THE UNIVERSITY OF MICHIGAN

COLLEGE OF ENGINEERING
Department of Engineering Mechanics
Tire and Suspension Systems Research Group

Technical Report No. 3

THE CONTACT OF CERTAIN ELASTIC SHELLS WITH RIGID FLAT SURFACES

Delmar H. Robbins

ORA Project 05608

under contract with:

NATIONAL AERONAUTICS AND SPACE ADMINISTRATION
GRANT NO. NsG-344
WASHINGTON, D.C.

administered through:

OFFICE OF RESEARCH ADMINISTRATION ANN ARBOR

September 1965

This report was also a dissertation submitted in partial fulfillment of the requirements for the degree of Doctor of Philosophy in The University of Michigan, 1965.

The work described in this report was sponsored by

THE NATIONAL AERONAUTICS AND SPACE ADMINISTRATION

under Grant NsG-344.

TABLE OF CONTENTS

TABLE OF CONTENTS (Concluded)

LIST OF TABLES

LIST OF TABLES (Concluded)

LIST OF FIGURES

ABSTRACT

An interesting and little discussed class of problems in the theory of beams, plates, and shells is suggested by the contact of a pneumatic tire with a roadway. It is a purpose of the present discussion to consider in detail the various stresses and deformations which occur in certain simple symmetric shells which are forced into contact with rigid flat surfaces. It is hoped that detailed solutions of relatively simple problems will aid in laying the foundation on which may be based discussions of more complicated problems.

Shear deformation has been included in the governing equations of linear, small deformation shell theory under the premise that the rapid changes of curvature which may occur near the contact region might lead to significant shear stresses and strains. It is assumed that surface normal stresses are applied in the contact region while surface shear stresses are neglected.

It has been shown that a possible means for analyzing such problems involves the solving of the system equations separately in the free region and in the contact region. The two solutions are then matched at the common boundary at the edge of the contact region.

This method has been applied to contact problems involving half-rings and spherical caps. Numerical results show that the effect of shear deformation should not be neglected when the radius-to-thickness ratio is 10 or less.

Experiments performed on rubber half-rings tended to verify the shear-deformation analysis. An exception is the prediction of the normal stress distribution in the contact region. Although the shear-deformation analysis offers a better approximation than the classical bending theory, it may be surmised from the results given here that transverse normal stresses and strains should be included for accurate prediction of this quantity.

CHAPTER I

INTRODUCTION

An interesting class of problems in the theory of beams, plates, and shells is suggested by the contact of a pneumatic tire with a roadway. Included in this class might be static and dynamic problems where some portion of the body being considered is forced to come into contact with another rigid or elastic surface. Thus a displacement condition is imposed on the contact region of the body and known normal stresses act on the remainder of the surface.

Only a few discussions on this subject are known to the current author. Basically two types of analytical approaches have been used in solving these problems. The most common approach has been to assume the Kirchhoff hypothesis of classical beam, plate, and shell theory. This leads to a shear stress resultant discontinuity at the boundary between the contact region of the body and the portion of the body which does not remain in contact. The other approach has been to include the effect of shear deformation under the premise that the rapid changes of curvature which may occur near the contact region might lead to significant shear strains and stresses.

Theories which do not include the effects of shear deformation have been used in many discussions. Most recently Wu and Plunkett[1] have considered the large deformation of two thin circular rings which are forced into contact by the moving together of two rigid surfaces. They indicated that the assumption of perfect slip between the contacting surfaces might be inaccurate since the major portion of the applied normal load is provided by the shear force discontinuity at the edge of the contact region. Gudramovich and Mossakovskii[2] have solved a problem involving an elastic ring which is loaded by a surface tangential stress applied sinusoidally along the circumference and, by this means, pressed into contact with a segment of a circular arc. Several bending problems involving straight beams which are forced into contact with rigid surfaces are discussed by Timoshenko.[3] In Ref. 4 he solves two problems involving circular plates.

The contact of pneumatic tires with a rigid surface has been the subject of several reports by S. K. Clark and associates. In Ref. 5 the Theorem of Minimum Potential Energy is used to determine the deformations and stresses in a tire statically pressed against a rigid flat surface. The length of the contact region is determined by applying the condition that the normal pressure must be zero at the edge of the region. An analog for a tire is an elastic ring on a weak elastic foundation.[6,7] In these analyses the shear stress resultant is again considered discontinuous.

Whether an approximation to the shear discontinuity is actually developed at the edge of a contact region is an open question at the present time. Mossakovskii was unsuccessful in obtaining detailed information using a photoelastic "rigid" surface. Some pressure measurements on automobile tires indicate no increase of pressure at the outer periphery of the contact region, while others show this effect. However there is some evidence that the highest pressures in the contact region of an aircraft tire are developed near the edge of contact. This may be due to the large deflections and incipient buckling. In the case of soft half-rings (see Figure 17), measurements indicate that peak pressure occurs at the center of contact. There is no evidence of the concentrated force. The magnitudes of quantities such as the ratio of radius to thickness, the elastic modulus, and the deformations may strongly influence the shape of the pressure distribution in the contact region.

Essenburg[8] uses Reissner plate theory to solve a problem involving circular plates. There is no shear discontinuity using this formulation. A discontinuity arises in the normal surface pressure which rises to a peak value at the edge of the contact region and immediately drops to zero outside the region.

The purposes of the current thesis may be stated as follows:

1. Development of a technique for applying shell theory including shear deformation to static contact problems involving simple rings and shells;

2. solution of some representative problems using this technique; and

3. experimentation designed to examine some of the important load and deformation quantities arising in the contact of half-rings with a rigid flat surface.

CHAPTER II

GENERAL THEORY

A. INTRODUCTION

The purpose of this chapter is to develop the general theory for the remainder of the thesis. To make the material relatively self-contained, the first section includes a listing of Naghdi's equations[9,10] for a linear shell theory which includes the effects of transverse shear deformation. In the next two sections a calculating procedure is developed for solving problems involving the contact of particular symmetric shells with rigid flat surfaces. The chapter is concluded with a discussion of boundary conditions occurring in such problems and comments concerning extension of the technique.

B. EQUATIONS OF LINEAR SHELL THEORY INCLUDING TRANSVERSE SHEAR DEFORMATION

If α and Θ are coordinates of a point on the middle surface of a shell and ζ is the distance measured along an outward normal to the middle surface, the square of the linear element of the resulting curvilinear orthogonal coordinate system is

$$ds^2 = A^2\left(1 + \frac{\zeta}{R_\alpha}\right)^2 d\alpha^2 + B^2\left(1 + \frac{\zeta}{R_\theta}\right)^2 + d\zeta^2 \qquad (2.1)$$

where R_α and R_Θ are the principal radii of curvature of the middle surface. For an axisymmetric shell, the metric coefficients A and B may be written as

$$A = R_\alpha(\alpha) \quad , \quad B = R_\theta(\alpha)\sin\alpha = r(\alpha) \qquad (2.2)$$

In surface of revolution coordinates the equilibrium equations of Ref. 9 reduce to the following three equations:

$$\frac{\partial(r N_\alpha)}{\partial \alpha} - N_\theta \frac{\partial r}{\partial \alpha} + r V = 0$$

$$\frac{\partial(r V)}{\partial \alpha} - R_\alpha\left(\frac{N_\alpha}{R_\alpha} + \frac{N_\theta}{R_\theta}\right) + r R_\alpha q^+ = 0$$

$$\frac{\partial(r M_\alpha)}{\partial \alpha} - r R_\alpha V = 0 \qquad (2.3)$$

where N_α and N_θ are the normal stress resultants in the α and θ directions; V_α is the shear stress resultant due to the transverse shear stress $\tau_{\alpha\zeta}$; and M_α is one of the stress couples. It should be noted that the non-zero stress couple M_θ does not appear in these equations. Quantities such as $N_{\alpha\theta}$, $N_{\theta\alpha}$, $M_{\alpha\theta}$, $M_{\theta\alpha}$, and V_θ are zero due to the symmetry of the problem.

The normal displacement W, which will be measured positive along an outward normal to the middle surface, and the tangential displacement U_α, which will be measured positive in the α-direction, are assumed to be of the form

$$U_\alpha = u + \zeta\beta \quad , \quad W = w \qquad (2.4)$$

where u and w are the components of middle surface displacements and β is the change of slope of a line which is initially normal to the middle surface. In terms of these quantities the middle surface strains become

$$\epsilon_\alpha^o = \frac{1}{R_\alpha}\left(\frac{\partial u}{\partial \alpha} + w\right) \quad , \quad \epsilon_\theta^o = \frac{u}{rR_\alpha}\frac{\partial r}{\partial \alpha} + \frac{w}{R_\theta}$$

$$K_\alpha = \frac{1}{R_\alpha}\frac{\partial \beta}{\partial \alpha} \quad , \quad K_\theta = \frac{\beta}{rR_\alpha}\frac{\partial r}{\partial \alpha}$$

$$\gamma_{\alpha\zeta}^o = \frac{1}{R_\alpha}\frac{\partial w}{\partial \alpha} - \left(\frac{u}{R_\alpha} - \beta\right) \qquad (2.5)$$

Finally the relations between stress resultants and midsurface strains, as derived from a shell form of a variational theorem due to E. Reissner,[9,11] are written as

$$N_\alpha = \left[(\epsilon_\alpha^o + \nu\epsilon_\theta^o) - \frac{h^2}{12}\left(\frac{1}{R_\alpha} - \frac{1}{R_\theta}\right)K_\alpha\right]K$$

$$N_\theta = \left[(\epsilon_\theta^o + \nu\epsilon_\alpha^o) - \frac{h^2}{12}\left(\frac{1}{R_\theta} - \frac{1}{R_\alpha}\right)K_\theta\right]K$$

$$M_\alpha = \left[(K_\alpha + \nu K_\theta) - \left(\frac{1}{R_\alpha} - \frac{1}{R_\theta}\right)\epsilon_\alpha^o\right]D \qquad (2.6)$$

$$M_\theta = \left[(K_\theta + \nu K_\alpha) - \left(\frac{1}{R_\theta} - \frac{1}{R_\alpha}\right)\epsilon_\theta^o\right]D$$

$$V_\alpha = \frac{5Gh}{6}\gamma_{\alpha\zeta}^o$$

where
$$K = \frac{Eh}{1-\nu^2} \quad , \quad D = \frac{Eh^3}{12(1-\nu^2)}$$

When shear deformation is neglected in an analysis, certain of the Equations (2.4), (2.5), and (2.6) must be rewritten as

$$U_\alpha = u \quad , \quad \gamma_{\alpha\zeta}^o = 0 \qquad (2.7)$$

The last of (2.6) is deleted indicating that V_α is determined completely by equilibrium. The rotation β must now be specified in terms of u and w and may be written

$$\beta = \frac{1}{R_\alpha}\left(u - \frac{\partial w}{\partial \alpha} \right) \tag{2.8}$$

C. THE SYSTEMS OF EQUATIONS FOR THE ANALYSIS OF A CONTACT PROBLEM

Figure 1 shows the geometry of the problems which are to be considered. The shell, which has its edge rigidly fixed at

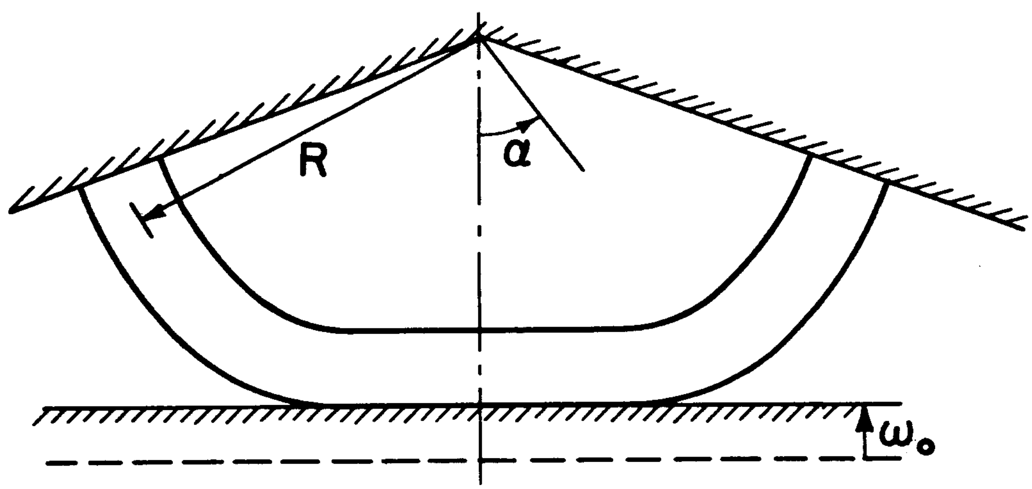

Figure 1. Contact of a circular ring on sphere with a rigid flat surface.

