A PROBLEM ON SUMMATION OVER HISTORIES IN QUANTUM MECHANICS *

by

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ABSTRACT

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The transition amplitude corresponding to a Dirac particle is evaluated as a sum over histories by a method based on unitary equivalence properties and the canonical formalism of quantum mechanics.

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1. Introduction

The sum over histories representation for the transition amplitude (or propagator) occurs in Tobocman's approach [5] to quantum mechanics. This approach is based on the canonical formalism of quantum mechanics and represents a generalization of the Feynman path-integral method [4] . A related canonical treatment has also been given by Davies [3] .

The difficulty with these functional integral methods is the actual evaluation of the expression for the sum over histories. Cases which have been explicitly evaluated are those corresponding to the free Schrödinger particle and the harmonic oscillator [2] , to the Dirac field [5] , and to classical wave motion [1] . One of these cases, the Dirac field, involved non-commuting operators in the exponent, and the evaluation of the transition amplitude was accomplished by means of a Hilbert hyperspace approach [5] . Essentially the same operators occur in the transition amplitude for a free Dirac particle, but this case can be treated by a different method based on unitary transformations in Hilbert space.

2. The transition amplitude

Consider a physical system with position coordinate $q(t)$ and canonical momentum $p(t)$. For the sake of simplicity we restrict ourselves to a system with one degree of freedom. The generalization to a system with n degrees of freedom can readily be made. Let $H(q,p)$ be the Hamiltonian for the system. Then the transition amplitude $G(q''t''; q't')$ connecting states at times t' and t''

may be written [5]

$$G(q''t''; q't') = \lim_{n\to\infty} G(q_n t_n ; q_0 t_0) , \tag{1}$$

where

$$G(q_n t_n ; q_0 t_0) = \int \frac{dp_1}{2\pi} dq_1 \frac{dp_2}{2\pi} dq_2 \cdots dq_{n-1} \frac{dp_n}{2\pi}$$
$$\times \exp\Big[i \sum_{r=1}^{n} \{ p_r (q_r - q_{r-1}) - H(q_r, p_r)(t_r - t_{r-1})\}\Big]. \tag{2}$$

Here, the time interval (t', t'') has been divided into a large number of parts

$$t' = t_0 < t_1 < t_2 < \cdots < t_n = t'', \tag{3}$$

and $q_r = q(t_r)$, $p_r = p(\tau_r)$ with $t_{r-1} \leq \tau_r < t_r$, and

$$q_0 = q' , \quad q_n = q''. \tag{4}$$

Equations (1) and (2) give the transition amplitude in the form of a sum over histories. These histories correspond to classical trajectories in that the coordinate and momentum are specified at each instant. However, these are not true classical trajectories since there is no relationship between velocity and momentum.

We may write the transition amplitude in terms of functional integrals as

$$G(q''t''; q't') = \int \delta(q(t)) \int \delta\left(\frac{p(t)}{2\pi}\right) \exp iS_{pq} \quad , \qquad (5)$$

with

$$S_{pq} = \int_{t'}^{t''} \{ p\,dq - H(q,p)\,dt \} . \qquad (6)$$

In (6), the subscripts p,q denote any history of the system specified by two arbitrary functions of time $q(t)$ and $p(t)$ subject to the restrictions

$$q(t') = q' , \quad q(t'') = q'' . \qquad (7)$$

Thus S_{pq} is the action (or, to be precise, Hamilton's principal function) for a history pq . In (5), the integrations, by their definition in (2), mean a sum over all paths in momentum space as well as in coordinate space, with p unrestricted and q subject to the conditions of equation (7).

Tobocman [5] has shown that the expression (2) reduces to the Feynman functional integralin the case when the Hamiltonian is classical in form and quantization is carried out in terms of commutators rather than anti-commutators.

