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COMPUTATION OF GENERAL PLANETARY PERTURBATIONS

PART III. AN EXPANSION OF THE DISTURBING FORCE

by Lloyd Carpenter

Goddard Space Flight Center
Greenbelt, Md.

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ABSTRACT

An expansion of the planetary disturbing force is given in a form convenient for a method of successive approximations in the application of the method of rectangular coordinates. The difficulties associated with slow convergence along powers of the planetary masses are absorbed in the iteration process.

CONTENTS

COMPUTATION OF GENERAL PLANETARY PERTURBATIONS, PART III. AN EXPANSION OF THE DISTURBING FORCE*

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INTRODUCTION

The present level of interest in general planetary perturbations emphasizes the importance of any developments which have strong computational advantages. One of the main difficulties in the computation of perturbation theories is the slow convergence along powers of the planetary masses. This slow convergence comes from small divisors associated with near resonance or possible close approaches of the planets or relatively large factors from the eccentricities. In such cases the convergence can become arbitrarily slow to the point where a perturbation theory is not appropriate. However, the convergence along powers of the masses is slow in many cases where the total perturbations are not too large. These ideas were borne out by the work of Hill (Reference 1a or 1b) where the higher order terms were much larger than expected. In the method of rectangular coordinates (Reference 2) this difficulty can be reduced by modifying the usual concepts of order and replacing the expansion in powers of the disturbing masses by an expansion in powers of the perturbations themselves.

In terms of the expansion this means that in the disturbing force the orders of terms are based on the powers of the perturbations and the mass factor, but after the integration all terms are treated alike or on the basis of the actual sizes of the coefficients. The equations are integrated by successive approximations since the unknown perturbations appear in the integrands. In the iteration process the start may be based on elliptic motion or any approximate or previous general theory. At any stage the coefficients may be modified to improve the convergence of successive approximations if necessary.

The results may be compared in part with Musen (Reference 3).

*In Part I (Reference 4) of this report, a description of a program for computing Hansen's Planetary Perturbations was given. Part II (Reference 5) contains a comparison of Hansen's components with the rectangular components in Musen's method.

THE BASIC EXPANSION

Take the equation of motion in the form

$$\frac{d^2\vec{r}}{dt^2} = -\mu^2 \frac{\vec{r}}{r^3} + \mu^2 \sum_{i=1}^{N} \frac{m_i}{M+m} \left[\frac{\vec{\rho}_i}{\rho_i^3} - \frac{\vec{r}_i}{r_i^3}\right]$$

The notations are

- $\vec{r}$ the position vector of the disturbed planet with respect to the sun
- $\vec{r}_i$ the position vector of the i^{th} disturbing planet with respect to the sun
- $\vec{\rho}_i$ the position vector of the i^{th} disturbing planet with respect to the disturbed planet ($\vec{\rho}_i = \vec{r}_i - \vec{r}$)
- M the mass of the sun
- m the mass of the planet
- m_i the mass of the i^{th} disturbing planet
- $\mu^2 = k^2 (M+m)$ where k is the Gaussian constant
- t the time.

It is understood that there are N disturbing planets. The equation of motion is decomposed into the equation of undisturbed motion

$$\frac{d^2\vec{r}_0}{dt^2} = -\mu^2 \frac{\vec{r}_0}{r_0^3}$$

plus the variational equation

$$\frac{d^2\vec{s}}{dt^2} + \mu^2\left(\frac{\vec{s}}{r_0^3} - 3\frac{\vec{r}_0 \cdot \vec{s}}{r_0^5}\vec{r}_0\right) = \mu^2 \vec{F}$$

with

$$\vec{F} = \left(-\frac{\vec{r}}{r^3} + \frac{\vec{r}}{r_0^3} - 3\frac{\vec{r}_0 \cdot \vec{s}}{r_0^5}\vec{r}_0\right) + \sum_{i=1}^{N} \frac{m_i}{M+m}\left(\frac{\vec{\rho}_i}{\rho_i^3} - \frac{\vec{r}_i}{r_i^3}\right)$$

and

$$\vec{r} = \vec{r}_0 + \vec{s},$$

where $\vec{r}_0$ is the elliptic motion and $\vec{s}$ is the perturbation. Each of the $\vec{r}_i$ and $\vec{\rho}_i$ are separated in the same way. To obtain a useful form for computing general perturbations, the terms $\vec{r}/r^3$, $\vec{\rho}_i/\rho_i^3$ and $\vec{r}_i/r_i^3$ are expanded in powers of the perturbations. Each of these terms has the same form. To simplify the development the standard binomial expansion is used in place of the elegant symbolic operators used by Musen (References 2 and 3).

Take the general vector $\vec{q}$ in the form

$$\vec{q} = \vec{q}_0 + \vec{\epsilon} \ .$$

Then

$$q^3 = q_0^3 \ [1 + \delta]^{3/2} ,$$

where

$$\delta = 2 \frac{\vec{q}_0 \cdot \vec{\epsilon}}{q_0^2} + \frac{\epsilon^2}{q_0^2} \ , \quad \epsilon = |\vec{\epsilon}| \text{ and } q = |\vec{q}| \ .$$

When δ is small the binomial expansion

$$\frac{1}{q^3} = \frac{1}{q_0^3} \left(1 - \frac{3}{2} \ \delta \left\{1 - \frac{5}{4} \ \delta \left[1 - \frac{7}{6} \ \delta \ \left(1 - \frac{9}{8} \ \delta \ . \ . \ .\right)\right]\right\}\right)$$

or

$$\frac{1}{q^3} = \frac{1}{q_0^3} - \frac{3}{2} \frac{\delta}{q_0^3} + \frac{15}{8} \frac{\delta^2}{q_0^3} - \frac{35}{16} \frac{\delta^3}{q_0^3} + \frac{315}{128} \frac{\delta^4}{q_0^3} + \ . \ . \ .$$

is useful. Substituting the expression for δ,

$$- \frac{\vec{q}}{q^3} = - \frac{\vec{q}_0}{q_0^3} + \left(- \frac{\vec{\epsilon}}{q_0^3} + 3 \frac{\vec{q}_0 \cdot \vec{\epsilon}}{q_0^5} \vec{q}_0\right)$$

$$+ \left[3 \frac{\vec{q}_0 \cdot \vec{\epsilon}}{q_0^5} \vec{\epsilon} + \frac{3}{2} \frac{\epsilon^2}{q_0^5} \vec{q}_0 - \frac{15}{2} \frac{(\vec{q}_0 \cdot \vec{\epsilon})^2}{q_0^7} \vec{q}_0\right]$$

$$+ \left\{\frac{3}{2} \frac{\epsilon^2}{q_0^5} \vec{\epsilon} - \frac{15}{2} \frac{(\vec{q}_0 \cdot \vec{\epsilon})^2}{q_0^7} \vec{\epsilon} - \frac{15}{2} \frac{\vec{q}_0 \cdot \vec{\epsilon} \epsilon^2}{q_0^7} \vec{q}_0 + \frac{35}{2} \frac{(\vec{q}_0 \cdot \vec{\epsilon})^3}{q_0^9} \vec{q}_0\right\}$$

$$+ \left\{\left[- \frac{15}{8} \frac{\epsilon^4}{q_0^7} + \frac{105}{4} \frac{(\vec{q}_0 \cdot \vec{\epsilon})^2 \ \epsilon^2}{q_0^9} - \frac{315}{8} \frac{(\vec{q}_0 \cdot \vec{\epsilon})^4}{q_0^{11}}\right] \vec{q}_0 + \frac{35}{2} \frac{(\vec{q}_0 \cdot \vec{\epsilon})^3}{q_0^9} \vec{\epsilon} - \frac{15}{2} \frac{\vec{q}_0 \cdot \vec{\epsilon} \ \epsilon^2}{q_0^7} \vec{\epsilon}\right\} + \ \cdots \ .$$

The terms have been separated according to the various powers of $\vec{\epsilon}$.