$\alpha = \alpha_e$, is symmetric about $\alpha = 0$ and possesses a constant undeformed radius R. This restricts the problems which may be solved to contact of a segment of a sphere or contact of a segment of a ring beam. A rigid flat frictionless surface, initially in contact only at $\alpha = 0$, has been moved upward a distance w_0. Some length of the beam's surface bounded by $\bar{\alpha}$ is now assumed to be in contact. It is clear that a displacement downward of the edge $\alpha = \alpha_e$ of the shell constitutes the same problem.

The calculating procedure which will be used to solve these problems consists of three parts. The first two parts are completed with the solution of systems of equations in the contact region and the free region. The third part involves satisfaction of fixity conditions at the shell's edge $\alpha = \alpha_e$, examination of symmetry requirements at $\alpha = 0$, and matching of the solutions at the edge of the contact region $\alpha = \bar{\alpha}$.

In the free region the system of equations to be solved consists of (2.3), (2.5), and (2.6). As the surface normal stress is assumed to be zero, $q^+ = 0$ in the second of (2.3). The resulting equations form a linear ordinary sixth-order system.

In the contact region the system will be slightly more complicated because of the non-zero normal surface stress q^+ which appears in the second of (2.3). This new unknown may be balanced by a restraint equation which arises because points on the undeformed surface of the shell must move to points on the rigid surface of contact. This equation, which is discussed in detail in the next section, is a relation between the three displacements u, w, and β. The resulting system of equations, consisting of (2.3), (2.5), (2.6), and the restraint Equation (2.11) is of the fourth order.

D. THE RESTRAINT EQUATION FOR THE CONTACT REGION

When the rigid surface is moved through the distance w_0, a point A on the surface of the initially undeformed shell will move to a point A' on the rigid surface. The accompanying displacements u, w, and β

are shown in the exaggerated drawing, Figure 2. From geometry the motion of A, as related to the motion w_0 of the plane, may be written

$$w_0 - \left(R + \frac{h}{2}\right)(1 - \cos\alpha)$$

$$= \frac{h}{2}\cos\alpha - w\cos\alpha + u\sin\alpha - \frac{h}{2}\cos(\beta+\alpha) \qquad (2.9)$$

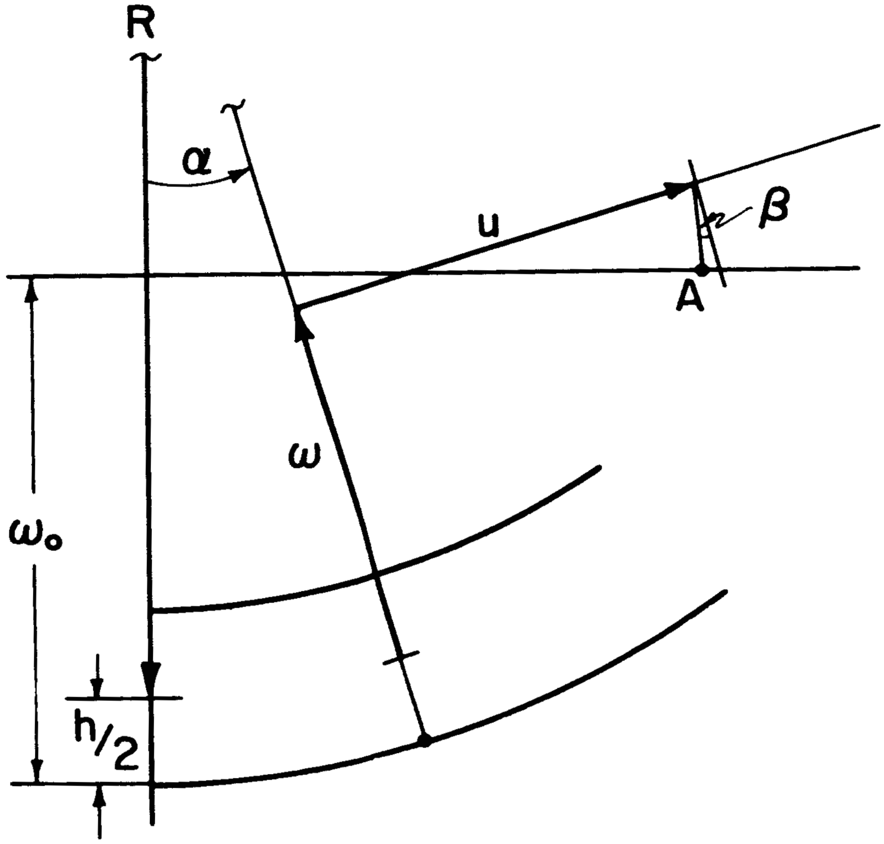

Figure 2. Motion of a point on the surface of the shell to a point on the contact plane.

where h is the thickness of the shell and w is a negative displacement. If w or u had been assumed in the opposite direction for some other A', they would have yielded the same geometrical relationship. This equation may be made dimensionless by dividing through by R giving

$$\bar{w}_o - \left(1 + \frac{h}{2R}\right)\left(1 - \cos\alpha\right)$$
$$= \frac{h}{2R}\cos\alpha - \bar{w}\cos\alpha + \bar{u}\sin\alpha - \frac{h}{2R}\cos(\beta+\alpha) \tag{2.10}$$

where

$$\bar{w}_o = \frac{w_o}{R} \quad , \quad \bar{w} = \frac{w}{R} \quad , \quad \bar{u} = \frac{u}{R} \tag{2.11}$$

Equation (2.10) will be linearized for the present work under the assumptions of small deformation linear shell theory and the small contact length defined by $\bar{\alpha}$. Terms will be ordered as follows: $\cos\alpha$ and $\cos\beta$ are Order 1; $\bar{w}_O$, $h/2R$, $\bar{u}$, $\sin\alpha$, $\sin\beta$, α, and β are Order 2; products of first order terms are Order 3; etc. An expansion of (2.10) gives

$$\bar{w}_o - 1 + \cos\alpha - \frac{h}{2R} + \frac{h}{2R}\cos\alpha$$
$$= \frac{h}{2R}\cos\alpha - \bar{w}\cos\alpha + \bar{u}\sin\alpha$$
$$- \frac{h}{2R}\left(\cos\alpha\cos\beta - \sin\alpha\sin\beta\right) \tag{2.12}$$

When the series expansions of the sine and cosine functions are used where appropriate and if terms of third order or higher are neglected, the restraint equation may be simplified to

$$\bar{w}_o - \left(1 - \cos\alpha\right) = -\bar{w}\cos\alpha + \bar{u}\sin\alpha \tag{2.13}$$

It is felt that this expression is consistent with Naghdi's shear deformation theory in that h/2R and terms of similar order are not neglected with respect to 1.

E. CONDITIONS AT THE BOUNDARIES

The edge $\alpha = \alpha_e$ is assumed to be fixed forcing the requirement that

$$W(\alpha_e) = 0 \quad , \quad U(\alpha_e) = 0 \tag{2.14}$$

which implies that

$$\bar{w}(\alpha_e) = 0 \quad , \quad \bar{u}(\alpha_e) = 0, \quad \beta(\alpha_e) = 0 \tag{2.15}$$

As the system of equations for the free region is of the sixth order, only three of the unknown coefficients in the general solution remain after application of these conditions.

As a result of symmetry and the imposed motion w_0, the conditions which must be satisfied in the contact region at $\alpha = 0$ are

$$\bar{w}(0) = -\bar{w}_0 \quad , \quad \beta(0) = 0 \tag{2.16}$$

If the first of these relations is substituted into the linear restraint Equation (2.13), it is apparent that

$$\bar{u}(0) = 0 \tag{2.17}$$

follows as a result. When (2.16) and (2.17) are satisfied, it follows that the change in slop of $\bar{w}$ must vanish, hence

$$\left.\frac{\partial \bar{w}}{\partial \alpha}\right|_{\alpha = 0} = 0 \tag{2.18}$$

If all these conditions are applied to the last of (2.6), the quantity V must vanish indicating that

(2.19)

Thus it is seen that only two of the five boundary conditions at $\alpha = 0$ are independent. For the two problems which are solved in the following chapters two coefficients remain in the general solution for the contact region after these boundary conditions are applied to the fourth order system.

At the boundary $\alpha = \bar{\alpha}$ six stress resultant and displacement quantities may be matched in order to solve for the six unknown coefficients consisting of three from the free region, two from the contact region, and either $\bar{\alpha}$ or $\bar{w}_0$. The matching of $\bar{u}$, $\bar{w}$, β, N_α, M_α, and V_α effectively establishes the continuity of the displacements and their first derivatives. It follows that the quantities N_θ and M_θ will also be continuous at $\alpha = \bar{\alpha}$. A discontinuity occurs in the normal pressure q^+ as it involves a higher order derivative of displacement.

F. COMMENTS ON THE EXTENSION OF THE CURRENT TECHNIQUE TO GENERAL AXISYMMETRIC PROBLEMS

Conceptually it would not be difficult to extend the current work to the contact of an axisymmetric shell with a curved axisymmetric surface. The equilibrium Equations (2.3) and the stress resultant-displacement Equations (2.5) and (2.6) remain unchanged. A more general restraint equation would be necessary. The order of the system remains the same. The only possible change might occur in the number of quantities which could be matched at $\alpha = \bar{\alpha}$. It would be necessary to examine a general restraint equation to determine whether two or less unknown coefficients would remain in the general solution for the contact region after application of the boundary and symmetry conditions at $\alpha = 0$.