3. Dirac particle

Consider a particle of mass m moving with one degree of freedom according to the Dirac equation

$$H\psi = i\frac{\partial\psi}{\partial t}, \tag{8}$$

with

$$H = -\beta m - \alpha p, \tag{9}$$

where α and β are the Dirac matrices which satisfy the relations

$$\alpha\beta + \beta\alpha = 0, \quad \alpha^2 = I, \quad \beta^2 = I, \tag{10}$$

where I is the unit matrix. The action S_{pq} for this system may then be written as

$$\begin{aligned} S_{pq} &= \int_{t'}^{t''}(p\,dq - H\,dt) \\ &= \sum_{r=1}^{n}\{p_r(q_r - q_{r-1}) + \beta m(t_r - t_{r-1}) + \alpha p_r(t_r - t_{r-1})\} \\ &= \sum_{r=1}^{n} p_r(q_r - q_{r-1}) + \beta b + \alpha a, \end{aligned} \tag{11}$$

where

$$a = \sum_{r=1}^{n} p_r (t_r - t_{r-1}), \quad b = m \sum_{r=1}^{n} (t_r - t_{r-1}) = m(t'' - t'). \tag{12}$$

Hence

$$\begin{aligned} \exp iS_{pq} &= \exp\left\{ i \sum_{r=1}^{n} p_r (q_r - q_{r-1}) + i\beta b + i\alpha a \right\} \\ &= \exp\left\{ i \sum_{r=1}^{n} p_r (q_r - q_{r-1}) \right\} \cdot \exp\left\{ i\beta b + i\alpha a \right\}, \end{aligned} \tag{13}$$

where we have used the fact that the first term commutes with the second and third terms.

Now in order to evaluate the functional integral (5) exactly, we must be able to write the second exponential factor in (13) as a <u>product</u> of exponentials. From the operator identity given in equation (A8) of the Appendix we see that this cannot be done for (13) as it stands.

To get round this difficulty we shall transform the problem into a unitarily equivalent problem.

4. Unitary transformation

Let K be an Hermitian operator and consider the unitary transformation defined by

$$G' = e^{iK} G e^{-iK}. \tag{14}$$

Then it follows that the transition amplitude for the transformed problem is given by

$$G'(q''t'';q't') = \int\delta(q(t))\int\delta\left(\frac{p(t)}{2\pi}\right) \exp iS'_{pq}, \tag{15}$$

where

$$S'_{pq} = \int_{t'}^{t''}\{p\,dq - H'(q,p)\,dt\}, \tag{16}$$

with

$$H' = e^{iK} H e^{-iK}. \tag{17}$$

Equations (15) and (16) provide an alternative to (5) and (6) for the evaluation of transition amplitudes.

5. Transition amplitude for Dirac particle.

For the Dirac particle with Hamiltonian $H = -\beta m - \alpha p$ we choose

$$K = k\beta, \tag{18}$$

where k is some number to be determined later. Then

$$H' = e^{iK} H e^{-iK}$$

$$= e^{ik\beta}(-\beta m - \alpha p)\, e^{-ik\beta}$$

$$= -\beta m - \alpha p \cos 2k + i\alpha\beta\, p \sin 2k\,, \qquad (19)$$

since $\alpha\beta + \beta\alpha = 0$ and $\beta^2 = I$. The corresponding action is given by (16)

$$S'_{pq} = \sum_{r=1}^{n} \{ p_r(q_r - q_{r-1}) + \beta m(t_r - t_{r-1}) + \alpha p_r(t_r - t_{r-1}) \cos 2k$$

$$- i\alpha\beta\, p_r(t_r - t_{r-1}) \sin 2k \}.$$

This may be written as

$$S'_{pq} = \sum_{r=1}^{n} \{ p_r(q_r - q_{r-1}) + \beta m(t_r - t_{r-1}) + \alpha p_r(\lambda_r - \lambda_{r-1})$$

$$- i\alpha\beta\, p_r(\lambda_r - \lambda_{r-1}) \tan b \}\,, \qquad (20)$$

where we have put

$$k = b/2\,, \qquad t \cos b = \lambda, \qquad (21)$$

and where b is the quantity defined in equation (12). It follows that

$$S'_{pq} = \sum_{r=1}^{n} p_r(q_r - q_{r-1}) + \beta b + \alpha c - i\alpha\beta\, c \tan b\,, \qquad (22)$$

where

$$C = \sum_{r=1}^{n} p_r(\lambda_r - \lambda_{r-1}) , \tag{23}$$

and so

$$\exp iS'_{pq} = \exp\left\{ i\sum_{r=1}^{n} p_r(q_r - q_{r-1})\right\} \cdot \exp\left\{ i\beta b + i\alpha C + \alpha\beta C \tan b\right\}. \tag{24}$$