It is convenient to separate the perturbation $\vec{\epsilon}$ into classes of terms. For example in the next section these classes will represent terms factored by various powers of the time, and in the following section they will represent contributions from various disturbing planets. Put

$$\vec{\epsilon} = \vec{\epsilon}_1 + \vec{\epsilon}_2 + \vec{\epsilon}_3 + \cdots$$

and

$$-\frac{\vec{q}}{q^3} = \vec{K}_0 + \sum_i \vec{K}_i + \sum_{i,j} \vec{K}_{i,j} + \sum_{i,j,k} \vec{K}_{i,j,k} + \cdots$$

Each component $\vec{\epsilon}_k$ is treated as if it were of the first order (the same order as $\vec{\epsilon}$). Now

$$\vec{K}_0 = -\frac{\vec{q}_0}{q_0^3} ,$$

$$\vec{K}_i = \left(-\frac{\vec{\epsilon}_i}{q_0^3} + 3\frac{\vec{q}_0 \cdot \vec{\epsilon}_i}{q_0^5}\,\vec{q}_0\right) ,$$

$$\vec{K}_{i,j} = \left[3\frac{\vec{q}_0 \cdot \vec{\epsilon}_i}{q_0^5}\,\vec{\epsilon}_j + \frac{3}{2}\frac{\vec{\epsilon}_i \cdot \vec{\epsilon}_j}{q_0^5}\,\vec{q}_0 - \frac{15}{2}\frac{(\vec{q}_0 \cdot \vec{\epsilon}_i)(\vec{q}_0 \cdot \vec{\epsilon}_j)}{q_0^7}\,\vec{q}_0\right] ,$$

$$\vec{K}_{i,j,k} = \left[\frac{3}{2}\frac{\vec{\epsilon}_i \cdot \vec{\epsilon}_j}{q_0^5}\,\vec{\epsilon}_k - \frac{15}{2}\frac{(\vec{q}_0 \cdot \vec{\epsilon}_i)(\vec{q}_0 \cdot \vec{\epsilon}_j)}{q_0^7}\,\vec{\epsilon}_k - \frac{15}{2}\frac{\vec{q}_0 \cdot \vec{\epsilon}_i\ \vec{\epsilon}_j \cdot \vec{\epsilon}_k}{q_0^7}\,\vec{q}_0 + \frac{35}{2}\frac{(\vec{q}_0 \cdot \vec{\epsilon}_i)(\vec{q}_0 \cdot \vec{\epsilon}_j)(\vec{q}_0 \cdot \vec{\epsilon}_k)}{q_0^9}\,\vec{q}_0\right] ,$$

$$\vec{K}_{i,j,k,l} = \left\{\left[-\frac{15}{8}\frac{\vec{\epsilon}_i \cdot \vec{\epsilon}_j\ \vec{\epsilon}_k \cdot \vec{\epsilon}_l}{q_0^7} + \frac{105}{4}\frac{(\vec{q}_0 \cdot \vec{\epsilon}_i)(\vec{q}_0 \cdot \vec{\epsilon}_j)\,\vec{\epsilon}_k \cdot \vec{\epsilon}_l}{q_0^9} - \frac{315}{8}\frac{(\vec{q}_0 \cdot \vec{\epsilon}_i)(\vec{q}_0 \cdot \vec{\epsilon}_j)(\vec{q}_0 \cdot \vec{\epsilon}_k)(\vec{q}_0 \cdot \vec{\epsilon}_l)}{q_0^{11}}\right]\vec{q}_0\right.$$

$$\left. - \frac{15}{2}\frac{(\vec{q}_0 \cdot \vec{\epsilon}_i)(\vec{\epsilon}_j \cdot \vec{\epsilon}_k)}{q_0^7}\,\vec{\epsilon}_l + \frac{35}{2}\frac{(\vec{q}_0 \cdot \vec{\epsilon}_i)(\vec{q}_0 \cdot \vec{\epsilon}_j)(\vec{q}_0 \cdot \vec{\epsilon}_k)}{q_0^9}\,\vec{\epsilon}_l\right\} ,$$

where i, j, k, and l take on all positive integer values.

This gives the expansion of $-\vec{q}/q^3$ through the fourth order in $\vec{\epsilon}$, and with the proper substitutions all the terms in $\vec{F}$ through the fourth order are obtained. It is clear how these expansions are extended to any order by a straightforward process.

EXPANSION FOR TWO PLANETS

In many cases the major part of the perturbations comes from the mutual attractions of the disturbed planet and one disturbing planet. The motions of Jupiter and Saturn are the prime example, and the motion of a minor planet disturbed by Jupiter is a special case. The actual computation in such cases is greatly simplified by the appearance of only two mean anomalies in the arguments of the trigonometric terms so that numerical double harmonic analysis can be used extensively. The usual series multiplications can be replaced by the more efficient process of evaluating the periodic terms within the double harmonic analysis routine. By holding the powers of the time as factors the operations are performed on functions which are periodic in two variables.

The expansion is actually done only with respect to the secular terms, the periodic terms being accounted for in a direct manner. This feature provides a more complete result from the higher order affects of long period terms. It is necessary to make an extensive accounting for terms factored by rather large powers of the time. All terms through the sixth power of the time are given.