CHAPTER III

SOLUTION OF THE HALF-RING PROBLEM

A. THE SYSTEM EQUATIONS

The ring-beam problem which will be considered in detail involves the contact of a half-ring with a rigid flat surface. A half-ring is defined as a segment of a ring having its fixed edge at $\alpha_e = \pi/2$. This problem will be treated as a plane problem. Thus deformations and stresses in a direction normal to the α-ζ plane will not be considered. In cylindrical coordinates the squared linear element is

$$ds^2 = R^2\left(1+\frac{\zeta}{R}\right)^2 d\alpha^2 + dz^2 + d\zeta^2 \tag{3.1}$$

where z is the direction normal to the α-ζ plane. This corresponds to choosing

$$r = 1 \; , \quad R_\alpha = R \; , \quad R_\theta = \infty \tag{3.2}$$

in surface of revolution coordinates. The equilibrium Equations (2.3) become

$$\begin{aligned} \dot{N} + V &= 0 \\ \dot{V} - N + R\,q^+ &= 0 \\ \dot{M} - RV &= 0 \end{aligned} \tag{3.3}$$

where "$\cdot$" indicates differentiation with respect to α. It should be noted that the subscript α has been dropped from the stress resultants. Using similar conventions the strains are

$$\epsilon_\alpha^0 = \frac{1}{R}(\dot{u} + w) \quad , \quad \epsilon_z^0 = 0$$

$$K_\alpha = \frac{1}{R}\dot{\beta} \quad , \quad K_z = 0$$

$$\gamma_{\alpha\zeta}^0 = \frac{1}{R}\dot{w} - \left(\frac{u}{R} - \beta\right) \tag{3.4}$$

Substitution of these into the Equations (2.6) results in

$$\begin{aligned} N &= \frac{K}{R}\left(\dot{u} + w - \frac{h^2}{12R}\dot{\beta}\right) \\ M &= \frac{D}{R}\left[\dot{\beta} - \frac{1}{R}(\dot{u} + w)\right] \\ V &= \frac{5Gh}{6}\left(\frac{1}{R}\dot{w} - \frac{u}{R} + \beta\right) \end{aligned} \tag{3.5}$$

The restraint Equation (2.13) remains unchanged as does the Equation (2.8) for the rotation β which results when shear-deformation is neglected.

B. SOLUTION IN THE FREE REGION

1. Solution Including Transverse Shear Deformation

Because $q^+ = 0$ in the free region the equilibrium equations are easily solved. The second of (3.3) is substituted into the first yielding

$$\ddot{N} + N = 0 \tag{3.6}$$

the solution of which is

$$N = B_1 \cos\alpha + B_2 \sin\alpha. \tag{3.7}$$

The shear resultant may be written

$$V = B_1 \sin\alpha - B_2 \cos\alpha. \tag{3.8}$$

The first integral of the moment equation is

$$M = -RB_1 \cos\alpha - RB_2 \sin\alpha + B_3. \tag{3.9}$$

If the first two equations of (3.5) are solved for $\dot{u} + w$ and $\dot{\beta}$, there results

$$\dot{u} + w = \frac{B_5}{K\left(1 - \frac{h^2}{12R^2}\right)} \tag{3.10}$$

$$\dot{\beta} = \frac{1}{K\left(1 - \frac{h^2}{12R^2}\right)} N + \frac{R}{D\left(1 - \frac{h^2}{12R^2}\right)} M$$

The second of these equations may be integrated giving

$$\beta = -\frac{B_1 R^2}{D}\sin\alpha + \frac{B_2 R^2}{D}\cos\alpha + \frac{R}{D\left(1 - \frac{h^2}{12R^2}\right)} B_3 \alpha + B_4 \tag{3.11}$$

A differential equation in u results if the first of (3.10) is differentiated and substituted into the third of (3.5).

$$\ddot{u} + u = -\left(\frac{6R}{5Gh} + \frac{R^3}{D}\right) B_1 \sin\alpha$$

$$+\left(\frac{6R}{5Gh} + \frac{R^3}{D}\right) B_2 \cos\alpha + \frac{R^2 B_3 \alpha}{D\left(1 - \frac{h^2}{12R^2}\right)} + RB_4 \tag{3.12}$$

Using variation of parameters the general solution of this equation is found to be

$$u = B_5 \cos\alpha + B_6 \sin\alpha - B_1 \eta_1 (\sin\alpha - \alpha\cos\alpha)$$

$$+ B_2 \eta_1 \alpha \sin\alpha + \eta_2 B_3(\alpha - \sin\alpha) + RB_4(1 - \cos\alpha) \tag{3.13}$$

where

$$\eta_1 = \frac{3R}{5Gh} + \frac{R^3}{2D}, \quad \eta_2 = \frac{R^2}{D\left(1 - \frac{h^2}{12R^2}\right)} \tag{3.14}$$

The normal deflection w may now be found directly from the first of (3.10).

$$w = B_5 \sin\alpha - B_6 \cos\alpha + B_1 \eta_1 \alpha \sin\alpha - B_2 \eta_1 (\sin\alpha + \alpha\cos\alpha) + B_3 \left(-\frac{R^2}{D} + \eta_2 \cos\alpha\right) - R B_4 \sin\alpha \tag{3.15}$$

If the boundary conditions (2.15) at the fixed edge $\alpha_e = \pi/2$ are applied to the displacements u, w, and β, the constants B_4, B_5, and B_6 can be written in terms of B_1, B_2, and B_3. The solutions may then be written

$$\begin{aligned}
N &= B_1 \cos\alpha + B_2 \sin\alpha \\
V &= B_1 \sin\alpha - B_2 \cos\alpha \\
M &= -R B_1 \cos\alpha - R B_2 \sin\alpha + B_3 \\
\beta &= B_1 \frac{R^2}{D}(1-\sin\alpha) + B_2 \frac{R^2}{D}\cos\alpha + B_3 \frac{\eta_2}{R}\left(\alpha - \frac{\pi}{2}\right) \\
u &= B_1\left[\eta_1 \cos\alpha \left(\alpha - \frac{\pi}{2}\right) + \frac{R^3}{D}(1-\sin\alpha)\right] + \eta_2 B_2 \left[\cos\alpha + \left(\alpha - \frac{\pi}{2}\right)\sin\alpha\right] + B_3\left[\eta_2\left(\alpha-\frac{\pi}{2}\right) + \frac{R^2}{D}\cos\alpha\right] \\
w &= B_1\left[\eta_1\left(\alpha - \frac{\pi}{2}\right)\sin\alpha - \left(\eta_1 - \frac{R^3}{D}\right)\cos\alpha\right] - B_2\left[\eta_1\left(\alpha - \frac{\pi}{2}\right)\cos\alpha\right] + B_3 \frac{R^2}{D}(\sin\alpha - 1)
\end{aligned} \tag{3.16}$$

As expected there are three unknown coefficients in these solutions.

2. Solution for Classical Bending Theory

For the bending problem the shear resultant V is determined from equilibrium alone. Hence the rotation Equation (2.8) is substituted for the last of (3.5). The solution of the system is begun in the same manner, the Equations (3.6) through (3.11) being the same. If (2.8) is differentiated once and if the Equations (3.10) are substituted into this result, a differential equation for w is given by

$$\ddot{w} + w = \frac{R^3}{D} B_1 \cos\alpha + \frac{R^3}{D} B_2 \sin\alpha + \frac{B_3}{1 - \frac{h^2}{12R^2}} \left(\frac{1}{K} - \frac{R^2}{D} \right) \tag{3.17}$$

Using variation of parameters the general solution of this equation is found to be

$$\begin{aligned} w = {} & B_5 \cos\alpha + B_6 \sin\alpha + \frac{R^3 B_1}{2D} \alpha \sin\alpha \\ & - \frac{R^3 B_2}{2D} (\alpha \cos\alpha - \sin\alpha) - \frac{B_3 R^2}{D} (1 - \cos\alpha) \end{aligned} \tag{3.18}$$

The tangential displacement u may now be found directly from (2.8).

$$\begin{aligned} u = {} & \frac{B_1 R^3}{2D} (\alpha \cos\alpha - \sin\alpha) + \frac{B_2 R^3}{2D} (2\cos\alpha + \alpha \sin\alpha) \\ & + \frac{B_3 R^2}{D} \left(-\sin\alpha + \frac{\alpha}{1 - \frac{h^2}{12R^2}} \right) + R B_4 \\ & - B_5 \sin\alpha + B_6 \cos\alpha \end{aligned} \tag{3.19}$$

The boundary conditions at $\alpha_e = \pi/2$ may now be applied to the displacements u, w, and β resulting, as before, in relations between B_1, B_2, B_3 and B_4, B_5, B_6. Substitution of these quantities into the expressions for the general solutions results in

$$N = B_1 \cos\alpha + B_2 \sin\alpha$$

$$V = B_1 \sin\alpha - B_2 \cos\alpha$$

$$M = -RB_1 \cos\alpha - RB_2 \sin\alpha + B_3$$

$$\beta = \left[\frac{R^2}{D}(1 - \sin\alpha)\right] B_1 + \left[\frac{R^2}{D}\cos\alpha\right] B_2 + \left[\frac{R\left(\alpha - \frac{\pi}{2}\right)}{D\left(1 - \frac{h^2}{12R^2}\right)}\right] B_3 \tag{3.20}$$

$$u = \left\{\frac{R^3}{2D}\left[2(1 - \sin\alpha) + \left(\alpha - \frac{\pi}{2}\right)\cos\alpha\right]\right\} B_1 + \left\{\frac{R^3}{2D}\left[\cos\alpha + \left(\alpha - \frac{\pi}{2}\right)\sin\alpha\right]\right\} B_2 + \left\{\frac{R^2}{D}\left[\cos\alpha + \left(\alpha - \frac{\pi}{2}\right)\Big/\left(1 + \frac{h^2}{12R^2}\right)\right]\right\} B_3$$

$$w = \left\{\frac{R^3}{2D}\left[\cos\alpha + \left(\alpha - \frac{\pi}{2}\right)\sin\alpha\right]\right\} B_1 + \left\{-\frac{R^3}{2D}\left[\left(\alpha - \frac{\pi}{2}\right)\cos\alpha\right]\right\} B_2 + \frac{R^2}{D}(\sin\alpha - 1) B_3$$

C. SOLUTION IN THE CONTACT REGION

1. Solution Including Transverse Shear Deformation

The system of equations to be solved consists of the equilibrium Equations (3.3), the stress resultant-displacement Equations (3.5) and the restraint Equation (2.13). If the first and third of the equilibrium Equations (3.3) are written in terms of displacement and if the restraint equation is written

$$w = u \tan\alpha - g \tag{3.21}$$

$$\dot{w} = \dot{u} \tan\alpha + u \sec^2\alpha - \dot{g}$$

where

$$g = (w_0 - R)\sec\alpha + R, \tag{3.22}$$

the following two equations in β and u result:

$$\frac{K}{R}\left[\ddot{u} + \dot{u}\tan\alpha + u\sec^2\alpha - \dot{g} - \frac{h^2}{12R}\ddot{\beta}\right]$$

$$+ \frac{5Gh}{6}\left[\frac{1}{R}\dot{u}\tan\alpha + \frac{1}{R}u\sec^2\alpha - \frac{\dot{g}}{R} - \frac{u}{R} + \beta\right] = 0$$

$$\frac{D}{R}\left[\ddot{\beta} - \frac{1}{R}\ddot{u} - \frac{1}{R}\dot{u}\tan\alpha - \frac{1}{R}u\sec^2\alpha + \frac{\dot{g}}{R}\right] \tag{3.23}$$

$$- \frac{5Gh}{6R}\left[\frac{1}{R}\dot{u}\tan\alpha + \frac{1}{R}u\sec^2\alpha - \frac{\dot{g}}{R} - \frac{u}{R} + \beta\right] = 0$$

