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PROPAGATION OF EM WAVES IN LINEAR, PASSIVE, GENERALIZED MEDIA

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ABSTRACT

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Properties of linear passive media are investigated by using a phenomenological approach. Properties of fields that can propagate in a passive medium are postulated and from this properties of the constitutive relationship are deduced. A necessary positive real condition on the constitutive relationship is found and some of its implications are considered. Also, the causality condition which is necessary for realizable media is considered.

Next a general formulation of the spectrum of characteristic waves in lossless linear passive media is made. Because of an orthogonality condition for the characteristic waves of the medium, the fields due to an arbitrary source can be separated into components parallel to the characteristic waves. The components of the source field are dependent only upon the portion of the source parallel to their characteristic field and to their own sheet(s) of the dispersion surface. The theory is then applied to two particular problems, electric dipoles in a general time-dispersive uniaxial medium and in an isotropic compressible plasma. Finally, the radiation field of an arbitrary source in a lossless linear passive medium is investigated using the spectral decomposition of the fields. By normalizing the length of the Total Poynting vector (electromagnetic plus medium) to unity for each characteristic field, a concise and physically interpretable expression for the source fields is obtained. These results are then applied to an anisotropic compressible plasma and to a magneto-ionic plasma.

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1. INTRODUCTION

One of the most simplifying and useful properties of Maxwell's equations is the linearity property. However, the linearity property in the past has not been exploited to its fullest. Of course Maxwell's equations will not be linear if the medium is not linear. Thus, in all that follows a linear constitutive relationship will be assumed.

To the author's knowledge, the most general linear constitutive relationship that Maxwell's equations will allow has never been used. There is, however, considerable motivation to formulate such a theory. In most cases the permittivity, ϵ, and the permeability, μ, are considered to be scalar constants. This was probably the original formulation of the constitutive relationship. Then these scalars were found to be functions of the applied frequency which gave rise to time-dispersive media. From this, it is found that the group velocity and phase velocity are not necessarily equal but are in the same direction. In crystal optics the permittivity constitutive relationship is given by a diagonal matrix, ϵ, while the permeability, μ, is still a scalar. Two such media that occur in nature are uniaxial and biaxial crystals. More recently a considerable amount of the literature has been devoted to magnetoplasmas which occur in the ionosphere and can be created artificially in the laboratory. Magnetoplasmas are characterized by a matrix permittivity and a scalar permeability. They may also be space and time dispersive, i.e., the permittivity matrix a function of the Fourier wave vector, $\vec{k}$, and the Fourier time number, ω, respectively. Also, there exist materials such as ferrites that are characterized by a scalar permittivity and a matrix permeability. Further, uniformly moving media has an even more complicated constitutive relationship. In a moving medium, the electric flux density, D, is linearly related to both the electric intensity, E, and the magnetic intensity, H. A similar dependence results for the magnetic flux density, B. These are a few examples of well-known media of nature; however, the possibility of creating artificial dielectrics gives impetus to investigating the general properties of linear passive media. Moreover, the synthesis of media with given space and time-dispersive characteristics would be highly desirable.

A large number of workers in the area of magnetoplasmas attempt to derive a macroscopic constitutive relationship of the medium by formulating a microscopic model that is describable by dynamical equations and/or probability distributions. Thus, they attempt to deduce the macroscopic properties of the medium, and hence the fields that can propagate in them from a postulated microscopic model. This is not the only method of deducing properties of the medium. Another very effective method is to postulate properties of fields that can propagate in the medium and from this deduce the properties of the constitutive relationship of the medium. This phenomonological approach will be the method used to investigate the properties of passive media.

Since Maxwell's equations are linear, the spectral theory and representation of the operators that compose Maxwell's equations for a lossless medium will be useful. With this the spectral representation of both the source-free and the source fields can be derived. In this way, the specific role played by the sheets of the dispersion surface and their associated characteristic fields is easily seen. Also, the manner in which the source excites the spectrum of characteristic fields to form the composite source field can be determined.

In Section 4, by using the spectral representation, the exact expression for the fields of a dipole in a general time-dispersive uniaxial medium will be derived. These expressions are then compared with Clemmow's[1] fields in an uniaxial medium, which he obtained by a scaling procedure. Section 6 is concerned with the radiation field of an arbitrary antenna in a lossless medium.

The purpose of this thesis is to exploit the linearity of Maxwell's equations and thereby derive general relationships for linear passive media and for the fields that can propagate in such media. Further, the usefulness of the general relationships are demonstrated by specific examples.

2. PROPERTIES OF LINEAR PASSIVE MEDIA

2.1 Maxwell's Equations as Six-Vectors

Most commonly, Maxwell's equations in point form are expressed as two separate vector equations, Ampere's Law and Faraday s Law. The vectors involved have three components. However, this separation is by no means unique nor necessary. With equal ease, one can express Maxwell's equations as six-vector equations involving vectors with single components only.

Likewise Maxwell's equations may be expressed as a single vector equation involving vectors with six components. This is the formulation that will be introduced presently and will prove to be of an ideal form for many purposes, particularly for conservation of electromagnetic energy. Define the partial differential operator $\mathcal{O}$ and the six-vectors $\mathcal{F}$, $\mathcal{F}_f$ and $\mathcal{C}$ as follows:

$$\mathcal{O} = \begin{bmatrix} 0 & \nabla\times \\ -\nabla\times & 0 \end{bmatrix} \tag{2.1}$$

$$\mathcal{F} = \begin{bmatrix} \mathcal{E}, \mathcal{H} \end{bmatrix} \tag{2.2}$$

$$\mathcal{F}_f = \begin{bmatrix} \mathcal{D}, \mathcal{B} \end{bmatrix} \tag{2.3}$$

$$\mathcal{C} = \begin{bmatrix} \mathcal{J}_e, \mathcal{J}_m \end{bmatrix} \tag{2.4}$$

The subscript "f" is meant to signify flux since the units of $\mathcal{F}_f$ involves "per square meter." With these definitions, Maxwell's equations are simply

$$\mathcal{O}\mathcal{F} = \dot{\mathcal{F}}_f + \mathcal{C} \tag{2.5}$$

2.2 Definition of Constitutive Relationship

As easily seen, this vector equation has more unknowns than its order and hence is indeterminate. The constitutive relationship yields the remaining equations of the system. Essentially, the consitutive relationship is a statement of the influence of the material medium on a wave propagating through it as compared to the wave propagating through a vacuum. As such, the source is not involved in the constitutive relationship. One would like to formulate a mathematical expression for the constitutive relationship that would encompass the general class of linear media. In the real space-time domain, say R, such a constitutive relationship must be a matrix convolution operator, denoted $\underline{\underline{\mathcal{V}}}$. Then the general linear constitutive relationship is

$$\mathcal{F}_f = \underline{\underline{\mathcal{V}}} * \mathcal{F} \tag{2.6}$$

where the convolution is with respect to all four variables of space-time. It can be verified that this expression is capable of accounting for the scalar, uniaxial, biaxial, magneto-ionic, ferrite, and uniformly moving media that were previously mentioned. Furthermore, the general linear constitutive relationship applies to transient fields as well as steady-state fields whether sinusoidal or otherwise.

In light of the complexity of the convolution linear operator, which involves integrals of the "operand," $\mathcal{F}$, at times it is preferrable to deal with the system of equations in Fourier transform space, $\mathcal{T}$. Throughout the following, the definition of the Fourier transform and its inverse transform will be, respectively,

$$F(\vec{k},\omega) = \iiiint_{-\infty}^{\infty} e^{-j(\omega t - \vec{k}\cdot\vec{r})} \, \mathcal{F}(\vec{r},t) \, d^3r \, dt \tag{2.7}$$

and

$$\mathcal{Y}(\vec{r},t)=(2\pi)^{-4}\iiiint_{-\infty}^{\infty} e^{j(\omega t-\vec{k}\cdot\vec{r})}F(\vec{k},\omega)d^3k\,d\omega \tag{2.8}$$

where the transform variables are ω and $\vec{k}$. Let the Fourier transform of $\mathcal{Y}_f$, $\mathcal{C}$, $\underline{\underline{\mathcal{U}}}$, and $\mathcal{O}$ be, respectively, F_f, C, $\underline{\underline{U}}$, and O. Then in space $\mathcal{T}$ the system of Maxwell's equations and the constitutive relationship are

$$OF=\omega F_f+C \tag{2.9}$$

and

$$F_f=\underline{\underline{U}}F \tag{2.10}$$

What is gained by such a transformation is the following. The operators are simply square matrices of order six with elements which are functions of the transform variables ω and $\vec{k}$. Also make the distinction between space-dispersive and time-dispersive media as Allis[2] does for magneto-ionic media; the constitutive matrix $\underline{\underline{U}}$ as a function of $\vec{k}$ and ω implies that the medium is space-dispersive and time-dispersive, respectively. In truth the names space and time-dispersive media are misnomers since dispersion is a property of the whole system and its dependence cannot be separated among the space and time variables. A non-dispersive medium is one whose dispersion surface is hypercones with their vertices passing through the origin in space $\mathcal{T}$. Thus, it is possible for a space and/or time-dispersive media, as defined above, to be non-dispersive.

2.3 Waves in Passive Media

The general class of linear media is larger than necessary for most purposes. In fact the homogeneous media that commonly occur in distributed systems is passive. (A passive medium is defined to be a medium in which both the electromagnetic energy density stored and the power density dissipated for all fields are non-negative.) For this reason, in all that follows we will restrict our attention to passive media. First, however, it is necessary to recognize the properties of passive media in order that the two classes may be separated.

For either passive or active media, the totality of fields (waves) that can propagate through it uniquely characterizes the media. Even certain subsets (complete sets-subsets of the propagating waves that span the totality of fields for the medium) of the totality of fields may suffice to characterize the media. An example of a complete subset is the set of plane waves. Even so, all plane waves are not admissable in passive media. Thus, we would like to determine the properties of the waves that can propagate through passive media and from this determine certain characteristics of the constitutive relationship. Neither Maxwell's equations nor the constitutive relationship will aid in determining the class of fields we seek. The remaining equation that is essential to determining whether or not a wave is propagating in a passive or active media is the conservation of energy. Multiplying Equation (2.5) on the left by $\mathcal{F}$ and transposing terms gives the equation of conservation of electromagnetic energy,

$$\mathcal{F}^{T}\dot{\mathcal{F}}_{t} - \mathcal{F}^{T}\mathcal{O}\mathcal{F} = -\mathcal{F}^{T}\mathcal{C} \tag{2.11}$$

or

$$\frac{\partial w_e}{\partial t} + p_{de} + \nabla^{T}\mathcal{P}_e = \frac{\partial s_e}{\partial t} \tag{2.12}$$

where the superscript T signifies the transpose. Identification of the corresponding terms results in

$$-\mathcal{F}^T \mathcal{C} = \frac{\partial s_e}{\partial t}$$; rate of change of electromagnetic energy density supplied by the sources.

$$-\mathcal{F}^T \mathcal{O} \mathcal{F} = \nabla^T \mathcal{P}_e$$; divergence of the electromagnetic Poynting vector.

$$\mathcal{F}^T \dot{\mathcal{F}}_f = \frac{\partial w_e}{\partial t} + p_{de}$$; rate of change of stored electromagnetic energy density plus the dissipated power density.

A further identification of the rate of change of stored electromagnetic energy density and the dissipated power density can be made provided the operator $\underline{\underline{\mathcal{V}}}$ is split into the sum of two operators. In the transform space, $\mathcal{T}$, it can be shown that the rate of change of stored energy is associated with the Hermitian part of the operator $\underline{\underline{U}}$, while the dissipated power density is associated with the skew-Hermitian part. Thus, it is only necessary to transform the Hermitian and skew-Hermitian operators back to the space-time domain, R, in order to obtain the separation there, i.e.,

$$\underline{\underline{\mathcal{V}}}_s = \frac{1}{2}\left[\underline{\underline{\mathcal{V}}}(\vec{r},t) + \underline{\underline{\mathcal{V}}}^T(-\vec{r},-t)\right] \tag{2.13}$$

and

$$\underline{\underline{\mathcal{V}}}_d = \frac{1}{2}\left[\underline{\underline{\mathcal{V}}}(\vec{r},t) - \underline{\underline{\mathcal{V}}}^T(-\vec{r},-t)\right] \tag{2.14}$$

Then we have

$$\mathcal{F}^T(\underline{\underline{\mathcal{V}}}_s * \dot{\mathcal{F}}) = \frac{\partial w_e}{\partial t} \tag{2.15}$$

and

$$\mathcal{F}^T(\underline{\underline{\mathcal{V}}}_d * \dot{\mathcal{F}}) = p_{de} \tag{2.16}$$

Now the separation into passive or active media can be made by determining whether or not both the stored and dissipated electromagnetic energy densities of all possible fields in the media are non-negative. This will be an essential factor in proving a necessary condition "positive real" theorem for the constitutive relationship of passive media that will be considered later.

At this point let us determine a property that all plane waves in a passive medium must obey. For a source-free passive medium, the net energy flow into a closed region must be non-negative. Since the closed region is arbitrary, this implies that the integral with respect to time of minus the divergence of the Poynting vector for any field is non-negative, i.e.,

$$-\int^{t} \nabla^T \mathcal{P}_e \, dt \geq 0.$$

Also, since the field is arbitrary, the condition should apply to a plane wave of the type

$$\mathcal{F} = 2 \operatorname{Re} \mathcal{F}_1 \tag{2.17}$$

where

$$\mathcal{F}_1 = \mathcal{F}_0 e^{(s_0 t - \vec{\gamma}_0 \cdot \vec{r})} \tag{2.18}$$

$\mathcal{F}_0$ is an arbitrary complex vector field which is independent of space and time, $s_0 = \sigma_0 + j\omega_0$, and $\vec{\gamma}_0 = \vec{\alpha}_0 + j\vec{k}_0$

$$-\nabla^T \mathcal{P}_e = \mathcal{F}^T \sigma \mathcal{F} \tag{2.19}$$

But

$$\sigma \mathcal{F} = 2 \operatorname{Re} O_0 \mathcal{F}_1 \tag{2.20}$$

where

$$O_0 = \begin{bmatrix} 0 & \vec{\gamma}_0 \times \\ -\vec{\gamma}_0 \times & 0 \end{bmatrix} \tag{2.21}$$

Therefore,

$$-\nabla^T \mathcal{P}_e = \mathrm{Re}\left[\mathcal{F}_1^{\dagger} O_0 \mathcal{F}_1 + \mathcal{F}_1^T O_0 \mathcal{F}_1\right] \tag{2.22}$$

or

$$-\nabla^T \mathcal{P}_e = 2\,\mathrm{Re}\left[\mathcal{F}_0^{\dagger} O_0 \mathcal{F}_0 e^{2(\sigma_0 t - \vec{\alpha}_0 \cdot \vec{r})} + \mathcal{F}_0^T O_0 \mathcal{F}_0 e^{2(s_0 t - \vec{\gamma}_0 \cdot \vec{r})}\right] \tag{2.23}$$

Let $\sigma_0 > 0$ and time, t, be much larger than any characteristic time of the system; then the above inequality is approximately given by,

$$\int^t -\nabla^T \mathcal{P}_e \, dt \approx e^{2(\sigma_0 t - \vec{\alpha}_0 \cdot \vec{r})} \mathrm{Re}\left[\frac{1}{\sigma_0} \mathcal{F}_0^{\dagger} O_0 \mathcal{F}_0 + \frac{1}{s_0} \mathcal{F}_0^T O_0 \mathcal{F}_0 e^{2j(\omega_0 t - \vec{k}_0 \cdot \vec{r})}\right] \geq 0 \tag{2.24}$$

Since $$e^{2(\sigma_0 t - \vec{\alpha}_0 \cdot \vec{r})} \geq 0$$

then

$$\mathrm{Re}\left[\frac{1}{\sigma_0} \mathcal{F}_0^{\dagger} O_0 \mathcal{F}_0 + \frac{1}{s_0} \mathcal{F}_0^T O_0 \mathcal{F}_0 e^{2j(\omega_0 t - \vec{k}_0 \cdot \vec{r})}\right] \geq 0 \tag{2.25}$$

The second term of Equation (2.25) may take both positive and negative values; therefore, the first term must be non-negative. In other words,

$$\mathrm{Re}\left[\mathcal{F}_0^{\dagger} O_0 \mathcal{F}_0\right] \geq 0 \tag{2.26}$$

But $\mathrm{Re}\left[\mathcal{F}_0^{\dagger} O_0 \mathcal{F}_0\right] = 4\vec{\alpha}_0^T \bar{P}_e$ where $\bar{P}_e = \frac{1}{2}\mathrm{Re}\left[E_0 \times H_0^*\right]$ is the real part of the complex Poynting vector. Now we have the desired result that for Re $s_o \geq 0$ then $\vec{\alpha}_o^T \, \overline{P}_e \geq 0$. The result is as intuitively as one might expect; in a passive source-free medium the fields decrease in the direction of the average power flow.

In addition, since the conservation of total energy is also applicable, one has

$$\frac{\partial \mathcal{W}_T}{\partial t} + \mathcal{P}_{dT} + \nabla^T \mathcal{P}_T = \frac{\partial \mathcal{S}_T}{\partial t} \tag{2.27}$$

Therefore, the difference between conservation of total energy and the conservation of electromagnetic energy is also a conservation law. That is,

$$\frac{\partial \mathcal{W}_m}{\partial t} + \mathcal{P}_{dm} + \nabla^T \mathcal{P}_m = \frac{\partial \mathcal{S}_m}{\partial t} \tag{2.28}$$

where

$$\mathcal{W}_m = \mathcal{W}_T - \mathcal{W}_e \tag{2.29}$$

$$\mathcal{P}_{dm} = \mathcal{P}_{dT} - \mathcal{P}_{de} \tag{2.30}$$

$$\mathcal{P}_m = \mathcal{P}_T - \mathcal{P}_e \tag{2.31}$$

and

$$\mathcal{S}_m = \mathcal{S}_T - \mathcal{S}_e \tag{2.32}$$

The subscript m is to denote medium energy density, medium power flux, and so forth.

Naturally all of the conservation laws may be expressed in transform space, $\mathcal{T}$; however, it is sufficient to only deal with the transform of the conservation of electromagnetic energy. Then

$$s\tilde{\mathcal{W}}_e + \tilde{\mathcal{p}}_{de} - \vec{\gamma}^{\,T}\tilde{\mathcal{P}}_e = s\tilde{\mathcal{S}}_e \tag{2.33}$$

where the tilde indicates the transform of the quantity.

2.4 Lossless Property of Media

Consider a harmonic electromagnetic field of the form

$$\mathcal{F} = \mathrm{Re}\left[\mathcal{F}_0\, e^{j(\omega t - \vec{k}\cdot\vec{r})}\right] \tag{2.34}$$

with

$$\dot{\mathcal{F}}_f = \mathrm{Re}\left[j\omega\, \underline{\underline{U}}(\vec{k},\omega)\mathcal{F}_0\, e^{j(\omega t - \vec{k}\cdot\vec{r})}\right] \tag{2.35}$$

By averaging the quantity $\mathcal{F}^T\dot{\mathcal{F}}_f$ with respect to time and equating this steady change in energy to zero, we can arrive at the condition for lossless media. The terms involving $e^{\pm j\omega t}$ will have a zero average, leaving

$$\left[\mathcal{F}_0^\dagger\left(j\omega\underline{\underline{U}}(\vec{k},\omega)\right)\mathcal{F}_0 + \mathcal{F}_0^\dagger\left(j\omega\underline{\underline{U}}(\vec{k},\omega)\right)^\dagger\mathcal{F}_0\right] = 0 \tag{2.36}$$

or

$$j\omega\mathcal{F}_0^\dagger\left[\underline{\underline{U}}(\vec{k},\omega) - \underline{\underline{U}}^\dagger(\vec{k},\omega)\right]\mathcal{F}_0 = 0 \tag{2.37}$$

But $\mathcal{F}_0$ is arbitrary and hence the constitutive matrix is Hermitian for real ω and $\vec{k}$, i.e.,

$$\underline{\underline{U}}(\vec{k},\omega) = \underline{\underline{U}}^\dagger(\vec{k},\omega) \tag{2.38}$$

2.5 "Energy" Condition and Group Velocity[3]

Consider a lossless medium. Hence, for real ω and $\vec{k}$ we have,

$$\underline{\underline{U}}(\vec{k},\omega) = \underline{\underline{U}}^{\dagger}(\vec{k},\omega) \tag{2.39}$$

and

$$O(\vec{k}) = -O^{\dagger}(\vec{k}) \tag{2.40}$$

Therefore, the matrix operator $\mathcal{M}(\vec{k},\omega)$ defined as,

$$\mathcal{M}(\vec{k},\omega) = O(\vec{k}) - j\omega\,\underline{\underline{U}}(\vec{k},\omega) \tag{2.41}$$

is skew-Hermitian. A source-free solution to Maxwell's equations, F, in space $\mathcal{T}$ exists when

$$\mathcal{M}(\vec{k},\omega)\,F(\vec{k},\omega) = 0 \tag{2.42}$$

or

$$(\mathcal{M}F)^{\dagger} = F^{\dagger}\mathcal{M}^{\dagger} = -F^{\dagger}\mathcal{M} = 0 \tag{2.43}$$

Allow small perturbations to occur in $\omega,\vec{k}$ and the medium such that

$$\omega_1 = \omega + \delta\omega \tag{2.44}$$

and

$$\vec{k}_1 = \vec{k} + \delta\vec{k} \tag{2.45}$$

Then a new field solution $F_1(\vec{k}_1,\omega_1)$ may be obtained for the perturbed system such that,

$$\mathcal{M}_1(\vec{k}_1,\omega_1)\,F_1(\vec{k}_1,\omega_1) = 0 \tag{2.46}$$

$\mathcal{M}_1$ is not required to be skew-Hermitian. Because the perturbations are small, however, we can write

$$\mathcal{M}_1(\vec{k}_1, \omega_1) = \mathcal{M}(\vec{k}, \omega) + \delta\omega \frac{\partial}{\partial \omega} \mathcal{M}(\vec{k}, \omega) + \delta\vec{k} \cdot \nabla_k \mathcal{M}(\vec{k}, \omega) + \delta \mathcal{M}(\vec{k}, \omega) \tag{2.47}$$

where $\delta \mathcal{M}(\vec{k}, \omega)$ represents any perturbation of the medium not due to a change in ω or $\vec{k}$. Therefore, by using Equations (2.46) and (2.47) one obtains,

$$F^{\dagger}\left[\mathcal{M} + \delta\omega \frac{\partial}{\partial \omega} \mathcal{M} + \delta\vec{k} \cdot \nabla_k \mathcal{M} + \delta \mathcal{M}\right] F_1 = 0 \tag{2.48}$$

However, from Equation (2.43) it is seen that

$$F^{\dagger} \mathcal{M} F_1 = 0 \tag{2.49}$$

and hence

$$F^{\dagger}\left[\delta\omega \frac{\partial}{\partial \omega} \mathcal{M} + \delta\vec{k} \cdot \nabla_k \mathcal{M} + \delta \mathcal{M}\right] F_1 = 0 \tag{2.50}$$

Since all of the terms in the brackets are small, the difference between F_1 and F may be neglected, yielding,

$$F^{\dagger}\left[\delta\omega \frac{\partial}{\partial \omega} \mathcal{M} + \delta\vec{k} \cdot \nabla_k \mathcal{M} + \delta \mathcal{M}\right] F = 0 \tag{2.51}$$

Now let us consider each term of Equation (2.51).

$$F^{\dagger} \frac{\partial}{\partial \omega} \mathcal{M} F = -jF^{\dagger} \frac{\partial}{\partial \omega}(\omega \underline{\underline{U}}) F = -j4\overline{W} \tag{2.52}$$

$\overline{W}$ is a generalization of what is commonly called the average electromagnetic energy density due to a harmonic field. This is discussed further in Section

2.9. From the second term we have

$$F^{\dagger}\left[\nabla_k \mathcal{M}\right]F = 4j\left[\bar{P}_e + \bar{P}_m\right] \tag{2.53}$$

where,

$$\bar{P}_e = -\frac{1}{4}jF^{\dagger}\left[\nabla_k O\right]F = 1/2\,\mathrm{Re}\left[E \times H^*\right] \tag{2.54}$$

= electromagnetic Poynting vector for a harmonic field

and,

$$\bar{P}_m = -\frac{1}{4}\omega\, F^{\dagger}\left[\nabla_k \underline{\underline{U}}\right] F \tag{2.55}$$

= medium Poynting vector for a harmonic field.

The third term reduces to

$$F^{\dagger}\left[\delta \mathcal{M}\right]F = -j\omega F^{\dagger}\left[\delta \underline{\underline{U}}\right]F \tag{2.56}$$

Putting the component terms back into Equation (2.51) gives,

$$\overrightarrow{\delta k}\cdot\left(\bar{P}_e + \bar{P}_m\right) = \delta\omega \bar{W} + \frac{1}{4}\,\omega F^{\dagger}\left[\delta\, \underline{\underline{U}}\right]F \tag{2.57}$$

Now for an unperturbed medium $\delta \underline{\underline{U}} = 0$, and we see that the group velocity is given by,

$$\vec{V}_g = \frac{\delta\omega}{\overrightarrow{\delta k}} = \nabla_k \omega = \left(\bar{P}_e + \bar{P}_m\right)/\bar{W} \tag{2.58}$$

Also when $\delta\omega = 0$, i.e., when the refractive index surface is given for a fixed frequency, the group velocity and the total power flux vector $\bar{P}_T = \bar{P}_e + \bar{P}_m$ are normal to the dispersion surface. It should be emphasized that the only stipulation made upon the medium is that it be lossless.

2.6 Onsager's Property

For the sake of completeness let us mention Onsager's principle as applied to the constitutive relationship. If the sixth-order matrix $\underline{\underline{W}}$ is

$$\underline{\underline{W}} = \begin{bmatrix} \underline{\underline{I}} & 0 \\ 0 & -\underline{\underline{I}} \end{bmatrix} \tag{2.59}$$

the Onsager's property can be expressed as

$$\underline{\underline{U}}^T(\vec{k},\omega,\vec{H}_0) = \underline{\underline{W}}\,\underline{\underline{U}}(\vec{k},\omega,-\vec{H}_0)\underline{\underline{W}} \tag{2.60}$$

For a description of the details refer to Onsager,[4] Meixner,[5] Casimir,[6] or De Groot.[7]

2.7 Real Property of Media

Since both the field strength $\mathcal{F}$ and the flux field $\mathcal{F}_f$ are real valued vector fields, the definition of the transform and Equation (2.6) implies that there exists a real property of the medium, namely,

$$\underline{\underline{U}}^*(s,\vec{\gamma}) = \underline{\underline{U}}(s^*,\vec{\gamma}^*) \tag{2.61}$$

where the star, *, indicates the complex conjugate, and $s = \sigma + j\omega$ and $\vec{\gamma} = j\vec{k}$. More explicitly, $\mathcal{F}$ and $\mathcal{F}_f$ real implies that

$$F^*(s,\vec{\gamma}) = F(s^*,\vec{\gamma}^*) \tag{2.62}$$

and

$$F_f^*(s,\vec{\gamma}) = F_f(s^*,\vec{\gamma}^*) \tag{2.63}$$

respectively. But the constitutive relationship implies

$$F_f(s,\vec{\gamma}) = \underline{U}(s,\vec{\gamma})F(s,\vec{\gamma}) \tag{2.64}$$

Therefore,

$$F_f^*(s,\vec{\gamma}) = \underline{\underline{U}}^*(s,\vec{\gamma})F^*(s,\vec{\gamma}) \tag{2.65}$$

and

$$F_f(s^*,\vec{\gamma}^*) = \underline{\underline{U}}(s^*,\vec{\gamma}^*)F^*(s,\vec{\gamma}) \tag{2.66}$$

Subtracting Equation (2.66) from (2.65) and using Equations (2.62) and (2.63) results in

$$0 = \left[\underline{\underline{U}}^*(s,\vec{\gamma}) - \underline{\underline{U}}(s^*,\vec{\gamma}^*)\right] F^*(s,\vec{\gamma}) \tag{2.67}$$

The vector $F^*(s,\vec{\gamma})$ is arbitrary, however. Therefore,

$$\underline{\underline{U}}^*(s,\vec{\gamma}) = \underline{\underline{U}}(s^*,\vec{\gamma}^*) \tag{2.68}$$

An immediate consequence is that if the constitutive matrix $\underline{\underline{U}}$ is independent of both s and $\vec{\gamma}$, then it is a real matrix. This property does not seem to have been fully exploited in the literature.

2.8 Positive Real Property

It is quite well known that the necessary and sufficient conditions for a function to be the driving point impedance of a linear passive network is that the function be a positive real function as originally defined by Brune[8] in 1931. At that time Foster had already presented a method for synthesizing certain types of one port networks. This greatly reduced the difficulty of the sufficiency proof. The significance, however, of Brune's positive real theorem was that it established a logical foundation for and an impetus to the development of network synthesis. Also, it has been established that the impedance and admittance matrices of passive networks are positive real matrices. With this as a background, it would be desirable to establish similar necessary and sufficient conditions on the constitutive matrix or perhaps some function of the constitutive matrix for passive media. The degree of difficulty of such a proof over that of Brune's for networks will be enhanced for at least two reasons: (1) Very little is known about synthesis of media, particularly the synthesis of dispersive properties; (2) The proof will include the wave vector $\vec{k}$ as a parameter and thus will require extra consideration and conditions upon it. Heuristically, one might expect that in the limit as the wave vector $\vec{k}$ approaches zero or as the wavelength approaches infinity a correspondence exists between the properties for passive media and passive networks.

Let us now establish the necessary "positive real" property of passive media as a theorem.

Theorem 2.1: The function of the constitutive matrix $Z(s,\vec{\gamma}) = s\underline{\underline{U}}(s,\vec{\gamma})$ is a "positive real" matrix for a linear passive medium, where $s = \sigma + j\omega$ and $\vec{\gamma} = j\vec{k}$ are the complex transform time number and space vector, respectively; i.e.,

(1) $Z^{*}(s,\vec{\gamma}) = Z(s^{*},\vec{\gamma}^{*})$

(2) $\mathrm{Re}\, s \geq 0$ implies $\mathrm{Re}\,\mathcal{F}^{\dagger} Z \mathcal{F} \geq 0,\ \forall \mathcal{F}$

Note that the restriction of $\vec{\gamma} = j\vec{k}$ where $\vec{k}$ is real is necessary for the convolution operation in space-time which gives $\mathcal{F}_f = \underline{\underline{\mathcal{U}}} * \mathcal{F}$ to exist.

Proof: Part (1) follows immediately from Section 2.7 and the definition $Z(s,\vec{\gamma})$. That is,

$$Z^*(s,\vec{\gamma}) = s^* \underline{\underline{U}}^*(s,\vec{\gamma}) = s^* \underline{\underline{U}}(s^*,\vec{\gamma}^*) = Z(s^*,\vec{\gamma}^*)$$

Recall that

$$\frac{\partial \mathcal{W}_e}{\partial t} + \wp_{de} = \mathcal{F}^T \dot{\mathcal{F}}_f \tag{2.69}$$

Both $\mathcal{W}_e(t)$ and $\wp_{de}(t)$ are positive definite or semidefinite for fields in passive media; therefore,

$$\mathcal{W}_e(t) + \int_{t_0}^{t} \wp_{de}\, dt = \int_{t_0}^{t} \mathcal{F}^T \dot{\mathcal{F}}_f\, dt + \mathcal{W}_e(t_0) \geq 0 \tag{2.70}$$

This is true for any field; thus, it should apply to any particular field. Let a particular field be $\mathcal{F} = 2\,\mathrm{Re}\,\mathcal{F}_1$ where $\mathcal{F}_1 = \mathcal{F}_0 e^{(s_0 t - \vec{\gamma}_0 \cdot \vec{r})}$ and $\mathcal{F}_0$ is a complex field vector independent of space and time. The flux field is then

$$\mathcal{F}_f = \underline{\underline{\mathcal{U}}} * \mathcal{F} \tag{2.71}$$

or

$$F_f = \underline{\underline{U}} F \tag{2.72}$$

Now

$$F = \mathcal{F}_0 \delta(s-s_0)\delta(\vec{\gamma}-\vec{\gamma}_0) + \mathcal{F}_0^* \delta(s-s_0^*)\delta(\vec{\gamma}-\vec{\gamma}_0^*) \tag{2.73}$$

and

$$F_f = \underline{\underline{U}}(s,\vec{\gamma})\, \mathcal{F}_0 \delta(s-s_0)\delta(\vec{\gamma}-\vec{\gamma}_0) + \underline{\underline{U}}(s,\vec{\gamma})\, \mathcal{F}_0^* \delta(s-s_0^*)\delta(\vec{\gamma}-\vec{\gamma}_0^*) \tag{2.74}$$

Therefore,

$$\mathcal{F}_f = \underline{\underline{U}}(s_0,\vec{\gamma}_0)\, \mathcal{F}_0 e^{(s_0 t - \vec{\gamma}_0 \cdot \vec{r})} + \underline{\underline{U}}(s_0^*,\vec{\gamma}_0^*)\, \mathcal{F}_0 e^{(s_0^* t - \vec{\gamma}_0^* \cdot \vec{r})} \tag{2.75}$$

However, because of the real property, $\underline{\underline{U}}(s_0^*, \vec{\gamma}_0^*) = \underline{\underline{U}}^*(s_0, \vec{\gamma}_0)$, the flux field and its time derivative then becomes, respectively,

$$\mathcal{F}_f = 2\,\mathrm{Re}\left[\underline{\underline{U}}(s_0, \vec{\gamma}_0)\,\mathcal{F}_0 e^{(s_0 t - \vec{\gamma}_0 \cdot \vec{r})}\right] \tag{2.76}$$

and

$$\dot{\mathcal{F}}_f = 2\,\mathrm{Re}\left[Z(s_0, \vec{\gamma}_0)\,\mathcal{F}_0 e^{(s_0 t - \vec{\gamma}_0 \cdot \vec{r})}\right] \tag{2.77}$$

Now it is easily shown that

$$\mathcal{F}^T \dot{\mathcal{F}}_f = 2\,\mathrm{Re}\left[\mathcal{F}_1^{\dagger} Z(s_0, \vec{\gamma})\,\mathcal{F}_1 + \mathcal{F}_1^T Z(s_0, \vec{\gamma}_0)\,\mathcal{F}_1\right] \tag{2.78}$$

or

$$\mathcal{F}^T \dot{\mathcal{F}}_f = 2\,\mathrm{Re}\left[\mathcal{F}_0\, Z(s_0, \vec{\gamma}_0)\,\mathcal{F}_0\, e^{2\sigma_0 t} + \mathcal{F}_0^T\, Z(s_0, \vec{\gamma}_0)\,\mathcal{F}_0\, e^{2(s_0 t - \vec{\gamma}_0 \cdot \vec{r})}\right] \tag{2.79}$$

Using the expression given in Equation (2.79) as the integrand of Equation (2.70) and integrating results in,

$$\begin{aligned}
&\mathrm{Re}\left[\frac{1}{\sigma_0}\mathcal{F}_0^{\dagger} Z(s_0, \vec{\gamma}_0)\,\mathcal{F}_0 e^{2\sigma_0 t} + \frac{1}{s_0}\,\mathcal{F}_0^T Z(s_0, \vec{\gamma}_0)\,\mathcal{F}_0 e^{2(s_0 t - \vec{\gamma}_0 \cdot \vec{r})}\right] \\
&-\mathrm{Re}\left[\frac{1}{\sigma_0}\mathcal{F}_\theta^{\dagger} Z(s_0, \vec{\gamma}_0)\,\mathcal{F}_0 e^{2\sigma_0 t_0} + \frac{1}{s_0}\mathcal{F}_0^T Z(s_0, \vec{\gamma}_0)\,\mathcal{F}_0 e^{2(s_0 t_0 - \vec{\gamma}_0 \cdot \vec{r})}\right] \\
&+\mathcal{W}_e(t_0) \geq 0
\end{aligned} \tag{2.80}$$

For $\sigma_o > 0$ and for t large enough, the values at t_o, i.e., exp $(2\,\sigma_o t_o)$ and exp $(2\,s_o t_o)$, are small compared to those at t. Also the electromagnetic energy density at t_o, $\mathcal{W}_e(t_0)$, is small. Then

$$e^{2\sigma_0 t}\ \mathrm{Re}\left[\frac{1}{\sigma_0}\mathcal{F}_0^{\dagger} Z(s_0,\vec{\gamma}_0)\mathcal{F}_0 + \frac{1}{s_0}\mathcal{F}_0^{T} Z(s_0,\vec{\gamma}_0)\mathcal{F}_0 e^{2j(\omega_0 t - \vec{k}_0\cdot\vec{r})}\right] \geq 0 \quad (2.81)$$

and

$$\mathrm{Re}\left[\frac{1}{\sigma_0}\mathcal{F}_0^{\dagger} Z(s_0,\vec{\gamma}_0)\mathcal{F}_0 + \frac{1}{s_0}\mathcal{F}_0^{T} Z(s_0,\vec{\gamma}_0)\mathcal{F}_0 e^{2j(\omega_0 t - \vec{k}_0\cdot\vec{r})}\right] \geq 0 \quad . \quad (2.82)$$

Since the second term of Equation (2.82) takes both positive and negative values, the first term must be non-negative, i.e.,

$$\mathrm{Re}\ s_0 \geq 0 \ \text{ implies } \ \mathrm{Re}\left[\mathcal{F}_0^{\dagger} Z(s_0,\vec{\gamma}_0)\mathcal{F}_0\right] \geq 0,\ \forall \mathcal{F}_0 \quad (2.83)$$

Therefore, as was to be proven, the matrix $Z(s,\vec{\gamma}) = s\,\underline{U}(s,\vec{\gamma})$ is a "positive real" matrix.

