

On The Eigenvalues of a Singular
Nonself-Adjoint Differential Operator
of Second Order

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The nonself-adjoint operator $\mathcal{L}y = y'' + q(x)y$, where $q(x) = q_1(x) + iq_2(x)$, $q_1(x)$ and $q_2(x)$ are real valued, $\lim_{x \rightarrow a} q_2(x) = \delta$, $\lim_{x \rightarrow b} q_2(x) = \gamma$, was considered over an interval (a, b) in $[1]$. From $\mathcal{L}$ a nonself-adjoint operator L was defined in $L^2(a, b)$. The spectrum and adjoint of L were found, and a "spectral resolution" was derived.

If r is an arbitrary point in (a, b) , it was shown that when $\lambda = \mu + i\nu$, $\nu \neq \gamma$, $\mathcal{L}y = \lambda y$ has a solution $\psi(x, \lambda)$ in $L^2(r, b)$, and when $\nu \neq \delta$, $\mathcal{L}y = \lambda y$ has a solution $n(x, \lambda)$ in $L^2(a, r)$. λ is an eigenvalue of L if and only if the Wronskian of $\psi(x, \lambda)$ and $n(x, \lambda)$ is zero. ~~The problem~~ This paper wishes to ~~consider is to~~ characterize these eigenvalues, the zeros of $W[\psi, n]$.

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In what follows every expression is a function of the complex variable λ . In the interest of notational clarity, it has been suppressed.

Let $\theta(x)$ and $\phi(x)$ be the solutions of $\lambda y = \lambda y$ satisfying $\theta(r) = 1$, $\theta'(r) = 0$, $\phi(r) = 0$, $\phi'(r) = -1$. We choose s in $[r, b)$ such that $|q_2(x) - \gamma| < |\gamma - \gamma|/2$ for all x in $[s, b]$ when γ is finite, or such that $|q_2(x) - \gamma| > \epsilon$ for all x in $[s, b]$ for some $\epsilon > 0$ when γ is infinite.

We then define $y_{1b}(x) = \theta'(s)\phi(x) - \phi'(s)\theta(x)$ and $y_{2b} = \theta(s)\phi(x) - \phi(s)\theta(x)$.

In $[s, b]$ a sequence of nested circles $C(\beta)$ is found all containing a limit circle or limit point given by

$$M = \lim_{\beta \rightarrow b} - \frac{y_{1b}(\beta)z_b + y_{1b}'(\beta)}{y_{2b}(\beta)z_b + y_{2b}'(\beta)},$$

where z_b is any real number. $\psi(x)$ is then given by

$$\psi(x) = \theta(x)[- \phi'(s) + M\phi(s)] + \phi(x)[\theta'(s) + M\theta(s)].$$

(See [1] and [2].)

Similarly we choose t in (a, r) such that $|q_2(x) - \delta| < |\gamma - \delta|/2$ for all x in $(a, t]$ when δ is finite, or such that $|q_2(x) - \gamma| > \epsilon$ for all x in $(a, t]$ for some $\epsilon > 0$ when δ is infinite.

We define $y_{1a}(x) = \theta'(t)\phi(x) - \phi'(t)\theta(x)$ and $y_{2a}(x) = -\theta(t)\phi(x) + \phi(t)\theta(x)$.

In $(a, t]$ a sequence of nested circles $C(\alpha)$ is found,

all containing a limit circle or limit point given by

$$-m = \lim_{\alpha \rightarrow a} - \frac{y_{1a}(\alpha)z_a + y_{1a}'(\alpha)}{y_{2a}(\alpha)z_a + y_{2a}'(\alpha)}$$

where z_a is any real number. $n(x)$ is then given by

$$n(x) = -\theta(x)[\phi'(t) + m\phi(t)] + \phi(x)[\theta'(t) + m\theta(t)].$$

(Again see [1] and [2].)