Since there is only one disturbing planet in this case, in the previous section replace m_i by m', $\vec{r}_i$ by $\vec{r}'$, and $\vec{\rho}_i$ by $\vec{\rho}$. Write

$$\vec{r} = \vec{r}_I + \vec{s}_I$$

$$\vec{r}' = \vec{r}'_I + \vec{s}'_I$$

$$\vec{\rho} = \vec{\rho}_I + \vec{\sigma}_I$$

with

$$\vec{\sigma}_I = \vec{s}'_I - \vec{s}_I .$$

Also

$$\vec{s}_I = \vec{s}_1 T + \vec{s}_2 T^2 + \ldots$$

$$\vec{s}'_I = \vec{s}'_1 T + \vec{s}'_2 T^2 + \ldots$$

$$\vec{\sigma}_I = \vec{\sigma}_1 T + \vec{\sigma}_2 T^2 + \ldots .$$

In this form $\vec{r}_I$, $\vec{r}_I'$, $\vec{\rho}_I$, $\vec{s}_i$, $\vec{s}_i'$ and $\vec{\sigma}_i$ are all periodic vector functions of the two mean anomalies g and g'. The expansion of the previous section can now be used by simple substitutions. For example the expansion of $-\vec{r}/r^3$ is obtained by replacing $\vec{q}_0$ by $\vec{r}_I$, and $\vec{\epsilon}_i$ by $\vec{s}_i\,T^i$. In the corresponding expansions of $\vec{\rho}/\rho^3$ and $-\vec{r}'/r'^3$, the terms involving the fourth power of the perturbations can be dropped because of the mass factor $m'/(M+m)$. In the solar terms

$$-\frac{\vec{r}}{r^3} + \frac{\vec{r}}{r_0^3} - 3\frac{\vec{r}_0 \cdot \vec{s}}{r_0^5}\,\vec{r}_0$$

put

$$\vec{s} = \vec{s}_0 + \vec{s}_I\,,$$

$$\vec{r}_I = \vec{r}_0 + \vec{s}_0\,,$$

where $\vec{s}_0$ contains the purely periodic terms in $\vec{s}$. In the quantities $\vec{K}_0$ and $\vec{K}_i$ (previous section) resulting from the expansion of $-\vec{r}/r^3$ in powers of $\vec{s}_I$, it is desirable to perform a further expansion in powers of

$$\delta = 2\frac{\vec{r}_0 \cdot \vec{s}_0}{r_0^2} + \frac{s_0^2}{r_0^2}\,.$$

The resulting quantities, being of second order, are then given with greater numerical accuracy.

Omitting the details of the substitutions, with the disturbing force in the form

$$a^2\vec{F} = a^2\vec{f}_I + a^2\vec{f}_1\,T + a^2\vec{f}_2\,T^2 + \ldots$$

the first term is

$$a^2\vec{f}_I = \frac{3}{2}\frac{a^2\delta}{r_0^3}\left\{1 - \frac{5}{4}\delta\left[1 - \frac{7}{6}\delta\left(1 - \frac{9}{8}\delta\ldots\right)\right]\right\}\vec{s}_0$$

$$+\frac{a^2}{r_0^3}\left\{\frac{3}{2}\frac{s_0^2}{r_0^2} - \frac{15}{8}\delta^2\left[1 - \frac{7}{6}\delta\left(1 - \frac{9}{8}\delta\ldots\right)\right]\right\}\vec{r}_0 + \frac{a^2m'}{M+m}\left[\frac{\vec{\rho}_I}{\rho_I^3} - \frac{\vec{r}_I'}{r_I'^3}\right].$$

For the remaining terms

$$a^2\vec{f}_i = B_i^{***}\,\vec{s}_0 + B_i^{**}\,\vec{r}_0 + B_i^{*}\,\vec{r}_I + D_i^{*}\,\vec{r}_I' + E_i^{*}\,\vec{\rho}_I + \sum_{j=1}^{i}\left[B_i^{(j)}\,\vec{s}_j + D_i^{(j)}\,\vec{s}_j' + E_i^{(j)}\,\vec{\sigma}_j\right],$$

where

$$B_i^{***} = 3\,\frac{a^2\,\vec{r}_I \cdot \vec{s}_i}{r_I^5}\ ,$$

$$B_i^{**} = 3\,\frac{a^2\,\vec{s}_0 \cdot \vec{s}_i}{r_I^5} - \frac{15}{2}\,\frac{a^2\,\vec{r}_0 \cdot \vec{s}_i}{r_0^5}\ \delta\ \left\{1 - \frac{7}{4}\,\delta\ \left[1 - \frac{9}{6}\,\delta\ \left(1 - \frac{11}{8}\,\delta\ .\ .\ .\right)\right]\right\}\ ,$$

$$B_i^{(i)} = \frac{3}{2}\,\frac{a^2\,\delta}{r_0^3}\ \left\{1 - \frac{5}{4}\,\delta\ \left[1 - \frac{7}{6}\,\delta\ \left(1 - \frac{9}{8}\,\delta\ .\ .\ .\right)\right]\right\}\ ,$$

$$D_i^{(i)} = -\,\frac{m'}{M+m}\ \frac{a^2}{r_I'^3}\ ,$$

$$E_i^{(i)} = +\,\frac{m'}{M+m}\ \frac{a^2}{\rho_I^3}\ ,$$

$$B_i^{(i-1)} = +\ 3\,\frac{a^2\ \vec{r}_I \cdot \vec{s}_1}{r_I^5}\ ,$$

$$D_i^{(i-1)} = +\ 3\,\frac{m'}{M+m}\ \frac{a^2\,\vec{r}_I' \cdot \vec{s}_1'}{r_I'^5}\ ,$$

$$E_i^{(i-1)} = -\ 3\,\frac{m'}{M+m}\ \frac{a^2\,\vec{\rho}_I \cdot \vec{\sigma}_1}{\rho_I^5}\ ,$$

$$B_i^{(i-2)} = +\ 3\,\frac{a^2\,\vec{r}_I \cdot \vec{s}_2}{r_I^5} + \frac{3}{2}\,\frac{a^2\,s_1^2}{r_I^5} - \frac{15}{2}\,\frac{a^2\,(\vec{r}_I \cdot \vec{s}_1)^2}{r_I^7}\ ,$$

$$D_i^{(i-2)} = + \frac{m'}{M+m} \left[3 \frac{a^2 \vec{r}_I' \cdot \vec{s}_2'}{r_I'^5} + \frac{3}{2} \frac{a^2 s_1'^2}{r_I'^5} - \frac{15}{2} \frac{a^2 (\vec{r}_I' \cdot \vec{s}_1')^2}{r_I'^7} \right] ,$$

and

$$E_i^{(i-2)} = - \frac{m'}{M+m} \left[3 \frac{a^2 \vec{\rho}_I \cdot \vec{\sigma}_2}{\rho_I^5} + \frac{3}{2} \frac{a^2 \sigma_2^2}{\rho_I^5} - \frac{15}{2} \frac{a^2 (\vec{\rho}_I \cdot \vec{\sigma}_1)^2}{\rho_I^7} \right] .$$

The remaining factors $D_i^{(i-k)}$ and $E_i^{(i-k)}$ may be deduced from $B_i^{(i-k)}$ by inspection.