In practice it is difficult to prove or disprove that a medium satisfies the positive real condition from the definition alone. For this reason let us prove a lemma that will facilitate this work.

Lemma 2.1: Let $Q(A) = X^{\dagger} A X$ and $Q(A_D) = X^{\dagger} A_D X$ where X is an arbitrary vector, and A_D is the diagonalized matrix of A Then Re $s \geq 0$ implies Re $Q(\underline{\underline{A}}) \geq 0$, $\forall X$ if and only if Re $s \geq 0$ implies Re $Q(A_D) \geq 0$, $\forall X$.

Proof: Let T be a matrix whose columns are eigenvectors of A. Also, let T be normalized such that $T^{\dagger} T = I$. Then when the vector Y of the transformation $X = TY$ takes all possible values, X takes all possible values and vice versa. But

$$X^{\dagger}AX = (TY)^{\dagger}A(TY) = Y^{\dagger}(T^{\dagger}AT)\,Y = Y^{\dagger}A_D Y \qquad \forall X,Y \quad (2.84)$$

Therefore,

$$\mathrm{Re}\, s \geq 0 \Longrightarrow \mathrm{Re}\, Q(A) \geq 0,\ \forall X \quad \text{if and only if} \quad \mathrm{Re}\, s \geq 0 \Longrightarrow \mathrm{Re}\, Q(A_0) \geq 0,\ \forall Y \qquad (2.85)$$

Thus, to prove the positive part of the "positive real" theorem, it is sufficient to show that for Re s $\geq$ 0 the real parts of the eigenvalues of $Z(s,\vec{k})$ are greater than or equal to zero.

2.9 Implications of the Positive Real Property[9]

Several implications are a direct result of the positive real property. For instance, it is possible to derive a result analogous to Foster's reactance theorem. Before doing so, however, let us consider the average electric energy density in a lossless medium. It is quite commonly stated that the average electric energy density of a harmonic field at frequency ω is

$$(1/2)\, E^{+} \frac{\partial(\omega \underline{\underline{\epsilon}})}{\partial \omega} E \qquad (2.86)$$

This equation is only an approximation, which to be sure is a better approximation than $(1/2)E^{+} \underline{\underline{\epsilon}}\, E$. But the energy density is dependent upon the entire history of the fields in the medium. Therefore, one would expect that the energy density at frequency ω be a function of all of the derivatives with respect to ω, depending on the manner the amplitude rises from zero. It is not even intuitively obvious why the approximation given by Equation (2.86) should be a positive number. The following theorem will prove it to be the case for lossless passive medium.

Theorem 2.2: For a lossless passive medium,

$$F^{+} \frac{\partial}{\partial \omega}\left(\omega \underline{U}(\omega,\vec{k})\right) F \geq 0 \qquad \forall F \qquad (2.87)$$

Proof: Let $Q(s,\vec{k})$ be defined by $Q(s,\vec{k}) = F\, Z(s,\vec{k})\, F$. For $s = j\omega$, $Q(s,\vec{k})$ is imaginary since $\underline{\underline{U}}(\omega,\vec{k})$ is Hermitian. Expand $Q(s,\vec{k})$ in a Taylor series about a point $s = j\omega$ and evaluate it at the point in the right half plane $s = s_o$.

$$Q(s_o,\vec{k}) - Q(j\omega,\vec{k}) = \sum_{n=1} \frac{1}{n!} \left.\frac{\partial^n Q(s,\vec{k})}{\partial s^n}\right|_{s=j\omega} (s_o - j\omega)^n \tag{2.88}$$

As s_o approaches $j\omega$, the first term of the series will be the predominant term. Define the following terms,

$$\alpha = \arg\left[Q(s_o,\vec{k}) - Q(j\omega,\vec{k})\right] \tag{2.89}$$

$$\beta = \arg\left[s_o - j\omega\right] \tag{2.90}$$

$$\gamma = \arg\left[\left.\frac{\partial Q(s,\vec{k})}{\partial s}\right|_{s=j\omega}\right] \tag{2.91}$$

Therefore, in the limit Equation (2.88) requires,

$$\lim_{s_o\to j\omega} \alpha = \lim_{s_o\to j\omega} \beta + \gamma \tag{2.92}$$

The "positive real" condition requires that $|\alpha| \leq \pi/2$ for $|\beta| \leq \pi/2$. Therefore, for Equation (2.92) to be satisfied, γ must be zero. Hence,

$$\left.\frac{\partial Q(s,\vec{k})}{\partial s}\right|_{s=j\omega} \geq 0 \tag{2.93}$$

or

$$\frac{\partial}{\partial\omega}\left[F^{\dagger}\left(\omega U(\omega,\vec{k})\right)F\right] \geq 0 \tag{2.94}$$

But since F is arbitrary, we have the final result

$$F^{\dagger}\left[\frac{\partial}{\partial\omega}\left(\omega \underline{\underline{U}}(\omega,\vec{k})\right)\right]F \geq 0, \qquad \forall F \tag{2.95}$$

Now it is a trivial matter to see that Equation (2.86) is a special case of Theorem 2.2.

A few important residue conditions that are a result of the "positive real" condition are as follows:

<u>Theorem 2.3:</u> (1) There are no poles of $F^{\dagger} Z(s,\vec{k}) F$ in the half plane $\mathrm{Re}\, s > 0$.

(2) The poles of $F^{\dagger} Z(s,\vec{k}) F$ on the $j\omega$ axis are simple and the residues are positive real.

Proof: Assume that there is a pole of order n at $s = s_o$, $\mathrm{Re}\ s_o > 0$. The Laurent expansion of $Q(s) = F^{\dagger} Z(s,\vec{k}) F$ in the neighborhood of the pole s_o is of the form

$$Q(s) = \sum_{i=-n}^{\infty} a_i (s-s_0)^i \tag{2.96}$$

If s is sufficiently close to s_o, the term corresponding to $i = -n$ is predominate. Then

$$\mathrm{Re}\, Q(s) \approx \frac{|a_{-n}|}{|s-s_o|^n} \cos\left[\arg(a_{-n}) - n \arg(s-s_o)\right] \tag{2.97}$$

Since $\arg(a_{-n})$ is independent of s and $\arg(s - s_o)$ ranges from 0 to 2π for $\mathrm{Re}\ s > 0$, then the dominant part of the Laurent expansion changes sign 2n times in the neighborhood of s_o. Therefore, there exists and s such that $\mathrm{Re}\ s > 0$ and $\mathrm{Re}\ Q(s) < 0$. But this contradicts the "positive real" theorem and thus there can be no poles of $F^{\dagger} Z(s,\vec{k}) F$ in the half plane $\mathrm{Re}\ s > 0$. If $s_o = j\omega_o$, then the dominant part of the Laurent expansion satisfies the "positive real" theorem when $\arg(a_{-n}) = 0$ and $n = 1$. Therefore, the poles of $F^{\dagger} Z(s,\vec{k}) F$ on the $j\omega$ axis are simple and the residues are positive real.

Another theorem follows from Part (2) of Theorem 2.3.

Theorem 2.4: The matrix of residues of $Z(s,\vec{k})$ at any poles on the $j\omega$ -axis must be positive definite or positive semidefinite.

Proof: Let the vector F be real and $s = j\omega_o$ be an arbitrary pole of $F^{\dagger} Z(s,\vec{k}) F$ on the $j\omega$ -axis. From Part (2) of the previous theorem, such a pole must be simple and the residue of $F^{\dagger} Z(s,\vec{k}) F$ be positive real. Therefore,

$$\text{Residue}\left[F^T Z(s,k) F\right] = \lim_{s \to j\omega_0} (s - j\omega_o) F^T Z(s,\vec{k}) F \geq 0 \qquad \forall \text{ real } F \tag{2.98}$$

$$= F^T \left[\lim_{s \to j\omega_o} (s - j\omega_o) Z(s,\vec{k})\right] F \geq 0 \qquad \forall \text{ real } F$$

But $\lim_{s \to j\omega_o} (s-j\omega_o) Z(s,\vec{k})$ is the matrix of residues of $Z(s,\vec{k})$ at $s = j\omega_o$. Therefore, the theorem is complete.

One other condition upon the constitutive relationship, which is implied by the "positive real" theorem, is the following:

Theorem 2.5: The matrix of the real part of $Z(s,\vec{k})$ must be positive definite or semidefinite for $\text{Re } s \geq 0$.

Proof: Since $\text{Re}\,[F^{\dagger} Z(s,\vec{k}) F] \geq 0$ $\forall F$ for $\text{Re } s \geq 0$, it will certainly be valid for F real. But for real F, the Re operator commutes with F,

$$\text{Re}\left[F^{\dagger} Z(s,\vec{k}) F\right] = F^T \left[\text{Re}\, Z(s,\vec{k})\right] F \tag{2.99}$$

Therefore,

$$\text{Re } s \geq 0 \text{ implies } F^T\left[\text{Re}\, Z(s,\vec{k})\right]F \geq 0 \tag{2.100}$$

and the theorem is complete.

Let us verify that the cold magneto-ionic medium satisfies the "positive real" conditions. For such a medium the constitutive relationship may be given by a matrix of the form,

$$\underline{\underline{U}} = \begin{bmatrix} \underline{\underline{\epsilon}} & 0 \\ 0 & \mu_0 I \end{bmatrix} \tag{2.101}$$

where

$$\underline{\underline{\epsilon}} = \begin{bmatrix} \epsilon' & \epsilon'' & 0 \\ -\epsilon'' & \epsilon' & 0 \\ 0 & 0 & \epsilon^o \end{bmatrix} \tag{2.102}$$

and

$$\epsilon' = (\epsilon_0/s)\left(\frac{s^3 + 2\nu s^2 + (\nu^2 + \omega_N^2 + \omega_H^2)s + \nu\omega_H^2}{s^2 + 2\nu s + (\nu^2 + \omega_H^2)}\right) \tag{2.103}$$

$$\epsilon'' = (\epsilon_0/s)\left(\frac{\omega_H^2\omega_N^2}{s^2 + 2\nu s + (\nu^2 + \omega_H^2)}\right) \tag{2.104}$$

$$\epsilon^o = (\epsilon_0/s)\left(\frac{s^2 + \nu s + \omega_N^2}{s + \nu}\right) \tag{2.105}$$

Since all of the parameters of Z(s) are real, it is readily apparent that the real condition is satisfied, i.e., $Z(s) = Z^*(s^*)$. The six eigenvalues of $Z(s) = s\,\underline{\underline{U}}(s)$ are,

$$\lambda_{1,2} = \epsilon_0 s + \frac{\epsilon_0 \omega_N^2}{s+(\nu \pm j\omega_H)} \qquad (2.106)$$

$$\lambda_3 = \epsilon_0 s + \frac{\epsilon_0 \omega_N^2}{s+\nu} \qquad (2.107)$$

$$\lambda_{4,5,6} = \mu_0 s \qquad (2.108)$$

while the real parts of the eigenvalues are

$$\mathrm{Re}\left[\lambda_{1,2}\right] = \epsilon_0 \sigma + \frac{\epsilon_0 \omega_N^2 (\sigma+\nu)}{(\sigma+\nu)^2 + (\omega \pm \omega_H)^2} \geq 0 , \quad \sigma \geq 0 \qquad (2.109)$$

$$\mathrm{Re}\left[\lambda_3\right] = \epsilon_0 \sigma + \frac{\epsilon_0 \omega_N^2 \sigma}{\sigma^2 + \omega^2} \geq 0 , \quad \sigma \geq 0 \qquad (2.110)$$

$$\mathrm{Re}\left[\lambda_{4,5,6}\right] = \mu\, \sigma \geq 0 , \quad \sigma \geq 0 \qquad (2.111)$$

Therefore, the cold magneto-ionic medium satisfies the positive real condition.

2.10 Causality Property

The energy condition is not sufficient for a constitutive matrix to represent a realizable passive medium. Even though there exists a causality condition for circuits, it does not play as important a role as the causality condition does for distributed systems, the reason being that the velocity of propagation is assumed to be infinite in circuits. In distributed systems in which characteristic lengths may be large compared to a wavelength, causality implies that the wave front can travel no faster than the velocity of light in a vacuum. Hence, causality for distributed systems should be stronger than causality for circuits. And one might even expect that if the velocity of light in a vacuum were mathematically forced to approach infinity the causality condition for distributed systems would approach the causality condition for circuits. Thus, the problem is to use the causality condition to derive extra necessary conditions upon the constitutive matrix for realizable passive media.

Consider an infinite homogeneous passive medium in which a Dirac delta electric or magnetic current source both in space and time is placed at the origin of a coordinate system. Then causality states that for the field $\mathcal{F}(\vec{r},t)$,

$$\mathcal{F}(\vec{r},t) = 0 \quad \text{for} \quad t - \frac{r}{c} < 0 \tag{2.112}$$

where c is the velocity of light in a vacuum. $\mathcal{F}(\vec{r},t)$ may be represented as a Laplace transform in time,

$$\mathcal{F}(\vec{r},t) = (1/2\pi j) \int_{\sigma_o - j\infty}^{\sigma_o + j\infty} F(\vec{r},s)\, e^{st} ds \tag{2.113}$$

where σ_o is a finite constant for which $F(\vec{r},s)$ has no singularities in the half plane $\text{Re}\ s \geq \sigma_o \geq 0$. Evaluate Equation (2.113) for $t - r/c < 0$ by closing the contour in the right half plane. Then we have

$$\mathcal{F}(\vec{r},t) = (-1/2\pi j) \lim_{R \to \infty} \int_{C_R} F(\vec{r},s)\, e^{\frac{sr}{c}} e^{s(t-\frac{r}{c})} ds = 0\,, \quad t - \frac{r}{c} < 0 \tag{2.114}$$

where C_R is a semicircular path in the clockwise direction of radius R. Assume that the integrand satisfies Jordan's lemma, i.e.,

$$\left|F(\vec{r},s)e^{\frac{sr}{c}}\right| \leq \frac{M(\vec{r})}{|s|^k} \text{ when } |s|>R_0 \text{ for } \mathrm{Re}\, s>\sigma_0 \tag{2.115}$$

where K is a positive constant and $\overrightarrow{M(r)}$ is some vector independent of s. Then

$$\left|F(\vec{r},s)\right| \leq \frac{\overrightarrow{M(r)}}{|s|^k} e^{-\frac{r}{c}\mathrm{Re}\, s} \tag{2.116}$$

when $|s| > R_o$, Re $s \geq \sigma_o$, for all $\vec{r}$. Therefore, we expect that for any realizable medium the Laplace time transform of the field due to a Green's (Dirac delta) source should obey this necessary condition. Note that it is easy to verify that a vacuum medium satisfies this condition.

Since it is difficult to find a more explicit condition upon the constitutive relationship for an arbitrary medium, assume that the medium is isotropic. Also assume that the elements of the constitutive matrix are ratios of two polynomials in the variables s, and $\vec{k}$. Let,

$$\det(O-s\underline{\underline{U}}) = D(s,k) \tag{2.117}$$

$$\det \underline{\underline{U}} = D_1(s,k)/D_2(s,k) \tag{2.118}$$

$$(O-s\underline{\underline{U}})^{-1} = \frac{\underline{\underline{N}}(s,\vec{k})}{D(s,k)} \tag{2.119}$$

where $D_1(s,k)$ and $D_2(s,k)$ are polynomials in the variables. Now we have,

$$(O-s\underline{\underline{U}})^{-1} = \frac{D_2(s,k)\,\underline{\underline{N}}(s,\vec{k})}{D_2(s,k)\,D(s,k)} \tag{2.120}$$

with the elements of $D_2(s,k)\ \underline{\underline{N}}(s,\vec{k})$ as polynomials in the variables. Therefore, the field due to an arbitrary Green's source $\mathcal{C}(\vec{r},t) = C\ \delta(\vec{r})\ \delta(t)$ is

$$F(\vec{r},s) = (2\pi)^{-3}\iiint_{-\infty}^{\infty}\frac{D_2(s,k)\underline{\underline{N}}(s,\vec{k})C}{D_2(s,k)D(s,k)}e^{-j\vec{k}.\vec{r}}d^3k \tag{2.121}$$

or

$$=(2\pi)^{-3}\left[D_2(s,j\nabla)\underline{\underline{N}}(s,j\nabla)\right]C\iiint_{-\infty}^{\infty}\frac{e^{-j\vec{k}.\vec{r}}}{D_2(s,k)D(s,k)}d^3k$$

since the elements of $D_2(s,k)\ \underline{\underline{N}}(s,\vec{k})$ are polynomials. By integrating in polar coordinates over the two angles one obtains,

$$F(r,s) = j(2\pi)^{-2}\left[D_2(s,j\nabla)\underline{\underline{N}}(s,j\nabla)\right]Cr^{-1}\int_{-\infty}^{\infty}\frac{e^{-jkr}\,k\,dk}{D_2(s,k)D(s,k)} \tag{2.122}$$

And finally a contour integration yields,

$$F(\vec{r},s) = \left[D_2(s,j\nabla)\underline{\underline{N}}(s,j\nabla)\right]C\sum_n \text{Residue}\left\{\frac{k}{D_2(s,k)D(s,k)}\right\}\frac{e^{-jk_n(s)r}}{2\pi r} \tag{2.123}$$

where $k_n(s)$ is a root of $D_2(s,k)\ D(s,k) = 0$ that satisfies the radiation condition. Therefore, by comparing Equations (2.116) and (2.123) we find that for an isotropic medium,

$$\text{Re}\left[jk_n(s)\right] \geq c^{-1}\text{Re } s\ ,\ |s|>R_0\ ,\ \text{Re } s \geq \sigma_0\ ,\ \forall n \tag{2.124}$$

where $k_n(s)$ is a root of the determinantal equation

$$\det\left[O(\vec{k}) - s\underline{\underline{U}}(s,\vec{k})\right] = 0 \tag{2.125}$$

that satisfies the radiation condition.

Now consider the special case of an isotropic medium in which both the permittivity and permeability are scalar functions of s only and of the form of the ratio of two polynomials. Then the causality condition implies,

$$\mathrm{Re}\left[jk_n(s)\right] = \mathrm{Re}\left[s\left(\mu(s)\epsilon(s)\right)^{1/2}\right] \geq c^{-1}\mathrm{Re}\, s\,,\quad |s| > R_0\,,\ \mathrm{Re}\, s \geq \sigma_0 \tag{2.126}$$

But, for both the permittivity and permeability, the positive real condition requires that the degree of the numerator minus the degree of the denominator polynomials is either, 0, -1, or -2. Therefore, for both the positive real condition and causality to be satisfied, the degree of the numerator equals the degree of the denominator for both $\mu(s)$ and $\epsilon(s)$.

2.11 Example of a Bandpass Waveguide

Finally, let us give an example of a medium that, if realized, can be advantagenously used to produce a bandpass waveguide. Consider a lossless medium whose permittivity and permeability are,

$$\epsilon(\omega) = K\frac{\omega_2^2-\omega^2}{\omega_1^2\ \omega^2}\,,\quad \mu = \mu_0 \tag{2.127}$$

Then s $\epsilon(s)$ satisfies the positive real condition if and only if ω_1 is less than or equal to ω_2. But

$$\lim_{s\to\infty}\left(\mu_0\epsilon(s)\right)^{1/2} = (\mu_0 K)^{1/2} \geq c^{-1} \tag{2.128}$$

implies that $K \geq \epsilon_o$. Suppose $K = \epsilon_o$. It is obvious that the permeability satisfies the necessary conditions. Now the medium inherently has a stop band between ω_1 and ω_2. Fill a uniform waveguide with this medium. For a uniform waveguide the propagation constant $\bar{\gamma}$ is

$$\bar{\gamma} = \left[\bar{c}^{2}\omega_{gc} - \omega^{2}\mu_{o}\epsilon_{o}(\omega)\right]^{\frac{1}{2}} \tag{2.129}$$

where ω_{gc} is the guide cutoff for a particular mode when the waveguide is filled with a vacuum. Then propagation exists for $\bar{\gamma}$ imaginary, or for frequencies ω such that

$$\omega^{2}\mu_{o}\epsilon(\omega) - \bar{c}^{2}\omega_{gc}^{2} > 0 \tag{2.130}$$

or $\omega^{2}(\omega_{2}^{2}-\omega^{2})(\omega_{1}^{2}-\omega^{2})^{-1} - \omega_{gc}^{2} > 0$. Figure 2.1 gives the regions of propagation. And the cutoff frequencies for the waveguide modes are

$$\omega_{c} = \pm\left[\frac{(\omega_{2}^{2}+\omega_{gc}^{2})}{2} \pm \left(\frac{(\omega_{2}^{2}+\omega_{gc}^{2})^{2}}{4} - \omega_{gc}^{2}\omega_{1}^{2}\right)^{\frac{1}{2}}\right]^{\frac{1}{2}} \tag{2.131}$$

Also observe that $\pm\,\omega_1$ and $\pm\,\infty$ are cluster points for the cutoff frequencies.

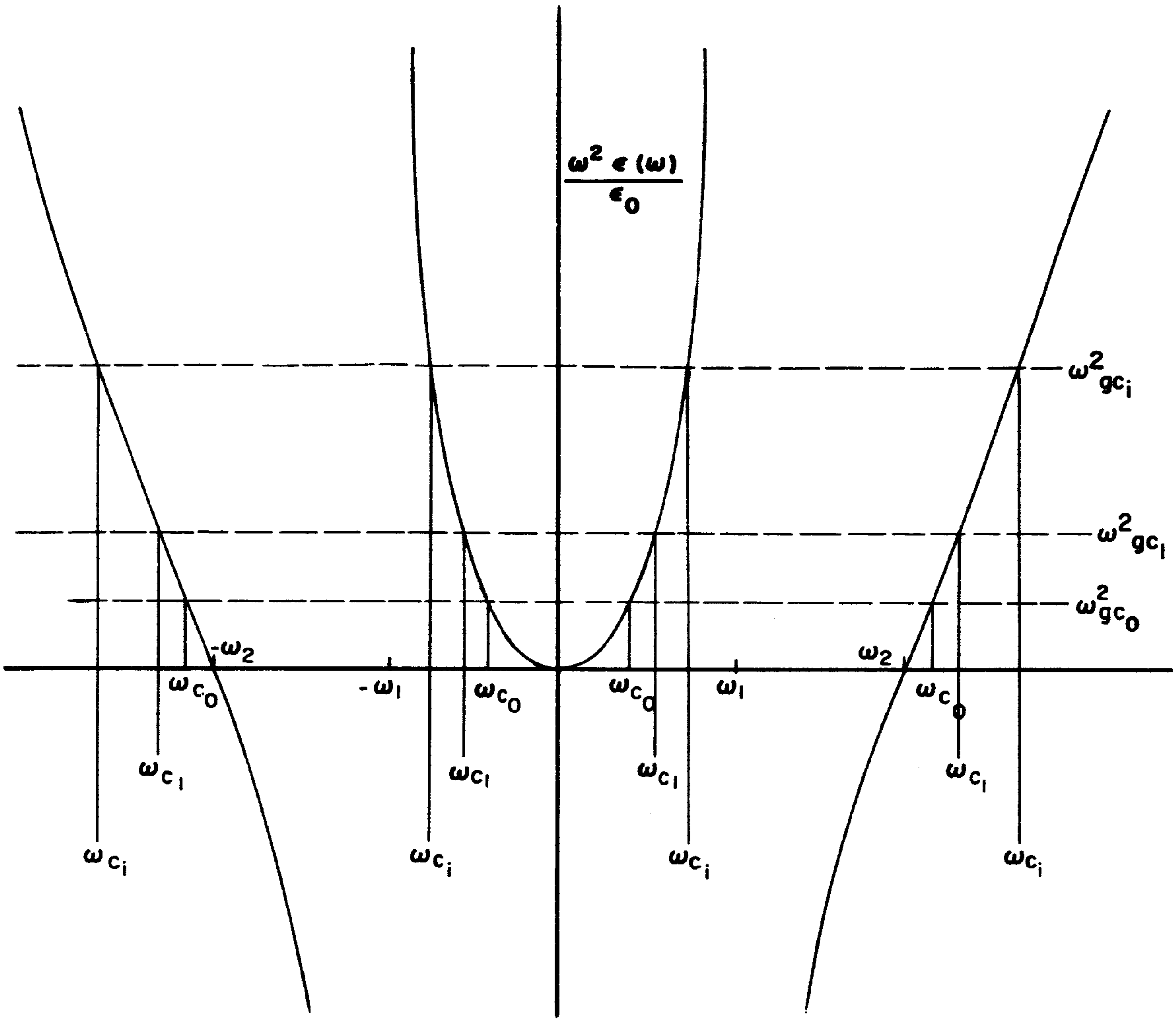

Figure 2.1. Cutoff frequencies and regions of propagation for modes in a uniform bandpass waveguide.

3. GENERAL FORMULATION OF THE SPECTRUM OF CHARACTERISTIC WAVES

3.1 Field of an Arbitrary Source in a General Anisotropic $(\underline{\underline{\mu}},\underline{\underline{\epsilon}})$ Lossless Medium as a Spectrum of Characteristic Waves

Let us consider the problem in which the medium in Fourier transform space is described by matrix permittivity, $\underline{\underline{\epsilon}}(\vec{k},\omega)$, and permeability, $\underline{\underline{\mu}}(\vec{k},\omega)$, even though no known natural media has such a constitutive relationship. Also assume that the elements of the matrices are arbitrary functions of ω and $\vec{k}$. With this formulation Maxwell's equations display a great deal of symmetry. And the symmetry is capable of revealing much as far as the form of the expected results are concerned.

The Fourier transformed Maxwell's equations are

$$
\begin{aligned}
-j\vec{k}\times E(\vec{k},\omega) &= -j\omega\mu_0 N H(\vec{k},\omega) - J_m(\vec{k},\omega) \\
-j\vec{k}\times H(\vec{k},\omega) &= j\omega\epsilon_0 K E(\vec{k},\omega) + J_e(\vec{k},\omega)
\end{aligned}
\tag{3.1}
$$

where $\underline{\underline{\epsilon}} = \epsilon_0 K(\vec{k},\omega)$ and $\underline{\underline{\mu}} = \mu_0 N(\vec{k},\omega)$. Eliminating **E** in Equation (3.1) gives the equation for **H**.

$$
(-\vec{k}\times K^{-1}\vec{k}\times - k_0^2 N)H(\vec{k},\omega) = -j\omega\epsilon_0 M_m(\vec{k},\omega) \tag{3.2}
$$

where

$$
M_m(\vec{k},\omega) = J_m(\vec{k},\omega) - (1/j\omega\epsilon_0)\left[-j\vec{k}\times K^{-1} J_e(\vec{k},\omega)\right] \tag{3.3}
$$

Then Equation (3.2) may be rewritten as,

$$
G_H(\vec{k},\omega)H(\vec{k},\omega) = -j\omega\epsilon_0 M_m(\vec{k},\omega) \tag{3.4}
$$

in terms of the matrix operator, $G_H(\vec{k},\omega)$,

$$G_H(\vec{k},\omega) = (-\vec{k} \times K^{-1}\vec{k} \times - k_o^2 N) \tag{3.5}$$

The source-free solutions may be found from Equations (3.4) and (3.5) with the source term equal to zero. Alternately, one can find the eigenvalues and eigenvectors[10] that correspond to the problem

$$-\vec{k} \times K^{-1}\vec{k} \times H_i = \lambda_i N H_i \qquad (i = 1,2,3) \tag{3.6}$$

and then equate the eigenvalue λ_i to k_o^2. This means that the source-free equations will be satisfied for the propagation vector, $\vec{k}$, on certain surfaces in Fourier space. These surfaces are prescribed by the equation det $[G_H(\vec{k},\omega)] = 0$. In terms of the eigenvalues of Equation (3.6) the det $[G_H(\vec{k},\omega)]$ is

$$(\det N)(\lambda_1-k_o^2)(\lambda_2-k_o^2)(\lambda_3-k_o^2) \text{ or } (\det N)S_1S_2S_3.$$

$(\lambda_i-k_o^2) = S_i = 0$ is a portion of the dispersion surface. For a cold plasma $(N = I,\ K \neq K(\vec{k}))$, $S_i = 0$ is one sheet of the dispersion surface; however, for a warm plasma, $(N = I,\ K = K(\vec{k}))$, $S_i = 0$ $(i = 1,2)$ may be more than one sheet.

Assuming that the medium is lossless, i.e., $K = K^{\dagger}$, $N = N^{\dagger}$, there exist certain orthogonality relationships between the eigenvectors H_i. First, however, it is necessary to show that the eigenvalues λ_i are real. Define A as $A = -\vec{k} \times K^{-1}\vec{k}\times$.

<u>Lemma 3.1</u>: The eigenvalues of the matrix equation $AH_i = \lambda_i NH_i$ $(i = 1,2,3)$ are real.

Proof:
$$A H_i = \lambda_i N H_i$$
$$H_i^\dagger A H_i = \lambda_i H_i^\dagger N H_i$$
$$H_i^\dagger A^\dagger H_i = \lambda_i^* H_i^\dagger N^\dagger H_i$$
$$H_i^\dagger A H_i = \lambda_i^* H_i^\dagger N H_i$$
$$(\because A \text{ and } N \text{ are Hermitian for real } \vec{k} \text{ and } \omega)$$

But
$$H_i^\dagger A H_i = \lambda_i H_i^\dagger N H_i$$

Therefore,
$$(\lambda_i - \lambda_i^*) H_i^\dagger N H_i = 0$$

Hence,
$$\text{i)} \quad \lambda_i = \lambda_i^*$$
or
$$\text{2)} \quad H_i^\dagger N H_i = 0$$

<u>Theorem 3.1</u>: The eigenvectors of the matrix equation $AH_i = \lambda_i NH_i$ $(i = 1,2,3)$ satisfy the orthogonality condition $H_j^\dagger NH_i = 0$ for $\lambda_i \neq \lambda_j$.

Proof:
$$A H_j = \lambda_j N H_j$$
$$H_i^\dagger A H_j = \lambda_j H_i^\dagger N H_j$$
$$H_j^\dagger A H_i = \lambda_j H_j^\dagger N H_i$$
$$(\because A, \underline{\underline{N}} \text{ are Hermitian and } \lambda_j \text{ is real})$$

But
$$H_j^\dagger A H_i = \lambda_i H_j^\dagger N H_i$$

Therefore,
$$(\lambda_i - \lambda_j) H_j^\dagger N H_i = 0$$

Hence,
$$H_j^\dagger N H_i = 0 \quad \text{for} \quad \lambda_i \neq \lambda_j$$

We observe that for lossless media, since the operator $-\vec{k} \times K^{-1}\vec{k} \times N^{-1}$ is the adjoint of $-N^{-1}\vec{k} \times K^{-1}\vec{k}\times$, then the set of eigenvectors, B_i, are the reciprocal basis[11] to the set of eigenvectors, H_i.

One of the eigenvalues, say λ_3, is zero. This fact immediately follows from the relationship

$$\det\,(-N^{-1}\vec{k}\times K^{-1}\vec{k}\times) = \lambda_1\lambda_2\lambda_3 \tag{3.7}$$

since the determinant of a product is equal to the product of the determinants of two matrices and since det $(\vec{k}x) = 0$. The eigenvector, H_3, corresponding to the eigenvalue $\lambda_3 = 0$ then may be chosen as $H_3 = \vec{k}$. From this it is evident that H_3 is longitudinal and that E_3 corresponding to H_3 is zero.

Choose the components of the eigenvectors, H_i and B_i to be polynomials in the transform variables ξ, η, ζ, ω with no common factors. This is always possible when the elements of K and N are the ratios of rational polynomials of the wavevector $\vec{k}$ and ω. Then the identity matrix, I, in terms of the eigenvectors is

$$I = \sum_{i=1}^{3} (H_i^{\dagger} N H_i)^{-1} N H_i H_i^{\dagger} \tag{3.8}$$

$$I = \sum_{i=1}^{3} (H_i^{\dagger} N H_i)^{-1} H_i H_i^{\dagger} N \tag{3.9}$$

The operator of Equation (3.8) operating on a vector splits the vector into its components that are parallel to B_i. Similarly, the operator of Equation (3.9) splits the vector into its components parallel to H_i. Equation (3.8) will be used in the interpretation of the field due to an arbitrary source.

