar{Q} \\ \bar{M} \end{Bmatrix}_2 \tag{2.6}$$

where λ is a diagonal matrix used to dimensionalize (see Vol. V) and change signs if necessary. In a similar manner the deformations δ can be transformed by

$$\begin{Bmatrix} u_\eta \\ u_\zeta \\ w \\ \omega_\eta \end{Bmatrix}_1 = \Phi \mu \begin{Bmatrix} u_\eta \\ u_\zeta \\ w \\ \omega_\eta \end{Bmatrix}_2 \tag{2.7}$$

where μ is a diagonal matrix used to dimensionalize and change signs.

With the previous definitions the constitutive boundary relations for an element can be written as

$$\hat{F} = \Phi \lambda k \mu^{-1} \Phi^T \hat{\delta} + \hat{F}^f \tag{2.8}$$

where it is understood that $\hat{F}$, $\hat{\delta}$, and $\hat{F}^f$ are in the transformed direction and k is the element stiffness matrix in its untransformed and nondimensionalized state. For future reference $\bar{k}$ corresponds to the transformed element stiffness matrix and is defined by

$$\bar{k} = \Phi \lambda k \mu^{-1} \Phi^T \tag{2.9}$$

where μ^{-1} is the inverse of μ and Φ^T is the transpose of Φ .

2.3 Compatibility and Equilibrium Along Juncture Line

At a shell juncture it is required that the displacement and rotation vector be equal for each element being joined together. The transformation matrix Φ in Eq. (2.5) provides a means by which the deformation of the various elements can be rotated to the same direction. All that must be done to guarantee compatibility is to equate the components of the deformation vector of the elements. If $\mu_1 {}^1\delta$ is the dimensional deformation of Element 1 along a juncture line and $\mu_2 {}^2\delta$ is the deformation of Element 2 along the same juncture line, then compatibility requires that

$$\mu_1 {}^1\delta = \Phi \mu_2 {}^2\delta \tag{2.10}$$

Similar equations can be written for other elements having the same juncture line.

Equilibrium along a juncture line requires that the summation of forces be zero. For m number of elements being joined together along a common juncture line, equilibrium can be written as

$$\sum_{\ell=1}^{m} {}^{\ell}F = 0 \tag{2.11}$$

where the superscript refers to the element number and it is implied that the forces are oriented in the same direction by use of the proper transformation matrices.

2.4 The Overall Stiffness Matrix of a Shell Structure

The major problem in forming or even thinking about the overall stiffness matrix is devoted to the notation and the ordering of the elements which form the shell structure.

The scheme to obtain the overall stiffness matrix can best be explained by a simple example. For this purpose the structure shown in Fig. 5 is divided into four elements denoted by 1, 2, 3, and 4. Edges which are not junctures are considered to be fixed. This simplifies the problem since the boundary conditions are already taken into account. It is assumed that the stiffness matrices

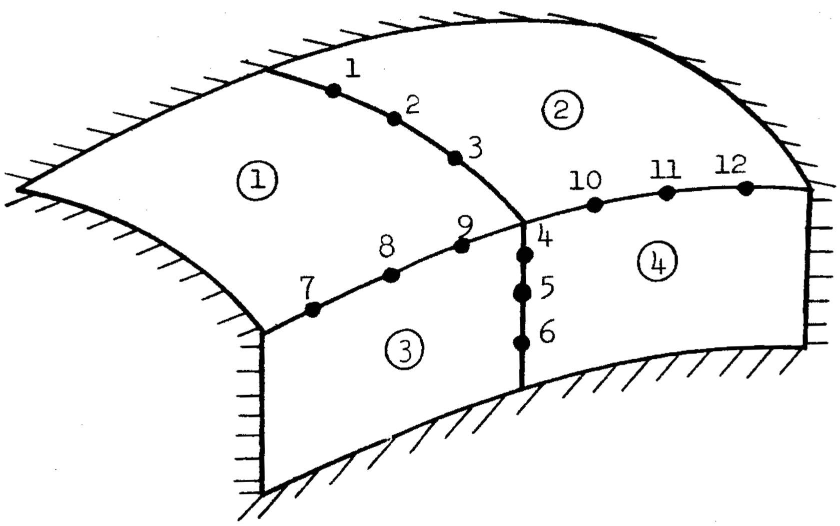


Fig. 5 Shell Structure

and fixed-edge forces have been developed for each of the shell elements and that the forces and displacements have been transformed to a compatible coordinate system. The next problem is to number the juncture points. The arrangement shown in Fig. 5 is one possible ordering. The constitutive edge relation for each element is written as