Lemma. Let $D_b(\beta) = y_{2b}(\beta)z_b + y_{2b}'(\beta)$,

$D_a(\alpha) = y_{2a}(\alpha)z_a + y_{2a}'(\alpha)$. Then $W[\psi, n] =$

$$\lim_{\substack{\alpha \rightarrow a \\ \beta \rightarrow b}} \{[\phi(\beta)z_b + \phi'(\beta)][\theta(\alpha)z_a + \theta'(\alpha)]$$

$$- [\theta(\beta)z_b + \theta'(\beta)][\phi(\alpha)z_a + \phi'(\alpha)]\} / D_a(\alpha)D_b(\beta).$$

Proof. We observe that $W[\psi, n]$ is independent of x .

Computing $W[\psi, n]$ at $x = r$ we see

$$W[\psi, n] = [\phi'(s) + M\phi(s)][\theta'(t) + m\theta(t)] -$$

$$- [\theta'(s) + M\theta(s)][\phi'(t) + m\phi(t)].$$

Inserting the expressions for m and M and taking limits completes the proof.

Theorem 1. If $D_a(\alpha)$ and $D_b(\beta)$ approach finite limits as

$\alpha \rightarrow a$ and $\beta \rightarrow b$, then $W[\psi, n] = 0$ if and only if

$$\lim_{\substack{\alpha \rightarrow a \\ \beta \rightarrow b}} \{[\phi(\beta)z_b + \phi'(\beta)][\theta(\alpha)z_a + \theta'(\alpha)]$$

$$- [\theta(\beta)z_b + \theta'(\beta)][\phi(\alpha)z_a + \phi'(\alpha)]\} = 0.$$

Note that this last equation in no way depends upon the points $x = s$ or $s = t$.

In the limit point cases the value of M (or m) is independent of the choice of z_b (or z_a). Only in the limit circle cases does M (or m) vary with the choice of z_b (or z_a).

Theorem 2. If the limit point case holds at a and b ,
then under the conditions of theorem 1

$W[\psi, n] = 0$ if and only if

$$\lim_{\substack{\alpha \rightarrow a \\ \beta \rightarrow b}} [\phi(\beta)\theta(\alpha) - \theta(\beta)\phi(\alpha)] = 0,$$

$$\lim_{\substack{\alpha \rightarrow a \\ \beta \rightarrow b}} [\phi(\beta)\theta'(\alpha) - \theta(\beta)\phi'(\alpha)] = 0,$$

$$\lim_{\substack{\alpha \rightarrow a \\ \beta \rightarrow b}} [\phi'(\beta)\theta(\alpha) - \theta'(\beta)\phi(\alpha)] = 0,$$

$$\lim_{\substack{\alpha \rightarrow a \\ \beta \rightarrow b}} [\phi'(\beta)\theta'(\alpha) - \theta'(\beta)\phi'(\alpha)] = 0.$$

Proof. Multiply the expression in theorem one out and collect the coefficients of $z_a z_b$, z_a and z_b together. Since z_a and z_b are arbitrary the result follows.

Theorem 3. If the limit point case holds at b , and the limit circle case holds at a , then under the conditions of theorem 1, $W[\psi, n] = 0$ if and only if

$$\lim_{\substack{\alpha \rightarrow a \\ \beta \rightarrow b}} \{z_a [\phi(\beta)\theta(\alpha) - \theta(\beta)\phi(\alpha)] + [\phi(\beta)\theta'(\alpha) - \theta(\beta)\phi'(\alpha)]\} = 0,$$

$$\lim_{\substack{\alpha \rightarrow a \\ \beta \rightarrow b}} \{z_a [\phi'(\beta)\theta(\alpha) - \theta'(\beta)\phi(\alpha)] + [\phi'(\beta)\theta'(\alpha) - \theta'(\beta)\phi'(\alpha)]\} = 0.$$

A similar statement is valid if the limit circle case holds at b , and the limit point case holds at a .

These results are valid only in nonself-adjoint problems. They are not applicable in the self-adjoint case, since, in that instance, all eigenvalues lie on the real axis. There the existence of ψ and η cannot be established in general.

References

- [1] Allan M. Krall, "On the expansion problem for nonself-adjoint ordinary differential operators of second order," submitted for publication.
- [2] _____, "On nonself-adjoint ordinary differential operators of second order," Doklady Akademii Nauk, to appear.