$$B_i^{(i-3)} = + 3 \frac{a^2 \vec{r}_I \cdot \vec{s}_3}{r_I^5} + 3 \frac{a^2 \vec{s}_1 \cdot \vec{s}_2}{r_I^5} - 15 \frac{a^2 (\vec{r}_I \cdot \vec{s}_1)(\vec{r}_I \cdot \vec{s}_2)}{r_I^7} + \frac{35}{2} \frac{a^2 (\vec{r}_I \cdot \vec{s}_1)^3}{r_I^9} - \frac{15}{2} \frac{(\vec{r}_I \cdot \vec{s}_1) s_1^2}{r_I^7} ,$$

$$B_i^{(i-4)} = + 3 \frac{a^2 \vec{r}_I \cdot \vec{s}_4}{r_I^5} + 3 \frac{a^2 \vec{s}_1 \cdot \vec{s}_3}{r_I^5} - 15 \frac{a^2 (\vec{r}_I \cdot \vec{s}_1)(\vec{r}_I \cdot \vec{s}_3)}{r_I^7} + \frac{3}{2} \frac{a^2 s_2^2}{r_I^5} - \frac{15}{2} \frac{a^2 (\vec{r}_I \cdot \vec{s}_2)^2}{r_I^7}$$

$$+ \frac{105}{2} \frac{a^2 (\vec{r}_I \cdot \vec{s}_1)^2 (\vec{r}_I \cdot \vec{s}_2)}{r_I^9} - \frac{15}{2} \frac{(\vec{r}_I \cdot \vec{s}_2) s_1^2}{r_I^7} - 15 \frac{(\vec{r}_I \cdot \vec{s}_1)(\vec{s}_1 \cdot \vec{s}_2)}{r_I^7} ,$$

$$B_i^{(i-5)} = + 3 \frac{a^2 \vec{r}_I \cdot \vec{s}_5}{r_I^5} + 3 \frac{a^2 \vec{s}_1 \cdot \vec{s}_4}{r_I^5} - 15 \frac{a^2 (\vec{r}_I \cdot \vec{s}_1)(\vec{r}_I \cdot \vec{s}_4)}{r_I^7}$$

$$+ 3 \frac{a^2 \vec{s}_2 \cdot \vec{s}_3}{r_I^5} - 15 \frac{a^2 (\vec{r}_I \cdot \vec{s}_2)(\vec{r}_I \cdot \vec{s}_3)}{r_I^7} + \frac{105}{2} \frac{a^2 (\vec{r}_I \cdot \vec{s}_1)^2 (\vec{r}_I \cdot \vec{s}_3)}{r_I^9} + \frac{105}{2} \frac{a^2 (\vec{r}_I \cdot \vec{s}_1)(\vec{r}_I \cdot \vec{s}_2)^2}{r_I^9}$$

$$- \frac{15}{2} \frac{(\vec{r}_I \cdot \vec{s}_1) s_2^2}{r_I^7} - 15 \frac{(\vec{r}_I \cdot \vec{s}_2)(\vec{s}_1 \cdot \vec{s}_2)}{r_I^7} - \frac{15}{2} \frac{(\vec{r}_I \cdot \vec{s}_3) s_1^2}{r_I^7} - 15 \frac{(\vec{r}_I \cdot \vec{s}_1)(\vec{s}_1 \cdot \vec{s}_3)}{r_I^7} ,$$

$$B_1^* = 0 , \qquad D_1^* = \frac{m'}{M+m} \cdot 3 \frac{a^2 \vec{r}_I' \cdot \vec{s}_1'}{r_I'^5} , \qquad E_1^* = - \frac{m'}{M+m} \, 3 \frac{a^2 \vec{\rho}_I \cdot \vec{\sigma}_1}{\rho_I^5} ,$$

$$B_2^* = + \frac{3}{2} \frac{a^2 s_1^2}{r_I^5} - \frac{15}{2} \frac{a^2 (\vec{r}_I \cdot \vec{s}_1)^2}{r_I^7} ,$$

$$D_2^* = \frac{m'}{M+m} \left\{ 3 \frac{a^2 \vec{r}_I' \cdot \vec{s}_2'}{r_I'^5} + \frac{3}{2} \frac{a^2 s_1'^2}{r_I'^5} - \frac{15}{2} \frac{a^2 (\vec{r}_I' \cdot \vec{s}_1')^2}{r_I'^7} \right\} ,$$

$$E_2^* = -\frac{m'}{M+m}\left\{3\frac{a^2\vec{\rho}_I\cdot\vec{\sigma}_2}{\rho_I^5} + \frac{3}{2}\frac{a^2\sigma_1^2}{\rho_I^5} - \frac{15}{2}\frac{a^2(\vec{\rho}_I\cdot\vec{\sigma}_1)^2}{\rho_I^7}\right\},$$

$$B_3^* = +3\frac{a^2\vec{s}_1\cdot\vec{s}_2}{r_I^5} - 15\frac{a^2\vec{r}_I\cdot\vec{s}_1\,\vec{r}_I\cdot\vec{s}_2}{r_I^7} - \frac{15}{2}\frac{a^2\vec{r}_I\cdot\vec{s}_1 s_1^2}{r_I^7} + \frac{35}{2}\frac{a^2(\vec{r}_I\cdot\vec{s}_1)^3}{r_I^9},$$

and

$$D_3^* = \frac{m'}{M+m}\left\{3\frac{a^2\vec{r}_I'\cdot\vec{s}_3'}{r_I'^5} + 3\frac{a^2\,\vec{s}_1'\cdot\vec{s}_2'}{r_I'^5} - 15\frac{a^2\vec{r}_I'\cdot\vec{s}_1'\vec{r}_I'\cdot\vec{s}_2'}{r_I'^7} - \frac{15}{2}\frac{a^2\vec{r}_I'\cdot\vec{s}_1' s_1'^2}{r_I'^7} + \frac{35}{2}\frac{a^2(\vec{r}_I'\cdot\vec{s}_1')^3}{r_I'^9}\right\}.$$

The remaining factors, D_i^* and E_i^*, can be supplied by inspection from B_i^* taking care not to omit the terms factored by $\vec{s}_i'$ and $\vec{\sigma}_i$.

$$B_4^* = 3\frac{a^2\vec{s}_1\cdot\vec{s}_3}{r_I^5} - 15\frac{a^2(\vec{r}_I\cdot\vec{s}_1)(\vec{r}_I\cdot\vec{s}_3)}{r_I^7} + \frac{3}{2}\frac{a^2 s_2^2}{r_I^5} - \frac{15}{2}\frac{a^2(\vec{r}_I\cdot\vec{s}_2)^2}{r_I^7}$$

$$- 15\frac{a^2\vec{r}_I\cdot\vec{s}_1\vec{s}_1\cdot\vec{s}_2}{r_I^7} - \frac{15}{2}\frac{a^2\vec{r}_I\cdot\vec{s}_2 s_1^2}{r_I^7} + \frac{105}{2}\frac{a^2(\vec{r}_I\cdot\vec{s}_1)^2\,\vec{r}_I\cdot\vec{s}_2}{r_I^9}$$

$$- \frac{15}{8}\frac{a^2 s_1^4}{r_I^7} + \frac{105}{4}\frac{a^2(\vec{r}_I\cdot\vec{s}_1)^2 s_1^2}{r_I^9} - \frac{315}{8}\frac{a^2(\vec{r}_I\cdot\vec{s}_1)^4}{r_I^{11}}.$$