$$ {}^{\ell}F_i = {}^{\ell}\bar{k}_{ij}\,\delta_j + {}^{\ell}F^f_i \qquad i = 1,\ 2 \ldots\ldots 12 \tag{2.12} $$

where the superscript ℓ refers to the element number and the subscript refers to the juncture point. Since compatibility is already taken into account, all that is necessary is to write the equilibrium equations

$$ \sum_{\ell=1}^{4} {}^{\ell}F_i = 0 \qquad i = 1,\ 2,\ \ldots\ldots 12 \tag{2.13} $$

Substituting Eq. (2.12) into Eq. (2.13) yields the system of equations

$$ ({}^{1}\bar{k}_{ij} + {}^{2}\bar{k}_{ij} + {}^{3}\bar{k}_{ij} + {}^{4}\bar{k}_{ij})\,\delta_j = -({}^{1}F^f_i + {}^{2}F^f_i + {}^{3}F^f_i + {}^{4}F^f_i) \tag{2.14} $$

$$ i = 1,\ 2 \ldots\ldots\ldots 12 $$

For the particular problem under consideration, these equations in expanded form become

$$ \begin{bmatrix} x & x & x & & & & x & x & x & x & x & x \\ x & x & x & & & & x & x & x & x & x & x \\ x & x & x & & & & x & x & x & x & x & x \\ & & & x & x & x & x & x & x & x & x & x \\ & & & x & x & x & x & x & x & x & x & x \\ & & & x & x & x & x & x & x & x & x & x \\ x & x & x & x & x & x & x & x & x & & & \\ x & x & x & x & x & x & x & x & x & & & \\ x & x & x & x & x & x & x & x & x & & & \\ x & x & x & x & x & x & & & & x & x & x \\ x & x & x & x & x & x & & & & x & x & x \\ x & x & x & x & x & x & & & & x & x & x \end{bmatrix} \begin{Bmatrix} \delta_1 \\ \delta_2 \\ \delta_3 \\ \delta_4 \\ \delta_5 \\ \delta_6 \\ \delta_7 \\ \delta_8 \\ \delta_9 \\ \delta_{10} \\ \delta_{11} \\ \delta_{12} \end{Bmatrix} = \begin{Bmatrix} \bar{F}_1 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \bar{F}_{12} \end{Bmatrix} \tag{2.15} $$

where the x denotes a non zero term. Hence this is the desired set of simultaneous equations which in matrix form is

$$\overline{K}\,\delta^R = \overline{F}$$

For this particular ordering it is seen that the $\overline{K}$ matrix is full. When a large number of points are involved it is advantageous to order the $\overline{K}$ matrix to take advantage of the zeros.

A "natural" ordering system is to consider points 1, 2, 3 to comprise one line, say line 1, and points 4, 5, 6 to comprise line 2, etc. Then with the notation that the superscript on the right implies the line number δ^1 signifies the deformation of line 1.

With this notation the expanded matrix Eq. (2.15) can be written in block form as

$$\begin{bmatrix} x & 0 & x & x \\ 0 & x & x & x \\ x & x & x & 0 \\ x & x & 0 & x \end{bmatrix} \begin{pmatrix} \delta^1 \\ \delta^2 \\ \delta^3 \\ \delta^4 \end{pmatrix} = \begin{pmatrix} \overline{F}^1 \\ \overline{F}^2 \\ \overline{F}^3 \\ \overline{F}^4 \end{pmatrix} \tag{2.16}$$

By reordering the equations, the following matrix equation can be obtained

$$\begin{bmatrix} x & x & x & 0 \\ x & x & 0 & x \\ x & 0 & x & x \\ 0 & x & x & x \end{bmatrix} \begin{pmatrix} \delta^1 \\ \delta^3 \\ \delta^4 \\ \delta^2 \end{pmatrix} = \begin{pmatrix} \overline{F}^1 \\ \overline{F}^3 \\ \overline{F}^4 \\ \overline{F}^2 \end{pmatrix} \tag{2.17}$$

For this small problem the ordering is not important; however, for larger systems the advantage is evident.