Using the operator identity given in equation (A8) we then have

$$\exp iS'_{pq} = \exp\left\{ i\sum_{r=1}^{n} p_r(q_r - q_{r-1})\right\} \cdot \exp\{i\beta b\} \cdot \exp\{i\alpha\, 2bC \operatorname{cosec} 2b\}$$

$$= \exp\{i\beta b\} \prod_{r=1}^{n} \Big[e^{ip_r(q_r - q_{r-1})} \Big\{ I\cos\left(p_r(\lambda_r - \lambda_{r-1})\, 2b \operatorname{cosec} 2b\right)$$

$$+ i\alpha \sin\left(p_r(\lambda_r - \lambda_{r-1})\, 2b \operatorname{cosec} 2b\right)\Big\}\Big]. \tag{25}$$

The summation over p-histories is now obtained from

$$\int \delta\left(\frac{p(t)}{2\pi}\right) \exp iS'_{pq} = \int \frac{dp_1}{2\pi}\frac{dp_2}{2\pi} \cdots \frac{dp_n}{2\pi} \exp iS'_{pq}$$

$$= e^{i\beta b} \prod_{r=1}^{n} \int_{-\infty}^{\infty} \frac{dp_r}{2\pi} \Big[\tfrac{1}{2}(I+\alpha) \exp\left\{ ip_r\left(q_r - q_{r-1} + (t_r - t_{r-1})\, b \operatorname{cosec} b\right)\right\}$$

$$+ \tfrac{1}{2}(I-\alpha) \exp\left\{ ip_r\left(q_r - q_{r-1} - (t_r - t_{r-1})\, b \operatorname{cosec} b\right)\right\}\Big]$$

$$= e^{i\beta b} \prod_{r=1}^{n} \left\{ \tfrac{1}{2}(I+\alpha)\ \delta(q_r - q_{r-1} + (t_r - t_{r-1})\, b \operatorname{cosec} b) \right.$$

$$\left. + \tfrac{1}{2}(I-\alpha)\ \delta(q_r - q_{r-1} - (t_r - t_{r-1})\, b \operatorname{cosec} b) \right\}, \quad (26)$$

where we have used

$$\int_{-\infty}^{\infty} dp \exp ipq = 2\pi\delta(q) . \quad (27)$$

The summation over histories is completed by integrating (26) over $q_1, q_2, \ldots, q_{n-1}$, so that

$$G'(q''t''; q't') = \int \delta(q(t)) \int \delta\left(\frac{p(t)}{2\pi}\right) \exp iS'_{pq}$$

$$= \int_{-\infty}^{\infty} dq_1 \ldots \int_{-\infty}^{\infty} dq_{n-1}\ e^{i\beta b}$$

$$\times \prod_{r=1}^{n} \left\{ \tfrac{1}{2}(I+\alpha)\ \delta(q_r - q_{r-1} + (t_r - t_{r-1})\, b \operatorname{cosec} b) \right.$$

$$\left. + \tfrac{1}{2}(I-\alpha)\ \delta(q_r - q_{r-1} - (t_r - t_{r-1})\, b \operatorname{cosec} b) \right\} .$$

Performing the integration over q_1 we get

$$\int_{-\infty}^{\infty} dq_1 \left\{ \tfrac{1}{2}(I+\alpha)\ \delta(q_1 - q_0 + (t_1 - t_0)\, b \operatorname{cosec} b) + \tfrac{1}{2}(I-\alpha)\ \delta(q_1 - q_0 - (t_1 - t_0)\, b \operatorname{cosec} b) \right\}$$

$$\times \left\{ \tfrac{1}{2}(I+\alpha)\ \delta(q_2 - q_1 + (t_2 - t_1)\, b \operatorname{cosec} b) + \tfrac{1}{2}(I-\alpha)\ \delta(q_2 - q_1 - (t_2 - t_1)\, b \operatorname{cosec} b) \right\}$$