Both $G_H(\vec{k},\omega)$ and its inverse $G_H^{-1}(\vec{k},\omega)$ can be expressed in terms of the eigenvalues and eigenvectors of Equation (3.6).

$$G_H(\vec{k},\omega) = \sum_{i=1}^{3} S_i \, (H_i^\dagger \, NH_i)^{-1} \, NH_i \, H_i^\dagger \, N \tag{3.10}$$

$$G_H^{-1}(\vec{k},\omega) = \sum_{i=1}^{3} S_i^{-1} (H_i^\dagger \, NH_i)^{-1} \, H_i \, H_i^\dagger \tag{3.11}$$

To verify that the expression for the inverse of $G_H(\vec{k},\omega)$ given in Equation (3.11) is correct, multiply $G_H(\vec{k},\omega)G_H^{-1}(\vec{k},\omega)$ on the right by the vector NH_j. Then

$$\begin{aligned} G_H G_H^{-1} NH_j &= G_H \sum_{i=1}^{3} S_i^{-1} (H_i^\dagger \, NH_i)^{-1} H_i \, H_i^\dagger NH_j \\ &= S_j^{-1} G_H H_j \\ &= S_j^{-1} S_j \, NH_j \qquad (\because \; G_H H_j = S_j \, NH_j) \\ &= NH_j \end{aligned}$$

Since any vector may be represented in terms of the three eigenvectors, $H_i (i = 1,2,3)$, Equation (3.11) is true in general. The solution for the magnetic intensity, $H(\vec{r},\omega)$, for the arbitrary source is obtained by the inverse Fourier transform of $H(\vec{k},\omega)$ deduced from Equations (3.4) and (3.11).

$$H(\vec{r},\omega) = \; j\omega\epsilon_0 (2\pi)^{-3} \iiint_{\infty}^{\infty} \sum_{i=1}^{3} S_i^{-1} (H_i^\dagger NH_i)^{-1} H_i H_i^\dagger M_m(\vec{k},\omega) e^{-j\vec{k}\cdot\vec{r}} d^3k \tag{3.12}$$

Multiplying Equation (3.12) by $\underline{\underline{\mu}}$ inside the integral sign gives the expression for the magnetic flux density, $B(\vec{r},\omega)$.

$$B(\vec{r},\omega) = -j\omega\mu_0\epsilon_0 (2\pi)^{-3} \iiint_{-\infty}^{\infty} \sum_{i=1}^{3} S_i^{-1} (H_i^\dagger NH_i)^{-1} NH_i H_i^\dagger M_m(\vec{k},\omega) e^{j\vec{k}\cdot\vec{r}} d^3k \tag{3.13}$$

Recall that Equation (3.8) operating on a vector, $M_m(\vec{k},\omega)$, splits the vector into its components that are parallel to the magnetic flux density, B_i, that corresponds to the eigenvector, H_i. Therefore, $M_m(\vec{k},\omega) = \sum_{i=1}^{3} M_{mi}(\vec{k},\omega)$

where
$$M_{mi}(\vec{k},\omega) = (H_i^\dagger N H_i)^{-1} N H_i H_i^\dagger M_m(\vec{k},\omega)$$

Similarly
$$B(\vec{k},\omega) = \sum_{i=1}^{3} B_i(\vec{k},\omega)$$

where
$$B_i(\vec{k},\omega) = (H_i^\dagger N H_i)^{-1} N H_i H_i^\dagger B(\vec{k},\omega)$$

Using these facts result in

$$B(\vec{r},\omega) = -j\omega\mu_0\epsilon_0(2\pi)^{-3}\iiint_{-\infty}^{\infty}\sum_{i=1}^{3} S_i^{-1} M_{mi}(\vec{k},\omega) e^{-j\vec{k}\cdot\vec{r}} d^3k \tag{3.14}$$

Because of the orthogonality condition, $H_j^\dagger N H_i = 0\ (H_j^\dagger B_i = 0)\ \lambda_i \neq \lambda_j$, the component of $B(\vec{k},\omega)$ parallel to B_i is entirely due to the component of $M_m(\vec{k},\omega)$ parallel to B_i, i.e.,

$$B_i(\vec{k},\omega) = -j\omega\mu_0\epsilon_0 S_i^{-1} M_{mi}(\vec{k},\omega) \tag{3.15}$$

Denote the inverse Fourier transform of S_i^{-1} and $(H_i^\dagger N H_i)^{-1}$ by Q_i and G_i, respectively. Then

$$B_i(\vec{r},\omega) = -j\omega\mu_0\epsilon_0 Q_i * M_{mi}(\vec{r},\omega) \tag{3.16}$$

and

$$B(\vec{r},\omega) = -j\omega\mu_0\epsilon_0\sum_{i=1}^{3} Q_i * M_{mi}(\vec{r},\omega) \tag{3.17}$$

$M_{mi}(\vec{r},\omega)$, the inverse transform of $M_{mi}(\vec{k},\omega)$ is

$$M_{mi}(\vec{r},\omega) = \mu_0^{-1} G_i * B_i'(j\nabla,\omega)\delta * H_i'^{\dagger}(j\nabla,\omega)\delta * M_m(\vec{r},\omega) \quad (3.18)$$

$$= \mu_0^{-1} G_i * B_i'(j\nabla,\omega) H_i'^{\dagger}(j\nabla,\omega) M_m(\vec{r},\omega) \quad (3.19)$$

The prime symbol is used to emphasize that the quantities are eigenvectors, not components of source fields. The reason for the choice of the normalization for H_i and B_i now becomes apparent in that the inverse transform of $H_i'(\vec{k},\omega)$ and $B_i'(\vec{k},\omega)$ are $H_i'(j\nabla,\omega)$ and $B_i'(j\nabla,\omega)$, respectively. Also, because of the isomorphism between polynomials in Fourier space and partial differential operators operating on the Dirac delta function δ, an orthogonality condition equivalent to $H_j^{\dagger} N H_i = 0, \lambda_i \neq \lambda_j$, exists in x,y,z space, i.e.,

$$H_j'^{\dagger}(j\nabla,\omega)\delta * B_i'(j\nabla,\omega)\delta = 0 \qquad \text{for } \lambda_i \neq \lambda_j \quad (3.20)$$

or

$$H_j'^{\dagger}(j\nabla,\omega) B_i'(j\nabla,\omega)\delta = 0 \quad (3.21)$$

If in Equation (3.1) H was eliminated instead of E, analogous results would occur. However, invoking duality produces the same results in a more enlightening manner. Duality implies that one can replace the quantity (I) by the quantity (II) in the previous formulation to give the desired results.

(I)	(II)	
E	H	
H	$-E$	
J_e	J_m	
J_m	$-J_e$	(3.22)
μ_o	ϵ_o	
ϵ_o	μ_o	
N	K	
K	N	

Using duality, with the dual operator $\mathcal{D}$, Equations (3.2) through (3.5) result in,

$$(-\vec{k}xN^{-1}\vec{k}x-k_o^2K)E(\vec{k},\omega)=-j\omega\mu_o M_e(\vec{k},\omega) \tag{3.23}$$

$$M_e(\vec{k},\omega)=-\mathcal{D}\left[M_m(\vec{k},\omega)\right]=J_e(\vec{k},\omega)+(1/j\omega\mu_o)(-j\vec{k}x)N^{-1}J_m(\vec{k},\omega) \tag{3.24}$$

$$G_E(\vec{k},\omega)E(\vec{k},\omega)=-j\omega\mu_o M_e(\vec{k},\omega) \tag{3.25}$$

$$G_E(\vec{k},\omega)=\mathcal{D}\left[G_H(\vec{k},\omega)\right]=-\vec{k}xN^{-1}\vec{k}x-k_o^2K \tag{3 26}$$

Since the operators det and $\mathcal{D}$ commute, det $[G_E(\vec{k},\omega)]$ is easily seen to be

$$(\det\ K)\left[\mathcal{D}(\lambda_1-k_o^2)\right]\left[\mathcal{D}(\lambda_2-k_o^2)\right]\left[\mathcal{D}(\lambda_3-k_o^2)\right]$$

$\lambda_3 = 0$; therefore, $\mathcal{D}\left[S_3\right]=S_3$. At this point, we will prove some relationships between the eigenvalues and eigenvectors, that are necessary in order to continue the discussion of the electric field using the duality principle.

<u>Lemma 3.2</u>: The dual of eigenvalue λ_i is λ_j, $i, j = 1, 2$, $i \neq j$, if

$$\mathcal{D}\left[\lambda_i\right] \neq \lambda_i. \qquad \left(\mathcal{D}\left[\lambda_i\right] = \lambda_j, i \neq j; i, j \neq 3, \mathcal{D}\left[\lambda_i\right] \neq \lambda_i\right)$$

Proof: Assume that an eigenvalue, λ_i, and its eigenvector, H_i, satisfy the equation

$$G_H(\vec{k}, \omega, \lambda_i) H_i = 0 \tag{3.27}$$

The dual of Equation (3.27) is

$$\mathcal{D}\left[G_H(\vec{k}, \omega, \lambda\ ) H_i\right] = G_E\left(\vec{k}, \omega, \mathcal{D}\left[\lambda_i\right]\right) \mathcal{D}\left[H_i\right] = 0 \tag{3.28}$$

Therefore, $\mathcal{D}[\lambda_i]$ is an eigenvalue of $G_E(\vec{k}, \omega, \lambda) E = 0$ with $\mathcal{D}[H_i]$ as its eigen-vector. Since

$$(\det N) \det\left[G_E(\vec{k}, \omega, \lambda)\right] = (\det K) \det\left[G_H(\vec{k}, \omega, \lambda)\right] = 0 \tag{3.29}$$

$\mathcal{D}[\lambda_i]$ is also an eigenvalue of $G_H(\vec{k}, \omega, \lambda) H = 0$. Now $\mathcal{D}[\lambda_i] \neq 0$ since it is assumed that $\lambda_i \neq 0$. Therefore, either $\mathcal{D}[\lambda_i] = \lambda_j$, $j = 1,2$. But, since $\mathcal{D}[\lambda_i] \neq \lambda_i$, $\mathcal{D}[\lambda_i] = \lambda_j$; $i,j = 1,2$, $i \neq j$. Note that if $\mathcal{D}[\lambda_i] = \lambda_i$ for either i equal to one or two, then the other nonzero eigenvalue must also be equal to its dual.

The duality relationships among the eigenvalues and eigenvectors are summerized in Figure 3.1.

Now with the aid of Lemma 3.2, we are able to discuss the implication of duality upon the dispersion surfaces. First let us note that the dispersion surface for the electric field is not necessarily equal to the dispersion surface for the magnetic field. By this is meant that the zeros of det G_E are not

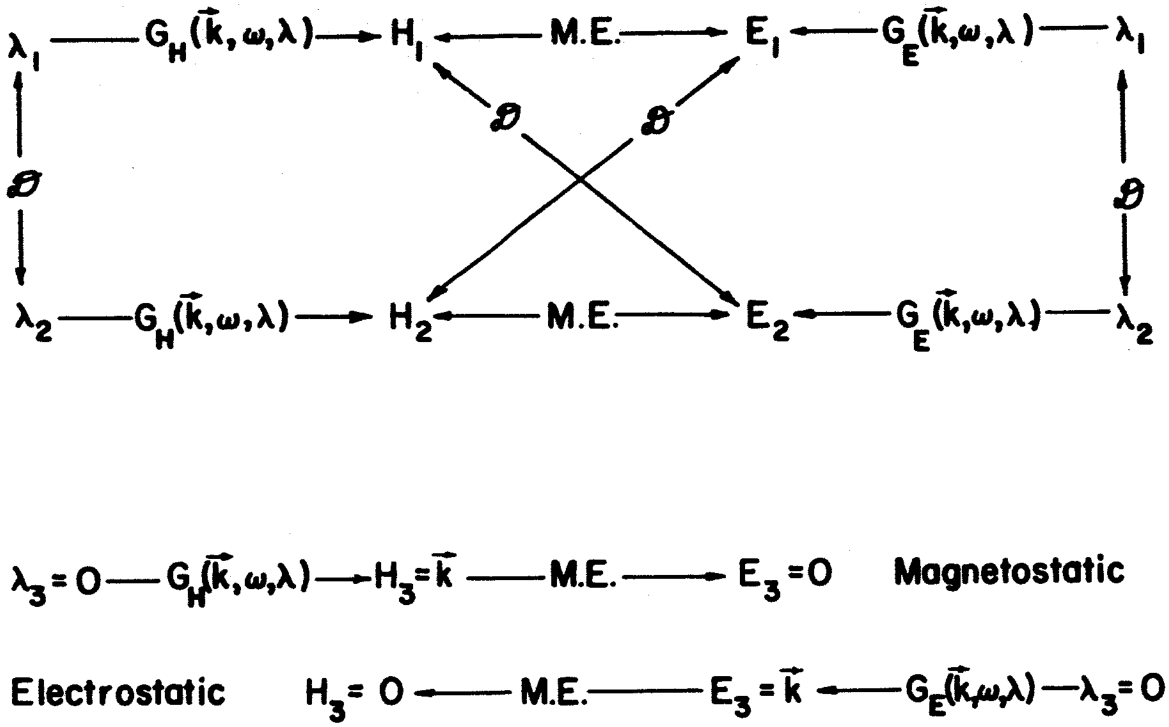

Figure 3.1. Duality relationship among the eigenvalues and eigenvectors of $G_E(\vec{k},\omega,\lambda)$ and $G_H(\vec{k},\omega,\lambda)$. ($G_E(k,\omega,\lambda) = -\vec{k} \times N^{-1} \vec{k} \times - \lambda K$; $G_H(k,\omega,\lambda) = -\vec{k} \times K^{-1} \vec{k} \times - \lambda N$; $\mathcal{D}$ = dual; M.E. = Maxwell's Equations).

necessarily equal to the zeros of det G_H. A warm plasma is an appropriate example. Hence the phrase, "the dispersion surface of an electromagnetic field," is not precise. For the dispersion surfaces to be equal would imply that the zeros of det $\underline{\underline{\epsilon}}$ equal the zeros of det $\underline{\underline{\mu}}$. In truth, the relation between the two dispersion surfaces is duality. Lemma 3.2 can be used to explain a finer structure among certain sheet(s) of either dispersion surface. Consider the dispersion surface for the electric field, i.e., det $G_E = (\det K) \prod_{i=1}^{3} S_i = 0$. The sheet(s) defined by $S_3 = 0$ is equal to the dual of itself. If the sheet(s) $S_i = 0$, $(i = 1,2)$ is not equal to the dual of itself, then the dual of sheet(s) $S_i = 0$ is equal to the sheet(s) $S_j = 0$, $(i \neq j,\ j = 1,2)$. This characteristic is exemplified by the general time-dispersive uniaxial problem of Section 4. Similar results apply to the dispersion surface for the magnetic field, det $G_H = 0$.

After this discussion of the implications of duality upon the dispersion surfaces, eigenvalues and eigenvectors, the procedure to obtain the electric field from the first part of this section by duality should be self-evident.

From the previous formulation, the field $\mathcal{F} = \left[\mathcal{E}, \mathcal{H}\right]$ could be found in three ways: (1) by the equation $G_H(\vec{k},\omega)H(\vec{k},\omega) = -j\omega\epsilon_o M_m(\vec{k},\omega)$ and Amperes law, (2) by the equation $G_E(\vec{k},\omega)E(\vec{k},\omega) = -j\omega\mu_o M_e(\vec{k},\omega)$ and Faraday's law, and (3) by the equations $G_H(\vec{k},\omega)H(\vec{k},\omega) = -j\omega\epsilon_o M_m(\vec{k},\omega)$ and $G_E(\vec{k},\omega)E(\vec{k},\omega) = -j\omega\mu_o M_e(\vec{k},\omega)$. Only the last method is symmetric in the field components. All three methods depend on either G_H or G_E. That is, in all three methods, Maxwell's equations, with an assumed form of constitutive relations ($D = \underline{\underline{\epsilon}}E$, $B = \underline{\underline{\mu}}H$), were reduced to an equation envolving only one field component, $E(\vec{k},\omega)$ or $H(\vec{k},\omega)$. However, this relatively simple reduction may not always be possible. If the flux quantities, D and B, are linearly related to both the field intensities, E and

H, then the simple reduction is not possible. An example of such a coupled constitutive relation occurs for the fields in a moving medium. In general, the linear constitutive relationship may be written as

$$F_f = \underline{\underline{U}}F \text{ where } F_f = [D,B] \text{ and } F = [E,H].$$

Thus, it seems desirable to formulate a method to find the field due to a source in terms of the characteristic fields in a symmetric manner that envolves both E and H, and that circumvents the difficulties that arose in the previous formulation.

3.2 Definitions of Symbols

Let us define some symbols of quantities (vectors, matrices, functions) in the Fourier transform domain.

$$F = \begin{bmatrix} E \\ \\ H \end{bmatrix} \text{ (six-vector of the electromagnetic field elements)} \qquad (3.30)$$

$$C = \begin{bmatrix} J_e \\ \\ J_m \end{bmatrix} \text{ (six-vector of the electromagnetic source elements)} \qquad (3.31)$$

$$O = \begin{bmatrix} O & -j\vec{k}x \\ j\vec{k}x & O \end{bmatrix} \qquad (3.32)$$

$\underline{\underline{U}}$ (sixth-order constitutive matrix) (3.33)

$$\mathcal{M}(\vec{k},\omega) = O - j\omega\underline{\underline{U}} \qquad (3.34)$$

$$F_f = \begin{bmatrix} D \\ \\ B \end{bmatrix} = \underline{\underline{U}}F \text{ (six-vector of the electromagnetic flux elements)} \quad (3.35)$$

$$I \quad \text{(sixth-order identity matrix)} \quad (3.36)$$

F_i, ν_i (the eigenvectors and eigenvalues corresponding to the matrix equation $OF_i = \nu_i\,\underline{\underline{U}}F_i$) (3.37)

$$F_{fi} = \underline{\underline{U}}F_i \quad (3.38)$$

$$I_i^f = (F_i^\dagger \underline{\underline{U}} F_i)^{-1} \underline{\underline{U}} F_i F_i^\dagger \quad (3.39)$$

$$I_i = (I_i^f)^\dagger = (F_i^\dagger \underline{\underline{U}} F_i)^{-1} F_i F_i^\dagger \underline{\underline{U}} \quad (3.40)$$

$V_{fi} = I_i^f V$ (V an arbitrary vector; V_{fi}, component of V parallel to F_{fi}) (3.41)

$V_i = I_i V$ (V an arbitrary vector; V_i, component of V parallel to F_i) (3.42)

3.3 Characteristic Waves

By definition, characteristic waves are the fields which are described by the eigenvector, F_i, and its associated eigenvalue, ν_i, that are the solutions the equation

$$\underline{\underline{O}}F_i = \nu_i \underline{\underline{U}} F_i \tag{3.43}$$

Since O and $\underline{\underline{U}}$ are sixth-order matrices, there will be six characteristic waves, i.e., the index i will run from one to six. When the eigenvalue, ν_i, is equal to $j\omega$, then the eigenvector, F_i, will be a source-free solution to Maxwell's equations. However, the field, F_i, corresponding to $\nu = \nu_i$ may not encompass all of the solutions to the characteristic equation. Field vectors for the characteristic equation exist, if and only if

$$\det\left[O - \nu \underline{\underline{U}}\right] = 0 \tag{3.44}$$

But

$$\det\left[O - \nu \underline{\underline{U}}\right] = (\det \underline{\underline{U}}) \prod_{i=1}^{6} (\nu_i - \nu) \tag{3.45}$$

Hence, other solutions will exist when det $\underline{\underline{U}} = 0$. It is not necessary that the determinant of $\underline{\underline{U}}$ equal zero identically. In general, $\underline{\underline{U}}$ will be a function of the transform variables ω and $\vec{k}$. Also, $\underline{\underline{U}}$ will be dependent upon other parameters. Thus for certain ω, $\vec{k}$ and other parameters of $\underline{\underline{U}}$, the condition det $\underline{\underline{U}} = 0$ may be satisfied.

At this point it should be emphasized that the eigenvalue problem that concerns us is in distinct contrast to the usual eigenvalue problem. The usual eigenvalue problem deals with the question: Given a medium, normally isotropic and homogeneous, contained in certain boundaries with "walls" that may in general be described by an impedance, what are the source-free solutions or modes that may exist? These eigenvalue problems encompass both the bound case (cavities) and the unbounded case (waveguides). They are in the time-space domain. The modes then are a superposition of the source-free waves that may exist in the unbounded medium in such a way that the boundary conditions are satisfied. For example, in a vacuum filled perfectly conducting rectangular waveguide, four appropriately chosen plane waves comprise a mode. In a perfectly conducting circular waveguide, an infinitely nondenumerable number of appropriately chosen plane wave comprise a mode. The eigenvalues in this type of problem are largely determined by the boundaries. In our problem, however, the medium is of infinite extent. Also the domain of consideration is the Fourier domain in contrast to the real space-time domain. The eigenvalues and eigenvectors are entirely a function of the matrix operators O and $\underline{\underline{U}}$. The operator O is a result of the form of Maxwell's equation only and is independent of the medium. The operator $\underline{\underline{U}}$, however, is the constitutive relationship and hence comprises the entire electromagnetic description of the medium. Thus, it can be said that eigenvalues and eigenvectors are a function of the medium only. In fact, the constitutive relationship $\underline{\underline{U}}$ can be completely expressed in terms of the eigenvalues and eigenvectors.

Proof:

$$\underline{\underline{U}} = (1/2j\omega)\left[\eta_{-} - \eta\right] \tag{3.46}$$

where

$$\eta_{-} = O + j\omega U \tag{3.47}$$

$$\underline{\underline{U}} = \sum_{i=1}^{6} (F_i^{\dagger} \underline{\underline{U}} F_i)^{-1} \underline{\underline{U}} F_i F_i^{\dagger} \underline{\underline{U}} \tag{3.48}$$

$$\underline{\underline{U}} = \sum_{i=1}^{6} \nu_i^{-1} (F_i^{\dagger} O F_i)^{-1} O F_i F_i^{\dagger} O \tag{3.49}$$

Since for $i = 3$ and 6 both ν_i and OF_i are zero, the limit must be taken for these terms. It must be emphasized, however, that knowledge of ν_i and F_i at one frequency is not sufficient; ν_i and F_i must be known as functions of $\vec{k}$ and ω. Equivalently, a knowledge of the characteristic waves at every point on the dispersion surface in four-space $(\vec{k},\omega)$ completely determines the electromagnetic properties of the medium.

Certain properties of the eigenvalues and eigenvectors may be determined by only knowing the symmetry properties of the operators 0 and $\underline{\underline{U}}$. It is easily shown that the operator 0 is skew-Hermitian for real $\vec{k}$, i.e., $0 = -0^{\dagger}$. Assume that the medium is lossless. This is equivalent to saying that the operator $\underline{\underline{U}}$ is Hermitian, i.e., $\underline{\underline{U}} = \underline{\underline{U}}^{\dagger}$. With these two facts it is easily shown that ν_i is imaginary. This follows from the fact that $(F_i^{\dagger}\, OF_i)$ is imaginary, $(F_i^{\dagger}\, \underline{\underline{U}}F_i)$ is real and that $\nu_i = (F_i^{\dagger}\, OF_i)(F_i^{\dagger}\, \underline{\underline{U}}F_i)^{-1}$. Also, there exists an orthogonality property among the eigenvectors. The conjugate transpose of the equation $F_i^{\dagger}\, OF_j = \nu_j\, F_i^{\dagger}\, \underline{\underline{U}}F_j$ is the equation $F_j^{\dagger}\, OF_i = \nu_j F_j^{\dagger}\underline{\underline{U}}F_i$. Use has been made of the symmetry of 0 and $\underline{\underline{U}}$, and that ν_j is imaginary. But we independently know that $F_j^{\dagger}\, OF_i = \nu_i F_j^{\dagger}\underline{\underline{U}}F_i$. Hence, $(\nu_i - \nu_j)F_j^{\dagger}\underline{\underline{U}}F_i$. Therefore, for $\nu_i \neq \nu_j$, $F_j^{\dagger}\underline{\underline{U}}F_i = 0$ or $F_j^{\dagger}F_{fi} = 0$.

To better understand these characteristic fields for an arbitrary Hermitian constitutive relationship, it may be profitable to digress for the moment to correlate the characteristic fields from the six-vector and previously considered three-vector methods. Naturally such a correlation implies that the constitutive relationship is "diagonal," i.e., $\underline{\underline{U}} = \begin{bmatrix} \underline{\underline{\epsilon}} & 0 \\ 0 & \underline{\underline{\mu}} \end{bmatrix}$ since only then is the simple decomposition into three-vectors possible.

3.4 Relationship Between $\mathcal{M}(\nu)$, $G_E(\lambda)$, and $G_H(\lambda)$

It has already been established that

$$G_E(\lambda) = -\vec{k} \times \mu_0 \underline{\underline{\mu}}^{-1} \vec{k} \times -\lambda \epsilon_0^{-1} \underline{\underline{\epsilon}} \tag{3.50}$$

$$G_H(\lambda) = -\vec{k} \times \epsilon_0 \underline{\underline{\epsilon}}^{-1} \vec{k} \times -\lambda \mu_0^{-1} \underline{\underline{\mu}} \tag{3.51}$$

and

$$\mathcal{M}(\nu) = \begin{bmatrix} \nu \underline{\underline{\epsilon}} & -j\vec{k}\times \\ j\vec{k}\times & -\nu \underline{\underline{\mu}} \end{bmatrix} \text{ for } \underline{\underline{U}} = \begin{bmatrix} \underline{\underline{\epsilon}} & 0 \\ 0 & \underline{\underline{\mu}} \end{bmatrix} \tag{3.52}$$

Further define

$$G_E'(\lambda) = \mu_0^{-1} G_E(\lambda) \tag{3.53}$$

$$G_H'(\lambda) = \epsilon_0^{-1} G_E(\lambda) \tag{3.54}$$

Using these definitions it may be shown that

$$-\mathcal{M}(-\nu) U^{-1} \mathcal{M}(\nu) = \begin{bmatrix} G_E'(-\mu_0 \epsilon_0 \nu^2) & 0 \\ 0 & G_H'(-\mu_0 \epsilon_0 \nu^2) \end{bmatrix} \tag{3.55}$$

Since

$$\mathcal{M}(-\nu) = \begin{bmatrix} -I & 0 \\ 0 & I \end{bmatrix} \mathcal{M}(\nu) \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix} \tag{3.56}$$

and

$$\begin{bmatrix} G'_E & 0 \\ 0 & G'_H \end{bmatrix} = \begin{bmatrix} G'_E & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & G'_H \end{bmatrix} \tag{3.57}$$

then,

$$\det \mathcal{M}(-\nu) = \det \mathcal{M}(\nu) \quad \text{and} \quad \det \begin{bmatrix} G'_E & 0 \\ 0 & G'_H \end{bmatrix} = (\det G'_E)(\det G'_H)$$

Therefore,

$$\begin{aligned} \det\left[-\mathcal{M}(-\nu)U^{-1}\mathcal{M}(\nu)\right] &= (\det U^{-1})\left[\det \mathcal{M}(\nu)\right]^2 \\ &= \det G'_E(-c^{-2}\nu^2)\det G'_H(-c^{-2}\nu^2) \end{aligned} \tag{3.58}$$

Or

$$\det \mathcal{M}(\nu) = \left\{\left[(\det \underline{\underline{\mu}})\det G'_E(-c^{-2}\nu^2)\right]\left[(\det \underline{\underline{\epsilon}})\det G'_H(-c^{-2}\nu^2)\right]\right\}^{1/2} \tag{3.59}$$

This expression can be simplified further by using the following lemma.

<u>Lemma 3.3:</u> $(\det \underline{\underline{\mu}})\det G'_E(-c^{-2}\nu^2) = (\det \underline{\underline{\epsilon}})\det G'_H(-c^{-2}\nu^2)$

Proof: Define

$$G''_E(-c^{-2}\nu^2) = A\underline{\underline{\mu}}^{-1}A + c^{-2}\nu^2\underline{\underline{\epsilon}} \tag{3.60}$$

and

$$G'_H(-C^{-2}\nu^2) = A\underline{\underline{\epsilon}}^{-1}A + \bar{C}^2\nu^2\underline{\underline{\epsilon}} \tag{3.61}$$

for an arbitrary matrix A

Then

$$A\underline{\underline{\epsilon}}^{-1}G''_E(-\bar{C}^2\nu^2) = G''_H(-\bar{C}^2\nu^2)\underline{\underline{\mu}}^{-1}A \tag{3.62}$$

$$\det A \det\underline{\underline{\epsilon}}^{-1}\det G''_E = \det G''_H \det\underline{\underline{\mu}}^{-1}\det A \tag{3.63}$$

Or

$$\det\underline{\underline{\mu}}\det G''_E(-C^{-2}\nu^2) = \det\underline{\underline{\epsilon}}\det G''_H(-C^{-2}\nu^2) \tag{3.64}$$

In particular, if $A = -\,j\vec{k}x$,

$$\det\underline{\underline{\mu}}\det G'_E(-\bar{C}^2\nu^2) = \det\underline{\underline{\epsilon}}\det G'_H(-\bar{C}^2\nu^2) \tag{3.65}$$

Now the determinant of $\mathcal{M}(\nu)$ can be expressed as

$$\det\mathcal{M}(\nu) = \pm\det\underline{\underline{\mu}}\det G'_E(-\bar{C}^2\nu^2) = \pm\det\underline{\underline{\epsilon}}\det G'_H(-\bar{C}^2\nu^2) \tag{3.66}$$

The sign ambiguity occurs since the square root has been taken. However, since the determinant of $\mathcal{M}(\nu)$ is set equal to zero to find the eigenvalues, the sign coefficient is of no great importance at this point.

A few observations concerning the preceding mathematics are in order, particularly about the relationships between the eigenvalues and eigenvector of the three-vector and six-vector methods. One apparent discrepency is contained in the question: Why does the three-vector and six-vector methods result in three and six eigenvalues and eigenvectors, respectively, particularly in light of the fact that both methods represent Maxwell's equations for the same medium and both sets of resulting eigenvalues and eigenvectors are sufficient to completely represent electromagnetic propagation in the medium? The answer to the question is found in Equations (3.55) and (3.66). Both Equations (3.55) and (3.66) indicate the relationship between the eigenvalues of the two methods, i.e., $\lambda = -c^{-2}\nu^2$. This shows that corresponding to each of the three eigenvalues λ there exists two eigenvalues $\nu = \pm jc(\lambda)^{1/2}$. If $F = [E,H]$ is an eigenvector, i.e., $\mathcal{M}(\nu)F = 0$, then from Equation (3.55), E and H are eigenvectors of G_E and G_H, respectively. Conversely, a pair of eigenvectors E and H, corresponding to the eigenvalue λ of G_E and G_H, is also an eigenvector of $\mathcal{M}(\nu)$. Moreover, the six eigenvectors F_i are not completely unrelated as one might surmise from the fact that corresponding to every eigenvalue ν_i, there exist another one, $-\nu_i$. There is in fact a relationship between eigenvectors corresponding to ν and $-\nu$.

Assume that corresponding to the eigenvalue ν, F is the characteristic field, i.e.,

$$OF = \nu \underline{\underline{U}}F$$

$$\begin{bmatrix} 0 & -j\vec{k}\times \\ j\vec{k}\times & 0 \end{bmatrix}\begin{bmatrix} E \\ H \end{bmatrix} = \nu\begin{bmatrix} \underline{\epsilon} & 0 \\ 0 & \underline{\mu} \end{bmatrix}\begin{bmatrix} E \\ H \end{bmatrix} \tag{3.67}$$

Now we claim that the characteristic field, $\check{F}$, corresponding to the eigenvalue $-\nu$ is $\check{F} = [E,-H]$, i.e.,

$$O\check{F} = -\nu\,\underline{\underline{U}}\check{F} \tag{3.68}$$

This is easily seen by distributing the negative sign of **-H** with the preceeding matrix elements and the multiplying on the left by $\begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix}$. The result is Equation (3.67) which was assumed to be valid.

The following table will summarize the relationship between the characteristic fields of the two methods.

TABLE I

Three-Vector Characteristic Fields			Six-Vector Characteristic Fields			
λ_1	E_1	H_1	ν_1	$F_1 = [E_1, H_1]$	$-\nu_1$	$\check{F}_1 = [E_1, -H_1]$
λ_2	E_2	H_2	ν_2	$F_2 = [E_2, H_2]$	$-\nu_2$	$\check{F}_2 = [E_2, -H_2]$
$\lambda_3 = 0$	$E_3 = \vec{k}$	$H_3 = 0$	$\nu_3 = 0$	$F_3 = [E_3 = \vec{k}, 0]$		
	$E_3 = 0$	$H_3 = \vec{k}$			$\nu_3 = 0$	$F_6 = [0, H_6 = \vec{k}]$

3.5 Completeness of Characteristic Fields

Before any attempt is made to represent a field due to a source distribution in terms of the characteristic fields, it first must be established whether such a set of fields can represent the desired field, i.e., does the characteristic fields form a basis for the vector space. Let us consider this question in terms of projections.

A projection I_i on a Hilbert space, H, is simply an idempotent ($I_i^2 = I_i$) linear transformation of H into itself. The range and null space of I_i are $M_i = [I_i X : X \in H]$ and $N_i = [I_i X = 0 : X \in H]$, respectively. Thus, the projections I_i partition the Hilbert range space. From this we see that I_i defined as $(F_i^{\dagger} \underset{=}{U} F_i)^{-1} F_i F_i^{\dagger} \underset{=}{U}$ (i = 1,2, ... 6) are indeed projections as defined above. Furthermore, they are orthogonal projections $I_i I_j = 0$ $(i \neq j)$. In our case, the projection I_i partitions the range space into the space of vectors parallel, M_i, and orthogonal, N_i, to the characteristic vector F_i. If an arbitrary vector F is to be expressed as the sum of characteristic vectors, then it is obvious that the sum of the ranges of the projections must span the space, $\sum_{i=1}^{6} M_i = H$. Now the sum of orthogonal projections is a projection. Thus, it is not surprising that the sum of the projections is the identity projection whose range is the Hilbert space, i.e., $\sum_{i=1}^{6} I_i = I$. Indeed, this is an equivalent statement as to the completeness of the characteristic vectors F_i.

Analogous reasoning applies to the set of projections $\left[I_i^f : i = 1,2, \ldots 6 \right]$. The main difference is that the set $\left[I_i^f : i = 1,2, \ldots 6 \right]$ partitions the Hilbert space into sets of vectors parallel to the characteristic vectors F_{fi} not F_i. Therefore, the latter set would be more profitably used with flux field vectors, whereas the former should be used with the field intensity vectors.

3.6 Spectral Representation of Fields Due to a Source

One of the most important purposes of representing the field due to a source as a spectrum of characteristic waves is to gain physical insight into the process of propagation in the medium. Most often the source problem is attacked in the Fourier domain by inverting a complicated coefficient matrix followed by a Fourier transform inversion. Since all of the sheets of the dispersion surface will occur as singularities of the Fourier inversion integral, the integration process is extremely difficult. Also, the mathematics reveals little physical insight other than the singularities, i.e., the dispersion surface which implies the source free solutions, are a prime determining factor in the source problem.

A number of questions may be asked concerning the relationships between the source free problem and the source problem. A very significant question is: what is the explicit relationship of each sheet of the dispersion surface and its associated characteristic wave to the source field? Also, in what manner does the source excite the spectrum of characteristic waves to form the source field? These and other questions will be answered in the process of finding the spectral representation of fields due to a source.

There is at least two ways of deriving the spectral representation of the source fields. The first and perhaps the simplest is to express the source in terms of and assume that the source field may be expressed in terms of the characteristic fields. This is possible since it has already been established that the characteristic fields are complete and may be used as a basis. Also, an important factor is the orthogonality of the eigenvectors. Actually the source will be expressed in terms of the characteristic flux fields, F_{fi}, since the source itself is a flux as can be seen from Maxwell's equations. That is, let

$$C = \sum_{i=1}^{6} C_{fi} = \sum_{i=1}^{6} \beta_i F_{fi} \tag{3.69}$$

C_{fi} is the component of the source that is parallel to the characteristic flux field F_{fi}. Then use the orthogonality condition to determine the coefficients β_i. Therefore, $F_i^{\dagger} C = \beta_i F_i^{\dagger} F_{fi}$ or $\beta_i = (F_i^{\dagger} F_{fi})^{-1} F_i^{\dagger} C$. Hence,

$$C = \sum_{i=1}^{6} (F_i^{\dagger} F_{fi})^{-1} F_{fi} F_i^{\dagger} C \tag{3.70}$$

Similarly,

$$F(\vec{k},\omega) = \sum_{i=1}^{6} F_i(\vec{k},\omega) = \sum_{i=1}^{6} \alpha_i F_i \tag{3.71}$$

$F_i(\vec{k},\omega)$ is the component of the source field that is parallel to the characteristic field F_i.

$$\mathcal{M}\sum_{i=1}^{6}\alpha_i F_i = \sum_{i=1}^{6}\beta_i F_{fi} \tag{3.72}$$

But

$$\mathcal{M} F_i = (\nu_i - j\omega) F_{fi} \tag{3.73}$$

and

$$F_j^{\dagger}\mathcal{M} F_i = (\nu_i - j\omega) F_i^{\dagger} F_{fi}\,\delta_{ij} \tag{3.74}$$

where δ_{ij} is the Kronecker delta. Therefore,

$$\alpha_i (\nu_i - j\omega) F_i^{\dagger} F_{fi} = \beta_i F_i^{\dagger} F_{fi} \tag{3.75}$$

and

$$\alpha_i = (\nu_i - j\omega)^{-1}\beta_i \tag{3.76}$$

Equation (3.76) shows an interesting fact. The component of the source field parallel to the characteristic field F_i is completely and entirely due to the component of the source that is parallel to the flux field F_{fi}. Moreover, this component is only affected by its own sheet of the dispersion surface and not by the sheets of the other characteristic fields. Now the component of the source field $F_i(\vec{k},\omega)$ parallel to the characteristic field F_i may be expressed in its spectral form,

$$F_i(\vec{k},\omega) = (\nu_i - j\omega)^{-1}(F_i^{\dagger} F_{fi})^{-1} F_i F_i^{\dagger} C \tag{3.77}$$

The total source field as a spectral representation becomes,

$$F(\vec{k},\omega) = \sum_{i=1}^{6} (\nu_i - j\omega)^{-1} (F_i^{\dagger} F_{fi})^{-1} F_i F_i^{\dagger} C \tag{3.78}$$

The second method of deriving the spectral representation in contrast to the first is to directly express the matrix operators $\mathcal{M}$ and $\mathcal{M}^{-1}$ in terms of the characteristic values and vectors without making any assumptions about the source or the field. If the first method had been determined at the outset, then it would give a motivation for the expression for $\mathcal{M}^{-1}$. Let us postulate the expressions for $\mathcal{M}$ and $\mathcal{M}^{-1}$ in terms of the characteristic values and fields, and then show that they are indeed valid. The appropriate expressions for $\mathcal{M}$ and $\mathcal{M}^{-1}$ are,

$$\mathcal{M}(j\omega) = \underline{\underline{U}}\left[\sum_{i=1}^{6} (\nu_i - j\omega) I_i\right] \tag{3.79}$$

and

$$\mathcal{M}^{-1}(j\omega = \underline{\underline{U}}^{-1}\left[\sum_{i=1}^{6} (\nu_i - j\omega)^{-1} I_i^f\right] \tag{3.80}$$

The notations for I_i and I_i^f have previously been defined. To verify the expression for $\mathcal{M}(j\omega)$, it must be shown that

$$\mathcal{M}(j\omega)V = \underline{\underline{U}}\left[\sum_{i=1}^{6} (\nu_i - j\omega) I_i\right] V \tag{3.81}$$

for an arbitrary vector V. Since, however, the characteristic fields are complete, it is only necessary to show that the equation is satisfied for vector

V to be any characteristic field. Assume that

$$\mathcal{M}(j\omega)F_j = \underline{\underline{U}}\left[\sum_{i=1}^{6}(\nu_i - j\omega)I_i\right]F_j \tag{3.82}$$

Then

$$\mathcal{M}(j\omega)F_j = \underline{\underline{U}}(\nu_j - j\omega)F_j \qquad (\because \text{ orthogonality}) \tag{3.83}$$

$$= (\nu_j - j\omega)\underline{\underline{U}}F_j \tag{3.84}$$

$$= \mathcal{M}(\nu_j)F_j + (\nu_j - j\omega)\underline{\underline{U}}F_j \qquad (\because \mathcal{M}(\nu_j)F_j = 0) \tag{3.85}$$

$$= \mathcal{M}(j\omega)F_j \qquad (\because \mathcal{M}(\nu_j) = 0 - \nu_j\underline{\underline{U}}) \tag{3.86}$$

Hence, the expression for $\mathcal{M}(j\omega)$ is verified. Perhaps the simplest way of verifying the expression for $\mathcal{M}^{-1}(j\omega)$ is to directly show that $\mathcal{M}(j\omega)\mathcal{M}^{-1}(j\omega) = I$, i.e.,

$$\mathcal{M}(j\omega)\underline{\underline{U}}^{-1}\left[\sum_{i=1}^{6}(\nu_i - j\omega)^{-1}I_i^f\right] = I \tag{3.87}$$

Now

$$\mathcal{M}(j\omega)\left\{\underline{\underline{U}}^{-1}\left[\sum_{i=1}^{6}(\nu_i - j\omega)^{-1}I_i^f\right]\right\} = \sum_{i=1}^{6}(\nu_i - j\omega)^{-1}\mathcal{M}(j\omega)\underline{\underline{U}}^{-1}I_i^f \tag{3.88}$$

$$= \sum_{i=1}^{6}(\nu_i - j\omega)^{-1}\mathcal{M}(j\omega)I_i\underline{\underline{U}}^{-1} \qquad (\because \text{ definition of } I_i \text{ and } I_i^f) \tag{3.89}$$

$$= \sum_{i=1}^{6}\underline{\underline{U}}I_i\underline{\underline{U}}^{-1} \qquad (\because \mathcal{M}(j\omega)I_i = (\nu_i - j\omega)\underline{\underline{U}}I_i) \tag{3.90}$$

$$= \sum_{i=1}^{6} I_i^f \tag{3.91}$$

$$= I \tag{3.92}$$

Therefore, the expression for $\mathcal{M}^{-1}(j\omega)$ is also verified.