$$B_5^* = 3\frac{a^2\vec{s}_1\cdot\vec{s}_4}{r_I^5} - 15\frac{a^2(\vec{r}_I\cdot\vec{s}_1)(\vec{r}_I\cdot\vec{s}_4)}{r_I^7} + 3\frac{a^2\vec{s}_2\cdot\vec{s}_3}{r_I^5} - 15\frac{a^2(\vec{r}_I\cdot\vec{s}_2)(\vec{r}_I\cdot\vec{s}_3)}{r_I^7}$$

$$- 15\frac{a^2\vec{r}_I\cdot\vec{s}_1\vec{s}_1\cdot\vec{s}_3}{r_I^7} - \frac{15}{2}\frac{a^2\vec{r}_I\cdot\vec{s}_3 s_1^2}{r_I^7} + \frac{105}{2}\frac{a^2(\vec{r}_I\cdot\vec{s}_1)^2\vec{r}_I\cdot\vec{s}_3}{r_I^9}$$

$$- 15\frac{a^2\vec{r}_I\cdot\vec{s}_2\vec{s}_1\cdot\vec{s}_2}{r_I^7} - \frac{15}{2}\frac{a^2\vec{r}_I\cdot\vec{s}_1 s_2^2}{r_I^7} + \frac{105}{2}\frac{a^2(\vec{r}_I\cdot\vec{s}_2)^2\vec{r}_I\cdot\vec{s}_1}{r_I^9}$$

$$- \frac{15}{2}\frac{a^2 s_1^2\vec{s}_1\cdot\vec{s}_2}{r_I^7} + \frac{105}{2}\frac{a^2(\vec{r}_I\cdot\vec{s}_1)^2\vec{s}_1\cdot\vec{s}_2}{r_I^9} + \frac{105}{2}\frac{a^2(\vec{r}_I\cdot\vec{s}_1)(\vec{r}_I\cdot\vec{s}_2)\,s_1^2}{r_I^9}$$

$$- \frac{315}{2}\frac{a^2(\vec{r}_I\cdot\vec{s}_1)^3\vec{r}_I\cdot\vec{s}_2}{r_I^{11}}.$$

Finally,

$$
\begin{aligned}
B_6^* = {} & 3\,\frac{a^2\vec{s}_1\cdot\vec{s}_5}{r_I^5} - 15\,\frac{a^2(\vec{r}_I\cdot\vec{s}_1)(\vec{r}_I\cdot\vec{s}_5)}{r_I^7} + 3\,\frac{a^2\vec{s}_2\cdot\vec{s}_4}{r_I^5} \\
& - 15\,\frac{a^2(\vec{r}_I\cdot\vec{s}_2)(\vec{r}_I\cdot\vec{s}_4)}{r_I^7} + \frac{3}{2}\,\frac{a^2 s_3^2}{r_I^5} - \frac{15}{2}\,\frac{a^2(\vec{r}_I\cdot\vec{s}_3)^2}{r_I^7} \\
& - 15\,\frac{a^2\vec{r}_I\cdot\vec{s}_1\vec{s}_1\cdot\vec{s}_4}{r_I^7} - \frac{15}{2}\,\frac{a^2\vec{r}_I\cdot\vec{s}_4 s_1^2}{r_I^7} + \frac{105}{2}\,\frac{a^2(\vec{r}_I\cdot\vec{s}_1)^2\,\vec{r}_I\cdot\vec{s}_4}{r_I^9} \\
& - 15\,\frac{a^2\vec{r}_I\cdot\vec{s}_1\vec{s}_2\cdot\vec{s}_3}{r_I^7} - 15\,\frac{a^2\vec{r}_I\cdot\vec{s}_2\vec{s}_1\cdot\vec{s}_3}{r_I^7} - 15\,\frac{a^2\vec{r}_I\cdot\vec{s}_3\vec{s}_1\cdot\vec{s}_2}{r_I^7} + 105\,\frac{a^2(\vec{r}_I\cdot\vec{s}_1)\,\vec{r}_I\cdot\vec{s}_2\,\vec{r}_I\cdot\vec{s}_3}{r_I^9} \\
& - \frac{15}{2}\,\frac{a^2\vec{r}_I\cdot\vec{s}_2\, s_2^2}{r_I^7} + \frac{35}{2}\,\frac{a^2(\vec{r}_I\cdot\vec{s}_2)^3}{r_I^9} \\
& - \frac{15}{2}\,\frac{a^2 s_1^2\vec{s}_1\cdot\vec{s}_3}{r_I^7} + \frac{105}{2}\,\frac{a^2(\vec{r}_I\cdot\vec{s}_1)^2\vec{s}_1\cdot\vec{s}_3}{r_I^9} + \frac{105}{2}\,\frac{a^2\vec{r}_I\cdot\vec{s}_1\vec{r}_I\cdot\vec{s}_3\, s_1^2}{r_I^9} - \frac{315}{2}\,\frac{a^2(\vec{r}_I\cdot\vec{s}_1)^3\,\vec{r}_I\cdot\vec{s}_3}{r_I^{11}} \\
& - \frac{15}{4}\,\frac{a^2 s_1^2 s_2^2}{r_I^7} - \frac{15}{2}\,\frac{a^2(\vec{s}_1\cdot\vec{s}_2)^2}{r_I^7} + \frac{105}{4}\,\frac{a^2(\vec{r}_I\cdot\vec{s}_1)^2\, s_2^2}{r_I^9} + \frac{105}{4}\,\frac{a^2(\vec{r}_I\cdot\vec{s}_2)^2\, s_1^2}{r_I^9} \\
& + 105\,\frac{a^2\vec{r}_I\cdot\vec{s}_1\,\vec{r}_I\cdot\vec{s}_2\,\vec{s}_1\cdot\vec{s}_2}{r_I^9} - \frac{945}{4}\,\frac{a^2(\vec{r}_I\cdot\vec{s}_1)^2\,(\vec{r}_I\cdot\vec{s}_2)^2}{r_I^{11}}\,.
\end{aligned}
$$

In forming the D and E factors, the terms involving the fourth power of the perturbations may be dropped since the mass factor will reduce these to the fifth order.

These are the final formulas in the form in which they have been programmed except that the fourth order terms have not yet been included in $a^2\,\vec{f}_5$ and $a^2\,\vec{f}_6$. The importance of the higher powers of the secular terms depends on the time interval for which the results are to be accurate. The fourth and higher order terms make no contribution to $\vec{f}_I$, $\vec{f}_1$ or $\vec{f}_2$ and the fifth order terms will make no contribution to $\vec{f}_3$ so these formulas are complete to all orders. If fifth order terms are added then $\vec{f}_4$ will be complete, etc. In these considerations the order is given by the power of the secular terms which appears, the mass factor $m'/(M+m)$ increasing the order by one. The essential difference from the usual approach is that here all of the terms in the perturbations are taken to be of the first order. It is only assumed that the terms for higher powers of the time decrease so that the process can be terminated at some point, and this assumption is satisfied in the applications with an appropriate time scale.