$$= \tfrac{1}{4}(I+\alpha)^2 \delta(q_2 - q_0 + (t_2 - t_0)\, b \operatorname{cosec} b) + \tfrac{1}{4}(I-\alpha)^2 \delta(q_2 - q_0 - (t_2 - t_0)\, b \operatorname{cosec} b), \quad (28)$$

since

$$(I+\alpha)(I-\alpha) = 0, \tag{29}$$

and

$$\int_{-\infty}^{\infty} dq\, \delta(q+v)\,\delta(w-q) = \delta(v+w) . \tag{30}$$

The integrals over q_2 , q_3 , . . . , q_{n-1} can be evaluated in the same way and we obtain

$$G'(q''t''; q't') = e^{i\beta b}\Big\{ \frac{1}{2^n}(I+\alpha)^n\, \delta(q''-q'+(t''-t')\, b\operatorname{cosec} b) + \frac{1}{2^n}(I-\alpha)^n\, \delta(q''-q'-(t''-t')\, b\operatorname{cosec} b)\Big\} . \tag{31}$$

Using the relations

$$(I\pm\alpha)^n = 2^{n-1}(I\pm\alpha) , \tag{32}$$

and the inverse transformation

$$G = e^{-iK} G' e^{iK} , \tag{33}$$

where

$$K = \tfrac{1}{2}\beta b\,, \tag{34}$$

we have finally

$$G(q''t''; q't') = e^{i\beta b/2}\left\{\tfrac{1}{2}(I+\alpha)\,\delta(q''-q'+(t''-t')\,b\,\mathrm{cosec}\,b)\right.$$
$$\left.+\tfrac{1}{2}(I-\alpha)\,\delta(q''-q'-(t''-t')\,b\,\mathrm{cosec}\,b)\right\} e^{i\beta b/2}, \tag{35}$$

with $b = m(t''-t')$.

This completes the determination of the transition amplitude for a free Dirac particle.

APPENDIX

1. The operator $\exp A \exp B$.

The operator identity

$$\exp A \cdot \exp B = \exp (A+B) \tag{A1}$$

is valid when the operators A and B commute. We wish to derive the corresponding result when A and B do not commute. To do this we define

$$f(x) = e^{Ax} \cdot e^{Bx}, \tag{A2}$$

where x is a real variable. Then

$$\frac{df}{dx} = A e^{Ax} \cdot e^{Bx} + e^{Ax} B e^{Bx}$$

$$= (A + e^{Ax} B e^{-Ax}) f .$$

Integration then gives

$$\exp(Ax) \cdot \exp(Bx) = \exp\left\{ \int_0^x (A + e^{A\lambda} B e^{-A\lambda}) d\lambda \right\} . \tag{A3}$$

For $x = 1$ it follows that

$$\exp A \cdot \exp B = \exp\left\{ \int_0^1 (A + e^{A\lambda} B e^{-A\lambda}) d\lambda \right\} , \tag{A4}$$

which is the required generalization of the identity (A1).

2. Case $AB + BA = 0$.

When the operators A and B anti-commute, that is $AB + BA = 0$, the identity (A4) assumes a simpler form, since in that case

$$e^{A\lambda} B e^{-A\lambda} = e^{2A\lambda} B \;, \tag{A5}$$

and

$$\int_0^1 e^{2A\lambda} B \, d\lambda = (2A)^{-1} \left(e^{2A} - I\right) B \,. \tag{A6}$$

Hence when $AB + BA = 0$ (A4) becomes

$$\exp A \cdot \exp B = \exp\left\{A + (2A)^{-1}\left(e^{2A} - I\right) B\right\} . \tag{A7}$$

We now apply (A7) to the case when

$A = i\beta b$, and $B = i\alpha\, 2bc \operatorname{cosec} 2b$,

where b and c are constants and α and β are the Dirac matrices. It can then be readily shown that

$$\exp(i\beta b) . \exp(i\alpha\, 2bc \operatorname{cosec} 2b) = \exp(i\beta b + i\alpha c + \alpha\beta\, c \tan b) . \tag{A8}$$

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