Section 3

APPLICATION OF THE STIFFNESS METHOD TO A MULTICELLULAR STRUCTURE

3.1 General

The problem considered in this section is concerned with developing the overall stiffness matrix of the multicellular shell structure shown in Fig. 1. This structure is subjected to internal pressure, body forces, and thermal gradients. It is assumed that the loads are symmetric with respect to the diametrical plane bisecting each pair of cells located opposite each other. Thus in a structural analysis only one-half of a cell need be considered. Each cell is composed of four basic structural elements, namely, flat plate, cylindrical, conical and spherical segments. These elements have been investigated in Volumes II through V for the analysis of intermediate loads and the effect of boundary deformations in computing the element influence coefficients.

Techniques to determine the overall stiffness matrix of shell structures discussed in Section 2 are applied to the analysis of a structural model of the multicell. It is not the purpose in this section to present detailed equation systems but is orientated to the application of special ordering methods and techniques to arrange the stiffness matrix in a form by which the set of simultaneous equations can be solved by methods discussed in Vol. 1.

In some cases it is expedient to have a method by which the structural behavior can be obtained by simple means. From this standpoint a successive approximation technique is presented which by-passes the necessity of solving a very large set of equations at one time. Its advantage lies in its inherent simplicity and the fact that local behavior can be assessed during the analysis. The scheme is based on line relaxation in which the elements with a common juncture along one line are solved at one time.

3.2 Overall Stiffness Matrix

The structural model shown in Fig. 6 represents one way in which one-half of a cell can be subdivided into basic elements. As shown in this figure there are eleven elements joined together along eighteen boundary lines. The ordering of elements and lines shown is not unique. For this particular numbering system the juncture lines around Element 9 are even numbers, while the transverse lines between elements such as the sphere and cone are odd numbers. Lines 19 and 20 can be used to input known data such as displacements or loads. If deformations are specified along these lines, this information will become part of the right hand vector $\overline{F}$.

Because of symmetry all of the elements have three distinct boundary lines except for Element 9 which forms a connection with all other elements. This plate can be considered to have eight boundary lines which are even numbers. The nomenclature used in this section corresponds to line ordering; thus, superscripts are used on forces F , deformations δ , and influence coefficients to denote the element and line number. Values on the upper left denote the element number while the values on the upper right denote the line segment. For the deformation a number on the upper right only corresponds to the transformed deformation vector along the specified line, i.e., δ^4 is the deformation along line 4. With this notation the constitutive boundary equations for Element 1 which is a cone becomes

$$\begin{Bmatrix} {}^1F^1 \\ {}^1F^2 \\ {}^1F^3 \end{Bmatrix} = \begin{bmatrix} {}^1\overline{K}_{11} & {}^1\overline{K}_{12} & {}^1\overline{K}_{13} \\ {}^1\overline{K}_{21} & {}^1\overline{K}_{22} & {}^1\overline{K}_{23} \\ {}^1\overline{K}_{31} & {}^1\overline{K}_{32} & {}^1\overline{K}_{32} \end{bmatrix} \begin{Bmatrix} \delta^1 \\ \delta^2 \\ \delta^3 \end{Bmatrix} + \begin{Bmatrix} {}^1F^{f1} \\ {}^1F^{f2} \\ {}^1F^{f3} \end{Bmatrix} \qquad (3.1)$$

where ${}^1F^m$ and ${}^1F^{fm}$ are dimensionalized and transformed to specific coordinate systems. Equation (3.1) does not take advantage of the possible small coupling between lines 1 and 3. The constitutive relations for all the other elements have the same form as Eq. (3.1) except for Element 9. This element can in general have coupling all around the plate. Because of the symmetrical loading it is not subjected to fixed edge forces unless it is thermally loaded.