To derive the spectral representation for the source field $F(\vec{k},\omega)$, it is only necessary to use the spectral representation of the inverse of the field operator $\mathcal{M}(j\omega)$ which was given above. Thus

$$F(\vec{k},\omega) = \underline{\underline{U}}^{-1} \sum_{i=1}^{6} (\nu_i - j\omega)^{-1} I_i^f C \tag{3.93}$$

$$= \underline{\underline{U}}^{-1} \sum_{i=1}^{6} (\nu_i - j\omega)^{-1} C_{fi} \tag{3.94}$$

or

$$F_f(\vec{k},\omega) = \sum_{i=1}^{6} (\nu_i - j\omega)^{-1} C_{fi} \tag{3.95}$$

Not only is this expression compact, but it is also quite revealing in physical insight. However, before the physical interpretation is given, let us show one more fact. Resolve the source flux field $F_f(\vec{k},\omega)$ into its components parallel to the characteristic flux fields.

By definition the component of $F_f(\vec{k},\omega)$ that is parallel to F_{fj} is $F_{fj}(\vec{k},\omega) = I_j^f F_f(\vec{k},\omega)$. Using Equation (3.95), this yields

$$F_{fj}(\vec{k},\omega) = \sum_{i=1}^{6} (\nu_i - j\omega)^{-1} I_j^f C_{fi} .$$

Now because of orthogonality $I_j^f C_{fi} = \delta_{ij} C_{fi}$. Therefore,

$$F_{fj}(\vec{k},\omega) = (\nu_j - j\omega)^{-1} C_{fj} \tag{3.96}$$

and

$$F_f(\vec{k},\omega) = \sum_{i=1}^{6} F_{fi}(\vec{k},\omega) \tag{3.97}$$

Again we arrive at the conclusion that the component of the source field that is parallel to the characteristic flux field, F_{fj}, is solely excited by the component of the source that is parallel to the characteristic flux field, F_{fj}. Now using Equations (3.69), (3.96), and (3.97), a physical interpretation to the source field may be given. Simply stated, the Fourier transform of the flux field $F_f(\vec{k},\omega)$ is the sum of the components of the source parallel to the characteristic fields, C_{fi}, each divided by its respective sheet(s) factor of the dispersion surface, $S_i = (\nu_i - j\omega)$. This fact is illustrated in Figure 3.2. It is also evident that there exist source distributions such that certain sheets of the dispersion surface and their respective characteristic fields play absolutely no part in the total field, namely, those source distributions which are orthogonal to the said characteristic fields, i.e., C's such that $F_j^{\dagger} C = 0$ for some index j.

In the real space-time domain, the flux field may be given as

$$\mathcal{F}_f(\vec{r},t) = \sum_{i=1}^{6} \mathcal{F}_{fi}(\vec{r},t) \tag{3.98}$$

where $\mathcal{F}_{fi}(\vec{r},t) = Q_i * \mathcal{C}_{fi}$ and Q_i and $\mathcal{C}_{fi}$ are the inverse Fourier transforms of $S_i^{-1} = (\nu_i - j\omega)^{-1}$ and C_{fi}, respectively. The convolution is a four-

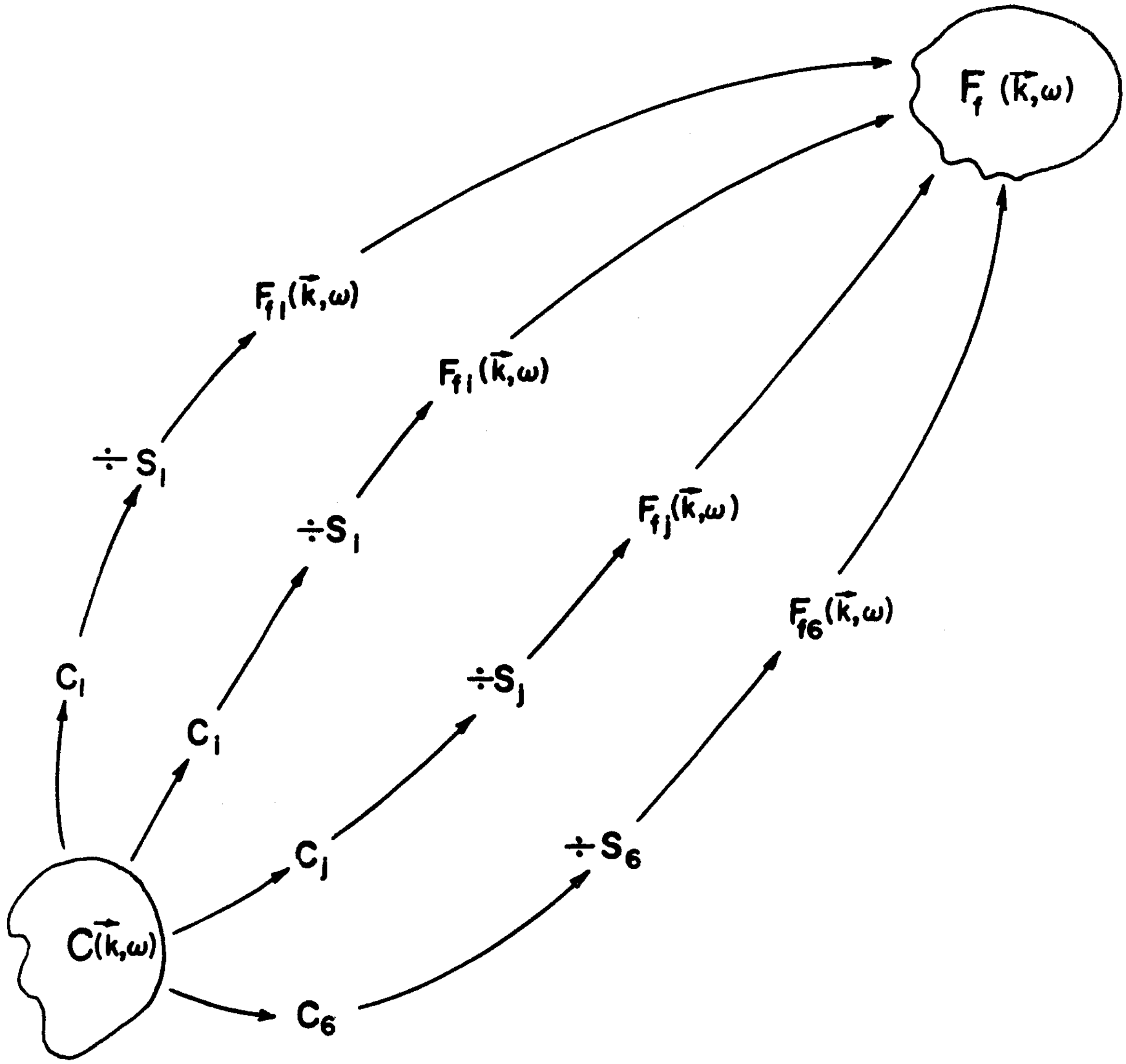

Figure 3.2. The Fourier transform of the flux field, $F_f(\vec{k},\omega)$, in terms of the "characteristic sources" and the sheets of the dispersion surface.

dimensional space-time convolution. In the space-time domain the source flux field $\mathcal{J}_f(\vec{r},t)$ in terms of the "characteristic sources" is illustrated in Figure 3.3.

At this point the matrix Green's function $\underline{\underline{\Gamma}}$ is easily determined. By definition the matrix Green's function $\underline{\underline{\Gamma}}(\vec{r},t)$ is an operator such that

$$\mathcal{J}_f(\vec{r},t) = \underline{\underline{\Gamma}}(\vec{r},t) * \mathcal{E}(\vec{r},t) \tag{3.99}$$

Now the "characteristic source" is $\mathcal{E}_{fi} = \mathcal{I}_i^f * \mathcal{E}$ where $\mathcal{I}_i^f$ is the inverse Fourier transform of the operator I_i^f. Therefore, from Equation (3.98) and since the convolution operations are associative, the source flux field is

$$\mathcal{J}_f(\vec{r},t) = \left\{\sum_{i=1}^{6} Q_i * \mathcal{I}_i^f\right\} * \mathcal{E} \tag{3.100}$$

Thus, the matrix Green's function in terms of the characteristic fields and the influence of the dispersion surface (the operator $\mathcal{I}_i^f$ is easily expressed in terms of the characteristic fields) is

$$\underline{\underline{\Gamma}}(\vec{r},t) = \sum_{i=1}^{6} Q * \mathcal{I}_i^f \tag{3.101}$$

Alternately, the matrix Green's function could have immediately been determined from the inverse Fourier transform of $\underline{\underline{U}}\mathcal{M}^{-1}(j\omega)$ which is equal to $\sum_{i=1}^{6} S_i^{-1} I_i^f$.

It should be emphasized that these results are general and that the only conditions that have been made are that the medium be homogeneous, linear and lossless, i.e., have a Hermitian constitutive relationship.

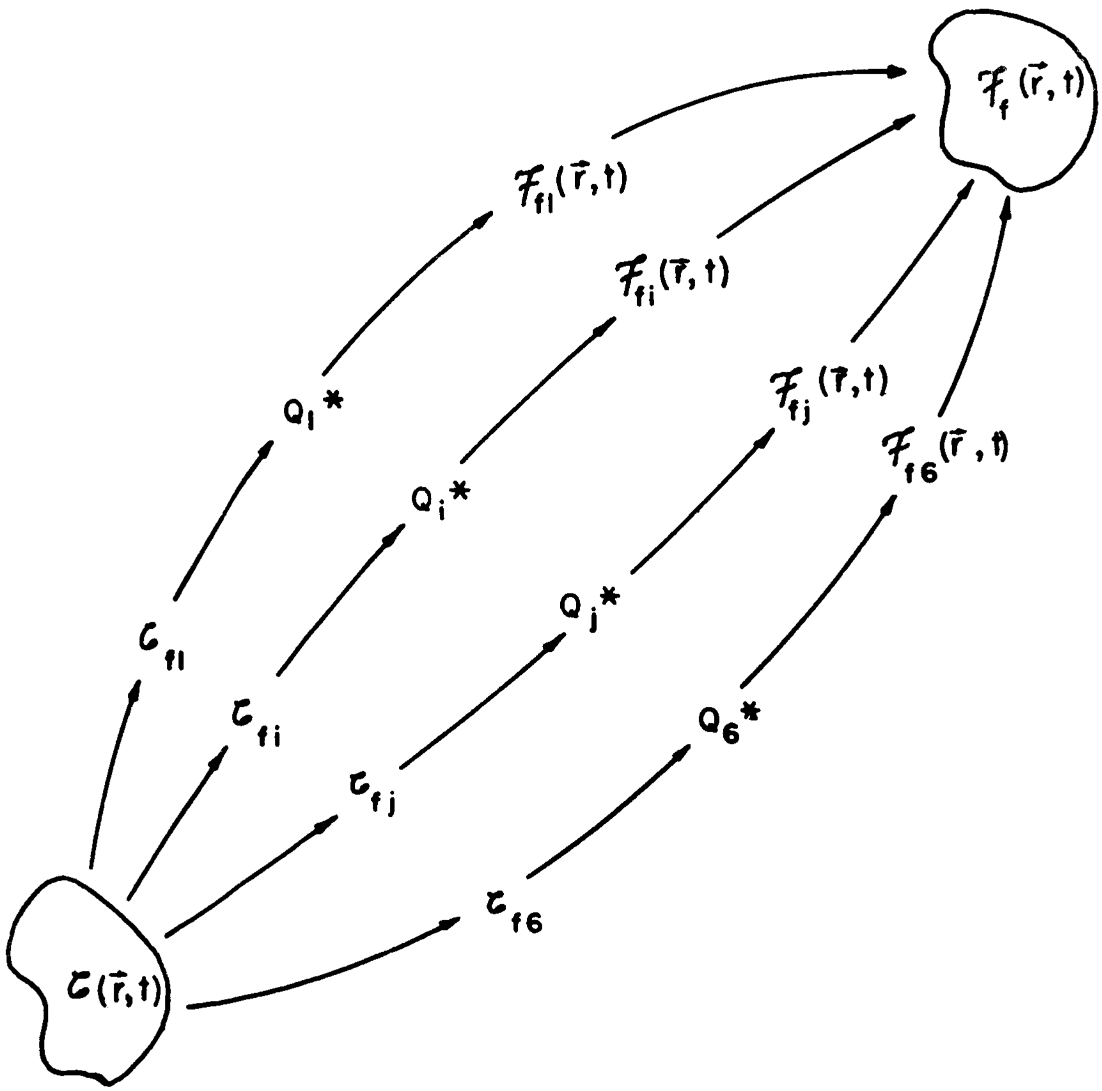

Figure 3.3. The source flux field, $\mathcal{F}_f(\vec{r},t)$, in terms of the "characteristic sources" and the influence of the sheets of the dispersion surface.

3.7 Equilvalence Between Formulations

Again it is instructive to show the equivalence between the six-vector and three-vector methods for the source problem. Let $\underline{\underline{U}} = \begin{bmatrix} \underline{\underline{\epsilon}} & 0 \\ 0 & \underline{\underline{\mu}} \end{bmatrix}$ and $\underline{\underline{U}} = \underline{\underline{U}}$. The off "diagonals" of $\underline{\underline{U}}$ must be zero for otherwise the simple decoupling between the **E** and **H** vectors does not apply.

$$\mathcal{M}\, F = C \tag{3.102}$$

Multiply both sides by $-\mathcal{M}_- U^{-1}$.

But

$$-\mathcal{M}_-\, \underline{\underline{U}}^{-1} C = -j\omega C' = -j\omega \begin{bmatrix} J_e - \frac{1}{\omega}\, \vec{k} \times \underline{\underline{\mu}}^{-1} \vec{k} \times J_m \\ J_m + \frac{1}{\omega}\, \vec{k} \times \underline{\underline{\epsilon}}^{-1} \vec{k} \times J_e \end{bmatrix} = -j\omega \begin{bmatrix} M_e \\ M_m \end{bmatrix} \tag{3.103}$$

Therefore,

$$-\mathcal{M}_-\, \underline{\underline{U}}^{-1} C = -j\omega C' \tag{3.104}$$

$$F(\vec{k},\omega) = (-\mathcal{M}^{-1}\, \underline{\underline{U}}\, \mathcal{M}_-^{-1})(-j\omega C') \tag{3.105}$$

$$\mathcal{M}^{-1} = \sum_{i=1}^{6} (\nu_i - j\omega)^{-1}\, I_i\, \underline{\underline{U}}^{-1} \tag{3.106}$$

$$\mathcal{M}_-^{-1} = \sum_{i=1}^{6} (\nu_i - j\omega)^{-1}\, I_i\, \underline{\underline{U}}^{-1} \tag{3.107}$$

Therefore,

$$F(\vec{k},\omega) = -\sum_{i=1}^{6} \sum_{n=1}^{6} (\nu_i - j\omega)^{-1} (\nu_n + j\omega)^{-1}\, I_i\, \underline{\underline{U}}^{-1} \underline{\underline{U}}\, I_n\, \underline{\underline{U}}^{-1} (-j\omega C') \tag{3.108}$$

$$= \sum_{i=1}^{6} (\nu_i^2 + \omega^2)^{-1} I_i \underline{\underline{U}}^{-1}(-j\omega C') \qquad \text{(orthogonality)} \quad (3.109)$$

$$F(\vec{k},\omega) = -j\omega\mu_0\epsilon_0 \sum_{i=1}^{6} (\lambda_i - k_0^2)^{-1} I_i \underline{\underline{U}} C' \quad ,(\because\ \lambda_i = -\nu_i^2 \mu_0 \epsilon_0) \quad (3.110)$$

Since $\lambda_1 = \lambda_4$, $\lambda_2 = \lambda_5$ and C' is independent of an index, we may sum these terms. Therefore,

$$F_i(\vec{k},\omega) + F_{i+3}(\vec{k},\omega) = -j\omega\mu_0\epsilon_0 (\lambda_i - k_0^2)^{-1} (I_i + I_{i+3}) \underline{\underline{U}}^{-1} C' \quad ,(i=1,2) \quad (3.111)$$

Now

$$I_i = (F_i^{\dagger} \underline{\underline{U}} F_i)^{-1} F_i F_i^{\dagger} \underline{\underline{U}} \qquad (3.112)$$

and

$$(F_i^{\dagger} \underline{\underline{U}} F_i) = (F_{i+3}^{\dagger} \underline{\underline{U}} F_{i+3}) \qquad (3.113)$$

for $i = 1,2$ since $F_{i+3} = \check{F}_i$

Therêfore,

$$I_i + I_{i+3} = 2 (F_i^{\dagger} \underline{\underline{U}} F_i)^{-1} \begin{bmatrix} E_i E_i^{\dagger} & 0 \\ 0 & H_i H_i^{\dagger} \end{bmatrix} \underline{\underline{U}} \qquad (3.114)$$

And
$$F(\vec{k},\omega) = -2j\omega\mu_0\epsilon_0\sum_{i=1}^{2}(\lambda_i - k_0^2)^{-1}(F_i^{\dagger}\underline{\underline{U}}F_i)^{-1}\begin{bmatrix} E_iE_i^{\dagger} & 0 \\ 0 & H_iH_i^{\dagger}\end{bmatrix}C'$$

$$-j\omega\mu_0\epsilon_0\sum_{i=3,6}(\lambda_i - k_0^2)^{-1}(F_i^{\dagger}\underline{\underline{U}}F_i)^{-1}F_iF_i^{\dagger}C' \qquad (3.115)$$

For i = 1,2 $\qquad (F_i^{\dagger}\underline{\underline{U}}F_i) = 2(E_i^{\dagger}\underline{\underline{\epsilon}}E_i) = 2(H_i^{\dagger}\underline{\underline{\mu}}H_i)$

For i = 3 $\quad F_3 = \left[E_3, 0\right]$ $\qquad$ and for i = 6, $\quad F_6 = \left[0, H_6\right]$

Using these simplifications, Equation (3.115) reduces to that of the earlier method through G_E' and G_H'. Hence,

$$E(\vec{k},\omega) = -j\omega\mu_0\epsilon_0\sum_{i=1}^{3}(\lambda_i - k_0^2)^{-1}(E_i^{\dagger}\underline{\underline{\epsilon}}E_i)^{-1}E_iE_i^{\dagger}M_e \qquad (3.116)$$

And

$$H(\vec{k},\omega) = -j\omega\mu_0\epsilon_0\sum_{i=1,2,6}(\lambda_i - k_0^2)^{-1}(H_i^{\dagger}\underline{\underline{\mu}}H_i)^{-1}H_iH_i^{\dagger}M_m \quad (3.117)$$

3.8 Equivalence Between Formulations By Direct Addition of Terms

An alternate and perhaps more illuminating correspondence between the two formulations may be made by the direct pairwise addition of the terms corresponding to ν_i and $-\nu_i$.

$$F_i(\vec{k},\omega) + F_{i+3}(\vec{k},\omega) = (\nu_i - j\omega)^{-1} I_i \underline{\underline{U}}^{-1} C + (-\nu_i - j\omega)^{-1} \check{I}_i \underline{\underline{U}}^{-1} C \quad (i=1,2) \tag{3.118}$$

$$= (\nu_i - j\omega)^{-1} (F_i^{\dagger} \underline{\underline{U}} F_i)^{-1} F_i F_i^{\dagger} C - (\nu_i + j\omega)^{-1} (F_i^{\dagger} \underline{\underline{U}} F_i)^{-1} \check{F}_i \check{F}_i^{\dagger} C \tag{3.119}$$

$$= (F_i^{\dagger} \underline{\underline{U}} F_i)^{-1} \left[(\nu_i - j\omega)^{-1} F_i F_i^{\dagger} - (\nu_i + j\omega)^{-1} \check{F}_i \check{F}_i^{\dagger} \right] C \tag{3.120}$$

$$= (F_i^{\dagger} \underline{\underline{U}} F_i)^{-1} \left\{ \frac{2j\omega}{(\nu_i^2 + \omega^2)} \begin{bmatrix} E_i E_i^{\dagger} & 0 \\ 0 & H_i H_i^{\dagger} \end{bmatrix} + \frac{2\nu_i}{(\nu_i^2 + \omega^2)} \begin{bmatrix} 0 & E_i H_i^{\dagger} \\ H_i E_i^{\dagger} & 0 \end{bmatrix} \right\} C \tag{3.121}$$

Lemma 3.4: $\nu_i H_i^{\dagger} J_m = j\omega E_i^{\dagger} \left(-\frac{1}{\omega} \vec{k} \times \underline{\underline{\mu}}^{-1} J_m \right)$

Proof:

$$-j\vec{k} \times E_i = -\nu_i \underline{\underline{\mu}} H_i$$

$$\nu_i H_i = j \underline{\underline{\mu}}^{-1} \vec{k} \times H_i$$

$$\nu_i H_i^{\dagger} = -j E_i^{\dagger} \vec{k} \times \underline{\underline{\mu}}^{-1}$$

$$\nu_i H_i^{\dagger} J_m = j\omega E_i^{\dagger} \left(-\frac{1}{\omega} \vec{k} \times \underline{\underline{\mu}}^{-1} J_m \right)$$

Lemma 3.5: $\nu_i E_i^{\dagger} J_e = j\omega H_i^{\dagger} \left(\frac{1}{\omega} \vec{k} \times \underline{\underline{\epsilon}}^{-1} J_e \right)$

Proof:

$$-j\vec{k} \times H_i = \nu_i \underline{\underline{\epsilon}} E_i$$

$$\nu_i E_i = -j \underline{\underline{\epsilon}}^{-1} \vec{k} \times H_i$$

$$\nu_i E_i^{\dagger} = j H_i^{\dagger} \vec{k} \times \underline{\underline{\epsilon}}^{-1}$$

$$\nu_i E_i^{\dagger} J_e = j\omega H_i^{\dagger} \left(\frac{1}{\omega} \vec{k} \times \underline{\underline{\epsilon}}^{-1} J_e \right)$$

Using the two lemmas it can be shown that

$$2\nu_i \begin{bmatrix} 0 & E_iH_i^{\dagger} \\ H_iE_i^{\dagger} & 0 \end{bmatrix} C = 2j\omega \begin{bmatrix} E_iE_i^{\dagger}(-\frac{1}{\omega}\vec{k}\times\underline{\underline{\mu}}^{-1}J_m) \\ H_iH_i^{\dagger}(\frac{1}{\omega}\vec{k}\times\underline{\underline{\epsilon}}^{-1}J_e) \end{bmatrix} \tag{3.122}$$

$$= 2j\omega \begin{bmatrix} E_iE_i^{\dagger} & 0 \\ 0 & H_iH_i^{\dagger} \end{bmatrix} \begin{bmatrix} (-\frac{1}{\omega}\vec{k}\times\underline{\underline{\mu}}^{-1}J_m) \\ (\frac{1}{\omega}\vec{k}\times\underline{\underline{\epsilon}}^{-1}J_e) \end{bmatrix} \tag{3.123}$$

Placing this in Equation (3.121) and collecting common factors we get

$$F_i(\vec{k},\omega)+F_{i+3}(\vec{k},\omega) = (F_i^{\dagger}\underline{\underline{U}}F_i)^{-1}\frac{2j\omega}{(\nu_i^2+\omega^2)} \begin{bmatrix} E_iE_i^{\dagger} & 0 \\ 0 & H_iH_i^{\dagger} \end{bmatrix} \left\{ C + \begin{bmatrix} (-\frac{1}{\omega}\vec{k}\times\underline{\underline{\mu}}^{-1}J_m) \\ (\frac{1}{\omega}\vec{k}\times\underline{\underline{\epsilon}}^{-1}J_e) \end{bmatrix} \right\}, \tag{3.124}$$

$$(i = 1,2)$$

However, the term in the braces is just C'.

Therefore,

$$F_i(\vec{k},\omega)+F_{i+3}(\vec{k},\omega) = -2j\omega\mu_0\epsilon_0(\lambda_i-k_0^2)^{-1}(F_i^{\dagger}\underline{\underline{U}}F_i)^{-1} \begin{bmatrix} E_iE_i^{\dagger} & 0 \\ 0 & H_iH_i^{\dagger} \end{bmatrix} C' , \tag{3.125}$$

$$(i = 1,2)$$

Since $\lambda_3 = 0$ and $H_3 = 0$

$$F_3(\vec{k},\omega) = (\nu_i - j\omega)^{-1}(F_3^{\dagger}\underline{\underline{U}}F_3)^{-1}F_3F_3^{\dagger}C \tag{3.126}$$

$$= -j\omega\mu_0\epsilon_0(\lambda_3-k_0^2)^{-1}(F_3^{\dagger}\underline{\underline{U}}F_3)^{-1}F_3F_3^{\dagger}C \tag{3.127}$$

$$F_3(\vec{k},\omega) = -j\omega\mu_0\epsilon_0(\lambda_3-k_0^2)^{-1}(F_3^{\dagger}\underline{\underline{U}}F_3)^{-1}F_3F_3^{\dagger}C' \quad , (\because\ E_3 \parallel \vec{k}) \tag{3.128}$$

Similarly,

$$F_6(\vec{k},\omega) = -j\omega\mu_0\epsilon_0(\lambda_6-k_0^2)^{-1}(F_6^{\dagger}\underline{\underline{U}}F_6)^{-1}F_6F_6^{\dagger}C \tag{3.129}$$

$$= -j\omega\mu_0\epsilon_0(\lambda_6-k_0^2)^{-1}(F_6^{\dagger}\underline{\underline{U}}F_6)^{-1}F_6F_6^{\dagger}C' \quad , (\because\ H_6 \parallel \vec{k}) \tag{3.130}$$

Now adding the terms gives the same equation which was found by the previous method, i.e.,

$$F(\vec{k},\omega) = -2j\omega\mu_0\epsilon_0 \sum_{i=1}^{2}(\lambda_i-k_0^2)^{-1}(F_i^{\dagger}\underline{\underline{U}}F_i)^{-1}\begin{bmatrix} E_iE_i^{\dagger} & 0 \\ 0 & H_iH_i^{\dagger}\end{bmatrix}C'$$
$$-j\omega\mu_0\epsilon_0 \sum_{i=3,6}(\lambda_i-k_0^2)^{-1}(F_i^{\dagger}\underline{\underline{U}}F_i)^{-1}F_iF_i^{\dagger}C' \tag{3.131}$$

which reduces to

$$E(\vec{k},\omega) = -j\omega\mu_0\epsilon_0\sum_{i=1}^{3}(\lambda_i-k_0^2)^{-1}(E_i^{\dagger}\underline{\underline{\epsilon}}E_i)^{-1}E_iE_i^{\dagger}M_e \tag{3.132}$$

and

$$H(\vec{k},\omega) = -j\omega\mu_0\epsilon_0\sum_{i=1,2,6}(\lambda_i-k_0^2)^{-1}(H_i^{\dagger}\underline{\underline{\mu}}H_i)^{-1}H_iH_i^{\dagger}M_m \tag{3.133}$$

4. SPECTRUM OF CHARACTERISTIC WAVES IN A GENERAL TIME-DISPERSIVE UNIAXIAL MEDIUM

The general time-dispersive uniaxial problem is described by a matrix permeability and permittivity of the form $\underline{\underline{\mu}} = \mu_0 \begin{bmatrix} N_1 & 0 & 0 \\ 0 & N_1 & 0 \\ 0 & 0 & N_0 \end{bmatrix}$ and $\mathcal{E} = \mathcal{E}_0 \begin{bmatrix} K_1 & 0 & 0 \\ 0 & K_1 & 0 \\ 0 & 0 & K_0 \end{bmatrix}$ respectively. Maxwell's equations for a lossless medium are,

$$\text{Curl } E(\vec{r}) = -j\omega\mu_0 NH(\vec{r}) - J_m(\vec{r})$$

$$\text{Curl } H(\vec{r}) = j\omega\mathcal{E}_0 KE(\vec{r}) + J_e(\vec{r}) \qquad (4.1)$$

Eliminating the electric field gives the equation

$$\text{Curl } K^{-1} \text{ Curl } H(\vec{r}) - k_0^2 NH(\vec{r}) = -j\omega\mathcal{E}_0 M_m(\vec{r}) \qquad (4.2)$$

where

$$M_m(\vec{r}) = J_m(\vec{r}) - \left(\frac{1}{j\omega\mathcal{E}_0}\right)\nabla \times K^{-1} J_e(\vec{r}) \qquad (4.3)$$

Taking the Fourier transform with respect to the space variables results in,

$$G_H(\vec{k})H(\vec{k}) = -j\omega\mathcal{E}_0 M_m(\vec{k}) \qquad (4.4)$$

where $\vec{k}$ is the wave vector,

$$G_H(\vec{k}) = -\vec{k}\times K^{-1}\vec{k}\times - \vec{k}_0^2 N \qquad (4.5)$$

Explicitly $G_H(\vec{k})$ is in terms of the wave vector $\vec{k} = (\xi,\eta,\zeta)$

$$G_H(\vec{k}) = \begin{bmatrix} (L_1\zeta^2+L_0\eta^2-k_0^2N_1) & -L_0\xi\eta & -L_1\xi\zeta \\ -L_0\xi\eta & (L_0\xi^2+L_1\zeta^2-k_0^2N_1) & -L_1\eta\zeta \\ -L_1\xi\zeta & -L_1\eta\zeta & (L_1\xi^2+L_1\eta^2-k_0^2N_0) \end{bmatrix} \tag{4.6}$$

The characteristic waves are obtained by determining the solution to Equation (4.4) when the source $M_m(\vec{k})$ is zero. This implies that the determinant of $G_H(\vec{k})$ is zero for nonzero solutions. Denote the determinant of $G_H(\vec{k})$ by $D(\vec{k})$. Then,

$$D(\vec{k}) = -k_0^2(L_0T_1\rho^2+L_1T_1\zeta^2-k_0^2)(L_1T_0\rho^2-L_1T_1\zeta^2-k_0^2)\,|N| \tag{4.7}$$

where $\rho = (\xi^2+\eta^2)^{\frac{1}{2}}$

Denote S_1, S_2, and S_3 by

$$\begin{aligned} S_1 &\equiv (L_0T_1\rho^2+L_1T_1\zeta^2-k_0^2), \\ S_2 &\equiv (L_1T_0\rho^2+L_1T_1\zeta^2-k_0^2), \quad S_3 \equiv -k_0^2 \end{aligned} \tag{4.8}$$

where $T = N^{-1}$ and $\det N = |N|$

Let

$$A = -\vec{k}\times\overleftrightarrow{K}\,\vec{k}\times \tag{4.9}$$

$$AH_i = \lambda_i N H_i \qquad (i=1,2,3) \tag{4.10}$$

or

$$(A-\lambda_i N)\, H_i = 0 \tag{4.11}$$

We claim that λ_i is real and $H_j^\dagger N H_i = 0$, $i \neq j$

Proof:

$$H_i^\dagger A H_i = \lambda_i H_i^\dagger N H_i \tag{4.12}$$

$$(H_i^\dagger A H_i)^\dagger = \lambda_i^* (H_i^\dagger N H_i)^\dagger \tag{4.13}$$

$$H_i^\dagger A H_i = \lambda_i^* H_i^\dagger N H_i \qquad \text{since A and N are Hermitian} \tag{4.14}$$

$$(\lambda_i - \lambda_i^*) H_i^\dagger N H_i = 0 \text{ implies } \lambda_i \text{ real if } H_i^\dagger N H_i \neq 0 \tag{4.15}$$

$$H_j^\dagger A H_i = \lambda_i H_j^\dagger N H_i \tag{4.16}$$

$$H_i^\dagger A H_j = \lambda_j H_i^\dagger N H_j \tag{4.17}$$

$$H_j^\dagger A H_j = \lambda_j H_j^\dagger N H_i \qquad \text{since } \lambda_i \text{ real and A, N Hermitian} \tag{4.18}$$

$$(\lambda_i - \lambda_j)(H_j^\dagger N H_i) = 0 \text{ implies } H_j^\dagger N H_i = 0 \text{ if } \lambda_i \neq \lambda_j \tag{4.19}$$

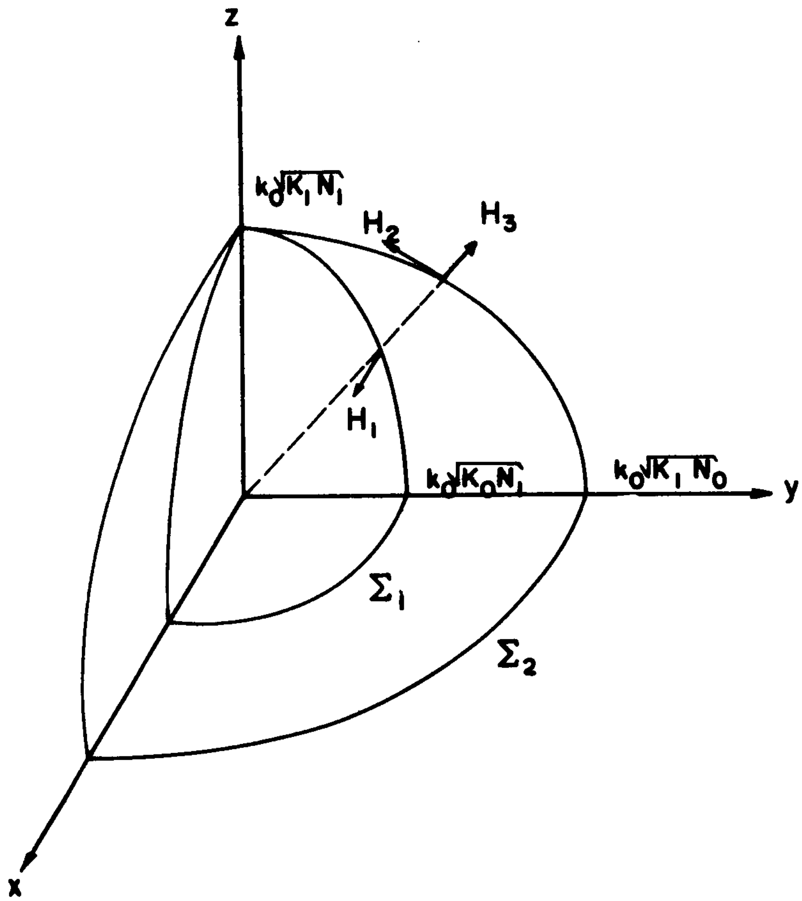

Figure 4.1. One quadrant of the three-space dispersion surface for a general time dispersive uniaxial medium.