The expansion was made as if the terms $\vec{s}_0$, $\vec{s}_1$, $\vec{s}_2$, . . . in the perturbations were already known. A method of successive approximations, or iteration, can be used starting with any approximate solution. If at first $\vec{s}$ and $\vec{s}'$ are put equal to zero then all terms in the expansion are zero except f_I and

$$a^2\vec{F} = \frac{a^2 m'}{M+m}\left[\frac{\vec{\rho}_0}{\rho_0^3} - \frac{\vec{r}_0'}{r_0'^3}\right],$$

which is the usual first order expression. Upon integrating, approximate values are obtained for $\vec{s}_0$ and $\vec{s}_1$ (Reference 2). If Musen's method is used, the perturbations can be obtained in the form

$$\vec{s} = \alpha\,\vec{r}_0 + \beta\left(\frac{1}{n}\frac{d\vec{r}_0}{dt}\right) + \gamma\,(a\vec{R}).$$

Then $\vec{s}_0$ is obtained from the periodic terms in α, β and γ, and $\vec{s}_1$ is obtained from the secular terms etc. If β contains large long period terms, it is at this point that the constant term and the term with twice the long period argument in α are adjusted (Reference 5). Letting θ be the long period argument, β contains the terms

$$C\cos\theta + S\sin\theta,$$

and $-\beta^2/2$ contains the terms

$$-\frac{1}{4}(C^2+S^2) - \frac{1}{4}(C^2-S^2)\cos 2\theta - \frac{1}{2}CS\sin 2\theta.$$

When the eccentricity is neglected, these terms give the main differences between α and Hansen's component ν. If these terms from the first order are added to α, the results obtained in the next approximation are much improved, as found from experience. Furthermore, in the higher approximations it is worthwhile to make the corrections corresponding to the changes in the coefficients C and S. In the case of Jupiter and Saturn this process will reduce the number of necessary iterations by about one half. In any particular case the importance of these corrections depends on the size of the long period terms.

The formulas given in this section have been found to be the most convenient for computing general planetary perturbations in rectangular coordinates. Several others have been tried. All the terms in the mutual attractions of two planets are computed using a single basic program with separate subroutines for evaluating the factors of the various powers of the time. The smaller cross-action terms which are functions of three mean anomalies are discussed in the next section.

CROSS-ACTION TERMS

For the mutual attractions of two planets an extensive development of the disturbing force, $\vec{F}$, has been given. Those formulas are sufficient for computing the separate affects of the individual disturbing planets. What remains is to give the cross actions of the separate contributions. In this category, for example, would be the modifications of the perturbations of Jupiter due to Saturn arising from the perturbations of Jupiter and Saturn due to Uranus. In addition, the solar attraction on Jupiter is modified by the interaction of the perturbations of Jupiter due to Saturn with the perturbations of Jupiter due to Uranus. In the higher orders the verbal description of the affects becomes more involved, but the formulas can be derived to any prescribed degree of accuracy by a straightforward and well defined procedure. Either Musen's approach (Reference 3) using symbolic operators or the binomial expansion method may be used. The point here is not to enter into an extensive expansion, but rather to give a form which has been found most suitable for computation.

In the mutual attractions of two planets complete advantage was taken of the powerful method of numerical double harmonic analysis by evaluating the trigonometric terms within the routine. Considering the periodic functions of three variables which occur in the cross action terms, one is not faced with any theoretical difficulty in performing numerical triple harmonic analyses, but rather the difficulty is that the operations cannot be performed within the computer memory. There are too many terms. The job could be done in parts but then there is no apparent advantage over the use of series multiplications.

The method given here takes a third approach wherein full use is made of double harmonic analysis, and the terms containing the third argument are held as factors until the final step.

It will be sufficient at first to give only those cross action terms which are of the second order. The term "disturbed planet" refers to that planet for which the disturbing force is being given. The notations are, for the present purpose,

$\vec{r}_0$	the position vector of the disturbed planet as obtained from the elliptic motion
$\vec{s}_0$	the perturbations of the disturbed planet
m_0	the mass of the disturbed planet
$\vec{r}_i$	the position vector of the i^{th} disturbing planet as obtained from its elliptic elements
$\vec{s}_i$	the perturbations of the i^{th} planet
m_i	the mass of the i^{th} planet
$\vec{\rho}_i$	the position vector of the i^{th} planet with respect to the disturbed planet. $\vec{\rho}_i = \vec{r}_i - \vec{r}_0$
M	the mass of the Sun.

With this notation the disturbed planet is also the zeroth planet. The second order terms in $\vec{F}$ are

$$\vec{F} = 3\,\frac{\vec{r}_0\cdot\vec{s}_0}{r_0^5}\,\vec{s}_0 + \frac{3}{2}\,\frac{\vec{s}_0\cdot\vec{s}_0}{r_0^5}\,\vec{r}_0 - \frac{15}{2}\,\frac{(\vec{r}_0\cdot\vec{s}_0)^2}{r_0^7}\,\vec{r}_0$$

$$+\sum_{i=1}^{N}\frac{m_i}{M+m_0}\left\{-\frac{1}{\rho_i^3}\,\vec{s}_0 + 3\,\frac{\vec{\rho}_i\cdot\vec{s}_0}{\rho_i^5}\,\vec{\rho}_i + \frac{1}{\rho_i^3}\,\vec{s}_i - 3\,\frac{\vec{\rho}_i\cdot\vec{s}_i}{\rho_i^5}\,\vec{\rho}_i - \frac{1}{r_i^3}\,\vec{s}_i + 3\,\frac{\vec{r}_i\cdot\vec{s}_i}{r_i^5}\,\vec{r}_i\right\},$$

where N is the number of disturbing planets. This formula gives all the terms of the second order, whereas only those involving cross actions are needed here. Put

$$\vec{s}_i = \sum_{\substack{j=0\\ j\neq i}}^{N}\vec{s}_{i,j} \qquad i = 0,\ 1,\ 2,\ \ldots,\ N$$

where $\vec{s}_{i,j}$ represents the perturbations of the i^{th} planet due to the j^{th}.