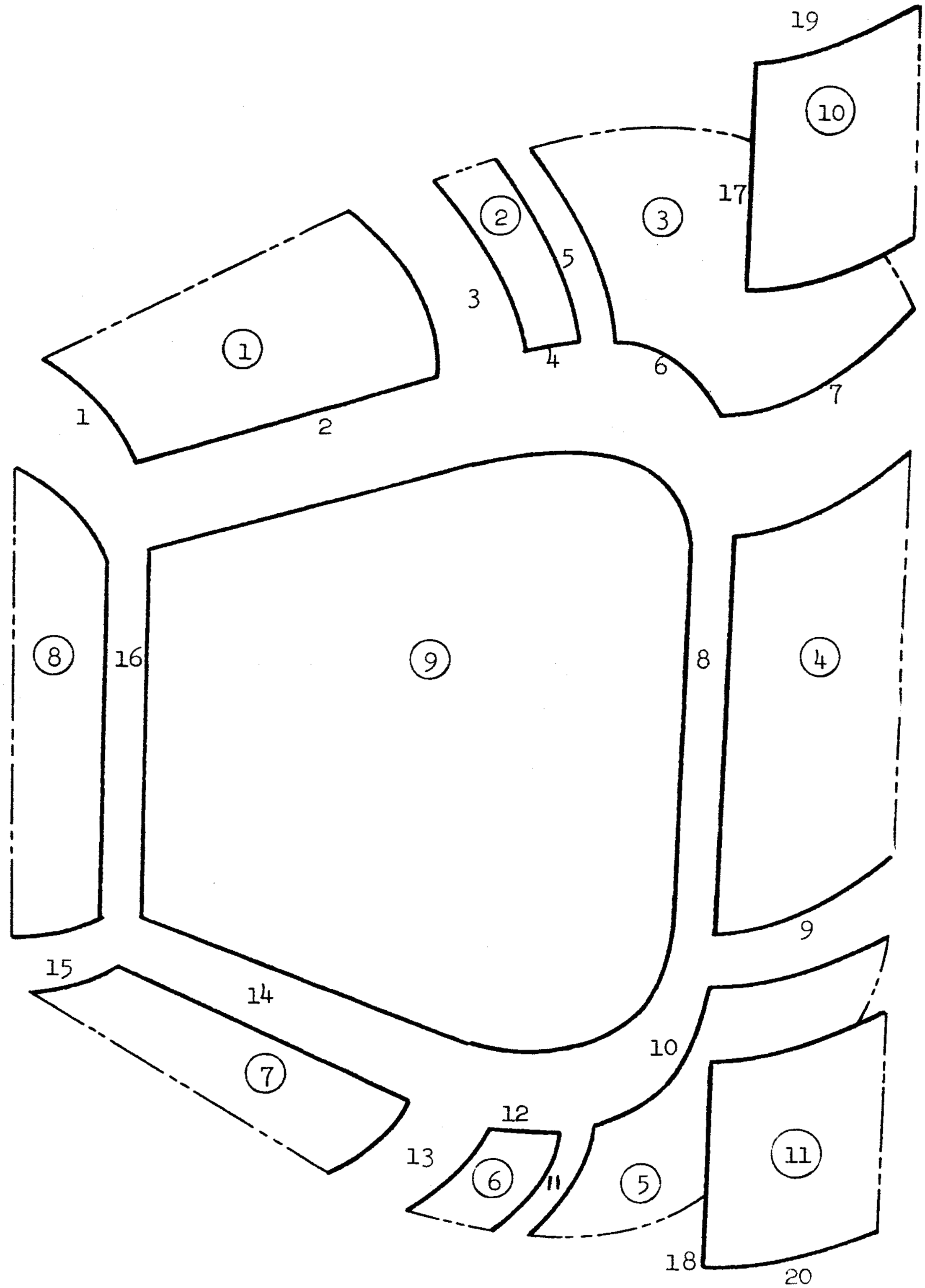


Fig. 6 Basic Structural Model of Multicellular Structure with a Specific Ordering of Elements and Juncture Lines

Including this possibility, the general form for the constitutive boundary relation of Element 9 of Fig. 6 is

$$\begin{Bmatrix} {}^9F^2 \\ {}^9F^4 \\ {}^9F^6 \\ {}^9F^8 \\ {}^9F^{10} \\ {}^9F^{12} \\ {}^9F^{14} \\ {}^9F^{16} \end{Bmatrix} = \begin{bmatrix} {}^9\overline{K}_{11} & \cdot & \cdot & \cdot & \cdot & {}^9\overline{K}_{18} \\ \cdot & & & & & \cdot \\ \cdot & & & & & \cdot \\ \cdot & & & & & \cdot \\ \cdot & & & & & \cdot \\ \cdot & & & & & \cdot \\ \cdot & & & & & \cdot \\ {}^9\overline{K}_{81} & \cdot & \cdot & \cdot & \cdot & {}^9\overline{K}_{88} \end{bmatrix} \begin{Bmatrix} \delta^2 \\ \delta^4 \\ \delta^6 \\ \delta^8 \\ \delta^{10} \\ \delta^{12} \\ \delta^{14} \\ \delta^{16} \end{Bmatrix} + \begin{Bmatrix} {}^9F^{f2} \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ {}^9F^{f16} \end{Bmatrix} \tag{3.2}$$