Although these equations appear to be formidably coupled, they are easily separated by substituting $\ddot{\beta}$ from the second into the first yielding the result

$$\ddot{u} + \dot{u}\tan\alpha + u\sec^2\alpha = \dot{g}, \tag{3.24}$$

the first integral of which is

$$\dot{u} + u\tan\alpha = g + C \tag{3.25}$$

where C is an arbitrary coefficient. The general solution is

$$u = (w_0 - R)\sin\alpha + C_1\lambda(\alpha)\cos\alpha + C_2\cos\alpha \tag{3.26}$$

where $\lambda(\alpha)$ is defined by

$$\lambda(\alpha) = \frac{1}{2}\log\frac{1+\sin\alpha}{1-\sin\alpha} \tag{3.27}$$

and

$$C_1 = C + R$$

Substitution of (3.26) into (3.23) gives the differential equation for β,

$$\ddot{\beta} - \chi^2\beta = \frac{C_1\chi^2}{R}\tan\alpha \tag{3.28}$$

where the large coefficient χ^2 is defined as

$$\chi^2 = \frac{5 G h R^2}{6 D} \tag{3.29}$$

Using variation of parameters the solution of (3.28) is found to be

$$\beta = C_3 e^{\chi\alpha} + C_4 e^{-\chi\alpha} + C_1 (I_1 + I_2) \tag{3.30}$$

where I_1 and I_2 are given by

$$\begin{aligned} I_1 &= -\frac{\chi}{2R} e^{-\chi\alpha} \int_0^\alpha e^{\chi t} \tan t \, dt \\ I_2 &= \frac{\chi}{2R} e^{\chi\alpha} \int_0^\alpha e^{-\chi t} \tan t \, dt \end{aligned} \tag{3.31}$$

Two independent boundary conditions may now be applied to the solutions for u and β at $\alpha = 0$ and it will be seen that the other three are automatically satisfied.

$$\beta(0) = 0 = C_3 + C_4$$

$$u(0) = 0 = (w_0 - R)(0) + C_2 \lambda(0)(1) + C_2$$

These imply that

$$C_3 = -C_4 \quad , \quad C_2 = 0 \tag{3.32}$$

The solutions for β and u become

$$\begin{aligned} \beta &= C_5 \sinh \chi\alpha + C_1 (I_1 + I_2) \\ u &= (w_0 - R) \sin\alpha + C_1 \lambda(\alpha) \cos\alpha \end{aligned} \tag{3.33}$$

The remaining solutions w, N, M, V and q^+ may now be calculated directly from the Equations (3.3) and (3.33).

$$
\begin{aligned}
w &= R(\cos\alpha - 1) - w_0\cos\alpha + C_1\lambda(\alpha)\sin\alpha \\
N &= \frac{K}{R}\left\{C_1\left[1 - \frac{h^2\chi}{12R}(-I_1 + I_2)\right] - \frac{C_5 h^2\chi}{12R}\cosh\chi\alpha - R\right\} \\
M &= \frac{D}{R}\left\{C_1\left[-\frac{1}{R} + \chi(-I_1 + I_2)\right] + C_5\chi\cosh\chi\alpha + 1\right\} \qquad (3.34) \\
V &= \frac{5Gh}{6}\left\{C_1\left[\frac{1}{R}\tan\alpha + (I_1 + I_2)\right] + C_5\sinh\chi\alpha\right\} \\
\gamma^+ &= \frac{1}{R}\left[\left[C_1\left\{\frac{K}{R}\left[1 - \frac{h^2\chi}{12R}(-I_1 + I_2)\right] - \frac{5Gh}{6}\left[\frac{1}{R}\sec^2\alpha\right.\right.\right.\right. \\
&\qquad \left.\left.\left.\left. + \chi(-I_1 + I_2)\right]\right\} - C_5\chi\left[\frac{Kh^2}{12R^2} + \frac{5Gh}{6}\right]\cosh\chi\alpha - K\right]\right]
\end{aligned}
$$

It is easily shown that the remaining boundary conditions

$$\dot{w}\Big|_{\alpha=0} \quad , \quad w(0) = -w_0 \quad , \quad V(0) = 0$$

are automatically satisfied.

2. Solution for Classical Bending Theory

The Equation (3.24) applies equally well to both the classical theory and the shear deformation theory, hence the solution u may be written as

$$u = (w_0 - R)\sin\alpha + C_1\lambda(\alpha)\cos\alpha + C_2\cos\alpha \qquad (3.35)$$

If this result is substituted into the restraint Equation (3.21), the solution for w may be written as

$$w = (R - w_0)\cos\alpha + C_1\lambda(\alpha)\sin\alpha + C_2\sin\alpha - R \qquad (3.36)$$

The major difference between the two theories arises in the formation of β. This quantity now is formed from the rotation Equation (2.8) and assumes the simple form

$$\beta = -\frac{C_1}{R}\tan\alpha \tag{3.37}$$

The quantities N and M may now be found directly from the stress resultant-displacement equations.

$$\begin{aligned} N &= \frac{K}{R}\left[C_1\left(1+\frac{h^2}{12R^2}\sec^2\alpha\right)-R\right] \\ M &= \frac{D}{R^2}\left[R - C_1\left(1+\sec^2\alpha\right)\right] \end{aligned} \tag{3.38}$$

Finally the shear resultant and the normal surface pressure in the contact region may be found from the equilibrium equations.

$$\begin{aligned} V &= -\frac{2DC_1}{R^3}\sec^2\alpha\tan\alpha \\ q^+ &= \frac{K}{R}\left\{\frac{C_1 h^2}{4R^3}\left[\frac{R^2}{4h^2}+\sec^2\alpha\left(\sec^2\alpha+2\tan^2\alpha\right)\right]-1\right\} \end{aligned} \tag{3.39}$$

The boundary conditions on the quantity u gives

$$u(0) = 0 = C_2 \tag{3.40}$$

Examination of the solutions β, w, and V show that the conditions

$$\beta(0)=0\ ;\ w(0)=-w_0\ ;\ \dot{w}\Big|_{\alpha=0}=0\ ;\ V(0)=0$$

are all automatically satisfied. Only one unknown coefficient remains in this solution. This fact arises because the restraint on the rotation β serves to lower the order of the system of equations.

D. BOUNDARY MATCHING OF THE TWO SOLUTIONS AND NUMERICAL RESULTS

Both classical and shear deformation theories have been used to find general solutions in the free and contact regions. The only remaining step is the matching of the stress resultant and deformation quantities at the common boundary, $\alpha = \bar{\alpha}$, between the two regions.

The shear deformation solutions (3.16) contain the unknown coefficients B_1, B_2, and B_3 while the solutions (3.33) and (3.34) contain C_1 and C_5. Neither w_o or $\bar{\alpha}$ have been specified to this point. One or the other may be arbitrarily chosen as the externally applied boundary condition. Thus the six equations of continuity which may be written at the boundary $\alpha = \bar{\alpha}$ are

$$\begin{aligned} u_c &= u_f \; , \quad w_c = w_f \; , \quad \beta_c = \beta_f \\ N_c &= N_f \; , \quad M_c = M_f \; , \quad V_c = V_f \end{aligned} \tag{3.41}$$

where the subscripts "c" and "f" indicate the contact and free regions. In detail these equations are

$$\begin{aligned} &\left\{\sin\bar{\alpha}\right\} w_o + \left\{\lambda(\bar{\alpha})\cos\bar{\alpha}\right\} C_1 - \Big\{\eta_1 \cos\bar{\alpha}\left(\bar{\alpha} - \tfrac{\pi}{2}\right) \\ &\quad + \frac{R^3}{D}(1-\sin\bar{\alpha})\Big\} B_1 - \left\{\eta_1\left[\cos\bar{\alpha} + \left(\bar{\alpha} - \tfrac{\pi}{2}\right)\sin\bar{\alpha}\right]\right\} B_2 \\ &\quad - \left\{\eta_2\left(\bar{\alpha} - \tfrac{\pi}{2}\right) + \frac{R^2}{D}\cos\bar{\alpha}\right\} B_3 = R\sin\bar{\alpha} \\ &\left\{-\cos\bar{\alpha}\right\} w_o + \left\{\lambda(\bar{\alpha})\sin\bar{\alpha}\right\} C_1 - \Big\{\eta_1\left(\bar{\alpha} - \tfrac{\pi}{2}\right)\sin\bar{\alpha} \\ &\quad - \left(\eta_1 - \frac{R^3}{D}\right)\cos\bar{\alpha}\Big\} B_1 + \left\{\eta_1\left(\bar{\alpha} - \tfrac{\pi}{2}\right)\cos\bar{\alpha}\right\} B_2 \\ &\quad - \left\{\frac{R^2}{D}(\sin\bar{\alpha} - 1)\right\} B_3 = R\left(1 - \cos\bar{\alpha}\right) \end{aligned} \tag{3.42}$$

$$\left\{I_1 + I_2\right\} C_1 + \left\{\sinh \chi\bar{\alpha}\right\} C_5 - \left\{\frac{R^2}{D}(1 - \sin\bar{\alpha})\right\} B_1$$

$$- \left\{\frac{R^2}{D} \cos\bar{\alpha}\right\} B_2 - \left\{\frac{m_2}{R}\left(\bar{\alpha} - \frac{\pi}{2}\right)\right\} B_3 = 0$$

$$\left\{\frac{K}{R}\left[1 - \frac{h^2\chi}{12R}\left(-I_1 + I_2\right)\right\} C_1 - \left\{\frac{h^2\chi}{12R} \cosh \chi\bar{\alpha}\right\} C_5$$

$$- \left\{\cos\bar{\alpha}\right\} B_1 - \left\{\sin\bar{\alpha}\right\} B_2 = K$$

$$\left\{\frac{D}{R}\left[-\frac{1}{R} + \chi\left(-I_1 + I_2\right)\right]\right\} C_1 + \left\{\chi \cosh \chi\bar{\alpha}\right\} C_5 + \left\{R \cos\bar{\alpha}\right\} B_1$$

$$+ \left\{R \sin\bar{\alpha}\right\} B_2 - \left\{1\right\} B_3 = -\frac{D}{R}$$

$$\left\{\frac{5Gh}{6}\left[\frac{1}{R} \tan\bar{\alpha} + \left(I_1 + I_2\right)\right]\right\} C_1 + \left\{\frac{5Gh}{6} \sinh \chi\bar{\alpha}\right\} C_5$$

$$- \left\{\sin\bar{\alpha}\right\} B_1 + \left\{\cos\bar{\alpha}\right\} B_2 = 0 \qquad (3.42)$$

Because of the complexity of this system it was decided to find the solutions for the unknown coefficients numerically on a digital computer rather than to develop specific expressions. This is most easily accomplished by specifying the length of the contact region $\bar{\alpha}$ and then solving the resulting six by six system of linear algebraic equations for B_1, B_2, B_3, C_1, C_5, and w_o. These results are then used to compute all stress and deformation quantities and the solution procedure is complete for the shear theory.