The eigenvectors are,

$$H_1 = \vec{k} \times \hat{z}, \qquad H_2 = N^{-1} \vec{k} \times \vec{k} \times \hat{z}, \qquad H_3 = \vec{k} \tag{4.20}$$

Then

$$I = \sum_{i=1}^{3} (H_i^{\dagger} N H_i)^{-1} H_i H_i^{\dagger} N = \sum_{i=1}^{3} (H_i^{\dagger} N H_i)^{-1} N H_i H_i^{\dagger} \tag{4.21}$$

$$G_H(\vec{k}) = \sum_{i=1}^{3} S_i (H_i^{\dagger} N H_i)^{-1} N H_i H_i^{\dagger} N \tag{4.22}$$

$$G_H^{-1}(\vec{k}) = \sum_{i=1}^{3} S_i^{-1} (H_i^{\dagger} N H_i)^{-1} H_i H_i^{\dagger} \tag{4.23}$$

$$H_1^{\dagger} N H_1 = N_1 \rho^2, \quad H_2^{\dagger} N H_2 = N_0^{-1} N_1^{-1} (N_1 \rho^2 + N_0 \zeta^2) \rho^2, \quad H_3^{\dagger} N H_3 = (N_1 \rho^2 + N_0 \zeta^2) \tag{4.24}$$

$$H_1 H_1^{\dagger} = (\vec{k} \times \hat{z})^{\dagger} (\vec{k} \times \hat{z}) = \begin{bmatrix} \eta^2 & -\xi\eta & 0 \\ -\xi\eta & \xi^2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{4.25}$$

$$H_2 H_2^{\dagger} = (N^{-1} \vec{k} \times \vec{k} \times \hat{z})^{\dagger} (N^{-1} \vec{k} \times \vec{k} \times \hat{z}) = N_0^{-2} N_1^{-2} \begin{bmatrix} N_0^2 \xi^2 \zeta^2 & N_0^2 \xi \eta \zeta^2 & -N_0 N_1 \xi \rho^2 \zeta \\ N_0^2 \xi \eta \zeta^2 & N_0^2 \eta^2 \zeta^2 & -N_0 N_1 \eta \rho^2 \zeta \\ -N_0 N_1 \xi \rho^2 \zeta & -N_0 N_1 \eta \rho^2 \zeta & N_1^2 \rho^4 \end{bmatrix} \tag{4.26}$$

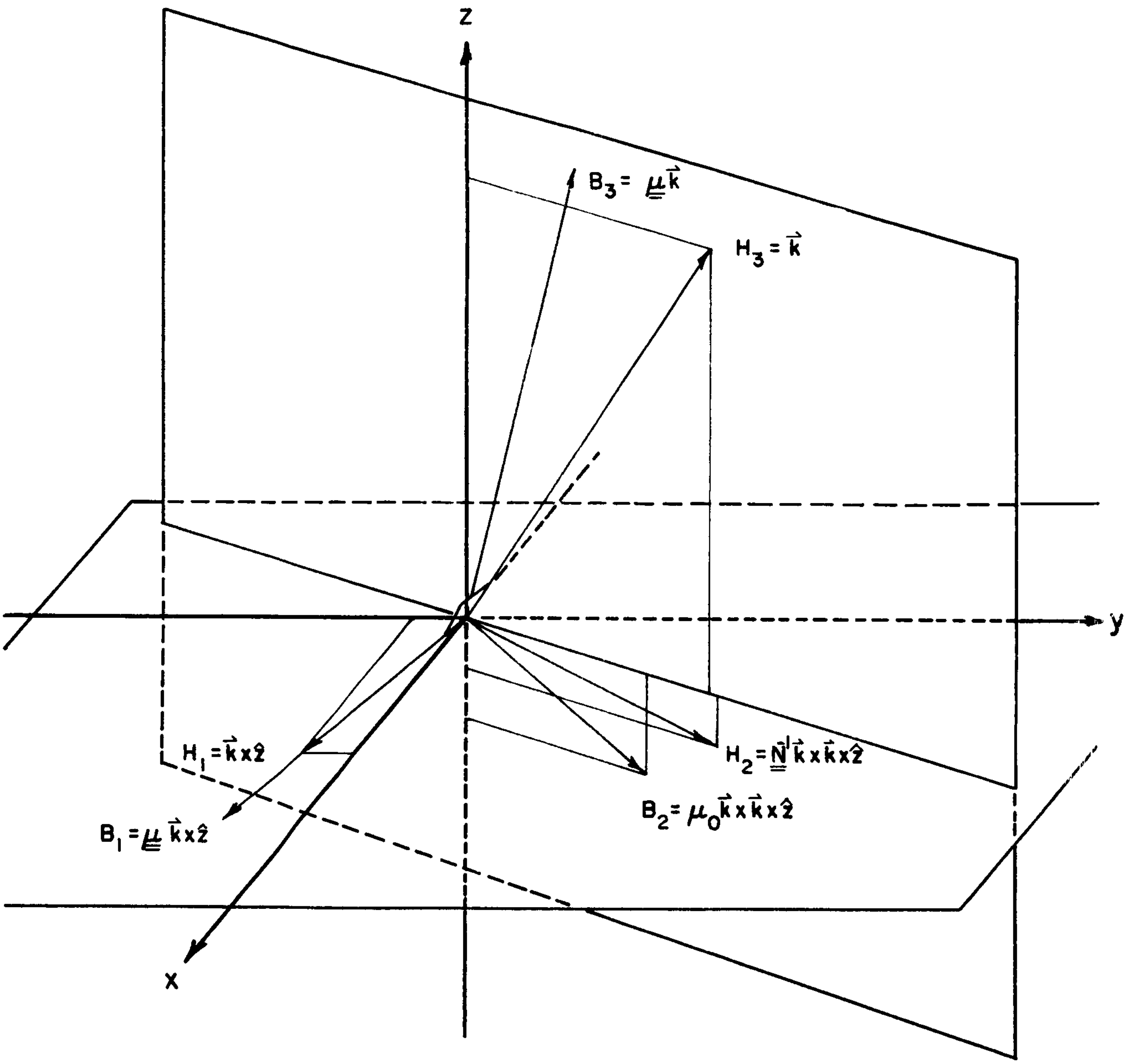

Figure 4.2. Eigenvectors for a general time dispersive uniaxial medium.

$$H_3H_3^{\dagger} = \vec{k}\,\vec{k}^{\dagger} = \begin{bmatrix} \xi^2 & \xi\eta & \xi\zeta \\ \xi\eta & \eta^2 & \eta\zeta \\ \xi\zeta & \eta\zeta & \zeta^2 \end{bmatrix} \tag{4.27}$$

By considering the known inverse transforms of ρ^2 and k^2 and the proper change of variables, one can determine the function G_i, i.e., the inverse transform of $(H_i^{\dagger} \times H_i)^{-1}$. The functions, G_i, that result are,

$$G_1(\vec{r}) = -\frac{1}{2\pi N_1}(\log P)\,\delta(z) \tag{4.28}$$

$$G_2(\vec{r}) = N_0 N_1^2 G_1 * G_3 \tag{4.29}$$

$$G_3(\vec{r}) = T_1 T_0^{\frac{1}{2}} \frac{1}{4\pi(T_1 P^2 + T_0 z^2)^{1/2}} \tag{4.30}$$

where

$$P = (x^2 + y^2)^{1/2} \tag{4.31}$$

Similarly, by considering the known inverse transform of $(k^2 - k_o^2)^{-1}$ and the proper change of variables, one can determine the function Q_i, i.e., the inverse transform of S_i^{-1}. The functions Q_i, so determined are,

$$Q_1 = \frac{K_0 K_1^{1/2} N_1^{3/2}}{4\pi} \frac{e^{-jk_o(K_0 N_1 P^2 + K_1 N_1 z^2)^{1/2}}}{(K_0 N_1 P^2 + K_1 N_1 z^2)^{1/2}} \tag{4.32}$$

$$Q_2 = \frac{K_1^{3/2} N_0 N_1^{1/2}}{4\pi} \frac{e^{-jk_0(K_1 N_0 P^2 + K_1 N_1 z^2)^{1/2}}}{(K_1 N_0 P^2 + K_1 N_1 z^2)^{1/2}} \tag{4.33}$$

$$Q_3 = -k_0^{-2}\,\delta(\vec{r}) \tag{4.34}$$

With the above functions it is possible to give an expression for the Green's matrix, $\underline{\underline{\Gamma}}$, and the magnetic field.

$$\underline{\underline{\Gamma}}(\vec{r},\omega) = -j\omega\epsilon_0 \sum_{i=1}^{3} Q_i * G_i * H_i(\nabla) H_i^{\dagger}(\nabla)\,\delta(\vec{r}) \tag{4.35}$$

$$\vec{H}(\vec{r},\omega) = \underline{\underline{\Gamma}}(\vec{r},\omega) * M_m(\vec{r}) \tag{4.36}$$

4.1 Field of an Electric Dipole with a Longitudinal Orientation

The Fourier transform of source $M_m(\vec{r})$ for the electric dipole $J_e(\vec{r}) = \delta(\vec{r})\,\hat{z}$, is $M_m(\vec{k}) = \frac{1}{(K_o \omega\, \epsilon_o)}\,\vec{k} \times \hat{z}$. The source $M_m(\vec{k})$ is orthogonal to both eigenvectors H_2 and H_3. Therefore,

$$M_m(\vec{k}) = M_{m1}(\vec{k}) \tag{4.37}$$

and

$$B(\vec{r}) = -j\omega\mu_0\epsilon_0 Q_1(\vec{r}) * M_m(\vec{r}) \tag{4.38}$$

$$B(\vec{r}) = -j\omega\mu_0\epsilon_0 Q_1(\vec{r}) * \left(\frac{-1}{j\omega\epsilon_0 K_0}\right) \nabla \times \delta(\vec{r})\,\hat{z} \tag{4.39}$$

$$B(\vec{r}) = L_0 \mu_0 \nabla \times Q_1(\vec{r}) \hat{z} \tag{4.40}$$

or

$$B(\vec{r}) = \frac{\mu_0 \sqrt{K_1}\, N_1^{3/2}}{4\pi} \nabla \times \frac{e^{-jk_0(K_0 N_1 P^2 + K_1 N_1 z^2)^{1/2}}}{(K_0 N_1 P^2 + K_1 N_1 z^2)^{1/2}} \hat{z} \tag{4.41}$$

Because $B(\vec{r})$ is transverse to the magnetic field and the particular form of the permeability matrix, the magnetic field intensity is,

$$H(\vec{r}) = \frac{1}{K_0 N_1} \nabla \times Q_1(\vec{r}) \hat{z} \tag{4.42}$$

It then follows that,

$$D(\vec{r}, \omega) = \frac{1}{j\omega} \nabla \times H(\vec{r}) - \delta(\vec{r}) \hat{z} \tag{4.43}$$

Notice that only one characteristic field, and accordingly one sheet of the dispersion surface, is involved in the source field.

4.2 Field of an Electric Dipole With a Transverse Orientation

The electric dipole is directed along the y axis in order to be able to check Clemmow's[1] results. The source is $M_m(\vec{r}) = -\left(\frac{1}{j\omega\epsilon_o}\right) \nabla \times K^{-1}\, \delta(\vec{r})\, \hat{y}$. In the transform domain this becomes $M_m(\vec{k}) = +\left(\frac{1}{\omega\epsilon_o K_1}\right) \vec{k} \times \hat{y}$.

$$M_{m1}(\vec{k}) = N_1^{-1} \rho^{-2}\, \vec{k} \times \hat{z}\, (\vec{k} \times \hat{z})^{+} \left(\frac{1}{\omega \epsilon_0 K_1}\right) \vec{k} \times \hat{y} \tag{4.44}$$

$$= \frac{1}{\omega\epsilon_0 K_1} N_1^{-1}\rho^{-2}\vec{k}\times\hat{z}\left[(\vec{k}\times\hat{z})^{\dagger}(\vec{k}\times\hat{y})\right] \tag{4.45}$$

$$= \frac{1}{\omega\epsilon_0 K_1} N_1^{-1}\rho^{-2}\eta\zeta(\eta, -\xi, 0) \tag{4.46}$$

$$= \frac{1}{\omega\epsilon_0 K_1} N_1^{-1}\rho^{-2}\eta\zeta\vec{k}\times\hat{z} \tag{4.47}$$

$$M_{m1}(\vec{r}) = \frac{1}{-j\omega\epsilon_0 K_1}\frac{\partial^2}{\partial y\partial z}\nabla\times\hat{z}G_1(\vec{r}) = \frac{-1}{j\omega\epsilon_0 K_1}\nabla\times\frac{\partial^2}{\partial y\partial z}G_1(\vec{r})\hat{z} \tag{4.48}$$

$$M_{m1x}(\vec{r}) = \left(\frac{1}{-j\omega\epsilon_0 K_1}\right)\frac{\partial^3 G_1}{\partial y^2\partial z} \tag{4.49}$$

$$M_{m1y}(\vec{r}) = \left(\frac{1}{j\omega\epsilon_0 K_1}\right)\frac{\partial^3 G_1}{\partial x\partial y\partial z} \tag{4.50}$$

$$M_{m1z}(\vec{r}) = 0 \tag{4.51}$$

$$M_{m2}(\vec{k}) = N_0 N_1 (N_1\rho^2 + N_0\zeta^2)^{-1}\rho^{-2}N^{-1}\vec{k}\times\vec{k}\times\hat{z}\left[(N^{-1}\vec{k}\times\vec{k}\times\hat{z})^{\dagger}\left(\frac{1}{\omega\epsilon_0 K_1}\right)\vec{k}\times\hat{y}\right] \tag{4.52}$$

$$= -\left(\frac{1}{\omega\epsilon_0 K_1}\right)\rho^{-2}\xi N^{-1}\vec{k}\times\vec{k}\times\hat{z} \tag{4.53}$$

$$M_{m2}(\vec{r}) = \left(\frac{N_1}{-j\omega\epsilon_0 K_1}\right)N^{-1}\nabla\times\nabla\times\hat{z}\frac{\partial G_1}{\partial x} \tag{4.54}$$

$$M_{m2x}(\vec{r}) = \left(\frac{1}{-j\omega\epsilon_0 K_1}\right)\frac{\partial^3}{\partial x^2\partial z}G_1 \tag{4.55}$$

$$M_{m2y}(\vec{r}) = \left(\frac{1}{-j\omega\epsilon_0 K_1}\right)\frac{\partial^3}{\partial x\partial y\partial z}G_1 \tag{4.56}$$

$$M_{m2z}(\vec{r}) = \left(\frac{1}{-j\omega\epsilon_0 K_1}\right)N_0^{-1}\frac{\partial}{\partial x}\delta(\vec{r}) \tag{4.57}$$

$$Q_1 * G_1 = (2\pi)^{-3} \iiint_{-\infty}^{\infty} \frac{e^{-j\vec{k}\cdot\vec{r}}\, d\xi\, d\eta\, d\zeta}{N_1 \rho^2 (L_0 T_1 \rho^2 + L_1 T_1 \zeta^2 - k_0^2)} \tag{4.58}$$

$$= (2\pi)^{-3} \int_0^{\infty}\int_0^{2\pi}\int_{-\infty}^{\infty} \frac{e^{-j\left[\rho P \cos(\phi' - \phi) + \zeta z\right]} \rho\, d\rho\, d\phi'\, d\zeta}{N_1 \rho^2 \left[L_0 T_1 \rho^2 + L_1 T_1 \zeta^2 - k_0^2\right]} \tag{4.59}$$

$$= (2\pi)^{-2} N_1^{-1} \int_0^{\infty}\int_{-\infty}^{\infty} \frac{J_0(\rho P) e^{-j\zeta z}\, d\rho\, d\zeta}{\rho \left[L_0 T_1 \rho^2 + L_1 T_1 \zeta^2 - k_0^2\right]} \tag{4.60}$$

$$= (2\pi)^{-2} (K_1 T_1)^{1/2} \int_0^{\infty}\int_{-\infty}^{\infty} \frac{J_0(\rho P) e^{-j\zeta' \sqrt{K_1 N_1}\, z}\, d\rho\, d\zeta'}{\rho \left[\zeta'^2 + L_0 T_1 \rho^2 - k_0^2\right]} \tag{4.61}$$

$$= \frac{(K_1 T_1)^{1/2}}{4\pi} \int_0^{\infty} \frac{J_0(\rho P) e^{-\sqrt{L_0 T_1 \rho^2 - k_0^2}\, \left|\sqrt{K_1 N_1}\, z\right|}\, d\rho}{\rho \sqrt{L_0 T_1 \rho^2 - k_0^2}} \tag{4.62}$$

$$\frac{\partial Q_1 * G_1}{\partial y} = \frac{-(K_1 T_1)^{1/2}}{4\pi} \frac{y}{P} \int_0^{\infty} \frac{J_1(\rho P) e^{-\sqrt{L_0 T_1 \rho^2 - k_0^2}\, \left|\sqrt{K_1 N_1}\, z\right|}\, d\rho}{\sqrt{L_0 T_1 \rho^2 - k_0^2}} \tag{4.63}$$

$$\frac{\partial^2 Q_1 * G_1}{\partial z\, \partial y} = \frac{K_1}{4\pi} \frac{y}{P} (\text{sgn. } z) \int_0^{\infty} J_1(\rho P) e^{-\sqrt{L_0 T_1 \rho^2 - k_0^2}\, \left|\sqrt{K_1 N_1}\, z\right|}\, d\rho \tag{4.64}$$

Integrate by parts

$$u = e^{-\sqrt{L_0 T_1 \rho^2 - k_0^2}\, \left|\sqrt{K_1 N_1}\, z\right|}$$

$$du = \frac{-\left|\sqrt{K_1 N_1}\, z\right| e^{-\sqrt{L_0 T_1 \rho^2 - k_0^2}\, \left|\sqrt{K_1 N_1}\, z\right|} L_0 T_1 \rho\, d\rho}{\sqrt{L_0 T_1 \rho^2 - k_0^2}}$$

$$dv = J_1(\rho P)d\rho \qquad\qquad v = \frac{-J_0(\rho P)}{P}$$

$$\frac{\partial^2 Q_1 * G_1}{\partial z\,\partial y} = \frac{K_1}{4\pi}\,\frac{y}{P}\,(\text{sgn. } z)\left\{\frac{-J_0(\rho P)}{P}\,e^{-\sqrt{L_0T_1\rho^2 - k_0^2}\,\left|\sqrt{K_1N_1}\,z\right|}\Bigg|_0^\infty\right.$$

$$\left.\frac{-\left|\sqrt{K_1N_1}\,z\right|L_0T_1}{P}\int_0^\infty \frac{J_0(\rho P)\,e^{-\sqrt{L_0T_1\rho^2 - k_0^2}}}{\sqrt{L_0T_1\rho^2 - k_0^2}}\,\rho\, d\rho\right\} \tag{4.65}$$

$$\frac{K_1}{4\pi}\,\frac{y}{P}\,(\text{sgn. } z)\left\{\frac{e^{-jk_0\left|\sqrt{K_1N_1}\,z\right|}}{P}\;\frac{-\left|\sqrt{K_1N_1}\,z\right|}{P}\right.$$

$$\left.\cdot\int_0^\infty \frac{J_0\!\left(\rho'\sqrt{K_1N_1}\,P\right)e^{-\sqrt{\rho'^2 - k_0^2}}}{\sqrt{\rho'^2 - k_0^2}}\,\rho'\, d\rho'\right\} \tag{4.66}$$

The integral of Equation (4.66) is Summerfeld's formula (page 34, <u>Functions of Mathematical Physics</u>, Magnus and Oberheitinger[12]).

$$\frac{\partial^2 Q_1 * G_1}{\partial z\,\partial y} = \frac{K_1}{4\pi}\,\frac{y}{P}\,(\text{sgn. } z)\left\{\frac{e^{-jk_0\left|\sqrt{K_1N_1}\,z\right|}}{P}\right.$$

$$\left.\frac{-\left|\sqrt{K_1N_1}\,z\right|}{P}\;\frac{e^{-jk_0\sqrt{K_0N_1P^2 + K_1N_1z^2}}}{\sqrt{K_0N_1P^2 + K_1N_1z^2}}\right\} \tag{4.67}$$

$$\left(\sqrt{K_0N_1}\,P \text{ and } \sqrt{K_1N_1}\,z \text{ real, } \frac{-\pi}{2} < \arg\sqrt{\rho'^2 - k_0^2} \le \frac{\pi}{2}\right)$$

Now

$$B_1(\vec{r}) = \frac{\mu_0}{K_1} \nabla x z \left(\frac{\partial^2 Q_1 * G_1}{\partial z \, \partial y} \right) \tag{4.68}$$

$$Q_2 * G_1 = (2\pi)^{-3} \iiint_{-\infty}^{\infty} \frac{e^{-j\vec{k}\cdot\vec{r}} \, d\xi \, d\eta \, d\zeta}{N_1 \rho^2 (L_1 T_0 \rho^2 + L_1 T_1 \zeta^2 - k_0^2)} \tag{4.69}$$

$$= (2\pi)^{-3} \int_0^{\infty} \int_0^{2\pi} \int_{-\infty}^{\infty} \frac{e^{-j\left[\rho P \cos(\phi' - \phi) + \zeta z\right]} \rho \, d\rho \, d\phi' \, d\zeta}{N_1 \rho^2 \left[L_1 T_0 \rho^2 + L_1 T_1 \zeta^2 - k_0^2\right]} \tag{4.70}$$

$$= (2\pi)^{-2} N_1^{-1} \int_0^{\infty} \int_{-\infty}^{\infty} \frac{J_0(\rho P) e^{-j\zeta z} \, d\rho \, d\zeta}{\rho \left[L_1 T_1 \rho^2 + L_1 T_1 \zeta^2 - k_0^2\right]} \tag{4.71}$$

$$= (2\pi)^{-2} (K_1 T_1)^{1/2} \int_0^{\infty} \int_{-\infty}^{\infty} \frac{J_0(\rho P) e^{-j\zeta' \sqrt{K_1 N_1} z} \, d\rho \, d\zeta'}{\rho \left[\zeta'^2 + L_1 T_0 \rho^2 - k_0^2\right]} \tag{4.72}$$

$$= \frac{(K_1 T_1)^{1/2}}{4\pi} \int_0^{\infty} \frac{J_0(\rho P) e^{-\sqrt{L_1 T_0 \rho^2 - k_0^2} \left|\sqrt{K_1 N_1} z\right|} \, d\rho}{\rho \sqrt{L_1 T_0 \rho^2 - k_0^2}} \tag{4.73}$$

$$\frac{\partial Q_2 * G_1}{\partial x} = \frac{-(K_1 T_1)^{1/2}}{4\pi} \frac{x}{P} \int_0^{\infty} \frac{J_1(\rho P) e^{-\sqrt{L_1 T_0 \rho^2 - k_0^2} \left|\sqrt{K_1 N_1} z\right|} \, d\rho}{\sqrt{L_1 T_0 \rho^2 - k_0^2}} \tag{4.74}$$

$$\frac{\partial^2 Q_2 * G_1}{\partial z \, \partial x} = \frac{K_1}{4\pi} \frac{x}{P} (\text{sgn. } z) \int_0^{\infty} J_1(\rho P) e^{-\sqrt{L_1 T_0 \rho^2 - k_0^2} \left|\sqrt{K_1 N_1} z\right|} \, d\rho \tag{4.75}$$

Integrate by parts.

$$u = e^{-\sqrt{L_1T_0\rho^2-k_0^2}\,\left|\sqrt{K_1N_1}\; z\right|}$$

$$du = \frac{-\left|\sqrt{K_1N_1}\,z\right| e^{-\sqrt{L_1T_0\rho^2-k_0^2}\,\left|\sqrt{K_1N_1}\,z\right|}\,L_1T_0\rho d\rho}{\sqrt{L_1T_0\rho^2-k_0^2}}$$

$$dv = J_1(\rho P)d\rho \qquad\qquad v = \frac{-J_0(\rho P)}{P}$$

$$\frac{\partial^2 Q_2 * G_1}{\partial z\,\partial x} = \frac{K_1}{4\pi}\frac{x}{P}(\text{sgn. } z)\left\{ \frac{-J_0(\rho P)}{P} e^{-\sqrt{L_1T_0\rho^2-k_0^2}\,\left|\sqrt{K_1N_1}\,z\right|}\Bigg|_0^\infty\right.$$

$$\left.\frac{-\left|\sqrt{K_1N_1}\,z\right| L_1T_0}{P}\int_0^\infty \frac{J_0(\rho P)e^{-\sqrt{L_1T_0\rho^2-k_0^2}\,\left|\sqrt{K_1N_1}\,z\right|}}{\sqrt{L_1T_0\rho^2-k_0^2}}\rho d\rho\right\} \tag{4.76}$$

$$= \frac{K_1}{4\pi}\frac{x}{P}(\text{sgn. } z)\left\{\frac{e^{-jk_0\left|\sqrt{K_1N_1}\,z\right|}}{P}\;\frac{-\left|\sqrt{K_1N_1}\,z\right|}{P}\right.$$

$$\left.\cdot\int_0^\infty \frac{J_0\!\left(\rho'\sqrt{K_1N_0}P\right)e^{-\sqrt{\rho'^2-k_0^2}\,\left|\sqrt{K_1N_1}\,z\right|}}{\sqrt{\rho'^2-k_0^2}}\rho' d\rho'\right\} \tag{4.77}$$

The integral of Equation (4.77) is Sommerfeld's formula (page 34, Functions of Mathematical Physics, Magnus and Oberheitinger[12]).

$$\frac{\partial^2 Q_2 * G_1}{\partial z \partial x} = \frac{K_1}{4\pi} \frac{x}{P} (\text{sgn. } z) \left\{ \frac{e^{-jk_0 \left| \sqrt{K_1 N_1}\, z \right|}}{P} \right.$$

$$\left. \frac{-\left|\sqrt{K_1 N_1}\, z\right|}{P} \frac{e^{-jk_0 \sqrt{K_1 N_0 P^2 + K_1 N_1 z^2}}}{\sqrt{K_1 N_0 P^2 + K_1 N_1 z^2}} \right\} \tag{4.78}$$

$$\left(\sqrt{K_1 N_0}\, P \text{ and } \sqrt{K_1 N_1}\, z \text{ real, } \frac{-\pi}{2} < \arg \sqrt{\rho'^2 - k_0^2} \leq \frac{\pi}{2} \right)$$

$$B_{2x}(\vec{r}) = \frac{\mu_0}{K_1} \frac{\partial}{\partial x} \left(\frac{\partial^2 Q_2 * G_1}{\partial z \partial x} \right) \tag{4.79}$$

$$B_{2y}(\vec{r}) = \frac{\mu_0}{K_1} \frac{\partial}{\partial y} \left(\frac{\partial^2 Q_2 * G_1}{\partial z \partial x} \right) \tag{4.80}$$

$$B_{2z}(\vec{r}) = \frac{\mu_0}{K_1 N_0} \frac{\partial}{\partial x} Q_2(\vec{r}) \tag{4.81}$$

$$B_1(\vec{r}) = \frac{\mu_0}{4\pi} (\text{sgn. } z) \nabla X \frac{y}{P^2} \left\{ e^{-jk_0 \left| \sqrt{K_1 N_1}\, z \right|} - \left| \sqrt{K_1 N_1}\, z \right| \frac{e^{-jk_0 R_1}}{R_1} \right\} \tag{4.82}$$

where

$$R_1 = (K_0 N_1 P^2 + K_1 N_1 z^2)^{1/2} \tag{4.83}$$

$$B_{1x}(\vec{r}) = \frac{\mu_0}{4\pi}(\text{sgn. } z)\frac{\partial}{\partial y}\left(\frac{y}{P^2}\right)\left\{e^{-jk_0\left|\sqrt{K_1N_1}\ z\right|} - \left|\sqrt{K_1N_1}\ z\right|\frac{e^{-jk_0R_1}}{R_1}\right\} \tag{4.84}$$

$$= \frac{\mu_0}{4\pi}(\text{sgn. } z)\left\{\frac{x^2-y^2}{P^4}\left[e^{-jk_0\left|\sqrt{K_1N_1}\ z\right|} - \left|\sqrt{K_1N_1}\ z\right|\frac{e^{-jk_0R_1}}{R_1}\right]\right.$$

$$\left. + \left|\sqrt{K_1N_1}\ z\right|\left(\frac{y}{P^2}\right)\left(\frac{jk_0K_0N_1\,y}{R_1^2} + \frac{K_0N_1\,y}{R_1^3}\right)e^{-jk_0R_1}\right\} \tag{4.85}$$

$$B_{1x}(\vec{r}) = \frac{\mu_0}{4\pi}\sqrt{K_1N_1}\quad\frac{z}{P^2}\left\{\frac{-(x^2-y^2)}{P^2}\frac{e^{-jk_0R_1}}{R_1} + \frac{jk_0K_0N_1y^2}{R_1}\left(1-\frac{j}{k_0R_1}\right)\frac{e^{-jk_0R_1}}{R_1}\right\}$$

$$+ \frac{\mu_0}{4\pi}(\text{sgn. } z)\frac{(x^2-y^2)}{P^4}\,e^{-jk_0\left|\sqrt{K_1N_1}\ z\right|} \tag{4.86}$$

$$B_{1y}(\vec{r}) = \frac{-\mu_0 y}{4\pi}(\text{sgn. } z)\frac{\partial}{\partial x}\left(\frac{1}{P^2}\right)\left\{e^{-jk_0\left|\sqrt{K_1N_1}\ z\right|} - \left|\sqrt{K_1N_1}\ z\right|\frac{e^{-jk_0R_1}}{R_1}\right\} \tag{4.87}$$

$$= \frac{\mu_0 y}{4\pi}(\text{sgn. } z)\left\{\frac{2x}{P^4}\left(e^{-jk_0\left|\sqrt{K_1N_1}\ z\right|} - \left|\sqrt{K_1N_1}\ z\right|\frac{e^{-jk_0R_1}}{R_1}\right)\right.$$

$$\left. - \frac{\left|\sqrt{K_1N_1}\,z\right|}{P^2}\left(\frac{jk_0K_0N_1\,x}{R_1^2} + \frac{K_0N_1\,x}{R_1^3}\right)e^{-jk_0R_1}\right\}$$

$$= \frac{\mu_0 xyz}{4\pi}\ \frac{\sqrt{K_1N_1}}{P^2}\left\{\left[\frac{2}{P^2} + \frac{jk_0K_0N_1}{R_1}\left(1-\frac{j}{k_0R_1}\right)\right]\frac{e^{-jk_0R_1}}{R_1}\right\}$$

$$+\frac{2\mu_0 xy(\text{sgn. } z)}{4\pi P^4}\, e^{-jk_0\left|\sqrt{K_1N_1}\ z\right|} \tag{4.89}$$

$$B_{1z}(\vec{r}) = 0 \tag{4.90}$$

$$B_{2x}(\vec{r}) = \frac{\mu_c\ (\text{syn. } z)}{4\pi}\frac{\partial}{\partial x}\left(\frac{x}{P^2}\right)\left\{e^{-jk_0\left|\sqrt{K_1N_1}\ z\right|} - \left|\sqrt{K_1N_1}\ z\right|\frac{e^{-jk_0R_2}}{R_2}\right\} \tag{4.91}$$

where

$$R_2 = (K_1N_0\,P^2 + K_1N_1\,z^2)^{1/2} \tag{4.92}$$

$$B_{2x}(\vec{r}) = \frac{\mu_0}{4\pi}(\text{sgn. } z)\left\{\frac{-(x^2-y^2)}{P^4}\left[e^{-jk_0\left|\sqrt{K_1N_1}\,z\right|} - \left|\sqrt{K_1N_1}\ z\right|\frac{e^{-jk_0R_2}}{R_2}\right]\right.$$

$$\left. + \left|\sqrt{K_1N_1}\ z\right|\frac{x}{P^2}\left(\frac{jk_0K_1N_0x}{R_2^2} + \frac{K_1N_0x}{R_2^3}\right)e^{-jk_0R_2}\right\} \tag{4.93}$$

$$= \frac{\mu_0}{4\pi}\sqrt{K_1N_1}\quad\frac{z}{P^2}\left\{\frac{(x^2-y^2)}{P^2}\frac{e^{-jk_0R_2}}{R_2} + \frac{jk_0K_1N_0x^2}{R_2}\left(1-\frac{j}{k_0R_2}\right)\frac{e^{-jk_0R_2}}{R_2}\right\}$$

$$-\frac{\mu_0}{4\pi}(\text{sgn. } z)\frac{(x^2-y^2)}{P^4}\, e^{-jk_0\left|\sqrt{K_1N_1}\ z\right|} \tag{4.94}$$

$$B_{2y}(\vec{r}) = \frac{\mu_0\, x(\text{sgn. } z)}{4\pi}\frac{\partial}{\partial y}\frac{1}{P^2}\left\{e^{-jk_0\left|\sqrt{K_1N_1}\ z\right|} - \left|\sqrt{K_1N_1}\ z\right|\frac{e^{-jk_0R_2}}{R_2}\right\} \tag{4.95}$$

$$= \frac{-\mu_0 x}{4\pi}(\text{sgn. } z)\left\{\frac{2y}{P^4}\left(e^{-jk_0\left|\sqrt{K_1N_1}\ z\right|} - \left|\sqrt{K_1N_1}\ z\right|\frac{e^{-jk_0R_2}}{R_2}\right)\right.$$

$$\left. - \frac{\left|\sqrt{K_1N_1}\,z\right|}{P^2}\left(\frac{jk_0K_1N_0\,y}{R_2^2} + \frac{K_0N_1\,y}{R_2^3}\right)e^{-jk_0R_1}\right\} \tag{4.96}$$

$$= \frac{\mu_0 xyz\ \sqrt{K_1N_1}}{4\pi P^2}\left\{\left[\frac{2}{P^2} + \frac{jk_0K_1N_0}{R_2}\right.\right.\left(1-\frac{j}{k_0R_2}\right)\left.\left.\frac{e^{-jk_0R_2}}{R_2}\right]\right\}$$

$$- \frac{2\mu_0 xy(\text{sgn. } z)}{4\pi P^4}\, e^{-jk_0\left|\sqrt{K_1N_1}\ z\right|} \tag{4.97}$$

$$B_{2z}(\vec{r}) = \frac{\mu_0\sqrt{K_1N_1}}{4\pi}\frac{\partial}{\partial x}\frac{e^{-jk_0R_2}}{R_2} \tag{4.98}$$

$$= \frac{-jk_0\mu_0 K_1^{3/2} N_0\sqrt{N_1}\ x}{4\pi}\left(1-\frac{j}{k_0R_2}\right)\frac{e^{-jk_0R_2}}{R_2^2} \tag{4.99}$$

Because of the form of this particular source, that is, a source that is orthogonal to the third characteristic magnetic field, $H_3(\vec{k})$, there will be no component of the field parallel to the third characteristic field, $B_3(\vec{k})$. This will be true of any electric source since then the Fourier transform of the equivalent source $M_m(\vec{k},\omega)$ will be transverse to the wave vector $\vec{k}$.

