On The Number of L^2 Solutions of $-y'' + q(t)y = \lambda y$ Where $q(t)$ Is Complex Valued.

by

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Let us consider the differential equation $-y'' + q(t)y = \lambda y$ on an interval with the right end at ∞. It has been shown [1] that when $q(t)$ is real valued; under certain conditions the differential equation does not possess two linearly independent solutions, both square summable toward ∞. ($\int^{\infty} |\cdot|^2 \, dt < \infty$). This is known as the classical limit point case of Hermann Weyl [6], its negation - the existence of two square summable solutions - being the limit circle case. The reason for these names is readily apparent from the theory [6].

While the correspondence between the number of square summable solutions and the limit point - circle cases is not quite so precise when $q(t) = q_1(t) + i\, q_2(t)$, where $q_1(t)$ and $q_2(t)$ are real valued and $q_2(t) \not\equiv 0$, a great many results can be extended. (See [2], [3], [4], and [5]). The purpose of this paper is to show that when $q_2(t) \not\equiv 0$, if $q_1(t)$ satisfies the same conditions stated for $q(t)$ when $q(t)$ is real valued, then the number of square summable solutions toward ∞ can still be restricted in the same way.

Theorem. *Suppose there exists a positive, differentiable function* $M(t)$ *satisfying* $M(t) \geq M_0 > 0$, $q_1(t) \geqq -k_1 M(t)$ *for some constant* k_1, $|M'(t) M^{-3/2}(t)| \leqq k_2$ *for some constant* k_2, $\int^{\infty} M^{-1/2}(t)\, dt = \infty$.

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Then the differential equation $-y'' + (q_1(t) + i\, q_2(t))y = \lambda y$ does not possess two linearly independent solutions, square summable toward ∞.

Proof. It has been shown ([1] or [2]) that if two square summable solutions exist for any value of λ, then two exist for all values of λ. Thus, without loss of generality, we can choose $\lambda = i\nu$ where ν is real.

Let X satisfy $-X'' + (q_1(t) + i\, q_2(t))X = i\nu X$ be square summable toward ∞. We then have

$$\int_c^t \frac{X''\,\overline{X}}{M}\,dt = \int_c^t \frac{q_1|X|^2}{M}\,dt + i\int \frac{[q_2-\nu]|X|^2 dt}{M} .$$

Integrating the left side by parts,

$$\int_o^t \frac{|X'|^2}{M}\,dt = \frac{X'\,\overline{X}}{M}\Big|_c^t + \int_c^t \frac{X'\,\overline{X}\,M'}{M}\,dt - \int_c^t \frac{q_1|X|^2}{M} + $$

$$- i\int_c^t \frac{[q_2-\nu]|X|^2}{M}\,dt .$$

Taking real part of this equation, we have

$$\int_c^t \frac{|X'|^2\,dt}{M} = \frac{1}{2}\,\frac{d}{dt}\,|X|^2\Big|_c^t + \mathrm{Re}\int_c^t \frac{X'\,\overline{X}\,M'}{M}\,dt - \int \frac{q_1|X|^2}{M}\,dt .$$

Using Cauchy's inequality on the second term of the right side.

$$\int_c^t \frac{|X'|^2}{M}dt \le \frac{1}{2}\,\frac{d}{dt}|X|^2\Big|_c^t + \sup\left|\frac{M'}{M^{3/2}}\right|\left(\int_c^t \frac{|X'|^2}{M}dt\right)^{1/2}\left(\int_c^t |X|^2 dt\right)^{1/2} -$$

$$- \int_c^t \frac{q_1|X|^2}{M}\,dt.$$

Letting $H(t) = \int_c^t \frac{|X'|^2}{M}\,dt$,

$$H(t) - \frac{1}{2}\frac{d}{dt}|X|^2\Big|_c^t - \sup_{t\geq c}\left|\frac{M'}{M^{3/2}}\right|\left(\int_c^t |X|^2 dt\right)^{1/2} H(t)^{1/2}$$

$$\leq \quad - \int_c^t \frac{q_1|X|^2}{M}\,dt.$$

Choose c sufficiently large so that

$k_2\left(\int_c^\infty |X|^2\,dt\right)^{1/2} < \frac{1}{2}$. Then $H(t) - \frac{1}{2}\frac{d}{dt}|X|^2(t) - \frac{1}{2}H(t)^{1/2} < k_3$,

for some k_3. If $\lim\limits_{t\to\infty} H(t) = \infty$, then eventually $\frac{d}{dt}|X|^2 > H(t)/2$.

Thus $|X|^2(t)$ is increasing, and $X \notin L^2(0, \infty)$. Thus $H(t)$ is bounded.

Now let θ, $\emptyset$ be solutions square summable toward ∞. Assume $W[\theta, \emptyset] = \theta\,\emptyset' - \emptyset\,\theta' = 1$ for all t. Then

$$\frac{\theta\,\emptyset'}{M^{1/2}} - \frac{\emptyset\,\theta'}{M^{1/2}} = \frac{1}{M^{1/2}}\,.$$

The left is summable by Schwarz's inequality. The right is not, giving us a contradiction.

We are now in a position to begin to characterize the roles of $q_1(t)$ and $q_2(t)$. [2] shows that if $\lim\limits_{t\to\infty} q_2(t) = \gamma$ and $\lim\limits_{t\to\infty} q_2(t) = \delta$, then with the exception of a countable number of eigenvalues, the lines $\lambda = i\,\gamma$ and $\lambda = i\,\delta$ contain the spectrum of the operator L defined by $L\,y = -\,y'' + q(t)y$. On the other hand $q_1(t)$ determines the nature of the spectrum. If two square summable solutions of $-\,y'' + q(t)y = \lambda y$ exist toward ∞, the spectrum on $\lambda = i\,\gamma$ consists of a countable number of eigenvalues.

If only one exists, the spectrum is still unspecified. Similarly if two square summable solutions exist toward $-\infty$, then the spectrum on $\lambda = i\delta$ consists of a countable number of eigenvalues. If only one exists, the spectrum is unspecified. (See [3].)

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