The classical solutions (3.20) contain unknown coefficients B_1, B_2, and B_3 while the solutions (3.35) through (3.39) contain C_1. As before, the final unknown is given by $\bar{\alpha}$ or w_o. The five continuity equations which may be written at the boundary $\alpha = \bar{\alpha}$ for this case are

$$u_c = u_f \; , \; w_c = w_f \; , \; \beta_c = \beta_f$$
$$N_c = N_f \; , \; M_c = M_f \tag{3.43}$$

which may be written in detail as

$$\begin{aligned}
&\{\sin\bar{\alpha}\} w_o + \{\lambda(\bar{\alpha})\cos\bar{\alpha}\} C_1 - \Big\{\frac{R^3}{2D}\Big[2(1-\sin\bar{\alpha}) \\
&\quad + (\bar{\alpha}-\tfrac{\pi}{2})\cos\bar{\alpha}\Big]\Big\} B_1 - \Big\{\frac{R^3}{2D}\Big[\cos\bar{\alpha} + (\bar{\alpha}-\tfrac{\pi}{2})\sin\bar{\alpha}\Big]\Big\} B_2 \\
&\quad - \Big\{\frac{R^2}{D}\Big[\cos\bar{\alpha} + (\bar{\alpha}-\tfrac{\pi}{2})\Big/\Big(1-\frac{h^2}{12R^2}\Big)\Big]\Big\} B_3 = R\sin\bar{\alpha} \\
&\{-\cos\bar{\alpha}\} w_o + \{\lambda(\bar{\alpha})\sin\bar{\alpha}\} C_1 - \Big\{\frac{R^3}{2D}\Big[\cos\bar{\alpha} \\
&\quad + (\bar{\alpha}-\tfrac{\pi}{2})\sin\bar{\alpha}\Big]\Big\} B_1 + \Big\{\frac{R^3}{2D}(\bar{\alpha}-\tfrac{\pi}{2})\cos\bar{\alpha}\Big\} B_2 \\
&\quad + \Big\{\frac{R^2}{D}(1-\sin\bar{\alpha})\Big\} B_3 = -R\cos\bar{\alpha} \\
&\Big\{-\frac{1}{R}\tan\bar{\alpha}\Big\} C_1 - \Big\{\frac{R^2}{D}(1-\sin\bar{\alpha})\Big\} B_1 - \Big\{\frac{R^2}{D}\cos\bar{\alpha}\Big\} B_2 \\
&\quad - \Big\{R(\bar{\alpha}-\tfrac{\pi}{2})\Big/D\Big(1-\frac{h^2}{12R^2}\Big)\Big\} B_3 = 0 \\
&\Big\{\frac{K}{R}\Big(1+\frac{h^2}{12R^2}\sec^2\bar{\alpha}\Big)\Big\} C_1 - \{\cos\bar{\alpha}\} B_1 - \{\sin\bar{\alpha}\} B_2 = K \\
&\Big\{-\frac{D}{R^2}(1+\sec^2\bar{\alpha})\Big\} C_1 + \{R\cos\bar{\alpha}\} B_1 + \{R\sin\bar{\alpha}\} B_2 - B_3 = -\frac{D}{R}
\end{aligned} \tag{3.44}$$

It should be noted that the shear stress resultant is not forced to be continuous as it was for the higher order shear deformation theory. Compatibility of displacements is assured by the Equations (3.44) while the quantities V and q^+ are given entirely by equilibrium. The same numerical procedures may be used to determine the unknown coefficients and finally to complete the solution.

A numerical study of the solutions is presented in Figures 3-8 and Tables 1-10. Appropriate nondimensional forms for the displacements are set forth in Equations (2.11). The form of Equations (3.5) indicate that convenient nondimensional stress resultants are

$$\bar{N} = \frac{N}{K} \; , \quad \bar{M} = \frac{h}{2R} \cdot \frac{R}{D} M \; , \quad \bar{V} = \frac{6V}{5Gh}$$
$$\bar{q}^+ = \frac{q^+ (1-\nu^2)}{E} \tag{3.45}$$

These quantities may also be related to the stresses using the stress-stress resultant equations of Naghdi [9]

$$\bar{N} = \frac{\sigma_{\alpha e} (1-\nu^2)}{E} \; , \quad \bar{M} = \frac{\sigma_{\alpha b} (1-\nu^2)}{E} \; , \quad \bar{V} = \frac{8\tau (1+\nu)}{5E}$$

where $\sigma_{\alpha e}$ is the normal membrane stress, $\sigma_{\alpha b}$ is the maximum stress due to bending at the outer surface of the shell, and τ is the transverse shear stress.

The figures show a direct comparison of the classical and shear deformation theories for a typical soft thick ring. The variables used in constructing the figures were chosen to bring out the maximum differences between the two theories. The basic shape of these curves remains unchanged with variation of R/h, $\bar{\alpha}$, and the modulus parameters. However the magnitudes of the different variables change as do the magnitudes of the differences between classical and shear deformation theories. The tables contain a parametric study of the solutions. In Tables 1 and 2 the effect of variation of $\overline{\alpha}$ or $\overline{w}_o$ is shown. The variations in the solutions for radius to thickness ratios of 10 and 50 are shown in Tables 3 and 4. All other quantities are held constant. Finally the material properties are considered in Tables 4 and 5 with thin rubber and steel rings being considered. Tables 6-10 are the same as 1-5 except that the classical solutions are used for computing the numbers.

The most interesting aspect of Figure 3 is the comparison of the rotation β for the shear deformation and classical theories. In the contact region $\alpha \leq .2$ the Kirchhoff hypothesis forces normals to the midsurface of the ring to become normal to the rigid contacting surface. Shear theory relaxes this restriction and a significant difference (25%) between the two theories is encountered for the thick ring. A study of the tables shows that a notable difference (13%) is still present for R/h = 10, but that the two theories give nearly identical results for R/h = 50.

The form of the dimensionless stress resultants shown in Figure 4 is similar for the two theories with one exception. The discontinuity in $\overline{V}$, which must be supplied as an externally applied load for the classical theory, and its effect on the pressure distribution in the contact region (see Figure 5) should be noted. As the rings are made thinner the distribution of $\overline{q}^+$ for shear theory tends to approach the distribution of the classical theory in that sharp peaks are developed

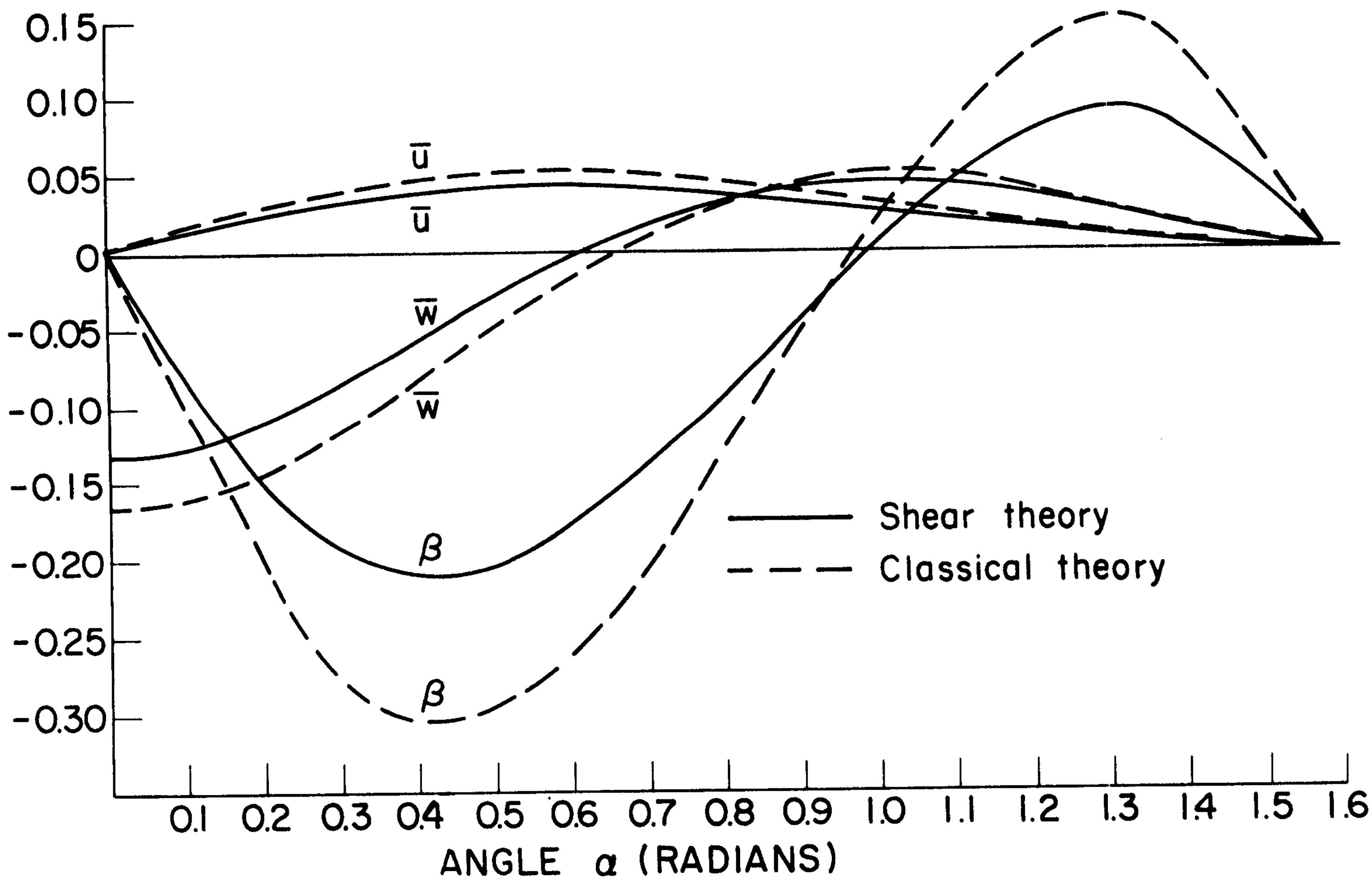

Figure 3. Deformations for ring. $\bar{\alpha} = 0.2$, R = 5 in., R/h = 5, E = 222 lb/in.^{2}, $\nu = 0.5$.