To arrive at the total magnetic field, $B(\vec{r})$, the components parallel to each characteristic field must be added.

$$B(\vec{r}) = \sum_{i=1}^{3} B_i(\vec{r}) \tag{4.100}$$

$$B_x(\vec{r}) = \frac{\mu_0\sqrt{K_1 N_1}}{4\pi}\,\frac{z}{P^2}\left[\frac{-(x^2-y^2)}{P^2}\left(\frac{e^{-jk_0R_1}}{R_1} - \frac{e^{-jk_0R_2}}{R_2}\right)\right.$$

$$\left. + jk_0K_0N_1y^2\left(1-\frac{j}{k_0R_1}\right)\frac{e^{-jk_0R_1}}{R_1^2} + jk_0K_1N_0x^2\left(1-\frac{j}{k_0R_2}\right)\frac{e^{-jk_0R_2}}{R_2^2}\right] \tag{4.101}$$

$$B_y(\vec{r}) = \frac{\mu_0\sqrt{K_1N_1}\,xyz}{4\pi P^2}\left[\frac{-2}{P^2}\left(\frac{e^{-jk_0R_1}}{R_1} - \frac{e^{-jk_0R_2}}{R_2}\right)\right.$$

$$\left. - jk_0K_0N_1\left(1-\frac{j}{k_0R_1}\right)\frac{e^{-jk_0R_1}}{R_1^2} + jk_0K_1N_0\left(1-\frac{j}{k_0R_2}\right)\frac{e^{-jk_0R_2}}{R_2^2}\right] \tag{4.102}$$

$$B_z(\vec{r}) = \frac{-jk_0\mu_0K_1^{3/2}N_0\sqrt{N_1}\,x}{4\pi}\left(1-\frac{j}{k_0R_2}\right)\frac{e^{-jk_0R_2}}{R_2^2} \tag{4.103}$$

To the author's knowledge these results for a general uniaxial medium of the type considered here have never been derived before. As they stand, they represent the Fourier time transform of a dipole field with a time dependence of a Dirac delta function in a time-dispersive medium; equivalently, they represent the space-time solution to a time harmonic source. Several remarks should be made concerning the above example. First, each component parallel to the characteristic fields is significantly different from the total field. Note that the terms with the exp $\left(-jk_0\left|\sqrt{K_1N_1}\,z\right|\right)$ dependence are not even evident in the total field. Although the interference phenomenon is well known, it is particularly emphasized in the above. Observe terms like

$$\left(\frac{e^{-jk_0R_1}}{R_1} - \frac{e^{-jk_0R_2}}{R_2} \right)$$

As R_1 approaches R_2, the beating phenomena becomes less rapid. R_1 may approach R_2 in either one of two ways. R_1 may approach R_2 along certain directions in space or the medium may be degenerate. When the medium is degenerate two of the eigenvalues are equal and the corresponding sheets of the dispersion surface are the same. For this uniaxial case the medium is degenerate when K_1/K_o equals N_1/N_o. Thus, it is immediately apparent that free space is a degenerate medium and for this reason there is a certain amount of arbitrariness in the characteristic fields.

For the moment, let $N_1 = N_2 = 1$ in order that a comparison can be made with the work of Clemmow. P. C. Clemmow published a paper entitled, "The Theory of Electromagnetic Waves in a Simple Anisotropic Medium," in the Proceedings of the IEE, Vol. 110, No. 1, January 1963. In this paper, he gives a method to find the exact fields due to a time-harmonic source in a nonspace-dispersive uniaxial medium. The above derived expressions for $\underline{\underline{N}} = \underline{\underline{I}}$ compare exactly with those of Clemmow. He shows that such a field is related by a scaling procedure to a corresponding vacuum field. The vacuum field is expressed as a superposition of a transverse magnetic field, in which the magnetic vector is everywhere perpendicular to the axis of symmetry of the anisotropic medium and a coplanar transverse electric field; and different scaling is applied separately to each partial field. Only in passing is it mentioned that these scaled transverse fields are two of the three characteristic fields in the uniaxial medium. Thus, the full signficance of the characteristic fields is not utilized. Since only two characteristic fields which cannot span the three-space are used, it seems questionable on the outset whether the field due to an arbitrary source (both electric and magnetic) can be described only in terms of these. However, the difficulty, which Clemmow does not mention, can be indirectly circumvented by using a superposition of the fields for each type of source. Another difficulty, the task of resolving the vacuum fields due to an arbitrary source into TM and TE fields, is not a trivial matter. The whole procedure depends upon the ability to split the source into components that excite each type of field.

In light of the preceeding discussion let us go back to Equation (1.3) and observe the explicit role played by the characteristic fields. We wish to show that by the use of the characteristic fields the source resolution is automatic and is not simply an artiface for a particular problem. Let us begin by saying that the source field $H(\vec{k})$ may be expressed as the weighted sum of the characteristic fields. Thus, in general, all three characteristic fields will be involved, although for some source distributions the weighting factor may be zero. Equation (4.4) now becomes,

$$\sum_{i=1}^{3} \alpha_i G_H H_i = -j\omega\epsilon_0 M_m \tag{4.104}$$

But $G_H H_i = S_i N H_i = \mu_0^{-1} S_i B_i$. Hence, we have

$$\sum_{i=1}^{3} \alpha_i S_i B_i = -j\omega\mu_0\epsilon_0 M_m(\vec{k}) \tag{4.105}$$

The weighting factors α_j are found by multiplying Equation (4.105) on the left by $H_j^\dagger$.

$$\alpha_j = -j\omega\mu_0\epsilon_0 S_j^{-1}(H_j^\dagger B_j)^{-1} H_j^\dagger M_m(\vec{k}) \tag{4.106}$$

At this point the prominant role played by orthogonality is easily seen. Also, the characteristic field H_j will not contribute to the total field if and only if $H_j^\dagger M_m(\vec{k})$ equals zero. This is exactly the case for H_3 and an electric source current, since H_3 equals $\vec{k}$ and $M_m(\vec{k})$ is transverse to $\vec{k}$. Since $\underline{\underline{N}} = \underline{\underline{I}}$, $H_1 = \vec{k} \times \hat{z}$ and $H_2 = \vec{k} \times \vec{k} \times \hat{z}$, then $H_1(\vec{r})$ and $H_2(\vec{r})$ correspond to the scaled TM and TE fields, respectively. The important factor to be remembered is that the TM and TE decomposition applies only because the two characteristic fields are TM and TE. However, the characteristic field decomposition (spectral decomposition) applies for any medium.

Let us go one step farther to produce another result. As was previously stated, Equations (4.101), (4.102) and (4.103) may be interpretated as the time Fourier transform field of a source $\vec{J}(\vec{r},t) = \delta(x,y,z,t)\,\hat{y}$. Now assume the constitutive relationship is independent of k and ω. This assumption makes the Fourier inversion possible. The results are almost immediate.

$$\begin{aligned} B_x(\vec{r},t) = \frac{\mu_0\sqrt{K_1N_1}}{4\pi}\,\frac{z}{p^2}\Bigg\{ & \frac{-(x^2-y^2)}{p^2}\left(\frac{\delta(t-\frac{1}{c}R_1)}{R_1} - \frac{\delta(t-\frac{1}{c}R_1)}{R_2}\right) \\ & + K_0N_1\frac{y^2}{R_1^2}\left(\frac{1}{c}\delta'(t-\frac{1}{c}R_1) + \frac{\delta(t-\frac{1}{c}R_1)}{R_1}\right) + \frac{K_1N_0}{R_2^2}\left(\frac{1}{c}\delta'(t-\frac{1}{c}R_2) + \frac{\delta(t-\frac{1}{c}R_2)}{R_2}\right)\Bigg\} \end{aligned} \tag{4.107}$$

$$B_y(\vec{r},t) = \frac{\mu_0\sqrt{K_1 N_1}}{4\pi}\,\frac{xyz}{P^2}\left\{\frac{-2}{P^2}\left(\frac{\delta(t-\frac{1}{c}R_1)}{R_i} - \frac{\delta(t-\frac{1}{c}R_2)}{R_2}\right)\right.$$

$$\left. - \frac{K_0 N_1}{R_1^2}\left(\frac{1}{c}\delta'(t-\frac{1}{c}R_1) + \frac{\delta(t-\frac{1}{c}R_1)}{R_1}\right) + \frac{K_\Phi N_0}{R_2^2}\left(\frac{1}{c}\delta'(t-\frac{1}{c}R_2) + \frac{\delta(t-\frac{1}{c}R_2)}{R_2}\right)\right\} \tag{4.108}$$

$$B_z(\vec{r},t) = \frac{-\mu_0 K_1^{3/2} N_0\sqrt{N_1}}{4\pi R_2^2}\left(\frac{1}{c}\delta'(t-\frac{1}{c}R_2) + \frac{\delta(t-\frac{1}{c}R_2)}{R_2}\right) \tag{4.109}$$

The support of the fields is two ellipsoids expanding with time in contrast to the support in free space which is one sphere expanding with time. Thus, an observer at an arbitrary point in space will be cognizant of two wave fronts not one. Although these ellipsoidal wave fronts are not of the same shape as the sheets of the index dispersion surface, they are related. The ellipsoidal surfaces are of course expanded or contracted ray surfaces, which may be given in parametric form by

$$\vec{V}_r = \frac{\nabla_k S}{\vec{k}\cdot\nabla_k S} \tag{4.110}$$

A problate ellipsoidal index surface will have oblate ellipsoidal wave fronts, and vice versa. Another point is made apparent by Equations (4.107), (4.108) and (4.109). No point on the wave front can travel faster than the velocity of light in a vacuum. Therefore, restrictions are placed on the arguments of the Dirac delta functions which in turn places restrictions on the permeability and permittivity matrices. Not only must the permeability and permittivity matrices be positive definite but also $1/N_o$, $1/N_1 < K_1$ and $1/N_1 < K_o$, K_1.

5. SPECTRUM OF CHARACTERISTIC WAVES IN AN ISOTROPIC COMPRESSIBLE PLASMA

Very few problems are solvable in closed form. As has been shown, the dipole in a nonspace-dispersive uniaxial medium is solvable. No doubt even some relaxation of the nonspace-dispersive restriction can be made. The biaxial and magneto-ionic problems, however, as yet have not been solved. Numerous attempts have been made, as the literature will testify, but all results have involved at least one unevaluated integral. The closed solutions for the latter two media have not been obtained even though they are nonspace-dispersive. The problem of a dipole in an isotropic compressible plasma of N mobile ion species is solvable in closed form even though the permittivity matrix is both nondiagonal and is space-dispersive. The reason why some problems are solvable and others are not seems to lie in the form of the determinantal equation or equivalently the characteristic equation. The eigenvalues of the solvable problems do not involve radicals whereas the eigenvalues of the unsolvable problems do. These observations should be more apparent after a comparison of several problems is made. For this reason and to further illustrate the usefulness of the spectral decomposition, it is instructive to obtain the solution of a dipole in an isotropic compressible plasma of N mobile ion species by the three-vector method.

First we must show that the permittivity matrix for the lossless plasma is Hermitian. The inclusion of a static magnetic field B_o will be made since the degree of difficulity for proof is not increased.

Lemma 5.1: The permittivity matrix of a lossless compressible plasma with a static magnetic field is Hermitian.

Proof: Assuming the fluid model for a plasma is applicable the force equation and the continuity equation can be written for each species of mobile ion (including electron).

$$m_j \frac{\partial \mathcal{V}_j}{\partial t} - q_j \mathcal{E} + (m_j a_j^2 / N_j) \nabla n_j + q_j B \; X \mathcal{V}_j = 0 \tag{5.1}$$

$$\frac{\partial n_j}{\partial t} + N_j \nabla \cdot \mathcal{V}_j = 0 \tag{5.2}$$

where a_j is the speed of sound for the j^{th} ion given by

$a_j^2 = \gamma_j K \, T_j / m_j$

γ_j = ratio of specific heats at constant pressure to that at constant volume

K = Boltzmanns constant

T_j = ion temperature

m_j = ion mass

Equations (5.1) and (5.2) with Ampere's equation is sufficient to derive the permittivity matrix. Eliminating the density n_j from Equations (5.1) and (5.2) yields

$$\frac{\partial^2 \mathcal{V}_j}{\partial t^2} - a_j^2 \nabla \nabla \cdot \mathcal{V}_j + \Omega_j \frac{\partial \mathcal{V}_j}{\partial t} = \frac{q_j}{m_j} \frac{\partial \mathcal{E}}{\partial t} \tag{5.3}$$

where

$$\Omega_j = \frac{q_j}{m_j} B_0 X \tag{5.4}$$

The temporal and spatial Fourier transforms of Ampere's law and Equation (5.3) are, respectively,

$$-j\vec{k}\times H(\vec{k},\omega) = j\omega\epsilon_0 E(\vec{k},\omega) + \sum_j q_j n_j V_j + J_e(\vec{k},\omega) \tag{5.5}$$

$$(\omega^2 - a_j^2 \vec{k}\vec{k}^T - j\omega\Omega_j)V_j(\vec{k},\omega) = -j\omega \frac{q_j}{m_j} E(\vec{k},\omega) \tag{5.6}$$

$$R_j V_j(\vec{k},\omega) = -j\omega \frac{q_j}{m_j} E(\vec{k},\omega) \tag{5.7}$$

Therefore,

$$V_j(\vec{k},\omega) = -j\omega \frac{q_j}{m_j} R_j^{-1} E(\vec{k},\omega) \tag{5.8}$$

$$-j\vec{k}\times H(\vec{k},\omega) = j\omega\epsilon_0 \left[I - \sum_j \frac{q_j^2 n_j}{m_j \epsilon_0} R_j^{-1}\right] E(\vec{k},\omega) + J_e(\vec{k},\omega) \tag{5.9}$$

From Equation (5.9) we see that the permittivity matrix is

$$\underline{\underline{\epsilon}} = \underline{\underline{\epsilon}}_0 K = \epsilon_0 \left[I - \sum_j \frac{q_j^2 n_j}{m_j \epsilon_0} R_j^{-1}\right] = \epsilon_0 \left[I - \sum_j \omega_j^2 R_j^{-1}\right] \tag{5.10}$$

where

$$\omega_j^2 = \frac{q_j^2 n_j}{m_j \epsilon_0} \tag{5.11}$$

The index j is summed over all species of compressible ions. As seen from Equations (5.7) and (5.8) and the fact that Ω_j is skew-Hermitian for real $\vec{k}$ and ω, R_j is Hermitian. Hence, the inverse of R_j is Hermitian. It then follows that the permittivity is a Hermitian matrix, i.e.,

$$\underline{\underline{K}} = \underline{\underline{K}}^{\dagger} \tag{5.12}$$

This derivation has assumed that collisions are negligible (lossless) and that the ion pressure for each species obey separate adiabatic relations.

From the previous derivation for the permittivity matrix it is observed that,

$$R_j = (\omega^2 - a_j^2 \vec{k}\vec{k}^T) \tag{5.13}$$

and then

$$R_j^{-1} = \frac{1}{\omega^2}\left(I - \frac{a_j^2 \vec{k}\vec{k}^T}{(a_j^2 k^2 - \omega^2)} \right) \tag{5.14}$$

Therefore, the normalized permittivity matrix is

$$\underline{\underline{K}} = \left(1 - \sum_j \frac{\omega_j^2}{\omega^2} \right) I + \left(\sum_j \frac{a_j^2 \omega_j^2}{\omega^2 (a_j^2 k^2 - \omega^2)} \right) \vec{k}\vec{k}^T \tag{5.15}$$

where ω_j is the angular plasma frequency of the j^{th} species, expressed by

$$\omega_j^2 = \frac{q_j^2 n_j}{m_j \epsilon_j}$$

5.1 Eigenvalues for an Isotropic Compressible Plasma

Since $G_E(\vec{k},\omega,\lambda)$ and $G_H(\vec{k},\omega,\lambda)$ have the same eigenvalues, it is sufficient and easier to evaluate them explicitly through $\det G_H(\vec{k},\omega,\lambda) = 0$. It will become apparent that the form of $\underline{\underline{K}}$ is the simplifying factor. $\underline{\underline{K}}$ is of the form $bk\,k^T + a$, and $\underline{\underline{K}}^{-1}$ is of the form $\underline{\underline{K}}^{-1} = 1/a\left[I - (k^2 + a/b)^{-1}\,\vec{k}\,\vec{k}^T\right]$. $G_H(\vec{k},\omega,\lambda) = (-\vec{k} \times \underline{\underline{K}}^{-1}\,\vec{k}x - \lambda)$. Therefore, the second term of $\underline{\underline{K}}^{-1}$ has no effect upon G_H, i.e.,

$$G_H(\vec{k},\omega,\lambda) = \left(-\frac{1}{a}\vec{k}x\vec{k}x - \lambda\right) = -\frac{1}{a}\left[\vec{k}\vec{k} - (k - a\lambda)\right] \tag{5.16}$$

Then

$$\det G_H(\vec{k},\omega,\lambda) = -\lambda\left(\lambda - \frac{k^2}{a}\right)^2 = 0 \tag{5.17}$$

From the previous section it is found that $a = \left(1 - \sum_j \frac{\omega_j^2}{\omega^2}\right)$ and $b = \left(\sum_j \frac{a_j^2\ \omega_j^2}{\omega^2(a_j^2\ k^2 - \omega^2)}\right)$. The eigenvalues are independent of the complicated term b. Since a is independent of $\vec{k}$, both eigenvalues λ_1 and λ_2 which are degenerate will have only one sheet, $S_1 = S_2 = \left(k^2/a - k_o^2\right)$. The sheet will be propagating for $\omega > \left(\sum_j \omega_j^2\right)^{1/2}$ and nonpropagating for $\omega < \left(\sum_j \omega_j^2\right)^{1/2}$.

In summary, there is only one degenerate transverse sheet due to the eigenvalues for an istropic compressible plasma.

5.2 Transverse Part of the Field Due to an Electric Dipole

Since the eigenvalues λ_1 and λ_2 are degenerate, arbitrarily choose,

$$E_1 = \vec{k}xz\,,\quad E_2 = \vec{k}x\vec{k}xz,\quad E_3 = \vec{k} \tag{5.18}$$

Also since the plasma is isotropic, arbitrarily choose the electric dipole source orientated in the $\hat{z}$ direction. A magnetic source cannot excite plasma waves; therefore, at this time we are not as interested in the fields due to it. With this choice of eigenvectors, E_1 does not enter into the picture.

$$E_2^{\dagger} \underline{\underline{K}} E_2 = (\vec{k} \times \vec{k} \times \hat{z})^{\dagger} (aI + b\vec{k}\,\vec{k}^T)(\vec{k} \times \vec{k} \times \hat{z}) = a\rho^2 k^2 \qquad (5.19)$$

$$E_2(\vec{k},\omega) = j\omega\mu_0 (k^2 - ak_0^2)^{-1} k^{-2} (\vec{k} \times \vec{k} \times \hat{z}) \qquad (5.20)$$

By partial fractions,

$$(k^2 - ak_0^2)^{-1} k^{-2} = -\left(\frac{1}{ak^2}\right)\left\{k^{-2} - (k^2 - ak_0^2)^{-1}\right\} \qquad (5.21)$$

Therefore,

$$E_2(\vec{r},\omega) = \frac{-j\omega\mu_0}{ak_0^2} \nabla \times \nabla \times \hat{z} \left\{\frac{e^{-j\sqrt{a}\,k_0 r}}{4\pi r} - \frac{1}{4\pi r}\right\} \qquad (5.22)$$

5.3 Plasma Waves

The eigenvalue for the longitudinal or plasma waves is zero. This implies that propagating plasma wave must come from the real zeros of $E_3^{\dagger} \underline{\underline{K}} E_3$. Since $E_3 = \vec{k}$,

$$E_3^{\dagger} \underline{\underline{K}} E_3 = k^2 \left\{\left(\sum_j \frac{a_j^2 \omega_j^2 k^2}{\omega^2 (a_j^2 k^2 - \omega^2)}\right) + \left(1 - \sum_j \frac{\omega_j^2}{\omega^2}\right)\right\} \qquad (5.23)$$

After obtaining a common denominator and simplification, one obtains

$$E_3^{\dagger} \underline{\underline{K}}\, E_3 = k^2 \, \frac{\prod_j \left(k^2 - \frac{\omega^2}{a_j^2}\right) + \sum_j \frac{\omega_j^2}{a_j^2} \prod_i^{(j)} \left(k^2 - \frac{\omega^2}{a_i^2}\right)}{\prod_j \left(k^2 - \frac{\omega^2}{a_j^2}\right)} \tag{5.24}$$

The superscript (j) indicates that the j^{th} term is omitted from the product. Except for the factor k^2 in Equation (5.24) the numerator of $E_3^{\dagger} \underline{\underline{K}}\, E_3$ is a polynomial in k^2 of order N. Therefore, $E_3^{\dagger} \underline{\underline{K}}\, E_3$ (discounting the k^2) will have N zeros. Since the numerator of Equation (5.24) is a polynomial in k^2 with real coefficients, the roots of k^2 will occur in conjugate pairs. Hence, it is possible if all of the roots are real to have N sheets to contribute to plasma waves. Restated, the maximum number of real sheets (that cause propagating waves) for an isotropic compressible plasma is one sheet per compressible ion species. This is further verified by obtaining the determinant of $\underline{\underline{K}}$ and comparing the zeros with the zeros of $E_3^{\dagger} \underline{\underline{K}}\, E_3$.

$$\det \underline{\underline{K}} = -\left(1 - \sum_j \frac{\omega_j^2}{\omega^2}\right) \left\{ \left(\sum_j \frac{a_j^2 \omega_j^2 k^2}{\omega^2 (a_j^2 k^2 - \omega^2)}\right) + \left(1 - \sum_j \frac{\omega_j^2}{\omega^2}\right) \right\} \tag{5.25}$$

The zeros of $E_3^{\dagger} \underline{\underline{K}}\, E_3$ are the zeros of

$$\prod_j \left(k^2 - \frac{\omega^2}{a_j^2}\right) + \sum_j \frac{\omega_j^2}{a_j^2} \prod_i^{(j)} \left(k^2 - \frac{\omega^2}{a_i^2}\right) = 0 \tag{5.26}$$

Equation (5.26) can be put into a form

$$\prod_j (k^2 - k_j^2) = 0 \tag{5.27}$$

where k_j is the zeros of (5.26). The product of the zeros is

$$\prod_j k_j^2 = \frac{\omega^{2N}}{\prod_j a_j^2}\left(1-\sum_j \frac{\omega_j^2}{\omega^2}\right) \tag{5.28}$$

The sum of the zeros is

$$\sum_j k_j^2 = \sum_j \frac{(\omega^2-\omega_j^2)}{a_j^2} \tag{5.29}$$

From Equation (5.28) one observes that for $\omega^2 < \sum_j \omega_j^2$, $\prod_j k_j^2 < 0$, implying that there will be at least one negative zero, i.e., at least one nonpropagating plasma sheet. For $\omega^2 > \sum_j \omega_j^2$ if N (number of compressible ion species) is odd, there is at least one propagating sheet.

Now consider the longitudinal fields,

$$(E_3^\dagger \underline{\underline{K}} E_3)^{-1} = \frac{\prod_j\left(k^2-\frac{\omega^2}{a_j^2}\right)}{k^2\left[\prod_j\left(k^2-\frac{\omega^2}{a_j^2}\right)+\sum_j \frac{\omega_j^2}{a_j^2}\prod_i^{(j)}\left(k^2-\frac{\omega^2}{a_i^2}\right)\right]}$$

$$= \frac{\left(\frac{1}{a}\right)}{k^2} + \frac{\left(1-\frac{1}{a}\right)\prod_j\left(k^2-\frac{\omega^2}{a_j^2}\right)-\frac{1}{a}\sum_j \frac{\omega_j^2}{a_j^2}\prod_i^{(j)}\left(k^2-\frac{\omega^2}{a_i^2}\right)}{k^2\left[\prod_j\left(k^2-\frac{\omega^2}{a_j^2}\right)+\sum_j \frac{\omega_j^2}{a_j^2}\prod_i^{(j)}\left(k^2-\frac{\omega^2}{a_i^2}\right)\right]} \tag{5.30}$$

$1/a$ is the residue of $(E_3^\dagger \underline{\underline{K}} E_3)^{-1}$ at $k^2 = 0$; therefore, the second term of Equation (5.30) does not have a pole at $k^2 = 0$.

$$E_3(\vec{k},\omega) = -j\omega\mu_0(-k_0^{-2})(E_3^\dagger \underline{\underline{K}} E_3)^{-1} E_3 E_3^\dagger M_e(\vec{k},\omega) \tag{5.31}$$

$$M_e(\vec{k},\omega) = \hat{z} \tag{5.32}$$

Let

$$E_3(\vec{k},\omega) = E_3'(\vec{k},\omega) + E_3''(\vec{k},\omega) \tag{5.33}$$

where

$$E_3'(\vec{r},\omega) = \frac{-j\omega\mu_o}{a k_o^2}\nabla\nabla^T\left(\frac{\hat{z}}{4\pi r}\right) \tag{5.34}$$

and

$$E_3''(\vec{r},\omega) = \frac{-j\omega\mu_o}{(2\pi)^3 k_o^2}\nabla\nabla^T \int_0^{\infty}\int_0^{\pi}\int_0^{2\pi} \frac{\left[\left(1-\frac{1}{a}\right)\pi_j\left(k^2-\frac{\omega^2}{a_j^2}\right) - \frac{1}{a}\sum_j \frac{\omega_j^2}{a_j^2}\pi_i^{(j)}\left(k^2-\frac{\omega^2}{a_i^2}\right)\right]}{\left[\pi_j\left(k^2-\frac{\omega^2}{a_j^2}\right) + \sum_j \frac{\omega_j^2}{a_j^2}\pi_i^{(j)}\left(k^2-\frac{\omega^2}{a_i^2}\right)\right]} \tag{5.35}$$

$$\cdot\, e^{-j\vec{k}\cdot\vec{r}} \sin\theta' \, dk \, d\theta' \, d\phi'$$

Since the medium is isotropic, the integral is independent of θ and ϕ. Therefore, for the sake of computation, let $\theta = 0$ and integrate over ϕ' and θ'.

$$E_3''(\vec{r},\omega) = \frac{\omega\mu_o}{(2\pi)^2 k_o^2}\nabla\nabla^T \frac{\hat{z}}{r}\int_{-\infty}^{\infty} \frac{\left[\left(1-\frac{1}{a}\right)\pi_j\left(k^2-\frac{\omega^2}{a_j^2}\right) - \frac{1}{a}\sum_j \frac{\omega_j^2}{a_j^2}\pi_i^{(j)}\left(k^2-\frac{\omega^2}{a_i^2}\right)\right]}{\left[\pi_j\left(k^2-\frac{\omega^2}{a_j^2}\right) + \sum_j \frac{\omega_j^2}{a_j^2}\pi_i^{(j)}\left(k^2-\frac{\omega^2}{a_i^2}\right)\right]k} \tag{5.36}$$

$$e^{-jkr}\, dk$$

The integral should be integrated over the closed contour, C, which is a path along the real k-axis and an arbitrarily large semicircle in the lower half k-plane. The path should be such that poles on the positive real axis are included in the contour, but poles on the negative real axis are excluded. Denote all of the zeros of

$$\left[\prod_j\left(k^2-\frac{\omega^2}{a_j^2}\right)+\sum_j\frac{\omega_j^2}{a_j^2}\prod_i^{(j)}\left(k^2-\frac{\omega^2}{a_i^2}\right)\right]=0 \tag{5.37}$$

on the positive real axis and in the lower half plane by k_j for $j = 1$ to N. Then the denomonator of the integral of Equation (5.36) is $k \prod_j^N (k^2 - k_j^2)$. Then Equation (5.36) becomes,

$$E_3''(\vec{r},\omega)=\frac{-j\omega\mu_0}{ak_0^2}\nabla\nabla^T\frac{\hat{z}}{4\pi r}\sum_{u=1}^{N}\frac{\left[(a-1)\prod_j\left(k_u^2-\frac{\omega^2}{a_j^2}\right)-\sum_j\frac{\omega_j^2}{a_j^2}\prod_i^{(j)}\left(k_u^2-\frac{\omega^2}{a_i^2}\right)\right]}{k_u\prod_j^{(u)}(k_u-k_j)\prod_j(k_u+k_j)} \tag{5.38}$$

$$\cdot\, e^{-jk_u r}$$

if all of the poles are simple. In general,

$$E_3''(\vec{r},\omega)=\frac{-j\omega\mu_0}{ak_0^2}\nabla\nabla^T\frac{\hat{z}}{2\pi r}\sum_C$$

$$\mathrm{Res}\left[\frac{(a-1)\prod_j\left(k^2-\frac{\omega^2}{a_j^2}\right)-\sum_j\frac{\omega_j^2}{a_j^2}\prod_i^{(j)}\left(k^2-\frac{\omega^2}{a_i^2}\right)}{k\left[\prod_j\left(k^2-\frac{\omega^2}{a_j^2}\right)+\sum_j\frac{\omega_j^2}{a_j^2}\prod_i^{(j)}\left(k^2-\frac{\omega^2}{a_i^2}\right)\right]}\;e^{-jkr}\right] \tag{5.39}$$

5.4 Total Field Due to an Electric Dipole

$$E(\vec{r},\omega)=E_2(\vec{r},\omega)+E_3'(\vec{r},\omega)+E_3''(\vec{r},\omega)$$

$$E(\vec{r},\omega)=\frac{-j\omega\mu_0}{ak_0^2}\nabla\times\nabla\times\hat{z}\,\frac{e^{-j\sqrt{a}\,k_0 r}}{4\pi r}+\frac{j\omega\mu_0}{ak_0^2}\,\delta(\vec{r})\,\hat{z} \tag{5.40}$$

$$-\frac{j\omega\mu_0}{a k_0^2}\nabla\nabla^T\left(\frac{\hat{z}}{2\pi r}\right)\sum_C$$

$$\mathrm{Res}\left[\frac{(a-1)\prod_j\left(k^2-\frac{\omega^2}{a_j^2}\right)-\sum_j\frac{\omega_j^2}{a_j^2}\prod_i^{(j)}\left(k^2-\frac{\omega^2}{a_i^2}\right)}{k\left[\prod_j\left(k^2-\frac{\omega^2}{a_j^2}\right)+\sum_j\frac{\omega_j^2}{a_j^2}\prod_i^{(j)}\left(k^2-\frac{\omega^2}{a_i^2}\right)\right]}e^{-jkr}\right] \tag{5.41}$$

The residues inside C are on the positive real axis and in the lower half k-plane.

Note that for N = 1 this result checks with Equation (28) of Hessel and Shmoys,[13] "Excitation of Plasma Waves by a Dipole in a Homogeneous Isotropic Plasma," Proceedings of the Symposium on Electromagnetics and Fluid Dynamics of Gaseous Plasma, Microwave Research Institute Symposia Series, Volume XI, Polytechnic Press, 1962, pp. 173. The "modal decomposition" of their paper is not the same as decomposing the field along its eigenvectors; however, the relationship between the two can easily be seen. The relationship is,

$$\hat{\underset{\sim}{E}}(\vec{r},\omega)=E_2(\vec{r},\omega)+E_3^{\prime}(\vec{r},\omega) \tag{5.42}$$

$$\frac{\omega_p^2}{n_0 e(\omega^2-\omega_p^2)}\nabla P(\vec{r},\omega)=E_3^{\prime\prime}(\vec{r},\omega) \tag{5.43}$$

$\hat{\underset{\sim}{E}}(\vec{r},\omega)$ is not entirely longitudinal or transverse, but $\nabla P \propto E_3^{\prime\prime}(\vec{r},\omega)$ is longitudinal.

5.5 Pressure in an Isotropic Compressible Plasma

The partial pressure variation for the j^{th} component is

$$P_j = a_j^2 m_j n_j = \gamma_j K T_j n_j \tag{5.44}$$

However, from Equations (5.2) and (5.8) one finds that

$$n_j(\vec{k},\omega) = \frac{-jq_jN_j}{m_j}\vec{k}^T R_j^{-1} E(\vec{k},\omega) \tag{5.45}$$

Using R_j^{-1} given in Equation (5.14), the partial pressure $P_j(\vec{k},\omega)$ is

$$P_j(\vec{k},\omega) = \frac{ja_j^2 q_jN_j}{(a_j^2k^2-\omega^2)}\vec{k}^T E(\vec{k},\omega) \tag{5.46}$$

whereas the inverse transform of $P_j(\vec{k},\omega)$ is

$$P_j(\vec{r},\omega) = -N_jq_j\left(\frac{e^{-j\frac{\omega}{a_j}r}}{4\pi r}\right) * \nabla^T E(\vec{r},\omega) \tag{5.47}$$

Thus, the total pressure variation $P(\vec{r},\omega)$ is

$$P(\vec{r},\omega) = \sum_j P_j(\vec{r},\omega) = -\left(\sum_j N_jq_j\frac{e^{-j\frac{\omega}{a_j}r}}{4\pi r}\right) * \nabla^T E(\vec{r},\omega) \tag{5.48}$$

Notice that only the longitudinal field contributes to the pressure

6. RADIATION FIELD OF AN ARBITRARY SOURCE IN A LOSSLESS LINEAR PASSIVE MEDIUM

6.1 Introduction

For the case of an isotropic medium, the radiation field for an arbitrary current source is well known. The problem of finding the radiation field of an antenna in a cold magnetoplasma such as an ionized gas in a constant magnetic field has been solved by Bunkin[14] and a number of workers.[15-19] However, such solutions have been limited to a particular nonspace-dispersive media. It is desirable to determine the radiation field for a general lossless linear space and time-dispersive medium. It is preferred to find the general field solution for any zone; however, as yet no one has achieved this for the cold magneto-plasma or even the biaxial medium. For a cold magnetoplasma with one mobile charged particle species, the problem is to solve nine first-order partial differential equations. If a harmonic field is assumed, six of the unknown variables may be eliminated to yield three second-order partial differential equations in say the field variable $\mathcal{E}$. Usually the system of three equations is attempted to be solved by the method of Fourier transforms. The solution can then be expressed as a volume integral in Fourier space. Thus, the difficulty is a triple integral of a function with a complicated singularity. Invariably, attempts at such a solution are expressed in at least one unevaluated integral.

With the results expressed in terms of an integral, it is difficult to make comparisons between fields of different sources and to interpret the physical processes. Thus, it is desirable, and for some purposes sufficient, to find an asymptotic solution for the radiation zone. Notably, two asymptotic integral methods have been used in the past, the steepest descent or saddle point method and the stationary phase method. To first-order, they produce essentially the same results. For our purposes the stationary phase method will be used since it will be seen to yield many physically interpretable results. Only lossless media will be considered throughout. This is not a severe restriction since in a lossy medium the concept of a radiation field is not very significant.

First, the principle of stationary phase is applied to a very general linear system which encompasses both space and time dispersive media. Then the results are particularized to some systems including warm plasmas, which are derivable from dynamical models. Notable physical interpretation of the mathematical results are made.

6.2 Stationary Phase Method for Arbitrary N-Vector System

Consider a system of equations whose Fourier transform is

$$\left[O - j\omega \underline{\underline{U}}\right] F = C \tag{6.1}$$

or

$$\mathcal{M} F = C \tag{6.2}$$

where the N-vectors C and F are the source and field vectors, respectively. Also, the N^{th} order square matrices O and $\underline{\underline{U}}$ are skew-Hermitian and Hermitian, respectively. All of the quantities may be a function of all the transform variables. Then similarly to Section 3 an eigenvalue equation may be defined.

$$O F_i = \nu_i \underline{\underline{U}} F_i \quad , i = 1, 2, \cdots, N \tag{6.3}$$

Analogous results concerning the eigenvalues and eigenvectors may be inferred. Thus,

$$F(\vec{k},\omega) = \sum_{i=1}^{N} F_i(\vec{k},\omega) = \underline{\underline{U}}^{-1} \sum_{i=1}^{N} S_i^{-1} I_i^f C \tag{6.4}$$

where $I_i^f = (F_i^\dagger \underline{\underline{U}} F_i)^{-1} \underline{\underline{U}} F_i F_i^\dagger$ and $S_i = \nu_i - j\omega$. In a form that we will use, the portion of the field that is parallel to the characteristic field F_i, is

$$F_i(\vec{k},\omega) = (F_i^\dagger \mathcal{M} F_i)^{-1} F_i F_i^\dagger C \tag{6.5}$$

with the normalization of F_i such that the components of F_i contain no singularities. Therefore, the inverse Fourier transform is

$$\mathcal{F}_i(\vec{r},t) = (2\pi)^{-4} \iiiint_{-\infty}^{\infty} (F_i^\dagger \mathcal{M} F_i)^{-1} F_i F_i^\dagger C e^{j(\omega t - \vec{k}\cdot\vec{r})} d^3k\, d\omega \tag{6.6}$$

The field solution as described above is not unique in that an arbitrary source free solution may be added to it. Furthermore, we wish the solution to describe the physical model. Thus, all waves must originate at the source and no source free solutions are permitted. In order to obtain such a unique field a "radiation condition" must be enforced upon the class of possible solutions. Now the solution is a frequency spectrum of time harmonic waves as the outer integral of Equation (6.6) stipulates. Let it be required that each of these component harmonic waves obey the "radiation condition." And let the Fourier frequency variable, ω, be slightly complex, i.e., replace $j\omega$ by $s = \sigma + j\omega$ where σ is small and positive. Denote the frequency component field by f_i.

$$f_i(s,\vec{r}) = (2\pi)^{-3} \iiint_{-\infty}^{\infty} (F_i^{\dagger}\,\eta\, F_i)^{-1} F_i F_i^{\dagger} C e^{(st - j\vec{k}\cdot\vec{r})} d^3k \tag{6.7}$$

With this, after the evaluation of $f_i(s,\vec{r})$, the required field for the physical problem is,

$$\mathcal{Y}(\vec{r},t) = (2\pi)^{-1} \int_{-\infty}^{\infty} \left[\lim_{s \to j\omega} f_i(s,\vec{r}) \right] d\omega \tag{6.8}$$

The reasoning for the mathematical manipulation of the Fourier frequency variable is as follows. With σ positive the harmonic frequency component $f_i(s,\vec{r})$ is increasing exponentially with time. Now if a source-free wave of finite amplitude is propagation from infinity then by the time it reaches a finite distance from the source it will be small compared to the exponentially increasing component $f_i(s,\vec{r})$. Thus, if only the exponentially increasing waves are sought, the source-free waves will be omitted. Taking the limit finally produces the Fourier frequency component satisfying the "radiation condition." Let us now asymptotically evaluate $f_i(s,\vec{r})$ for large r.