The most important feature in determining possible forms of the overall stiffness $\overline{K}$ matrix involves the coupling effect of the elements involved in the structure. The magnitude of the coupling can be determined from the influence coefficients by noting the effect of the deformations on forces away from the point at which the deformations are specified. If an element is coupled to at most three line elements, then the form of the $\overline{K}$ matrix can be arranged in a rather compact fashion. However, when there is full coupling very little sophistication is possible for efficient ordering. In many cases the coupling will be small; hence, for practical purposes the terms can be neglected. This simplification should necessarily be verified by simple meaningful checks such as joining elements together to form a larger structure of the same geometry.

Since there are eighteen line elements in the model shown in Fig. 6, the equilibrium equations are

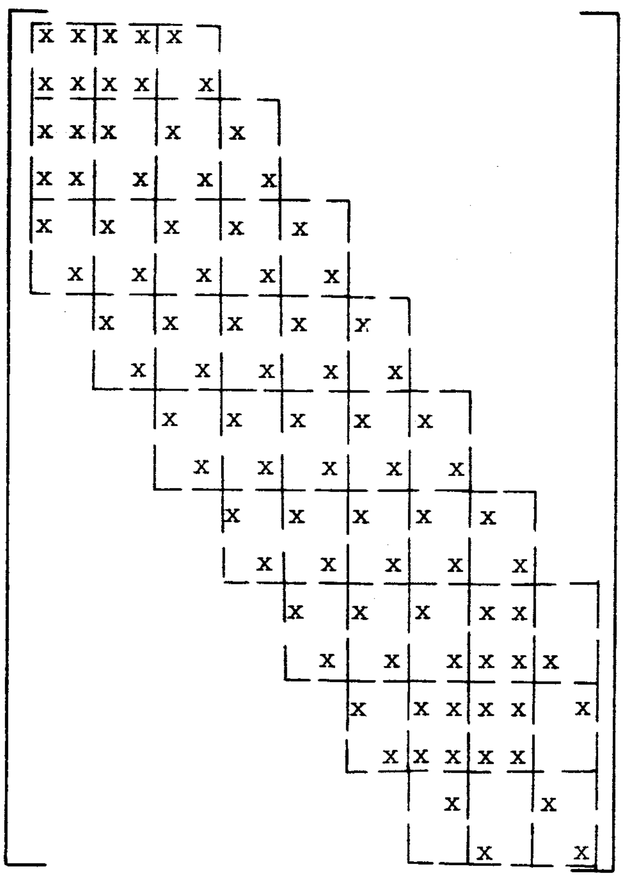
$$\sum_{\ell=i}^{j} {}^{\ell}F^m \qquad m = 1, 2, 3\ldots\ldots18 \tag{3.3}$$

where i through j are the elements with the same juncture line. For instance, line 3 is the juncture of Element 1 and 2, hence Eq. (3.3) for this line is

$${}^1F^3 + {}^2F^3 = 0 \tag{3.4}$$

On substituting equations of the form of Eqs. (3.1) and (3.2) into Eq. (3.3) yields the set of simultaneous equation system in terms of the unknown deformation δ^R . If the equilibrium equations are written in numerical order, i.e. $m = 1, 2 \ldots\ldots 18$ the $\bar{K}$ will be full. For 18 lines with 20 points each and 4 unknowns at each point yields a total of 1,440 equations in the overall stiffness matrix $\bar{K}$. Equation systems of this size can be solved by direct or iterative methods but time is generally prohibitive. If $\bar{K}$ can be arranged in block diagonal form, then the direct methods discussed in Vol. 1 can be used to solve the entire problem in a reasonable amount of computer time. In general $\bar{K}$ will be full if special coupling and ordering are not used.