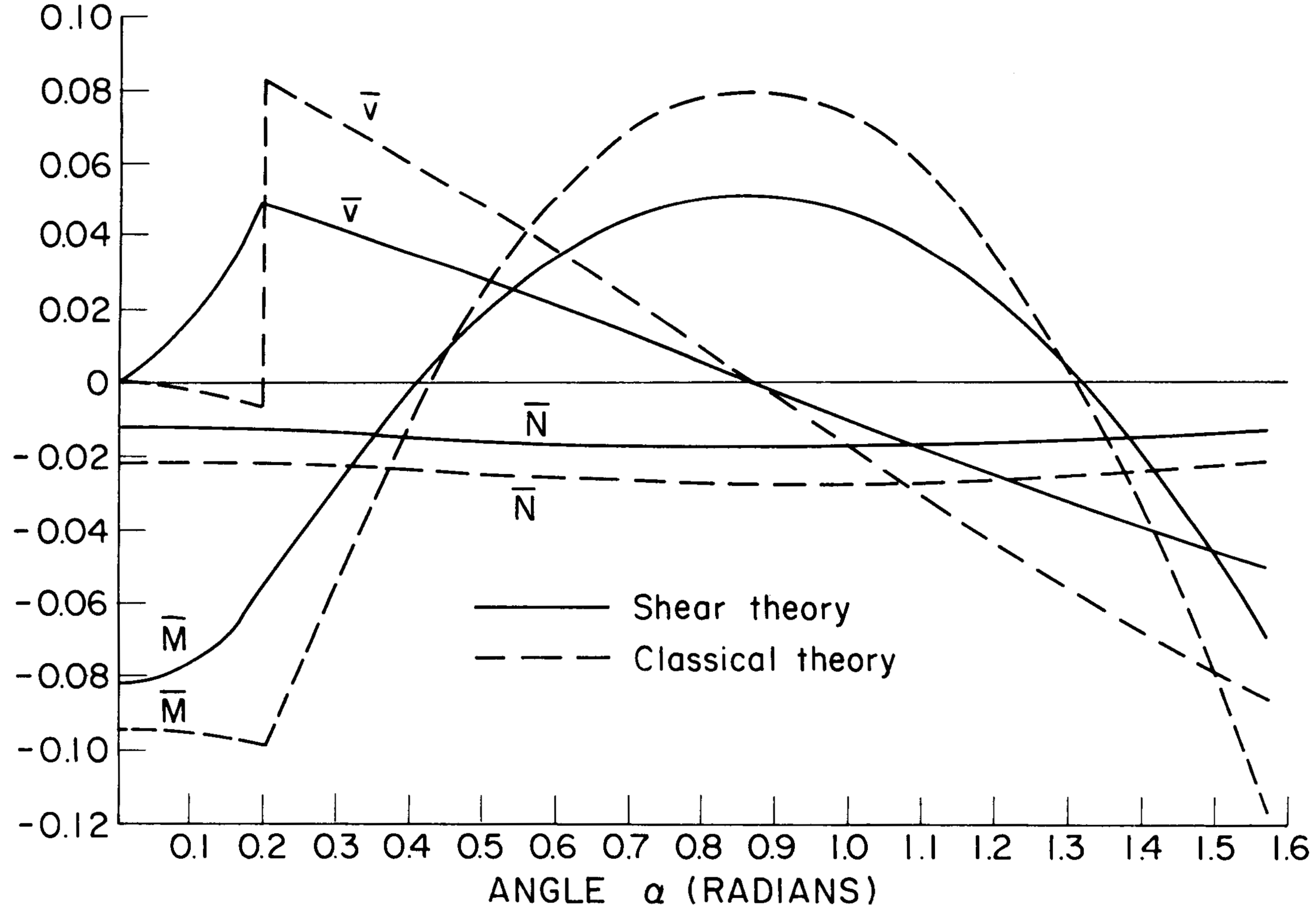

Figure 4. Stresses for ring. $\bar{\alpha} = 0.2$, R = 5 in., R/h = 5, E = 222 lb/in.^{2}, $\nu = 0.5$.

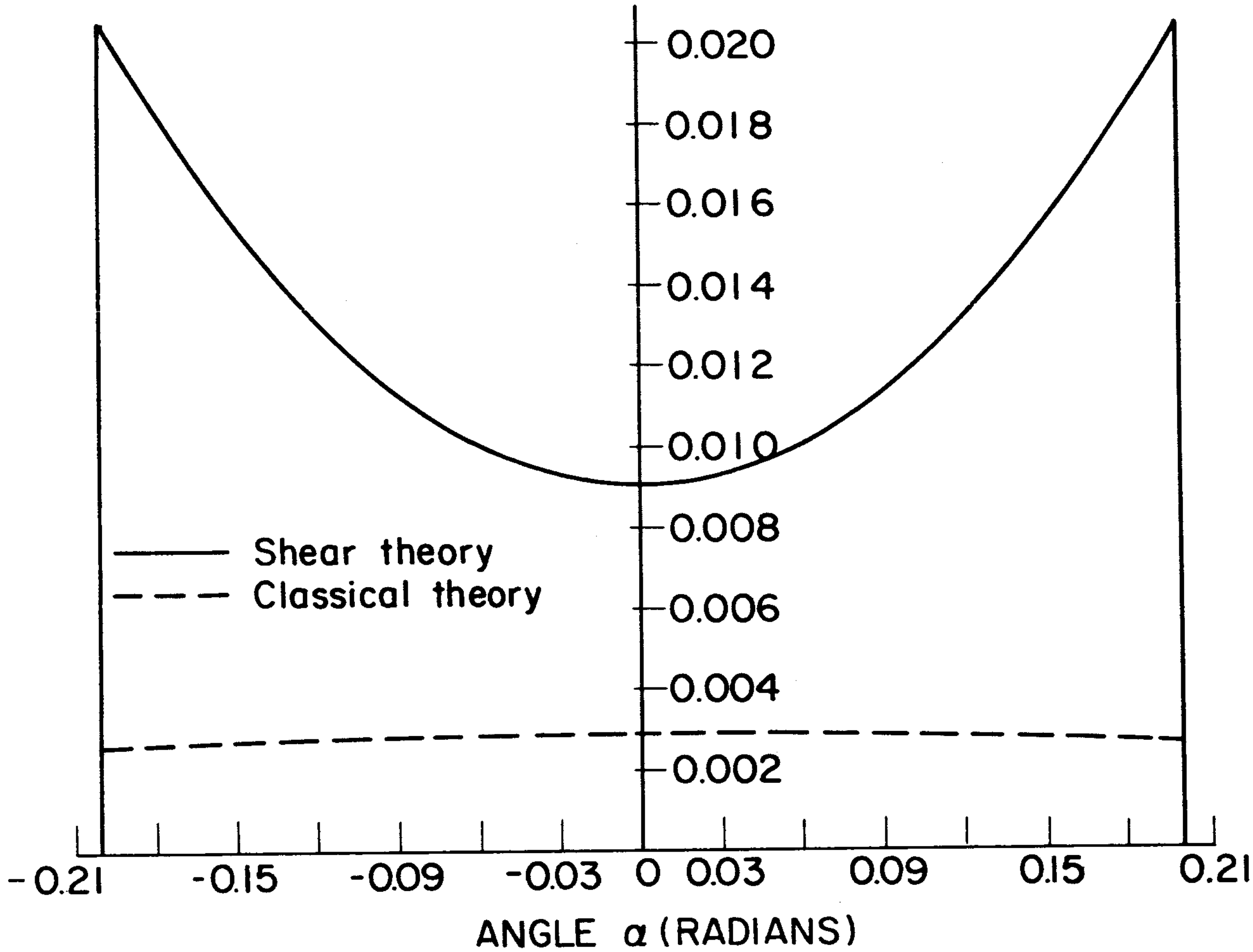

Figure 5. Normal compressive stress $\overline{q}^{+}$ for ring. $\overline{\alpha} = 0.2$, $R = 5$ in., $R/h = 5$, $E = 222$ lb/in.^{2}, $\nu = 0.5$.

near the edge of the contact region. However at the very edge of the contact region the pressure drops sharply as would be expected and desired! This behavior has been observed only in calculations for the thin rings and is demonstrated in Tables 4 and 5.

Figure 6 is a plot of the externally applied load versus the central deflection $\bar{w}_o$. Due to the restraining effect of the Kirchhoff hypothesis, the slope of the curve for the classical theory is steeper. As the rings are made thin the two curves become coincident.

Figure 7 points out another well-known difference between classical and shear deformation theories. It is seen that a finite point load must be applied to the classical ring before any contact region is developed. It is this point load which is propagated along the coordinate α to form the shear discontinuity. For the thin rings this particular difference between the two theories is lessened but has not been eliminated as can be seen by examining $\bar{N}$ in Tables 5 and 10.

In comparing the results for the steel and rubber rings it is seen that the deformation patterns remain the same as would be expected because of the boundary condition on $\bar{\alpha}$. The shapes of the stress resultant curves remain the same but are much magnified for the steel ring.

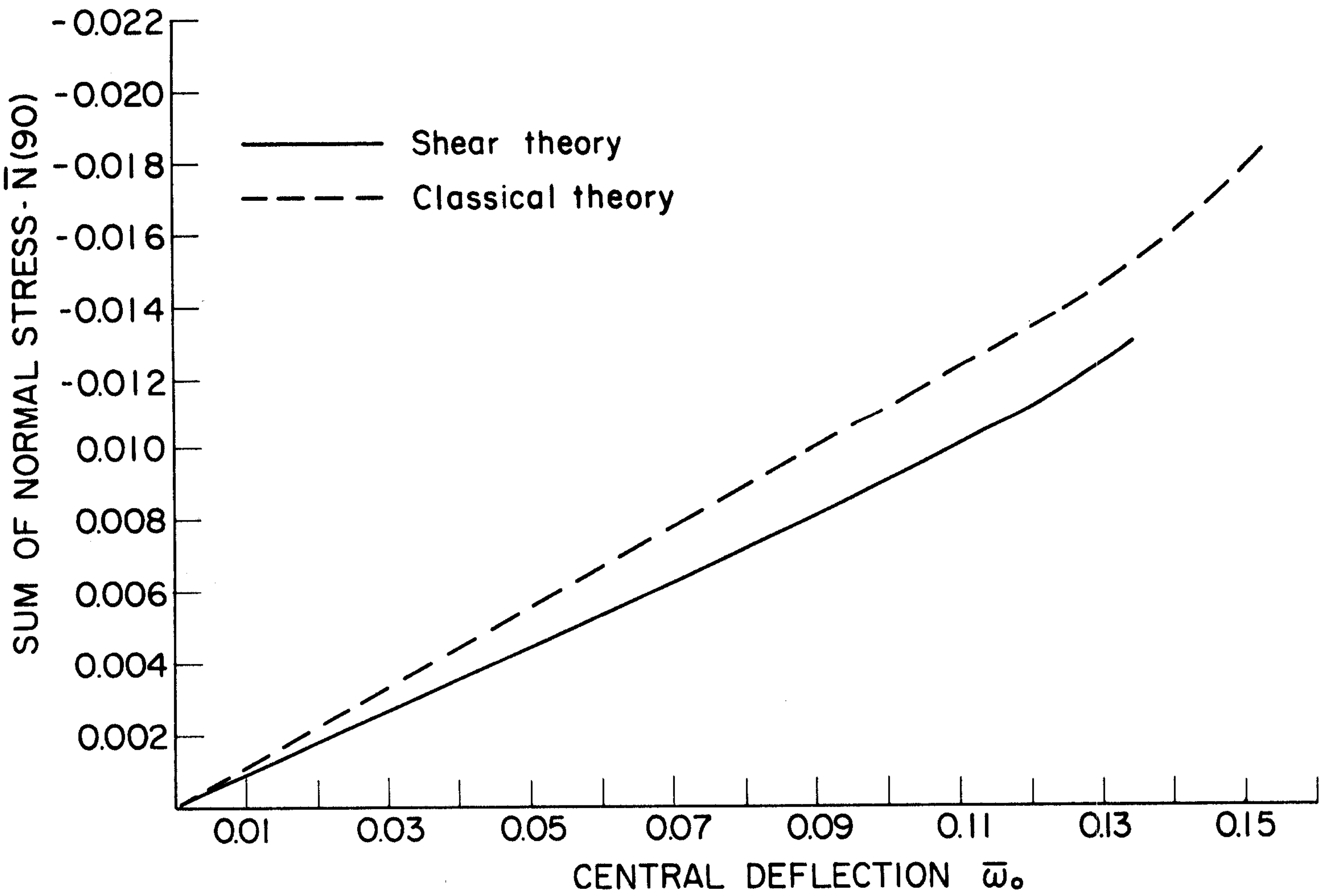

Figure 6. Load versus central deflection for ring. R = 5 in., R/h = 5, E = 222 lb/in.^{2}, ν = 0.5.