The denominator $(F_i^{\dagger}\,\eta\, F_i)$ is changed with the introduction of the new variable s, and its new form is approximately,

$$\begin{aligned} \left[F_i^{\dagger}\eta F_i\right](s,\vec{\gamma}) \approx \left[F_i^{\dagger}\eta F_i\right](s_0,\vec{\gamma}_0) + ds\frac{\partial}{\partial s}\left[F_i^{\dagger}\eta F_i\right](s,\vec{\gamma})\Big|_{(s_0,\vec{\gamma}_0)} \\ + d\vec{\gamma}\,\nabla_{\gamma}\left[F_i^{\dagger}\eta F_i\right](s,\vec{\gamma})\Big|_{s_0,\vec{\gamma}_0} \end{aligned} \tag{6.9}$$

If $(s_o, \vec{\gamma}_o)$ is a root, i.e., $\left[F_i^\dagger \mathcal{M} F_i\right](s_o, \vec{\gamma}_o) = 0$, then for $\left[F_i^\dagger \mathcal{M} F_i\right](s, \vec{\gamma})$ to be zero when $s = s_o + ds$ requires that

$$\vec{\gamma} = \vec{\gamma}_0 - d\vec{\gamma} = \vec{\gamma}_0 - ds \left(\frac{\partial}{\partial s}\left[F_i^\dagger \mathcal{M} F_i\right](s,\vec{\gamma})\,\hat{r} \Big/ \hat{r}\,\nabla_{\vec{\gamma}}\left[F_i^\dagger \mathcal{M} F_i\right](s,\vec{\gamma})\right)\Bigg|_{s_0,\,\vec{\gamma}_0} \tag{6.10}$$

Let $s_o = j\omega$, $\vec{\gamma} = j\vec{k}$ and $ds = \sigma$. Then the new root is shifted to

$$s = \sigma + j\omega,\ \vec{\gamma} = j\vec{k} - \sigma\left(\frac{\partial}{\partial \omega}\left[F_i^\dagger \mathcal{M} F_i\right](\omega,\vec{k})\,\hat{r} \Big/ \hat{r}\,\nabla_{\vec{k}}\left[F_i^\dagger \mathcal{M} F_i\right](\omega,\vec{k})\right) \tag{6.11}$$

Since the operator $\mathcal{M}$ is skew-Hermitian, the new root $\vec{\gamma}$ has been shifted off the imaginary axis. With this in mind, perform the first integration of $f_1(s,\vec{r})$ on a variable $k_\parallel = \vec{k}\cdot\hat{r}$ which is parallel to $\vec{r}$ with the transverse variables $\vec{k}_\perp$ held fixed. Evaluate the integral by the contour integration method. By the Cauchy residue theorem we have,

$$\begin{aligned}\int_{-jR}^{jR} (F_i^\dagger \mathcal{M} F_i)^{-1} F_i F_i^\dagger C e^{(st-\gamma_\parallel r)}\, d\gamma_\parallel + \int_L (F_i^\dagger \mathcal{M} F_i)^{-1} F_i F_i^\dagger C e^{(st-\gamma_\parallel r)}\, d\gamma_\parallel \\ = -2\pi j \sum \text{residues of poles in the contour}\end{aligned} \tag{6.12}$$

where L is a semicircular path of radius R in the right half plane in the clockwise direction. Then by substituting $-\gamma_\parallel$ for p and r for t in Lemma II of <u>Transform Calculus</u> by E. J. Scott,[20] we have

$$\lim_{R\to\infty} \int_L (F_i^\dagger \mathcal{M} F_i)^{-1} F_i F_i^\dagger C e^{(st-\gamma_\parallel r)}\, d\gamma_\parallel = 0 \quad (r > 0) \tag{6.13}$$

provided the components of $\left|(F_i^\dagger \mathcal{M} F_i)^{-1} F_i F_i^\dagger C e^{st}\right|$ are less than $M/\left|\gamma_\parallel\right|^K$ when $\left|\gamma_\parallel\right| > R_o$, where M, K are constants and $K > 0$. Therefore,

$$\lim_{R\to\infty}\int_{-jR}^{jR}(F_i^{\dagger}\mathcal{M}F_i)^{-1}F_iF_i^{\dagger}Ce^{(st-\gamma_{\parallel}r)}d\gamma_{\parallel}$$

$$= -2\pi j\sum \text{ residues in right half } \gamma_{\parallel} \text{ plane} \tag{6.14}$$

or

$$\int_{-\infty}^{\infty}(F_i^{\dagger}\mathcal{M}F_i)^{-1}F_iF_i^{\dagger}Ce^{(st-jk_{\parallel}r)}dk_{\parallel}$$

$$= -2\pi\sum \text{ residues in right half } \gamma_{\parallel} \text{ plane} \tag{6.15}$$

But it was previously noted that the poles in the right half plane are the poles such that $\left(\hat{r}\cdot\nabla_{\vec{k}}\left[F_i^{\dagger}\mathcal{M}F_i\right](\omega,\vec{k})\Big/\frac{\partial}{\partial\omega}\left[F_i^{\dagger}\mathcal{M}F_i\right](\omega,\vec{k})\right)<0$ for $\sigma > 0$. Since the source, C, is assumed to contain no singularities, then for simple poles the residue at a pole is

$$(\hat{r}\cdot\nabla_{\vec{\gamma}}F_i^{\dagger}\mathcal{M}F_i)^{-1}F_iF_i^{\dagger}Ce^{(st-\vec{\gamma}\cdot r)} \tag{6.16}$$

Thus, we have

$$\lim_{s\to j\omega} f_i(s,\vec{r}) = j(2\pi)^{-2}\iint_{\Sigma_{i+}}\left(\hat{r}\cdot\nabla_k\left[F_i^{\dagger}\mathcal{M}F_i\right](\omega,\vec{k})\right)^{-1}F_iF_i^{\dagger}Ce^{j(\omega t-\vec{k}\cdot\vec{r})}d^2k_{\perp} \tag{6.17}$$

where Σ_{i+} is the portion of the dispersion surface in which $\left[F_i^{\dagger}\mathcal{M}F_i\right](\omega,\vec{k}) = 0$ and

$$\left(\hat{r}\cdot\nabla_{\vec{k}}\left[F_i^{\dagger}\mathcal{M}F_i\right](\omega,\vec{k})\Big/\frac{\partial}{\partial\omega}\left[F_i^{\dagger}\mathcal{M}F_i\right](\omega,\vec{k})\right) < 0$$

Now perform the last two integrals by the method of stationary phase. Hence, the major contribution to the integral for large r is from those points on Σ_{i+} where the exponent $\vec{k}\cdot\vec{r}$ is stationary. These are points where $\nabla_{\vec{k}}\,\vec{k}\cdot\vec{r} = 0$ or, equivalently, points where the normal to Σ_{i+} is parallel to $\vec{r}$. In the vicinity of a stationary point $\vec{k}_K$, we can express $\vec{k}\cdot\vec{r}$ as a second-order surface in the variables transverse to $\vec{r}$. Let the transverse variables $k_{\perp 1}$ and $k_{\perp 2}$

be in the principal directions. Becauee $\vec{k}\cdot\vec{r}$ is stationary and $k_{\perp 1}$ and $k_{\perp 2}$ are the principal directions, $k_{\parallel}$, $k_{\perp 1}$ and $k_{\perp 2}$ form an orthogonal coordinate system. Also, let ρ_{1K} and ρ_{2K} be the curvatures of the $\vec{k}$ surface at the stationary point $\vec{k}_K$ associated with the principal directions $\vec{k}_{\perp 1}$ and $\vec{k}_{\perp 2}$, respectively. A positive value of curvature implies a concave curvature to $\vec{r}$, whereas negative implies a convex curvature to $\vec{r}$. Then we have approximately,

$$\vec{k}\cdot\vec{r} = \vec{k}_K\cdot\vec{r} + \left[\frac{1}{2}\rho_{1K}(k_{\perp 1}-k_{\perp 1K})^2 + \frac{1}{2}\rho_{2K}(k_{\perp 2}-k_{\perp 2K})^2\right] r \tag{6.18}$$

By the principle of stationary phase the asymptotic form for r approaching infinity is,

$$\lim_{s\to j\omega} f_i(s,\vec{r}) \approx j(2\pi)^{-2}\sum_K \left(\hat{r}\cdot\nabla_{\vec{k}}\left[F_i^+ \mathcal{M}\, F_i\right](\omega,\vec{k})\right)^{-1} F_i F_i^+ C\Bigg|_{\vec{k}=\vec{k}_K} e^{j(\omega t-\vec{k}_K\cdot\vec{r})}$$
$$\cdot\iint_{-\infty}^{\infty} e^{-j\frac{1}{2}\left[\rho_{1K}(k_{\perp 1}-k_{\perp 1K})^2+\rho_{2K}(k_{\perp 2}-k_{\perp 2K})^2\right]r} \tag{6.19}$$
$$\cdot\, dk_{\perp 1}dk_{\perp 2}$$

where $\sum_K$ represents the sum of all stationary points on Σ_{i+}. The integral in Equation (6.19) can be evaluated after possibly slight modification from most integral tables. Hence,

$$\lim_{s\to j\omega} f_i(s,\vec{r}) \approx j(2\pi)^{-1}\sum_K \left(\hat{r}\cdot\nabla_{\vec{k}}\left[F_i^+ \mathcal{M}\, F_i\right]\right)^{-1} F_i F_i^+ C\,\frac{1}{r\sqrt{|\rho_1\rho_2|}}\, e^{-j\frac{\pi}{4}(\operatorname{sgn}\rho_1+\operatorname{sgn}\rho_2)} \tag{6.20}$$
$$\cdot\, e^{j(\omega t-\vec{k}\cdot\vec{r})}$$

But the product of the principal curvatures is the Gaussian[21] or total curvature, i.e., $\rho_1\rho_2 = \mathcal{K}$. Therefore, the asymptotic form of the field $\mathcal{F}_i(\vec{r},t)$

for r approaching infinity is

$$\mathcal{F}_i(\vec{r},t) \approx j(2\pi)^{-2} \int_{-\infty}^{\infty} \sum_K \left(\hat{r} \cdot \nabla_{\vec{k}} \left[F_i^{\dagger} \mathcal{M} F_i \right] \right)^{-1} F_i F_i^{\dagger} C \frac{1}{r\sqrt{|\mathcal{K}|}} e^{-j\frac{\pi}{4}(\operatorname{sgn} \rho_1 + \operatorname{sgn} \rho_2)} \cdot e^{j(\omega t - \vec{k} \cdot \vec{r})} d\omega \tag{6.21}$$

And of course the total field for the system is,

$$\mathcal{F}(\vec{r},t) = \sum_{i=1}^{N} \mathcal{F}_i(\vec{r},t) \tag{6.22}$$

For the special case of a time harmonic source $\mathcal{C}(\vec{r},t) = \mathcal{C}(\vec{r})\, e^{j\omega_o t}$, the Fourier transform is

$$C(\vec{k},\omega) = 2\pi C(\vec{k}) \delta(\omega - \omega_o) \tag{6.23}$$

where δ is the Dirac delta function. With this equation, Equation (6.21) is easily evaluated to give

$$\mathcal{F}(\vec{r},t) = \sum_K \left(\hat{r} \cdot \nabla_{\vec{k}} \left[F_i^{\dagger} \mathcal{M} F_i \right] (\omega_o, \vec{k}) \right)^{-1} F_i F_i^{\dagger} C(\vec{k}) \frac{j}{2\pi r \sqrt{|\mathcal{K}|}} e^{-j\frac{\pi}{4}(\operatorname{sgn} \rho_1 + \operatorname{sgn} \rho_2)} \cdot e^{j(\omega_o t - \vec{k} \cdot \vec{r})} \tag{6.24}$$

6.3 Application of Section 6.2 to an Arbitrary Six-Vector Electromagnetic System

The general description of an electrodynamic system can be expressed in terms of Maxwell's equations (Faraday's and Ampere's laws) and an auxiliary set of equations. The auxiliary set of equations may be partial differential, integral as the expectation of a Boltzmann's current, or simply an algebraic equation as the Lorentz equation for moving coordinates. However, the auxiliary set will be coupled to Maxwell's equations by common variables. Thus, often times when only the electromagnetic field is desired, it is convenient to eliminate all the variables except the electromagnetic field variables $\mathcal{F} = \left[\mathcal{E}, \mathcal{H}\right]$. In doing so the partial differential equations usually become of higher order and other complications set in. Alternately, the space-time Fourier transform of the general linear system of equations may be taken and the extraneous variables eliminated by algebraic means to produce a six-vector equation of the form,

$$OF - j\omega \underline{\underline{U}}F + C' \tag{6.25}$$

where O and F are the Fourier transforms of $\mathcal{O} = \begin{bmatrix} 0 & \nabla\times \\ -\nabla\times & 0 \end{bmatrix}$ and $\mathcal{F}$, respectively. C' is the Fourier transform of the equivalent source involving all of the sources of the original system of equations. And the sixth-order square matrix $\underline{\underline{U}}$ is defined to be the constitutive matrix for the relationship $F_f = \underline{\underline{U}}F$, where F_f is the Fourier transform of $\mathcal{F}_f = \left[\mathcal{D}, \mathcal{B}\right]$. Now Equation (6.25) is of the form Equation (6.1) of Part 6.2 if the electrodynamic system is lossless and the constitutive matrix, $\underline{\underline{U}}$, is Hermitian. Thus, the results of Section 6.2 follow. For this situation, however, as seen from Part 2.5 the quantity $\nabla_{\vec{k}}\left[F_i^{\dagger}\mathcal{M}F_i\right]$ can be reduced further, i.e.,

$$\nabla_{\vec{k}}\left[F_i^{\dagger}\mathcal{M}F_i\right] = 2j\,\overline{P}_{Ti} = 2j(\overline{P}_{ei} + \overline{P}_{mi}) \tag{6.26}$$

$\bar{P}_{Ti}$, $\bar{P}_{ei}$, and $\bar{P}_{mi}$ are the total, electromagnetic, and medium average power flux vectors, respectively, for the i^{th} characteristic field. The mathematical definitions of $\bar{P}_{ei}$ and $\bar{P}_{mi}$ are repeated for the sake of lucidity.

$$\bar{P}_{ei} = \mathrm{Re}\,(E \times H^*) \qquad (rms) \tag{6.27}$$

$$P_{mi} = -\frac{\omega}{2} F_i^{\dagger} (\nabla_{\vec{k}} \underline{\underline{U}}) F_i \qquad (rms) \tag{6.28}$$

Also from Section 2.5 it was noted that $\bar{P}_{Ti}$ is normal to the dispersion surface and is related to the group velocity of the i^{th} characteristic field. Hence, the radiation condition implies that

$$\hat{r} \cdot \vec{V}_{gi} > 0 \tag{6.29}$$

or that $\hat{r} \cdot \bar{P}_{Ti} = \left| \bar{P}_{Ti} \right| > 0$ at a stationary point. Further, normalize the characteristic field F_i such that the length of its total average power flux vector is equal to one. Define the number $h(\vec{k})$ to be as follows

$$h(\vec{k}) = \left(|\mathcal{K}| \right)^{-\frac{1}{2}} e^{-j \frac{\pi}{4} (\mathrm{sgn}\, \rho_1 + \mathrm{sgn}\, \rho_2 - 2)} \tag{6.30}$$

Using the above considerations the asymptotic expression for the characteristic source field $\mathcal{F}_i(\vec{r},t)$ for the time harmonic source as given by Equation (6.24) reduces to

$$\mathcal{F}_i(\vec{r},t) \approx \sum_K \frac{-jh(\vec{k}_K)}{4\pi r} F_i^{\dagger} C'(\vec{k}_K) F_i \, e^{j(\omega_0 t - \vec{k}_K \cdot \vec{r})} \tag{6.31}$$

The summation is over all stationary points on Σ_{i+}. And again the total field is given by

$$\mathcal{F}(\vec{r},t) = \sum_{i=1}^{6} \mathcal{F}_i(\vec{r},t) \tag{6.32}$$

Note that this derivation is very general in that the only assumptions made are that the electrodynamic system is lossless and that the medium and the source are such that the condition for the contour integration of Part 6.2 is satisfied. Also, note that Equation (6.31) gives the electromagnetic field due to any source of the original system of equations and not just due to the electromagnetic sources since $\mathcal{G}'$ is an equivalent source involving all of the sources of the system.

6.4 Case of a Compressible Plasma with N Species of Charged Particles

Assuming the fluid model for a plasma is applicable, the linearized equations that describe the system are

$$\nabla \times \mathcal{E} = \mu_0 \frac{\partial}{\partial t} \mathcal{H} - \mathcal{J}_m \tag{6.33}$$

$$\nabla \times \mathcal{H} = \epsilon_0 \frac{\partial}{\partial t} \mathcal{E} + \sum_{j=1}^{N} \rho_{qj} \mathcal{V}_j + \mathcal{J}_e \tag{6.34}$$

$$\rho_{mj} \frac{\partial}{\partial t} \mathcal{V}_j = \rho_{qj} \mathcal{E} + \rho_{qj} \mathcal{V}_j \times B_0 - \nabla \mathcal{P}_j - \mathcal{F}_j^e \quad , j=1 \text{ to } N \tag{6.35}$$

$$\frac{\partial}{\partial t} \mathcal{P}_j + \gamma_j P_{0j} \nabla^T \mathcal{V}_j = -\mathcal{Q}_j \quad , j=1 \text{ to } N \tag{6.36}$$

where the quantities and their Fourier transforms are

(Fourier transform)

E	$\mathcal{E}$	electric field
H	$\mathcal{H}$	magnetic field
J_e	$\mathcal{J}_e$	electric source current
J_m	$\mathcal{J}_m$	magnetic source current
	ρ_{mj}	mean mass density of the j^{th} charged component
	ρ_{qj}	mean charge density of the j^{th} charged component
V_j	$\mathcal{V}_j$	velocity field imparted to the j^{th} charged component by the sources
P_j	$\mathcal{P}_j$	variation of the j^{th} partial pressure imparted by the source (normalized by the number density)
	P_{oj}	mean partial pressure of the j^{th} component (norm.)
F_j	$\mathcal{F}_j$	source term for the j^{th} force equation
Q_j	$\mathcal{Q}_j$	source term for the j^{th} continuity equation
	γ_j	ratio of specific heats at constant pressure and constant temperature for the j^{th} charged component
	B_0	static magnetic field

Equation (6.35) is the force equation and Equation (6.36) is the normalized continuity equation since the pressure is assumed to be proportional to the number density to the γ_jth power. A number of assumptions are involved in arriving at the above system of equations. First, the equations have been linearized and hence will more accurately describe reality for small variations. The medium is macroscopically homogeneous and of infinite extent. A scalar pressure for each charged component is assumed; hence, shear wave will not be evident in the solutions. Separate adiabatic conditions for each species of charged particles are assumed. The latter two assumptions are equivalent to the conditions for truncating the moments of the Boltzmann's equation. Also, collisions are negligible. These assumptions are not unlike those used by many investigators, however crude the model may be.

Now let us show that this system can be developed into a special case of Part 6.3. Thus, one must find the constitutive relationship. After taking the Fourier transform of the system of equations, eliminate the velocity and pressure variation variables. In doing so, one arrives at the pair of vector equations,

$$j\vec{k}\times H = j\omega\left[\epsilon_0 I+\sum_{n=1}^{N}\rho_{qn}^{2}R_n^{1}\right]E+\left[J_e-j\sum_{n=1}^{N}\rho_{qn}R_n^{-1}\,\vec{k}\,Q_n - j\omega\sum_{n=1}^{N}\rho_{qn}R_n^{-1}F_n^{\circ}\right] \tag{6.37}$$

$$-jk\times E = -j\omega\mu_0 H - J_m \tag{6.38}$$

where the matrix R_n is defined as $R_n = (-\omega^2\rho_{mn} I + j\omega\rho_{qn} B_o \times + \gamma_n P_{on}\vec{k}\vec{k}^T)$ and I is the identity matrix. From this it is easy to see that the constitutive matrix and the equivalent source are

$$\underline{\underline{U}}(\vec{k},\omega)=\begin{bmatrix}\left[\epsilon_0 I+\sum_{n=1}^{N}\rho_{qn}^{2}R_n^{-1}\right] & 0\\ 0 & \mu_0 I\end{bmatrix} \tag{6.39}$$

$$C'(\vec{k},\omega) = \begin{bmatrix} J_e - j\sum_{n=1}^{N} \rho_{qn} R_n^{-1} \vec{k} Q_n - j\omega \sum_{n=1}^{N} \rho_{qn} R_n^{-1} F_n^{o} \\ J_m \end{bmatrix} \tag{6.40}$$

Since R_n is Hermitian for real $\vec{k}$ and ω, then $\underline{\underline{U}}$ is Hermitian and Equations (6.31) and (6.32) for the radiation field apply for a time harmonic source.

Refer to the section on General Formulation of Spectrum of Characteristic Waves for a discussion of the eigenvalues and eigenvectors.

It has been determined that the radiation field is due to the stationary points on the surface $F_i^{\dagger} \mathcal{M} F_i = 0$. However, this is also equal to

$$F_i^{\dagger} \mathcal{M} F_i = S_i F_i^{\dagger} \underline{\underline{U}} F_i = (\nu_i - j\omega) F_i^{\dagger} \underline{\underline{U}} F_i = 0 \tag{6.41}$$

Therefore, for the longitudinal waves when $\nu_i = 0$, the quantity $F_i^{\dagger} \underline{\underline{U}} F_i$ must be zero and its surface found. Another apparently added difficulty is that the quantity $F_i^{\dagger} C'$ may contain singularities whenever the matrix R_n becomes singular. However, after a bit of algebra it can be shown that

$$F_i^{\dagger} C' = E_i^{\dagger} J_e + H_i^{\dagger} J_m + \sum_{n=1}^{N} V_{ni}^{\dagger} F_n^{o} + \sum_{n=1}^{N} \left(P_{ni}/\gamma_n P_{on}\right)^{\dagger} Q_n \tag{6.42}$$

and thus can be made to contain no singularities with the proper normalization of the eigenvectors.

Greater consideration will be made for the case N = 1 in a later section by the ten-vector method which is in some respects more desirable.

Note also that when N = 1 and $P_{on} = P_n = 0$ the usual cold plasma model used by numerous investigators results. Observe that the constitutive matrix is not a function of the wave vector. Thus, $\overline{P}_T = \overline{P}_e$ or $\overline{P}_m = 0$ and the normalization is adjusted accordingly. This case, too, will be discussed further in a later section.

6.5 Case of Compressible Plasma with N Species of Charged Particles (6 + 4N-Vector Method)

Consider the same system of equations as described in Part 6.4. Naturally the assumptions involved in arriving at the equations are the same. Now, however, let us operate upon the equations in a different manner to obtain the solution. Instead of eliminating some of the unknowns, we will keep all of the unknowns and seek the solution of a matrix equation of order (6 + 4N). This manner of seeking the solution does not seem to have been considered by investigators before. For this reason and since there is a good correspondence between the mathematic and the physical behavior, the (6 + 4N)-vector method will be dwelt upon extensively. We wish to transform the given system of equations into a matrix equation that will be a particular case of Part 6.2. The ordering of the equations and the normalization of the unknowns will be critical although not unique. The motivation for doing so is to satisfy the conditions of Part 6.2, i.e., make O skew-Hermitian and $\underline{\underline{U}}$ Hermitian.

Order the equations in the following manner:

1. Ampere's Law
2. Faraday's Law
3. Force equations for particles 1 through N
4. Continuity equations for particles 1 through N

The continuity equations are to be in the same order as the force equations. In addition, place all the terms involving partial derivatives with respect to time and the sources on the right side of the equal sign with the remaining terms on the left. Moreover, define the normalized quantities, $\mathcal{U}_n$ as,

$$\mathcal{U}_n = (\mathcal{P}_n / \gamma_n P_{on}) \quad , n = 1, 2, \cdots, N \tag{6.43}$$

Then the above directions produce the 6 + 4Nth order matrix equation

$$\Theta \mathcal{F} = \frac{\partial}{\partial t} \underline{\underline{\mathcal{V}}} * \mathcal{F} + \mathcal{G} \tag{6.44}$$

which is given in Figure 6.1. The corresponding terms should be obvious. Notice that already we have gained a benefit, for the conservation of total energy is expressed as,

$$\mathcal{F}^T \Theta \mathcal{F} = \mathcal{F}^T \dot{\mathcal{F}}_f + \mathcal{F}^T \mathcal{G} \tag{6.45}$$

or

$$-\nabla^T \mathcal{P}_T = \frac{\partial}{\partial t} \mathcal{W}_T \quad \frac{\partial}{\partial t} \mathcal{S}_T \tag{6.46}$$

where

$$\mathcal{P}_T = \mathcal{E} \times \mathcal{H} + \sum_{n=1}^{N} \gamma_n P_{on} \mathcal{U}_n \mathcal{V}_n \tag{6.47}$$

$$\mathcal{W}_T = \frac{1}{2} \left[\epsilon_0 \mathcal{E}^T \mathcal{E} + \mu_0 \mathcal{H}^T \mathcal{H} + \sum_{n=1}^{N} \rho_{mn} \mathcal{V}_n^T \mathcal{V}_n + \sum_{n=1}^{N} \gamma_n P_{on} \mathcal{U}_n^2 \right] \tag{6.48}$$

and,

$$\mathcal{F}^T \mathcal{G} = \frac{\partial}{\partial t} \mathcal{S}_T = \left[\mathcal{E}^T \mathcal{J}_e + \mathcal{H} \mathcal{J}_m + \sum_{n=1}^{N} \mathcal{V}_n^T \mathcal{F}_n^o + \sum_{n=1}^{N} \mathcal{U}_n \mathcal{Q}_n \right] \tag{6.49}$$

$$
\left[\begin{array}{cccccccccccc}
0 & \nabla\times & -\rho_{q1} & -\rho_{q2} & -\rho_{q3} & \cdots & -\rho_{qN} & 0 & 0 & 0 & \cdots & 0 \\
-\nabla\times & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 \\
\rho_{q1} & 0 & -\rho_{q1}B_0\times & 0 & 0 & \cdots & 0 & -\gamma_1 P_{01}\nabla & 0 & 0 & \cdots & 0 \\
\rho_{q2} & 0 & 0 & -\rho_{q2}B_0\times & 0 & \cdots & 0 & 0 & -\gamma_2 P_{02}\nabla & 0 & \cdots & 0 \\
\rho_{q3} & 0 & 0 & 0 & -\rho_{q3}B_0\times & \cdots & 0 & 0 & 0 & -\gamma_3 P_{03} & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
\rho_{qN} & 0 & 0 & 0 & 0 & \cdots & -\rho_{qN}B_0\times & 0 & 0 & 0 & \cdots & -\gamma_N P_{0N}\nabla \\
0 & 0 & -\gamma_1 P_{01}\nabla^T & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 \\
0 & 0 & 0 & -\gamma_2 P_{02}\nabla^T & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 \\
0 & 0 & 0 & 0 & -\gamma_3 P_{03}\nabla^T & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & 0 & 0 & \cdots & -\gamma_N P_{0N}\nabla^T & 0 & 0 & 0 & \cdots & 0
\end{array}\right]
\left[\begin{array}{c} \mathcal{E} \\ \mathcal{H} \\ v_1 \\ v_2 \\ v_3 \\ \vdots \\ v_N \\ u_1 \\ u_2 \\ u_3 \\ \vdots \\ u_N \end{array}\right]
= \frac{\partial}{\partial t}
\left[\begin{array}{cccccccccccc}
\epsilon_0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 \\
0 & \mu_0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 \\
0 & 0 & \rho_{m1} & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 \\
0 & 0 & 0 & \rho_{m2} & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 \\
0 & 0 & 0 & 0 & \rho_{m3} & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & 0 & 0 & \cdots & \rho_{mN} & 0 & 0 & 0 & \cdots & 0 \\
0 & 0 & 0 & 0 & 0 & \cdots & 0 & \gamma_1 P_{01} & 0 & 0 & \cdots & 0 \\
0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & \gamma_2 P_{02} & 0 & \cdots & 0 \\
0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \gamma_3 P_{03} & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & \gamma_N P_{0N}
\end{array}\right]
\left[\begin{array}{c} \mathcal{E} \\ \mathcal{H} \\ v_1 \\ v_2 \\ v_3 \\ \vdots \\ v_N \\ u_1 \\ u_2 \\ u_3 \\ \vdots \\ u_N \end{array}\right]
+
\left[\begin{array}{c} \mathcal{J}_e \\ \mathcal{J}_m \\ \mathcal{F}_1 \\ \mathcal{F}_2 \\ \mathcal{F}_3 \\ \vdots \\ \mathcal{F}_N \\ \mathcal{L}_1 \\ \mathcal{L}_2 \\ \mathcal{L}_3 \\ \vdots \\ \mathcal{L}_N \end{array}\right]
$$

Figure 6.1. Matrix organization of the (6-4N)-vector system of equations, $\mathcal{O}\mathcal{Y} = \underline{\underline{\mathcal{U}}} * \mathcal{Y}$, for a lossless anisotropic compressible plasma.

Now the Fourier transform of Equation (6.44) is

$$OF = j\omega \underline{\underline{U}} F + C \tag{6.50}$$

where O is $\mathcal{O}$ with the del operator, ∇, replaced by $-j\vec{k}$ and $\underline{\underline{U}}$ is

$$\underline{\underline{U}} = \begin{bmatrix} \epsilon_o & & & & & & & \\ & \mu_o & & & & \bigcirc & & \\ & & \rho_{m1} & & & & & \\ & & & \rho_{m2} & & & & \\ & & & & \ddots & & & \\ & & & & & \gamma_1 P_{O1} & & \\ & \bigcirc & & & & & \ddots & \\ & & & & & & & \gamma_N P_{ON} \end{bmatrix} \tag{6.51}$$

Now it is easily verified that for a real wave vector $\vec{k}$ the matrix O is skew-Hermitian. Not only is the matrix $\underline{\underline{U}}$ Hermitian and diagonal but it is also positive definite. Therefore, the conditions of Part 6.2 are satisfied and the results for the radiation field follows. Because of the symmetry of O and $\underline{\underline{U}}$, the usual type of eigenvalue problem can be defined.

$$OF_i = \lambda_i \underline{\underline{U}} F_i \qquad , i = 1, 2, \cdots, 6+4N \tag{6.52}$$

Since O and $\underline{\underline{U}}$ are not a function of ω and are of order 6 + 4N, the 6 + 4N eigenvalues will be a function of $\vec{k}$ only. They will be imaginary for real $\vec{k}$. Define the real valued surface Σ_{λ_i} to be the locus of points $(\vec{k}, -j\lambda_i(\vec{k}))$ in four-space. Then for a given $\omega = \omega_o$ the intersection of the two surfaces Σ_{λ_i} and

$-j\lambda_i - \omega_o = 0$ in four-space is a curve. But this curve in four-space is equivalent to a surface in three-space ($\vec{k}$ space) which is commonly called a sheet of the dispersion surface at $\omega = \omega_o$. The totality of all these surfaces in three-space for all $i = 1,2,\ldots, 6 + 4N$ is called the dispersion surface at angular frequency $\omega = \omega_o$. However, at a given frequency the plane $-j\lambda_i - \omega_o = 0$ may not intersect Σ_{λ_i}; therefore, there may not be $6 + 4N$ real sheets to the dispersion surface at a given frequency. As ω takes all real values, the curve in four-space which is the intersection of Σ_{λ_i} and the plane $-j\lambda_i - \omega = 0$ generates the i^{th} sheet of the dispersion surface in four-space. This is equivalent to relabeling the $-j\lambda$ axis as ω.

Recall that for source free solutions it is required that

$$\det\left[0 - j\omega \underline{\underline{U}}\right] = (\det \underline{\underline{U}}) \prod_{i=1}^{6+4N} (\lambda_i - j\omega) = 0 \tag{6.53}$$

But $\det \underline{\underline{U}} > 0$; hence, all of the sheets of the dispersion surface are associated with the eigenvalues of the characteristic Equation (6.52). Therefore, there is a one-to-one correspondence between all sheets of the dispersion surface $S_i = \lambda_i - j\omega = 0$ and the eigenvalues of the characteristic equation. This is in direct contrast to the six-vector method in which the factor det $\underline{\underline{U}}(\vec{k},\omega)$ may be zero and contribute to sheets of the dispersion surface. Further remember that it was possible for $F_i^{\dagger}\, \underline{\underline{U}}(\vec{k},\omega)\, F_i = 0$, for some i of the six-vector method to coincide with a sheet(s) of the dispersion surface. But for the 6 + 4N-vector method, $F_i^{\dagger}\, \underline{\underline{U}}\, F_i = 0$ if and only if $F_i = 0$ since $\underline{\underline{U}}$ is positive definite. This one-to-one correspondence between the sheets of the dispersion surface and the eigenvalues is one of the advantages of the 6 + 4N-vector method.

Another property of the dispersion surface can be determined from the real valued property of the fields, i.e., both $\mathcal{F}$ and $\mathcal{O}\mathcal{F}$ are real valued vectors. The real valued property of the field implies that

$$O(\vec{k}) = O^{*}(-\vec{k}) \tag{6.54}$$

Therefore,

$$\det\left[O(\vec{k}) - \lambda(\vec{k})\underline{\underline{U}}\right] = \det\left[O^{*}(-\vec{k}) - \lambda(\vec{k})\underline{\underline{U}}\right] = 0 \tag{6.55}$$

Which implies that

$$\det\left[O(-\vec{k}) - \left(-\lambda(\vec{k})\right)\underline{\underline{U}}\right] = 0 \tag{6.56}$$

and finally

$$\det\left[O(\vec{k}) - \left(-\lambda(-\vec{k})\right)\underline{\underline{U}}\right] = 0 \tag{6.57}$$

Hence if $\lambda_i(\vec{k})$ is an eigenvalue, there exists a j such that $\lambda_j(\vec{k}) = -\lambda_i(-\vec{k})$ is also an eigenvalue. From this we have that the i^{th} and j^{th} sheets of the dispersion surface are $S_i = \lambda_i(\vec{k}) - j\omega = 0$ and $S_j = -\lambda_i(-\vec{k}) - j\omega = 0$, respectively. Thus, if either point $(\vec{k},\omega)$ or $(-\vec{k},-\omega)$ is on the dispersion surface, then both points are.