By assuming that the coupling of Element 9 is restricted to one line to the right and one line to the left of the line in question and ordering the unknowns in an alternate manner, the form of the overall stiffness matrix for the model shown in Fig. 6 is of a 9 block diagonal form given by



$$\begin{Bmatrix}\delta^{16}\\ \delta^{1}\\ \delta^{15}\\ \delta^{2}\\ \delta^{14}\\ \delta^{3}\\ \delta^{13}\\ \delta^{4}\\ \delta^{12}\\ \delta^{5}\\ \delta^{11}\\ \delta^{6}\\ \delta^{10}\\ \delta^{7}\\ \delta^{9}\\ \delta^{8}\\ \delta^{17}\\ \delta^{18}\end{Bmatrix} = \begin{Bmatrix}\bar{F}^{16}\\ \cdot\\ \cdot\\ \cdot\\ \cdot\\ \cdot\\ \cdot\\ \cdot\\ \cdot\\ \cdot\\ \cdot\\ \cdot\\ \cdot\\ \cdot\\ \cdot\\ \cdot\\ \cdot\\ \bar{F}^{18}\end{Bmatrix}$$

This system can also be considered as a block five diagonal matrix if two lines are considered at a time.

3.3 Successive Approximation

Another approach to solve the juncture problem of multicellular structures is an iterative technique of successive approximation. Instead of solving for displacements of all the junctures by one large matrix, the junctures can be "relaxed" one at a time to satisfy equilibrium and compatibility at every point of one juncture line while all other junctures remain fixed. When one juncture line is relaxed, for example line 5 in Fig. 6, the displacements of this juncture will in general result in additional boundary forces at all the boundaries of the shells which are bounded by line 5, i.e. shell segments 2 and 3. After one juncture is relaxed, other junctures can be relaxed in the same manner until all the junctures of the structure are relaxed to form one complete cycle. In the first cycle, the junctures can be relaxed to satisfy the equilibrium condition by considering the fixed-edge forces only, leaving the boundary forces due to the juncture displacements to the next cycle. The additional boundary forces due to the displacements found during a line relaxation can be computed by the constitutive boundary equations for the shell elements. These forces are added to the right hand vector for the second cycle. This, in effect, accounts for the coupling between various elements. If the coupling is small the iteration should converge rapidly. It is also possible to consider the accumulated boundary forces due to the displacements of other junctures simultaneously with the fixed-edge forces to accelerate convergency. This successive approximation technique will never give the exact solution as after relaxing the last juncture some boundary forces of the other junctures will violate the equilibrium condition established previously. However, the inconsistency, i.e. the boundary forces resulting from relaxing the last juncture, can be reduced to any desired degree by repeating the cycles.

There are a number of advantages to this method. When one juncture is relaxed, only two or three shell segments are involved, transformations will be simpler than for the overall system. Equal spacings can be used between the boundary points to reduce the complication of dimensionalization. Besides, it has been shown in Volumes II, III and IV that the accuracy of numerical analysis of shells can be increased by reducing the spacing near the juncture, thus the spacing between the rows are graded to reduce the total number of mesh points and amount of numerical computations without sacrificing accuracy. If the junctures are considered one at a time, only one side of the shell segments needs to be spaced closely, while nominal spacings can be maintained on the other sides. In carrying out the successive approximation, the procedure to relax one juncture is the same as that to relax all the junctures at one time except for the additional boundary forces due to relaxing of the other juncture lines.

3.4 Numerical Example

To demonstrate the direct stiffness matrix method, two orthotropic cylindrical panels of equal dimensions are joined together end to end to form a panel with twice the length of each of the original elements as shown in Fig. 6. For