Now substitute these expressions for the perturbations into the formula for $\vec{F}$.

$$\vec{F} = \sum_{i=1}^{N}\left\{3\,\frac{\vec{r}_0\cdot\vec{s}_{0,i}}{r_0^5}\,\vec{s}_{0,i} + \frac{3}{2}\,\frac{s_{0,i}^2}{r_0^5}\,\vec{r}_0 - \frac{15}{2}\,\frac{(\vec{r}_0\cdot\vec{s}_{0,i})^2}{r_0^7}\,\vec{r}_0\right\}$$

$$+\sum_{i=1}^{N}\frac{m_i}{M+m_0}\left\{-\frac{1}{\rho_i^3}\,\vec{s}_{0,i} + 3\,\frac{\vec{\rho}_i\cdot\vec{s}_{0,i}}{\rho_i^5}\,\vec{\rho}_i + \frac{1}{\rho_i^3}\,\vec{s}_{i,0} - 3\,\frac{\vec{\rho}_i\cdot\vec{s}_{i,0}}{\rho_i^5}\,\vec{\rho}_i - \frac{1}{r_j^3}\,\vec{s}_{i,0} + 3\,\frac{\vec{r}_i\cdot\vec{s}_{i,0}}{r_i^5}\,\vec{r}_i\right\}$$

$$+\sum_{\substack{i,j=1\\ i\neq j}}^{N}\left\{3\,\frac{\vec{r}_0\cdot\vec{s}_{0,i}}{r_0^5}\,\vec{s}_{0,j} + \frac{3}{2}\,\frac{\vec{s}_{0,i}\cdot\vec{s}_{0,j}}{r_0^5}\,\vec{r}_0 - \frac{15}{2}\,\frac{(\vec{r}_0\cdot\vec{s}_{0,i})(\vec{r}_0\cdot\vec{s}_{0,j})}{r_0^7}\,\vec{r}_0\right\}$$

$$+\sum_{\substack{i,j=1\\ i\neq j}}^{N}\frac{m_i}{M+m_0}\left\{-\frac{1}{\rho_i^3}\,\vec{s}_{0,j} + 3\,\frac{\vec{\rho}_i\cdot\vec{s}_{0,j}}{\rho_i^5}\,\vec{\rho}_i + \frac{1}{\rho_j^3}\,\vec{s}_{i,j} - 3\,\frac{\vec{\rho}_i\cdot\vec{s}_{i,j}}{\rho_i^5}\,\vec{\rho}_i - \frac{1}{r_i^3}\,\vec{s}_{i,j} + 3\,\frac{\vec{r}_i\cdot\vec{s}_{i,j}}{r_i^5}\,\vec{r}_i\right\}.$$

The terms in the single summations are taken into account in the previous section, and the double summations give the second order cross-action terms which are to be considered. These latter terms are most conveniently separated into classes according to the perturbations by which they are factored. The total is given by

$$\vec{F}_{I} + \vec{F}_{II} + \vec{F}_{III},$$

where

$$\vec{F}_I = \sum_{j=i+1}^{N} \left\{ \sum_{i=1}^{N-1} H_I^{(i)} \right\} \cdot \vec{s}_{0,j} ,$$

$$\vec{F}_{II} = \sum_{\substack{i,j=1 \\ i \neq j}}^{N} H_{II}^{(i)} \cdot \vec{s}_{0,j} ,$$

and

$$\vec{F}_{III} = \sum_{\substack{i,j=1 \\ i \neq j}}^{N} H_{III}^{(i)} \cdot \vec{s}_{i,j} ,$$

the dyadics being given by

$$H_I^{(i)} = 3 \frac{\vec{r}_0 \cdot \vec{s}_{0,i}}{r_0^5} I + 3 \frac{\vec{s}_{0,i} \vec{r}_0}{r_0^5} + 3 \frac{\vec{r}_0 \vec{s}_{0,i}}{r_0^5} - 15 \frac{\vec{r}_0 \cdot \vec{s}_{0,i}}{r_0^7} \vec{r}_0 \vec{r}_0 ,$$

$$H_{II}^{(i)} = \frac{m_i}{M+m_0} \left\{ -\frac{1}{\rho_i^3} I + 3 \frac{\vec{\rho}_i \vec{\rho}_i}{\rho_i^5} \right\} ,$$

and

$$H_{III}^{(i)} = \frac{m_i}{M+m_0} \left\{ \left(\frac{1}{\rho_i^3} - \frac{1}{r_i^3} \right) I - 3 \frac{\vec{\rho}_i \vec{\rho}_i}{\rho_i^5} + 3 \frac{\vec{r}_i \vec{r}_i}{r_i^5} \right\} ,$$

the unit dyadic being denoted by I . The practical value of these forms comes from the fact that the dyadics, H , are independent of the mean anomaly g_j of the j^{th} planet. It is best to carry the development further and consider the components of $\vec{F}$ as they enter the formulas for the perturbations in Musen's method (Reference 2). In dimensionless form with time as the independent variable

$$\alpha = \int (M_1 D + M_2 E) \, d(nt) ,$$

$$\beta = \int (M_4 - 2\alpha) \, d(nt) ,$$

and

$$\gamma = \int M_3 D \, d(nt) ,$$

to which the contributions from the constants of integration are to be added. The integrands are obtained from the formulas:

$$M_1 = \vec{S}_1 \cdot (a^2\vec{F}) \ ,$$

$$M_2 = \vec{S}_2 \cdot (a^2\vec{F}) \ ,$$

$$M_3 = \frac{r_0}{a} \vec{R} \cdot (a^2\vec{F}) \ ,$$

$$M_4 = \int M_2 d(nt) \ ,$$

$$\vec{S}_1 = \vec{P} \cos \epsilon + \vec{Q} \frac{\sin \epsilon}{\sqrt{1 - e^2}} \ ,$$

$$\vec{S}_2 = -\vec{P} \sin \epsilon + \vec{Q} \frac{\cos \epsilon - e}{\sqrt{1 - e^2}} \ ,$$

$$D = \frac{a}{r_0} \left[\sin (\eta - \epsilon) - e \sin \eta + e \sin \epsilon\right] \ ,$$

and

$$E = 2 \frac{a}{r_0} \left[1 - \cos (\eta - \epsilon)\right] \ ,$$

where the notations are

- a the semi-major axis of the reference ellipse
- e the eccentricity of the reference ellipse
- n the mean motion in the reference ellipse
- ϵ the eccentric anomaly in the reference ellipse
- η · the eccentric anomaly which is held constant during the integration and replaced by ϵ afterward
- $\vec{P}, \vec{Q}, \vec{R}$ the standard unit vectors of the orbit.

The perturbations are given by

$$\vec{s}_0 = \alpha_0 \vec{r}_0 + \beta_0 \vec{w} + \gamma_0 \, a\vec{R} \, ,$$

where

$$\vec{w} = \frac{1}{n} \frac{d\vec{r}_0}{dt} \; .$$

For purposes of computation ,

$$\vec{r}_0 = a\vec{P} \, (\cos \epsilon - e) + a \sqrt{1 - e^2} \, \vec{Q} \sin \epsilon$$

$$\vec{w} = \frac{a}{r_0} \left[- a\vec{P} \sin \epsilon + a\sqrt{1 - e^2} \, \vec{Q} \cos \epsilon \right]$$

$$r_0 = a \, (1 - e \cos \epsilon) .$$

With these formulas the contributions of $a^2\vec{F}_I$, $a^2\vec{F}_{II}$, and $a^2\vec{F}_{III}$ to the components M_1 , M_2 and M_3 can be given in a convenient form. Only M_1 and M_2 are given since M_3, which is of lesser importance, can be handled in the same way.