The 6 + 4N eigenvectors which are a function of $\vec{k}$ only also display an orthogonality property. This property is

$$F_i^{\dagger}\underline{\underline{U}}F_j = 0 \qquad \text{for} \qquad \lambda_i \neq \lambda_j \qquad i,j = 1,2,\cdots 6+4N \tag{6.58}$$

But even for a degenerate system one can define a set of orthogonal eigenvectors. Thus, it will always be assumed that the eigenvectors form a mutually orthogonal set, i.e.,

$$F_i^{\dagger} \underline{\underline{U}} F_j = 0 \qquad , i \neq j \qquad i,j = 1,2, \cdots 6+4N \tag{6.59}$$

Another property may be surmised by examining the original real system of equations. In a lossless system the property of time reversal essentially states that if the time axis were suddenly reversed then the system would retrace its path of operation. That is, time reversal is expected if the system is invariant under the following transformation,

$$\begin{aligned} t &\longrightarrow -t \\ \mathcal{E} &\longrightarrow \mathcal{E} \\ \mathcal{H} &\longrightarrow -\mathcal{H} \\ \mathcal{V}_n &\longrightarrow -\mathcal{V}_n \\ \mathcal{U}_n &\longrightarrow \mathcal{U}_n \\ B_o &\longrightarrow -B_o \end{aligned} \tag{6.60}$$

That time reversal is expected for our plasma model is easily verified. Thus, it is anticipated that time reversal should manifest itself in some manner in the eigenvalues, eigenvectors and the dispersion surface. If the Fourier transform of a function $f(\vec{r},t,B_o)$ is $g(\vec{k},\omega,B_o)$, then the Fourier transform of the time reversed function $\pm f(\vec{r},-t,-B_o)$ is $\pm g(\vec{k},-\omega,-B_o)$. Therefore, for a characteristic field the time reversed characteristic field is as follows,

TABLE II

	$E_i(\vec{k}, \lambda_i, B_o)$		$E_i(\vec{k}, \lambda_i, -B_o)$
λ_i	$H_i(\vec{k}, \lambda_i, B_o)$	$-\lambda_i$	$-H_i(\vec{k}, \lambda_i, -B_o)$
	$V_{ni}(\vec{k}, \lambda_i, B_o)$		$-V_{ni}(\vec{k}, \lambda_i, -B_o)$
	$u_{ni}(\vec{k}, \lambda_i, B_o)$		$u_{ni}(\vec{k}, \lambda_i, -B_o)$
	Characteristic Field		Time Reversed Characteristic Field

It is easily verified that if the characteristic field satisfies the eigenvalue equation, the time reversed characteristic field does also. Hence, if the characteristic fields are known for all i such that $-j\lambda_i > 0$, then the other half is known by the property of time reversal. With respect to the dispersion surface, time reversal implies that if point $(\vec{k},\omega)$ is on the dispersion surface then $(\vec{k},-\omega)$ is also. Time reversal together with the real valued property implies that if point $(\vec{k},\omega)$ is on the dispersion surface then point $(-\vec{k},\omega)$ is also. Therefore, the three-space dispersion surface at a given ω has the symmetry of reflection through the origin. In addition, time reversal and the real valued property implies that if any one point of the set $(\vec{k},\omega)$, $(\vec{k},-\omega)$, $(-\vec{k},\omega)$ and $(-\vec{k},-\omega)$ is on the dispersion surface then all points are.

Since the group velocity of the i^{th} characteristic wave is

$$V_{gi} = -\frac{\nabla_{\vec{k}} S_i}{\frac{\partial}{\partial\omega} S_i} = -\frac{\nabla_{\vec{k}}(\lambda_i - j\omega)}{\frac{\partial}{\partial\omega}(\lambda_i - j\omega)} \tag{6.61}$$

$$= -j\nabla_{\vec{k}}\lambda_i$$

then the group velocity of the time reversed characteristic wave is

$$V_{gitr} = -\frac{\nabla_{\vec{k}}(-\lambda_i - j\omega)}{\frac{\partial}{\partial\omega}(-\lambda_i - j\omega)} \tag{6.62}$$

$$= j\nabla_{\vec{k}}\lambda_i$$

$$= -V_{gi}$$

Therefore, if a characteristic wave at $(\vec{k},\omega)$ satisfies the radiation condition, then the time reversed characteristic wave at $(\vec{k},-\omega)$ does not. And $(-\vec{k},-\omega)$ will satisfy the radiation condition but $(-\vec{k},\omega)$ will not. Thus, when the exact total field is sought, the sum (integral) of characteristic waves must be taken in such a manner that points $(\vec{k},\omega)$ and $(-\vec{k},-\omega)$, which satisfy the radiation condition, are included and points $(\vec{k},-\omega)$ and $(-\vec{k},\omega)$ are excluded. This may mean that the contour of integration must be chosen appropriately.

6.6 Case of a Compressible Plasma with N = 1 Species of Charged Particles (10-Vector Method)

For the case of a mobile electron with an isotropic pressure, the 10^{th} order determinantal equation reduces to

$$\left\| \begin{matrix} -\lambda\epsilon_0 & -j\vec{k}\times & -\rho_q & 0 \\ j\vec{k}\times & -\lambda\mu_0 & 0 & 0 \\ \rho_q & 0 & -\rho_q B_0 \times - \lambda\rho_m & \gamma P_0 j\vec{k} \\ 0 & 0 & \gamma P_0 j\vec{k}^T & -\lambda\gamma P_0 \end{matrix} \right\| = 0 \tag{6.63}$$

After considerable algebraic manipulations it is found that the equation becomes

$$
\begin{aligned}
&(\gamma P_o)(\rho_m \mu_o \epsilon_o)^3 \Big[\lambda^{10} + \left\{(2+\delta)y^2 + 3\omega_N^2 + \omega_H^2\right\}\lambda^8 + \Big\{(1+2\delta)y^4 + (4+\delta)\omega_N^2 y^2 \\
&+ 2\omega_H^2 y^2 + \delta\omega_H^2 \zeta^2 + \omega_H^2\omega_N^2 + 3\omega_N^4\Big\}\lambda^6 + \Big\{\delta y^6 + 2\delta\omega_H^2 y^2 \zeta^2 \\
&+ \left[(1+2\delta)\omega_N^2 + \omega_H^2\right]y^4 + \left[(2+\delta)\omega_N^4 + \omega_H^2\omega_N^2\right]y^2 + \omega_H^2\omega_N^2\zeta^2 + \omega_N^6\Big\}\lambda^4 \\
&+ \left\{\omega_H^2\omega_N^2\zeta^2 y^2 + \delta\omega_H^2\zeta^2 y^4\right\}\lambda^2\Big] = 0
\end{aligned}
\tag{6.64}
$$

where $\omega_N^2 = \rho_q^2/\rho_m\epsilon_o \, , \omega_H = \rho_q B_o/\rho_m \, , \delta = \gamma P_o/\rho_m c^2$

and $\vec{y} = c\vec{k} = (\xi, \eta, \zeta)$.

The eigenvectors corresponding to the nonzero eigenvalues are

$$E = \rho_m(\lambda\rho_q)^{-1}(\lambda^2 + \lambda\vec{\omega}_H X + \delta\vec{y}\vec{y}^T)V \tag{6.65}$$

$$H = j\rho_q c(\lambda\omega_N^2)^{-1}\vec{y} X(\lambda + \vec{\omega}_H X)V \tag{6.66}$$

$$V = V \tag{6.67}$$

$$u = j(\lambda c)^{-1}\vec{y}^T V \tag{6.68}$$

or explicitly

$$E_x = (\rho_m/\lambda\rho_q)\left[(\lambda^2+\delta\xi^2)V_x+(-\lambda\omega_H+\delta\xi\eta)V_y+\delta\xi\zeta V_z\right] \tag{6.69}$$

$$E_y = (\rho_m/\lambda\rho_q)\left[(\lambda\omega_H+\delta\xi\eta)V_x+(\lambda^2+\delta\eta^2)V_y+\delta\eta\zeta V_z\right] \tag{6.70}$$

$$E_z = (\rho_m/\lambda\rho_q)\left[\delta\xi\zeta V_x+\delta\eta\zeta V_y+(\lambda^2+\delta\zeta^2)V_z\right] \tag{6.71}$$

$$H_x = (j\rho_q c/\lambda\omega_N^2)\left[-\zeta\omega_H V_x-\zeta\lambda V_y+\eta\lambda V_z\right] \tag{6.72}$$

$$H_y = (j\rho_q c/\lambda\omega_N^2)\left[\zeta\lambda V_x-\zeta\omega_H V_y-\xi\lambda V_z\right] \tag{6.73}$$

$$H_z = (j\rho_q c/\lambda\omega_N^2)\left[(\xi\omega_H-\eta\lambda)V_x+(\eta\omega_H+\xi\lambda)V_y\right] \tag{6.74}$$

$$\begin{aligned} V_x = \lambda^2\Big\{&\lambda^5-\left[(1-\delta)(\eta^2+\zeta^2)-2(y^2+\omega_N^2)\right]\lambda^3+\omega_H\xi\eta\lambda^2 \\ &-(y^2+\omega_N^2)\left[(1-\delta)(\eta^2+\zeta^2)-(y^2+\omega_N^2)\right]\lambda+\omega_H\xi\eta(y^2+\omega_N^2)\Big\} \end{aligned} \tag{6.75}$$

$$\begin{aligned} V_y = \lambda^2\Big\{&-\omega_H\lambda^4+(1-\delta)\eta\xi\lambda^3-\omega_H\left[2y^2-\eta^2-(1-\delta)\zeta^2+\omega_N^2\right]\lambda^2 \\ &+(1-\delta)\eta\xi(y^2+\omega_N^2)\lambda-\omega_H\left[(y^2-\eta^2)(y^2+\omega_N^2)-(1-\delta)y^2\zeta^2\right]\Big\} \end{aligned} \tag{6.76}$$

$$\begin{aligned} V_z = \zeta\lambda\Big\{&(1-\delta)\xi\lambda^4+\delta\omega_H\eta\lambda^3+\xi\left[(1-\delta)(y^2+\omega_N^2)+\omega_H^2\right]\lambda^2 \\ &+\omega_H\eta\left[\delta y^2+\omega_N^2\right]\lambda+\omega_H^2\xi y^2\Big\} \end{aligned} \tag{6.77}$$

$$u = j(\lambda c)^{-1}\left[\xi V_x+\eta V_y+\zeta V_z\right] \tag{6.78}$$

The remaining two eigenvectors corresponding to the eigenvalues which are identically zero are

$$F_9 = \left[0, \vec{k}, 0, 0 \right] \tag{6.79}$$

and

$$F_{10} = \left[\vec{k}, 0, 0, j\rho_q / \gamma P_o \right] \tag{6.80}$$

The actual process of computing the "patterns" at a given frequency ω and distance r is not difficult. Only simple algebraic and differentiation operations are involved. However, even for this simplest warm plasma model either a computer or fortitude must be used because of the complexity of the algebraic quantities. The following is a brief outline of the steps involved in computing the "pattern."

(1) Determination of the dispersion surface. From the determinantal Equation (6.64) it is seen that the dispersion surface in $\vec{k}$-space is a surface of revolution and has at most three real sheets. Therefore, the dispersion surface may be found by obtaining the six roots of y with $\theta = \cos^{-1} (\zeta/y)$ as a parameter ranging from zero to 2π.

(2) Determination of stationary points for a given $\vec{r}$. The stationary points are those points in which the plane tangent to the dispersion surface is perpendicular to r. This is equivalent to points on the dispersion surface such that $\nabla_{\vec{k}} (\vec{k} \cdot \hat{r}) = 0$, which reduces to $\frac{d}{d\theta} \left[k(\theta) \cos (\theta - \alpha) \right] = 0$, ($\alpha$ is the angle between $\hat{z}$ and $\hat{r}$) since the surface is a surface of revolution. Hence, the stationary points lie in the plane determined by $\hat{r}$ and $\hat{z}$. The stationary

points that satisfy the radiation condition must also satisfy the condition $\vec{k}\cdot\hat{r} > 0$. The stationary points may be found either by use of a computer or graphically.

(3) Determination of the Gaussian radius of curvature. The Gaussian curvature $\mathcal{K}$ is defined as the product of the two principal curvatures ρ_1 and ρ_2. Since Σ is a surface of revolution ρ_1 (meridian curvature) and ρ_2 (parallel curvature) are given by

$$\rho_1 = (k^2 + 2k'^2 - kk'')(k^2 + k'^2)^{-3/2} \tag{6.81}$$

$$\rho_2 = k^{-1} \sin\alpha \csc\theta \tag{6.82}$$

where the prime denotes the derivative with respect to θ. Point $\vec{k}$ on Σ is a stationary point; hence,

$$k^{-1}\cos(\alpha-\theta) = -(k')^{-1}\sin(\alpha-\theta) = (k^2 + k'^2)^{-1/2} \tag{6.83}$$

From Equation (6.83), one finds

$$\frac{d\alpha}{d\theta} = (k^2 + 2k'^2 - kk'')(k^2 + k'^2)^{-1} \tag{6.84}$$

and

$$\rho_1 = k^{-1}\cos(\alpha-\theta)\frac{d\alpha}{d\theta} \tag{6.85}$$

Then it immediately follows that

$$\mathcal{K} = k^{-1} k_a^{-1} \frac{d\alpha}{d\theta} \tag{6.86}$$

where

$$k_{\eta} = k \sin \theta \csc \alpha \sec (\alpha - \theta) \tag{6.87}$$

(4) Normalization of the eigenvectors. The normalization process

$$\left| \mathrm{Re} \left[E X H^{*} + \gamma P_o u V^{*} \right] \right| = 1 \tag{6.88}$$

as a general function is a rather formidable task, however, is simple pointwise on a computer.

Needless to say, the remainder of the pattern calculations are rather trivial.

6.7 Case of a Cold Plasma with N = 1

Although the cold plasma is a particular case of the previous warm plasma and may be found accordingly, drastic simplifications make it quite feasible to find the radiation "patterns" without the use of a high-speed computer. First the determinantal equation becomes quadratic in k^2 and the usual Appleton-Hartree equation results. The dispersion surface then can be found by a graphical method of Deschamps and Weeks.[22] Also both the stationary points and the radii of curvature can be found graphically. Furthermore, the polarization ellipse of H_i has one axis in the plane $(\vec{k}, H_o)$ and the other perpendicular to it.[23,24] If the axial ratio is tan β, the angle β is simply related to θ and to the usual ionospheric parameters $X = \omega_N^2/\omega^2$ and $Y = \omega_H/\omega$ by

$$\tan 2\beta = 2 \cos \theta \csc^2 \theta (1 - X) Y^{-1} \tag{6.89}$$

The two solutions give the polarization ratios to the "ordinary" and "extraordinary" waves. For the usual definition of the "ordinary" wave (the sheet of the dispersion surface for which $k = k_o \sqrt{1-X}$ when $\vec{k} \cdot H_o = 0$) the major axis of the H-ellipse is perpendicular to the plane $(\vec{k}, H_o)$. The major axis of the H-ellipse for the "extraordinary" wave lies in the plane $(\vec{k}, H_o)$. The normalization requires that

$$|H_i|^2 = (k/\omega\mu_0)\cos(\alpha-\theta) \tag{6.90}$$

The vector E_i follows form Maxwell's equations

$$E_i = -(1/\omega)\, \underline{\underline{\epsilon}}^{-1} \vec{k} \times H_i \tag{6.91}$$

The above brief description along with Equation (6.31) should prove sufficient to find the radiation field of an arbitrary antenna in a cold plasma.

The principal advantages of this method of obtaining the radiation fields of an arbitrary antenna, over that of previous authors are:

(1) A high-speed computer is not necessary.

(2) Functions of an indeterminant form at the principal directions such as in Bunkin's and Keuhl's solutions are not involved.

(3) All terms of the solution and the normalization are easy to interpret physically.

It would be interesting to display the dispersive property of the medium by showing a diagram of the four-space $(\vec{k}, -j\lambda)$ dispersion surface, however, such is not possible. Fortunately since the $\vec{k}$-space dispersion surface is a surface of revolution, the dispersive property of the medium may be illustrated by a diagram of $(k_\rho, k_z, -j\lambda)$. Figures (6.2), (6.3), (6.4) and (6.5) show the sheets of one quadrant of such a surface in which $\omega_H/\omega_N = 2$. The sheets of the dispersion surface in the other quadrants are mirror images in the quadrant planes. Since there are nine eigenvalues, the remaining sheet is $-j\lambda = 0$. With the dispersion surface represented in this manner, there are only two topologically different dispersion surfaces, $\omega_H/\omega_N \gtrless 1$. Notice also that the sheets touch only at $-j\lambda = 0, \pm\,\omega_N$. Because of the complexity of sheet Σ_{λ_2} in Figure (6.3), perhaps a verbal description is in order. There are four values of $-j\lambda$ that correspond to important points on the sheet. The sheet intersects the $-j\lambda$ axis at $-j\lambda = \frac{1}{2}\left[(\omega_H^2 + 4\omega_N^2)^{\frac{1}{2}} - \omega_H\right]$ which is called cutoff. At $-j\lambda = \omega_N$, Σ_{λ_2} intersects the light cone with the portion of the surface for $-j\lambda < \omega_N$ "inside" and for $-j\lambda > \omega_N$ "outside" the light cone. $-j\lambda = \omega_H$ is an asymptote for the intersection of Σ_{λ_2} with the $(k_z, -j\lambda)$ plane. And $-j\lambda = (\omega_H^2 + \omega_N^2)^{\frac{1}{2}}$ is an asymptote for the intersection of Σ_{λ_2} with the $(k_\rho, -j\lambda)$ plane. In the neighborhood of $-j\lambda = \omega_N$, Σ_{λ_2} changes most rapidly. There is a "plateau" at $-j\lambda = \omega_N$ which is widest in the k_z direction and disappears in the k_ρ direction.

The intersection of planes $-j\lambda = \omega$ with the dispersion surface is what is called a CMA diagram (Clemmow, Mulally, Allis). They are represented by the dotted lines in the figures. The intersection of a plane passing through the $-j\lambda$ axis with the dispersion surface yields a Stringer[25] diagram, which is represented by an alternate dot-dash line. The curves corresponding to either side of the removed sector is a Stringer diagram.

One of the advantages of the dispersion surface represented in this manner is that it is emphasized that there are more than one "ordinary" and "extraordinary" sheets. Sheets Σ_{λ_1} and Σ_{λ_3} correspond to the "ordinary" sheets

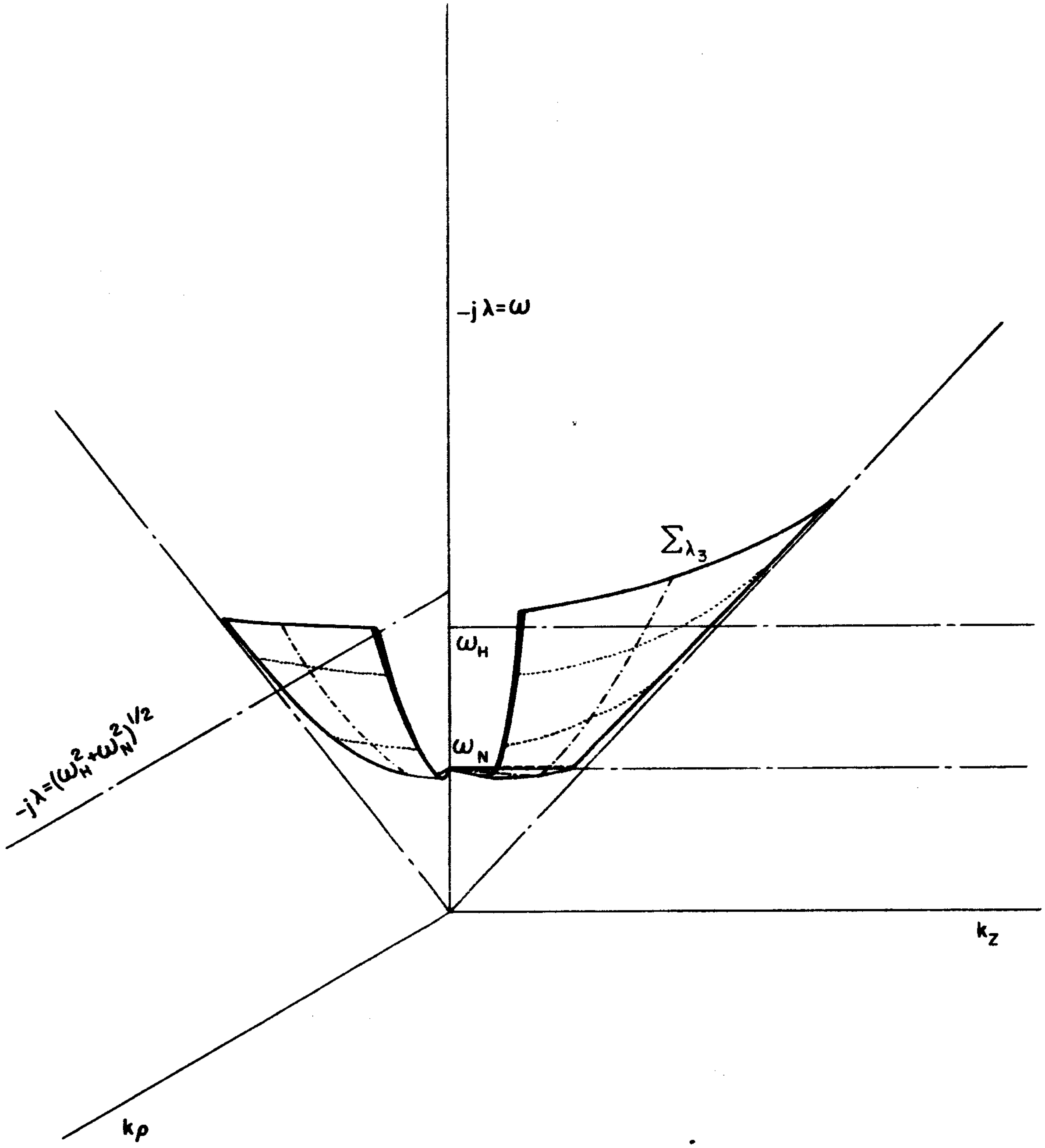

Figure 6.2. One quadrant of sheet Σ_{λ_1} of the dispersion surface for a magneto-ionic medium. $(\omega_H = 2\omega_N)$

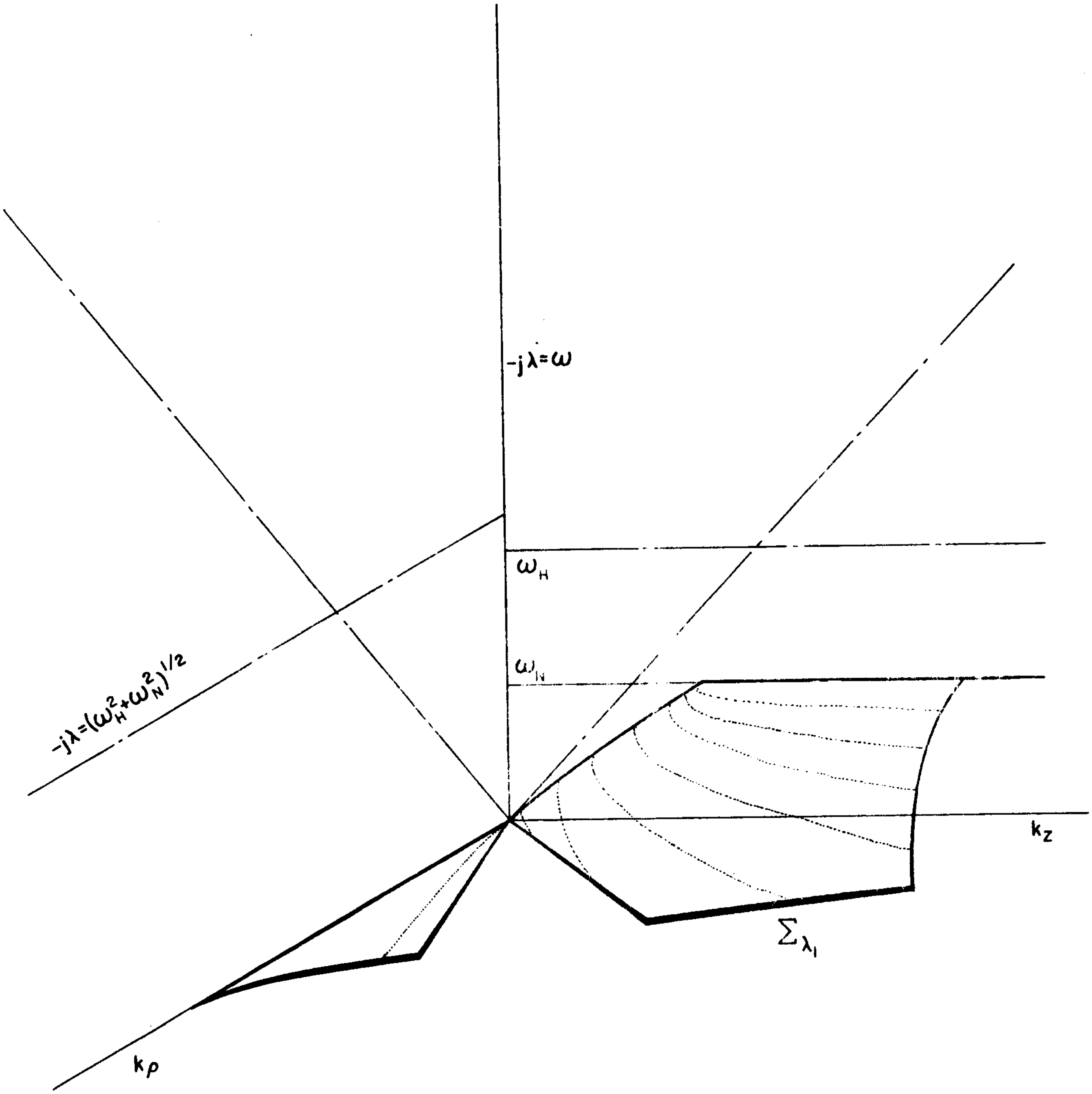

Figure 6.3. One quadrant of sheet Σ_{λ_2} of the dispersion surface for a magneto-ionic medium. $(\omega_H = 2\omega_N)$

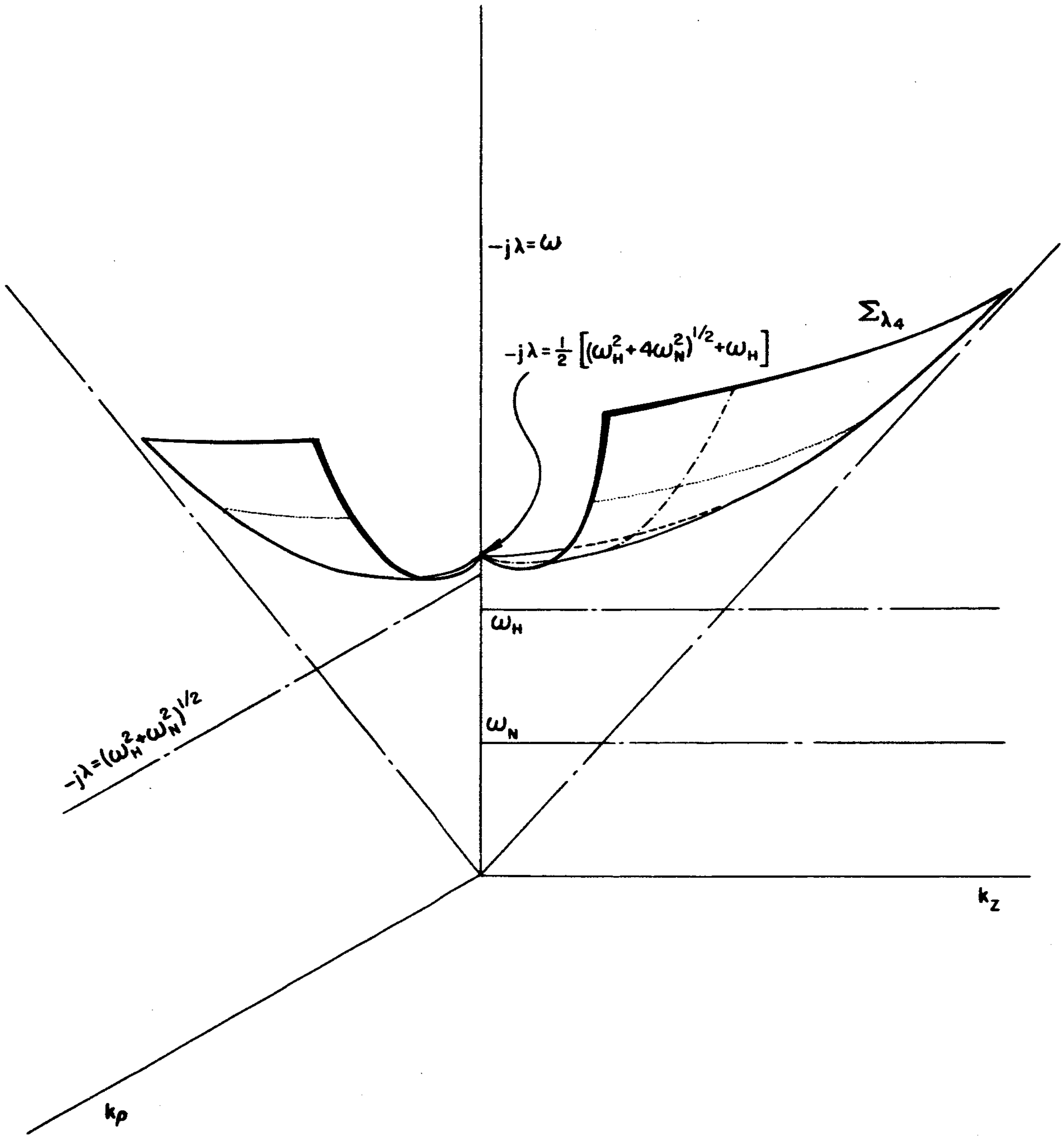

Figure 6.4 One quadrant of sheet Σ_{λ_3} of the dispersion surface for a magneto-ionic medium. $(\omega_H = 2\omega_N)$

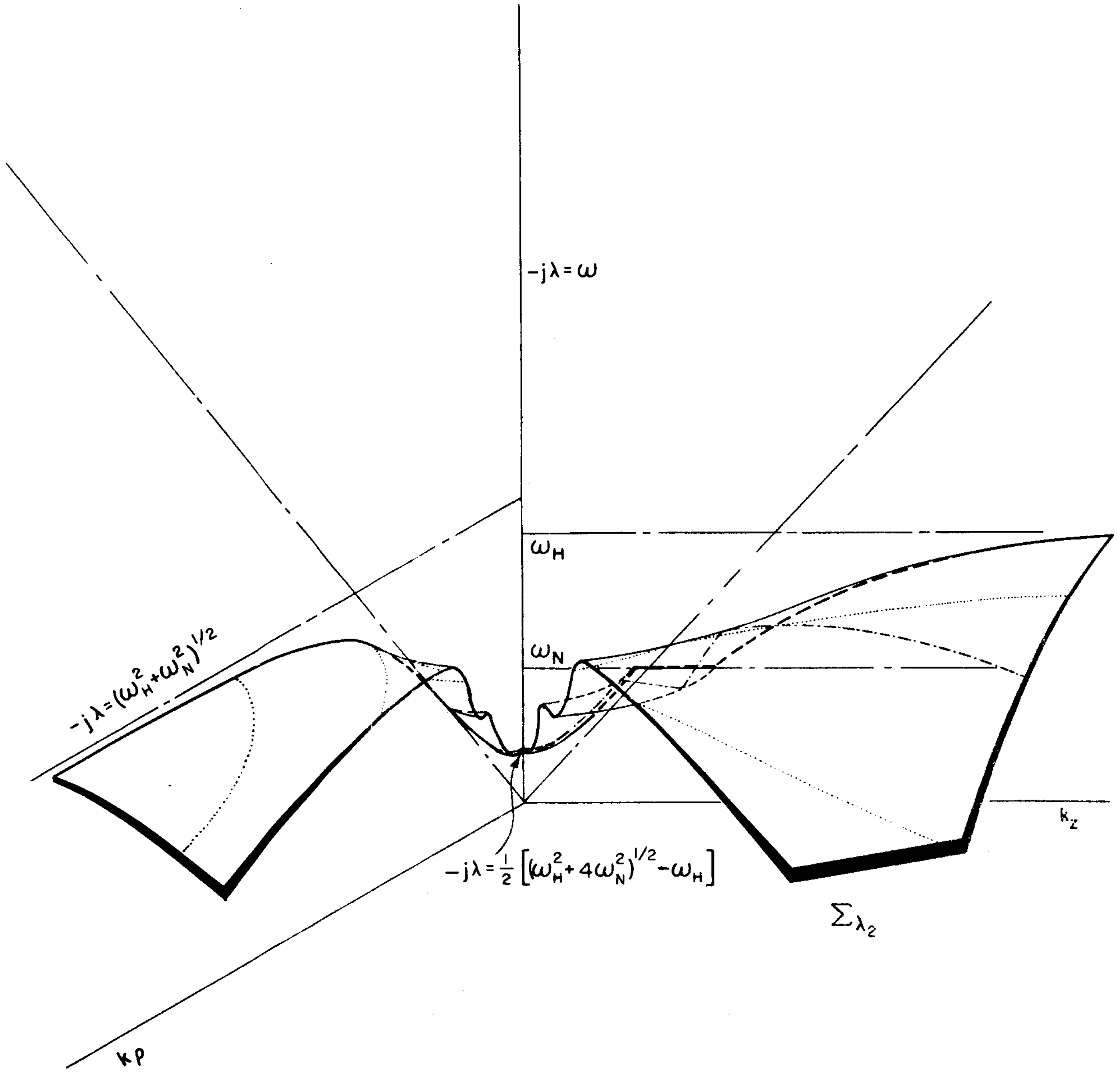

Figure 6.5. One quadrant of sheet Σ_{λ_4} of the dispersion surface for a magneto-ionic medium. $(\omega_H = 2\omega_N)$

while Σ_{λ_2} and Σ_{λ_4} correspond to the "extraordinary" sheets. Also sheets Σ_{λ_1} and Σ_{λ_4} are of left hand polarization while Σ_{λ_3} is of right hand polarization. Furthermore, the portion of Σ_{λ_2} "inside" the light cone ($-j\lambda < \omega_N$) corresponds to right hand polarization while the portion "outside" ($-j\lambda > \omega_N$) corresponds to left hand polarization.

Briefly, let us consider the major differences in the dispersion surfaces of the warm and cold plasmas with one mobile charged particle species. First the warm plasma system matrix can be expressed as a bordered cold plasma system matrix. Hence, the ten warm plasma eigenvalues will alternate with the nine cold plasma eigenvalues. Likewise the sheets of the warm plasma dispersion surface will alternate with the sheets of the cold plasma dispersion surface. The sheets of the warm plasma dispersion surface do not touch at $-j\lambda = \omega_N$ but may touch elsewhere. And the warm plasma sheet corresponding to Σ_{λ_2} is not limited to values of $-j\lambda$ between $\frac{1}{2}\left[(\omega_H^2 + 4\omega_N^2)^{\frac{1}{2}} - \omega_H\right]$ and $(\omega_H^2 + \omega_N^2)^{\frac{1}{2}}$ but extends to infinity in the $-j\lambda$ direction. That is, on this sheet for large k, $-j\lambda$ becomes large.

7. CONCLUSIONS

A phenomenological approach has been used to investigate some properties of linear passive media. Properties of fields that can propagate in linear passive media were postulated and from this properties of the media through the Fourier transform of the constitutive matrix were deduced. The concept of a positive real condition on the constitutive relationship for linear passive media was introduced and some of its implications were considered. Also, the concept of causality, which is more fundamental than the group velocity concept and which is necessary for realizable media, was considered, particularly for the case of isotropic media.

A general formulation of the spectrum of characteristic waves in lossless linear passive media has been made. The fields due to an arbitrary source can be separated into components parallel to the characteristic waves by using an orthogonality condition for the characteristic waves of the medium. The components of the source field are dependent only upon the portion of the source parallel to their characteristic field and to their own sheet(s) of the dispersion surface. The theory has been applied to the problems of longitudinal and transverse electric dipoles in a general time-dispersive uniaxial medium and to an electric dipole in an isotropic compressible plasma with N species of charged particles to obtain exact solutions.

Finally, the radiation field of an arbitrary source in a lossless linear passive medium has been obtained by using the spectral decomposition of the fields and the stationary phase method. It is shown that by normalizing the length of the Total Poynting vector (electromagnetic plus medium) to unity for each characteristic field, a concise and physically interpretable expression for the source fields is obtained. The radiation field for a time harmonic source was found to depend upon the Gaussian radius of curvature, the reaction of the source with the normalized characteristic field, and the characteristic field at each stationary point on the dispersion surface. These results have then been applied to an anisotropic compressible plasma and to a magneto-ionic plasma.

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