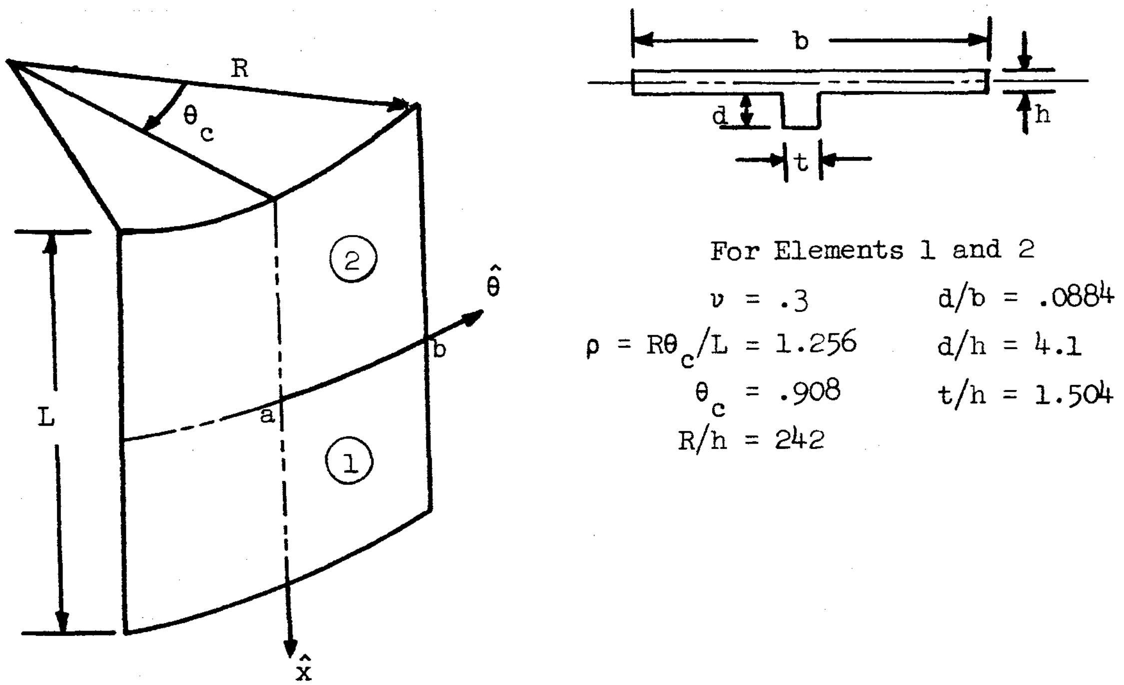


Fig. 6 Example Geometry for Juncture of Two Cylindrical Panels

this example one juncture line $\overline{ab}$ is involved. The complete structure is a cylindrical panel with fixed-edges subjected to internal pressure. This example is used to demonstrate the accuracy and feasibility of the technique since the results can be compared to the results of a complete cylindrical panel with symmetry.

A digital program has been written which prescribes equilibrium and compatibility between shell elements along juncture lines and generates the overall stiffness matrix $\overline{K}$ and the right hand vector $\overline{F}$, and solves the resulting set of equations

$$\overline{K}\,\delta^R = \overline{F}$$

for the deformations δ^R along the juncture lines. This program is used in conjunction with the influence coefficient program described in Vol. V, and through its own input data has the capability of forming the $\overline{K}$ matrix by calling the element stiffness matrices from tape, multiplying them by specified λ , μ and Φ matrices for dimensionalization and transformation as described by Eq. (2.9) and adding the new element stiffness matrices to $\overline{K}$ in a predetermined order. The $\overline{F}$ vector is formed in a similar manner.

For the example it is necessary to compute only the influence coefficients of Element 1 along its upper boundary $\overline{ab}$ since the coefficients of Element 2 on its lower boundary can be computed by multiplying the coefficients of Element 1 at a point in the following manner

$${}^2k = \begin{bmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix} {}^1k \begin{bmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix}$$

and the fixed-end forces of Element 2 are obtained by

$${}^2F^f = \begin{bmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix} {}^1F^f$$

If the boundary coordinates of Element 1 are used as a reference, then λ and μ for Element 2 are

$$\lambda = \mu = \begin{bmatrix} -1 & & & \\ & -1 & & \\ & & 1 & \\ & & & -1 \end{bmatrix}$$

and the angle of rotation ψ is zero for both elements. Thus Φ is a unity matrix. In the final analysis the influence coefficients and fixed-end forces of Element 2 can be found from the coefficients and fixed-end forces of Element 1 by using the following values of λ and μ

$$\lambda = \mu = \begin{bmatrix} 1 & & & \\ & -1 & & \\ & & 1 & \\ & & & -1 \end{bmatrix}$$

in Eq. (2.9). For Element 1 , λ , μ and Φ are unity matrices.

Four solutions of the problem were obtained using different spacings. This includes 5 and 10 columns with 5 equal spaced rows and 13 graded rows where the grading is fine near the juncture line. The results of this line relaxation are presented in Fig. 7 for the displacement components w and v . It is noted that the rotation and displacement component u are zero due to symmetry. The juncture matching is compared to results of a long cylindrical panel (ρ = .628) with 5 and 10 columns and 6 and 14 equally spaced rows. It is seen that the matching results are in good agreement with the results of the complete panel. The detailed distribution in the θ direction is revealed by using more columns. Thus it is essential to determine the accuracy required by using sufficiently close spacing. However, it is also seen that fair accuracy can be obtained by using a small number of rows and columns.