Class I Terms

The perturbations, $\vec{s}_{0,j}$, will be in the form

$$\vec{s}_{0,j} = \alpha_{0,j} \, \vec{r}_0 + \beta_{0,j} \, \vec{w} + \gamma_{0,j} \, a\vec{R} \, .$$

To hold the multiplications by terms involving g_j until the last, put

$$M_1 = K_{1,1} \, \alpha_{0,j} + K_{1,2} \, \beta_{0,j} + K_{1,3} \, \gamma_{0,j}$$

and

$$M_2 = K_{2,1} \, \alpha_{0,j} + K_{2,2} \, \beta_{0,j} + K_{2,3} \, \gamma_{0,j} \, .$$

The factors $K_{\ell,m}$ are given by the formulas

$$K_{1,1} = \vec{L}_1 \cdot \vec{r}_0 \, ,$$

$$K_{1,2} = \vec{L}_1 \cdot \vec{w} \; ,$$

$$K_{1,3} = \vec{L}_1 \cdot a\vec{R},$$

$$K_{2,1} = \vec{L}_2 \cdot \vec{r}_0,$$

$$K_{2,2} = \vec{L}_2 \cdot \vec{w},$$

and

$$K_{2,3} = \vec{L}_2 \cdot a\vec{R},$$

with

$$\vec{L}_1 = \vec{S}_1 \cdot a^2 H_I^{(i)} = a^2 \left\{ 3 \frac{\vec{r}_0 \cdot \vec{s}_{0,i}}{r_0^5} \vec{S}_1 + 3 \frac{\vec{S}_1 \cdot \vec{s}_{0,i}}{r_0^5} \vec{r}_0 + 3 \frac{\vec{S}_1 \cdot \vec{r}_0}{r_0^5} \vec{s}_{0,i} - 15 \frac{\vec{r}_0 \cdot \vec{s}_{0,i} \, \vec{S}_1 \cdot \vec{r}_0}{r_0^7} \vec{r}_0 \right\}$$

and

$$\vec{L}_2 = \vec{S}_2 \cdot a^2 H_I^{(i)} = a^2 \left\{ 3 \frac{\vec{r}_0 \cdot \vec{s}_{0,i}}{r_0^5} \vec{S}_2 + 3 \frac{\vec{S}_2 \cdot \vec{s}_{0,i}}{r_0^5} \vec{r}_0 + 3 \frac{\vec{S}_2 \cdot \vec{r}_0}{r_0^5} \vec{s}_{0,i} - 15 \frac{\vec{r}_0 \cdot \vec{s}_{0,i} \, \vec{S}_2 \cdot \vec{r}_0}{r_0^7} \vec{r}_0 \right\}.$$

Class II Terms

Terms in this class are computed from the same formulas as the terms in Class I except that

$$\vec{L}_1 = \vec{S}_1 \cdot a^2 H_{II}^{(i)} = \frac{a^2 m_i}{M + m_0} \left\{ -\frac{1}{\rho_i^3} \vec{S}_1 + 3 \frac{\vec{S}_1 \cdot \vec{\rho}_i}{\rho_i^5} \vec{\rho}_i \right\}$$

and

$$\vec{L}_2 = \vec{S}_2 \cdot a^2 H_{II}^{(i)} = a^2 \frac{m_i}{M + m_0} \left\{ -\frac{1}{\rho_i^3} \vec{S}_2 + 3 \frac{\vec{S}_2 \cdot \vec{\rho}_i}{\rho_i^5} \vec{\rho}_i \right\} .$$

Class III Terms

The perturbations of the disturbing planets will be in the form

$$\vec{s}_{i,j} = \alpha_{i,j} \vec{r}_i + \beta_{i,j} \vec{w}_i + \gamma_{i,j} a_i \vec{R}_i$$

corresponding to those of the disturbed planet. Now put

$$M_1 = K_{1,1} \alpha_{i,j} + K_{1,2} \beta_{i,j} + K_{1,3} \gamma_{i,j},$$

$$M_2 = K_{2,1}\alpha_{i,j} + K_{2,2}\beta_{i,j} + K_{2,3}\gamma_{i,j} ,$$

$$K_{1,1} = \vec{L}_1 \cdot \vec{r}_i ,$$

$$K_{1,2} = \vec{L}_1 \cdot \vec{w}_i ,$$

$$K_{1,3} = \vec{L}_1 \cdot a_i \vec{R}_i ,$$

$$K_{2,1} = \vec{L}_2 \cdot \vec{r}_i ,$$

$$K_{2,2} = \vec{L}_2 \cdot \vec{w}_i ,$$

and

$$K_{2,3} = \vec{L}_2 \cdot a_i \vec{R}_i ,$$

with

$$\vec{L}_1 = \vec{S}_1 \cdot a^2 H_{III}^{(i)} = a^2 \frac{m_i}{M + m_0} \left\{ \left(\frac{1}{\rho_i^3} - \frac{1}{r_i^3} \right) \vec{S}_1 - 3 \frac{\vec{S}_1 \cdot \vec{\rho}_i}{\rho_i^5} \vec{\rho}_i + 3 \frac{\vec{S}_1 \cdot \vec{r}_i}{r_i^5} \vec{r}_i \right\}$$

and

$$\vec{L}_2 = \vec{S}_2 \cdot a^2 H_{III}^{(i)} = a^2 \frac{m_i}{M + m_0} \left\{ \left(\frac{1}{\rho_i^3} - \frac{1}{r_i^3} \right) \vec{S}_2 - 3 \frac{\vec{S}_2 \cdot \vec{\rho}_i}{\rho_i^5} \vec{\rho}_i + 3 \frac{\vec{S}_2 \cdot \vec{r}_i}{r_i^5} \vec{r}_i \right\} .$$

In each class of terms the factors $K_{\ell,m}$ depend only on the two mean anomalies g_0 and g_i, so these factors can be expanded using double harmonic analysis. After these factors are formed, the coefficients in $\alpha_{0,j}$, $\beta_{0,j}$, $\gamma_{0,j}$ or $\alpha_{i,j}$, $\beta_{i,j}$, $\gamma_{i,j}$ are taken in groups corresponding to fixed multiples of the mean anomaly g_j. The resulting terms in M_1 and M_2 are obtained by series multiplications, and the terms in α and β are computed by the integration procedure and written out. Then the next group of terms is read into the computer. Using this procedure it is possible to work entirely within the computer memory, and this makes the execution much more efficient than would be possible otherwise. In some cases it may be desirable to combine the terms which have been separated into classes I and II, and this can easily be done.

The discussion has been limited to terms of the second order, but it should be clear how the same method can be applied to many of the most important higher order terms by modifying the vectors $\vec{L}_1$ and $\vec{L}_2$ to include powers of $\vec{s}_{0,i}$ and $\vec{s}_{i,0}$ and by taking terms with three or more arguments in the factors.

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