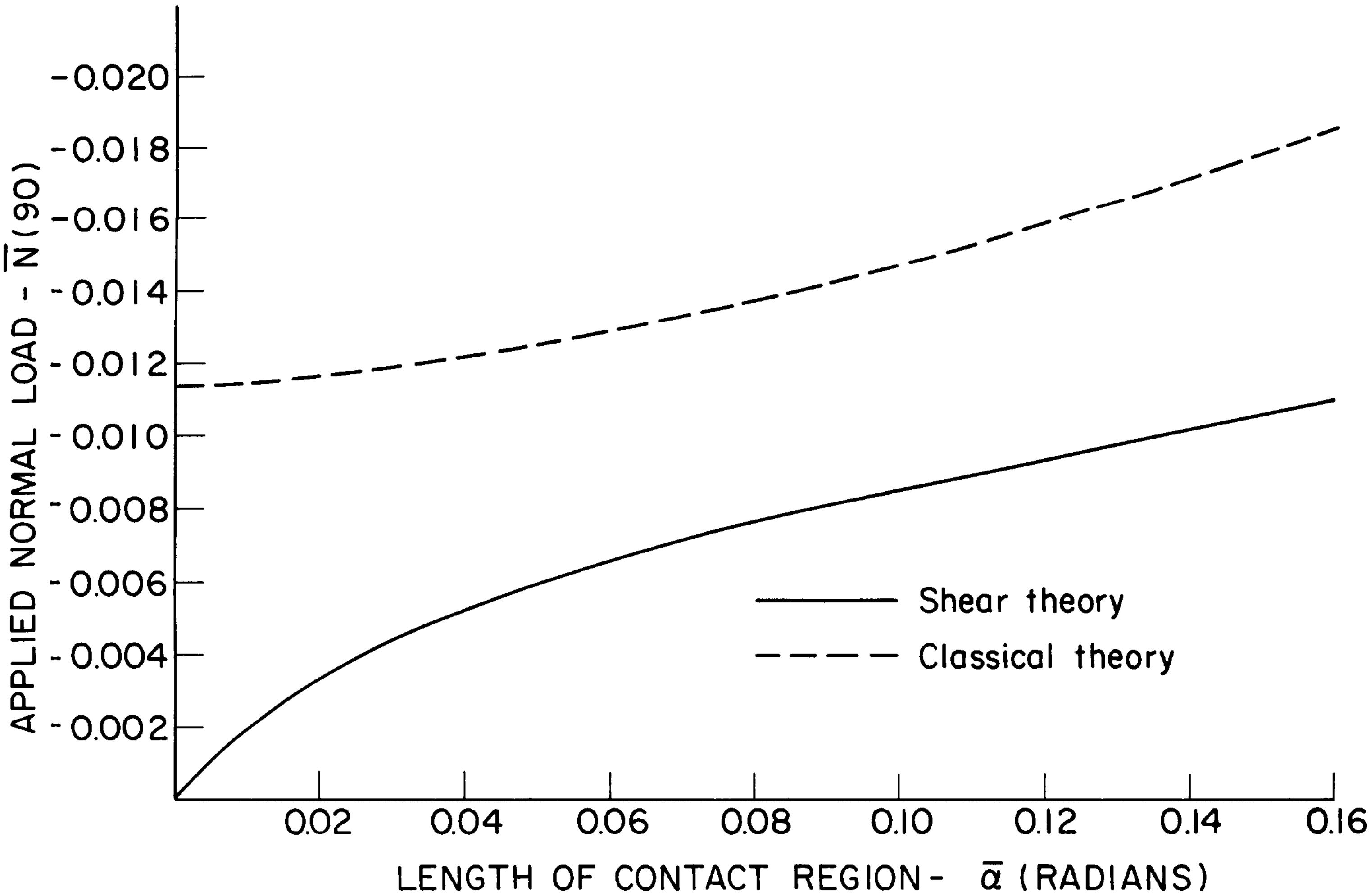

Figure 7. Load versus length of contact region for ring. $R = 5$ in., $R/h = 5$, $E = 222$ lb/in.^{2}, $\nu = 0.5$.

CHAPTER IV

SOLUTION OF THE CONTACT PROBLEM FOR A SPHERICAL CAP

A. THE SYSTEM EQUATIONS

The sphere problem which will be considered in detail in this chapter involves the contact of a spherical cap with a rigid flat surface. The maximum possible size of the cap as given by the location of its fixed edge α_e is determined by the relation

$$\alpha_e \left(\frac{R}{h} \left(1 - \nu^2 \right)^{1/2} \right)^{1/2} \leq 1.1 \tag{4.1}$$

where R is the radius, h is the thickness, and ν is Poisson's ratio. This rule for limiting the extent of the cap is based on the approximate nature of the shallow sphere solution in the free region.

In spherical coordinates the squared linear element given by (2.1) becomes

$$ds^2 = R^2 \left(1 + \frac{\zeta}{R} \right)^2 d\alpha^2 + R^2 \sin^2\alpha \left(1 + \frac{\zeta}{R} \right)^2 d\theta^2 + d\zeta^2 \tag{4.2}$$

where

$$R_\alpha = R \quad , \quad R_\theta = R \quad , \quad r = R \sin\alpha \tag{4.3}$$

The equilibrium Equations (2.3) are written

$$\begin{aligned} \dot{N}_\alpha + (N_\alpha - N_\theta) \operatorname{ctn}\alpha + V &= 0 \\ \dot{V} + V \operatorname{ctn}\alpha - (N_\alpha + N_\theta) \sin\alpha + R q^+ &= 0 \\ \dot{M}_\alpha + (M_\alpha - M_\theta) \operatorname{ctn}\alpha - RV &= 0 \end{aligned} \tag{4.4}$$

The relations between the stress resultants and the displacements become

$$
\begin{aligned}
N_\alpha &= \frac{K}{R}\left[\dot{u} + \nu u \operatorname{ctn}\alpha + (1+\nu) w\right] \\
N_\theta &= \frac{K}{R}\left[\nu \dot{u} + u \operatorname{ctn}\alpha + (1+\nu) w\right] \\
M_\alpha &= \frac{D}{R}\left[\dot{\beta} + \nu \beta \operatorname{ctn}\alpha\right] \\
M_\theta &= \frac{D}{R}\left[\nu \dot{\beta} + \beta \operatorname{ctn}\alpha\right] \\
V &= \frac{5Gh}{6}\left[\frac{\dot{w}}{R} - \frac{u}{R} + \beta\right]
\end{aligned}
\tag{4.5}
$$

if shear deformation is included in the analysis. The restraint equation given by (2.13) remains unchanged.

$$
w_0 = R(1-\cos\alpha) - w\cos\alpha + u\sin\alpha \tag{4.6}
$$

B. SOLUTION IN THE FREE REGION

A two-term asymptotic solution of the equations governing the deformation of a spherical shell has been given by DeSilva and Cohen.[12] Their work is based on the linear transverse-shear deformation theory of Naghdi.[10] The governing equation is

$$
\ddot{Y} + \left\{ i^3 \mu^2 \delta_2 + \left[\frac{-3/4}{\sin^2\alpha} + \frac{5}{4} + [\gamma]\right]\right\} Y = 0 \tag{4.7}
$$

where

$$
\begin{aligned}
Y &= (\sin\alpha)^{1/2}\left(\beta_r + i\,\delta\,\psi\right)\\
\mu^2 &= m\frac{R}{h}\\
m &= \left(12(1-\nu^2)\right)^{1/2}\\
\delta &= \delta_2 + i\,\delta_1\\
\delta_2 &= (1-\delta_1^2)^{1/2}\\
\delta_1 &= \frac{\nu + [\gamma]}{\mu^2}\\
[\gamma] &= -\frac{6}{5}(1+\nu)\\
i &= (-1)^{1/2}
\end{aligned}
\tag{4.8}
$$

and where ψ is related to a horizontal radial stress resultant P_H which is directed outward and is defined by

$$
\psi = \frac{mR}{Eh^2}P_h \sin\alpha \tag{4.9}
$$

The quantity β_r represents the rotation of normals to the midsurface of the shell. As the sign convention for many of the quantities in the paper by DeSilva and Cohen[12] is the reverse of the sign convention adopted in the greater portion of this thesis, a subscript "r" will be added to the reversed quantities to indicate this fact.

The two-term asymptotic solutions of Equation (4.7) are given as

$$
Y_1 \simeq \eta^{1/2} J_1(\eta)\left\{1 + \frac{1+[\gamma]}{2\rho^2 i}\pi\frac{\eta^2}{4}\left[2\frac{J_0(\eta)J_2(\eta)H_1^{(1)}(\eta)}{J_1(\eta)} + \right.\right. \tag{4.10}
$$

$$- J_0(\eta)\, H_2^{(1)}(\eta) - J_2(\eta)\, H_0^{(1)}(\eta)\Big]\Big\}$$

$$Y_2 \simeq \eta^{1/2} H_1^{(1)}(\eta)\Big\{1 + \frac{1+[\gamma]}{2\rho^2 i}\pi \frac{\eta^2}{4}\Big[-2\frac{J_1(\eta)H_0^{(1)}(\eta)\,H_2^{(1)}(\eta)}{H_1^{(1)}(\eta)}$$

$$+ H_0^{(1)}(\eta)\, J_2(\eta) + H_2^{(1)}(\eta)\, J_0(\eta)\Big]\Big\} \tag{4.10}$$

where

$$\eta = \rho\alpha \quad , \quad \rho = i^{3/2}\mu\, \delta_2^{1/2} \tag{4.11}$$

The limit of these solutions in a neighborhood of the origin $\alpha = 0$ is

$$Y_1 \simeq \eta^{1/2} J_1(\eta)\Big[1 + i\frac{1+[\gamma]}{2\mu^2\delta_2}\Big]$$
$$Y_2 \simeq \eta^{1/2} H_1^{(1)}(\eta)\Big[1 - i\frac{1+[\gamma]}{2\mu^2\delta_2}\Big] \tag{4.12}$$

These equations from De Silva and Cohen will be regarded as the solutions for the free region of the spherical cap in the work that follows.