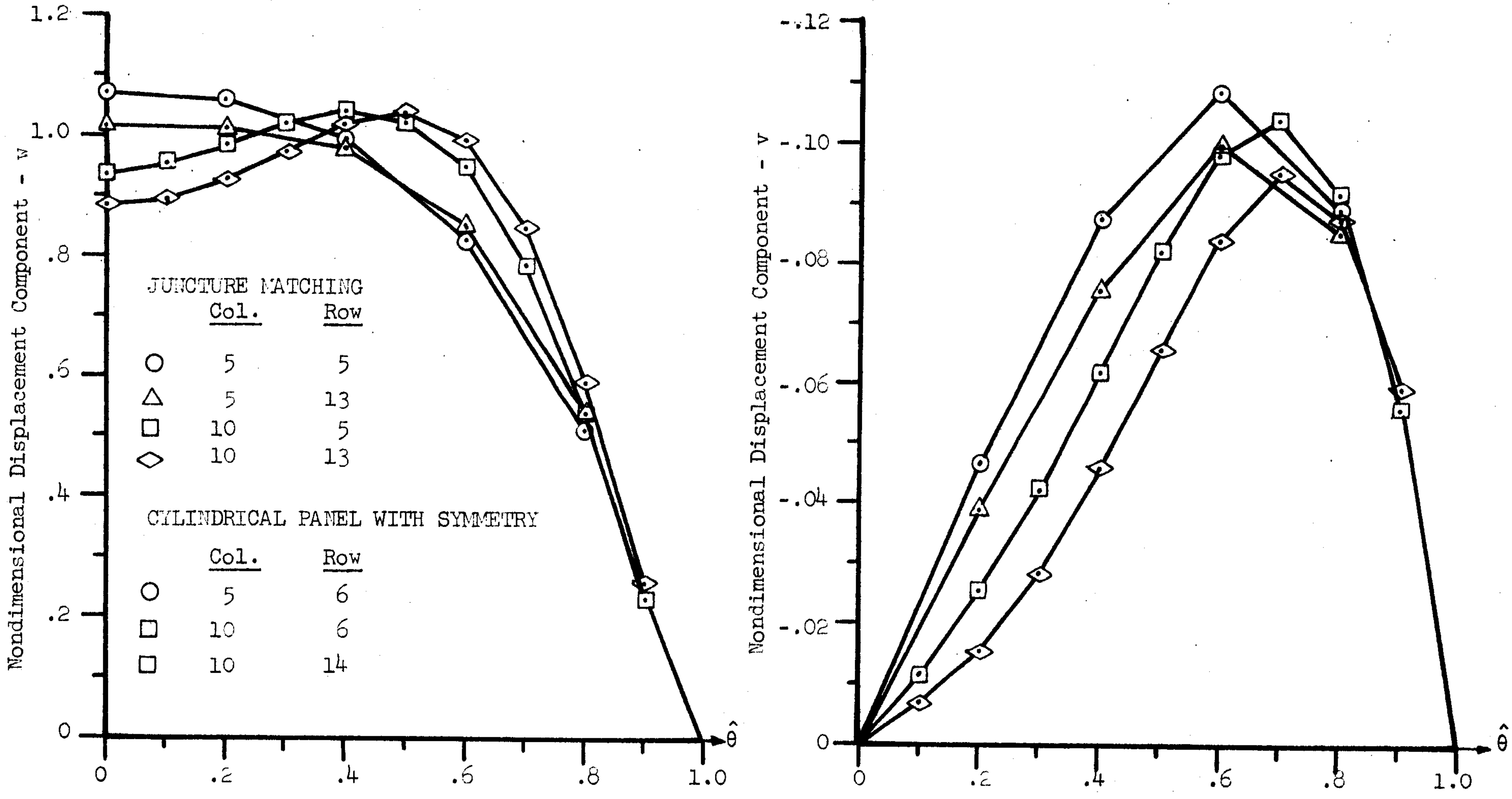


Fig. 7 Nondimensional Displacement Components along Juncture Line for Example