Equation (4.1) is based on a comparison of (4.12) with (4.10) for nonzero values of η. The largest difference between these two sets of solutions occurs when α_e and R/h satisfy

$$\alpha_e\left(\frac{R}{h}\left(12(1-\upsilon^2)\right)^{1/2}\right)^{1/2} = 1.1$$

This may be seen by evaluating Y_2 at α_e, subtracting Y_2 of Equation (4.10) from Y_2 of Equation (4.12), and factoring out the common quantity $\eta^{1/2}H_1^{(1)}(\eta)$. The imaginary part of the coefficient of $\eta^{1/2}H_1^{(1)}(\eta)$ is about 20% of the quantity $(1+[\gamma])/2\mu^2\delta_2$. The real part is less than 1% of one. If this procedure is carried out for Y_1, both the real and imaginary parts of the coefficient of $\eta^{1/2}J_1(\eta)$ are less than 1% of the integer 1 and the quantity $(1+[\gamma])/2\mu^2\delta_2$ respectively. It should be noted that the quantity $(1+[\gamma])/2\mu^2\delta_2$ is the correction of the classical shell theory of E. Reissner[13] due to the effect of transverse shear deformation. This term does not seem to dominate the solution quantities which will be developed in the following pages and hence the error will not be regarded as significant. In the numerical results which will be presented later, the ratio of radius to thickness and the location α_e of the edge of the shell are chosen so that the error in this small quantity does not exceed 20%. In many cases it is much less.

The solution Y may be written

$$Y = (\sin\alpha)^{1/2}\left[(\beta_r - \delta_1\psi) + i\,\delta_2\psi\right] \tag{4.13}$$

If the Equations (4.12) are substituted into this relation and if real and imaginary parts are equated it is possible to write the quantities β_r and ψ in terms of Kelvin functions as

$$\begin{aligned}\psi = \frac{1}{\delta_2}\left(\frac{\alpha}{\sin\alpha}\right)^{1/2}\Big\{&A_3(\mathrm{bei}_1 s + K_3\,\mathrm{ber}_1 s) + A_4(\mathrm{ber}_1 s - K_3\,\mathrm{bei}_1 s)\\ &- A_5\frac{2}{\pi}(\mathrm{ker}_1 s + K_3\,\mathrm{kei}_1 s) + A_6\frac{2}{\pi}(\mathrm{kei}_1 s - K_3\,\mathrm{ker}_1 s)\Big\}\end{aligned} \tag{4.14}$$

$$\begin{aligned}\beta_r = \left(\frac{\alpha}{\sin\alpha}\right)^{1/2}\Big\{&A_3(K_1\,\mathrm{ber}_1 s + K_2\,\mathrm{bei}_1 s) + A_4(K_2\,\mathrm{ber}_1 s - K_1\,\mathrm{bei}_1 s)\\ &+ A_5\frac{2}{\pi}(-K_5\,\mathrm{ker}_1 s + K_4\,\mathrm{kei}_1 s) + A_6\frac{2}{\pi}(K_4\,\mathrm{ker}_1 s + K_5\,\mathrm{kei}_1 s)\Big\}\end{aligned}$$

where

$$K_1 = 1 + \frac{\delta_1(1+[\gamma])}{2\mu^2\delta_2^2}, \quad K_2 = \frac{\delta_1}{\delta_2} - \frac{1+[\gamma]}{2\mu^2\delta_2} \qquad (4.15)$$
$$K_3 = \frac{1+[\gamma]}{2\mu^2\delta_2}, \quad K_4 = 1 - \frac{\delta_1(1+[\gamma])}{2\mu^2\delta_2^2}$$
$$K_5 = \frac{\delta_2}{\delta_1} + \frac{1+[\gamma]}{2\mu^2\delta_2}, \quad S = \mu\delta_2^{1/2}\alpha$$

and where A_3, A_4, A_5, and A_6 are arbitrary constants. The derivatives of ψ and β_r are given by

$$\dot{\psi} = A_3\left[\frac{1}{2\delta_2}\left(\frac{1}{\alpha\sin\alpha}\right)^{1/2}(1-\alpha\operatorname{ctn}\alpha)(\operatorname{bei}_1 S + K_3\operatorname{ber}_1 S) + \frac{\mu}{\delta_2^{1/2}}\left(\frac{\alpha}{\sin\alpha}\right)^{1/2}(\operatorname{bei}_1' S + K_3\operatorname{ber}_1' S)\right]$$
$$+ A_4\left[\frac{1}{2\delta_2}\left(\frac{1}{\alpha\sin\alpha}\right)^{1/2}(1-\alpha\operatorname{ctn}\alpha)(\operatorname{ber}_1 S - K_3\operatorname{bei}_1 S) + \frac{\mu}{\delta_2^{1/2}}\left(\frac{\alpha}{\sin\alpha}\right)^{1/2}(\operatorname{ber}_1' S - K_3\operatorname{bei}_1' S)\right]$$
$$+ A_5\left[-\frac{1}{\pi\delta_2}\left(\frac{1}{\alpha\sin\alpha}\right)^{1/2}(1-\alpha\operatorname{ctn}\alpha)(\operatorname{ker}_1 S + K_3\operatorname{kei}_1 S) - \frac{2\mu}{\pi\delta_2^{1/2}}\left(\frac{\alpha}{\sin\alpha}\right)^{1/2}(\operatorname{ker}_1' S + K_3\operatorname{kei}_1' S)\right]$$
$$+ A_6\left[\frac{1}{\pi\delta_2}\left(\frac{1}{\alpha\sin\alpha}\right)^{1/2}(1-\alpha\operatorname{ctn}\alpha)(\operatorname{kei}_1 S - K_3\operatorname{ker}_1 S) + \frac{2\mu}{\pi\delta_2^{1/2}}\left(\frac{\alpha}{\sin\alpha}\right)^{1/2}(\operatorname{kei}_1' S - K_3\operatorname{ker}_1' S)\right]$$

$$\dot{\beta}_r = A_3\left[\frac{1}{2}\left(\frac{1}{\alpha\sin\alpha}\right)^{1/2}(1-\alpha\operatorname{ctn}\alpha)(K_1\operatorname{ber}_1 S + K_2\operatorname{bei}_1 S) + \mu\delta_2^{1/2}\left(\frac{\alpha}{\sin\alpha}\right)^{1/2}(K_1\operatorname{ber}_1' S + K_2\operatorname{bei}_1' S)\right]$$
$$+ A_4\left[\frac{1}{2}\left(\frac{1}{\alpha\sin\alpha}\right)^{1/2}(1-\alpha\operatorname{ctn}\alpha)(K_2\operatorname{ber}_1 S - K_1\operatorname{bei}_1 S) + \mu\delta_2^{1/2}\left(\frac{\alpha}{\sin\alpha}\right)^{1/2}(K_2\operatorname{ber}_1' S - K_1\operatorname{bei}_1' S)\right]$$
$$+ A_5\left[\frac{1}{\pi}\left(\frac{1}{\alpha\sin\alpha}\right)^{1/2}(1-\alpha\operatorname{ctn}\alpha)(-K_5\operatorname{ker}_1 S + K_4\operatorname{kei}_1 S)\right.$$
$$\left. + \frac{2\mu\delta_2^{1/2}}{\pi}\left(\frac{\alpha}{\sin\alpha}\right)^{1/2}(-K_5\operatorname{ker}_1' S + K_4\operatorname{kei}_1' S)\right]$$
$$+ A_6\left[\frac{1}{\pi}\left(\frac{1}{\alpha\sin\alpha}\right)^{1/2}(1-\alpha\operatorname{ctn}\alpha)(K_4\operatorname{ker}_1 + K_5\operatorname{kei}_1 S) + \right. \qquad (4.16)$$

$$+\frac{2\mu\delta_2^{1/2}}{\pi}\left(\frac{\alpha}{\sin\alpha}\right)^{1/2}\left(K_4 \operatorname{ker}_1' S + K_5 \operatorname{kei}_1' S\right)\Big] \tag{4.16}$$

where a "prime" indicates differentiation of a Kelvin function with respect to α. It is possible to prove convergence of the asymptotic series for Y_1 and Y_2 by application of the work of Langer.[14] Convergence proofs have not been carried out for the derivatives of Y_1 and Y_2 but may be possible on the basis of the same work.

The stress resultants and displacements may now be written in terms of β_r, $\dot{\beta}_r$, ψ, and $\dot{\psi}$. The horizontal stress resultant as defined previously is given by,

$$P_H = \frac{Eh^2}{mR}\psi\csc\alpha \tag{4.17}$$

The vertical stress resultant may be found from the equilibrium equations and is given by

$$P_V = \frac{Eh^2}{mR}A_1\csc\alpha \tag{4.18}$$

where A_1 is an arbitrary constant of integration. The usual stress resultants N_α and V_r may be written if the components of P_H and P_V are computed in the α and θ directions. N_θ may be found from equilibrium.

$$\begin{aligned} N_\alpha &= P_H\cos\alpha + P_V\sin\alpha \\ V_r &= -P_H\sin\alpha + P_V\cos\alpha \\ N_\theta &= P_H\cos\alpha + \dot{P}_H\sin\alpha. \end{aligned} \tag{4.19}$$

In terms of ψ and $\dot{\psi}$ these quantities may be written

$$N_\alpha = \frac{Eh^2}{mR}\psi + \frac{Eh^2}{mR}A_1$$
$$V_r = -\frac{Eh^2}{mR}\psi + \frac{Eh^2}{mR}A_1 \operatorname{ctn}\alpha \tag{4.20}$$
$$N_\theta = \frac{Eh^2}{mR}\dot{\psi}$$

The stress couples which may be found from the relationships between the rotation β_r, $M_{\alpha r}$, and $M_{\theta r}$ may be written

$$M_{\alpha r} = \frac{D}{R}\left[\dot{\beta}_r + \nu\beta_r \operatorname{ctn}\alpha\right]$$
$$M_{\theta r} = \frac{D}{R}\left[\nu\dot{\beta}_r + \beta_r \operatorname{ctn}\alpha\right] \tag{4.21}$$

In terms of u and the reversed normal displacement w_r the Equations (2.5) relating extensional strains to displacements become

$$\epsilon_\alpha^\circ = \frac{1}{R}\left(\dot{u} - w_r\right)$$
$$\epsilon_\theta^\circ = \frac{1}{R}\left(\dot{u} \operatorname{ctn}\alpha - w_r\right) \tag{4.22}$$

The quantity w_r may be eliminated by subtracting ϵ_θ^0 from ϵ_α^0 giving

$$\epsilon_\alpha^\circ - \epsilon_\theta^\circ = \frac{1}{R}\left(\dot{u} - u \operatorname{ctn}\alpha\right) \tag{4.23}$$

Using the Equations (2.6) a relation between known stress resultants and the strains may be written

$$\epsilon_\alpha^\circ - \epsilon_\theta^\circ = \frac{1}{K}\left(N_\alpha - N_\theta\right) \tag{4.24}$$

The resulting nonhomogeneous first order differential equation in u may be solved giving

$$u = \frac{(1+\nu)h}{m}\left(A_1 \sin\alpha \log\left|\tan\frac{\alpha}{2}\right| - \psi + A_2 \sin\alpha\right) \tag{4.25}$$

where A_2 is the new arbitrary constant of integration. The only unknown left in (4.22) is w_r. This quantity may finally be written as

$$w_r = \frac{h}{m}\Big\{-\dot{\psi} - \psi\,\mathrm{ctn}\,\alpha + A_1\Big[(1+\nu)\cos\alpha \times \\ \times \log\left|\tan\frac{\alpha}{2}\right| + \nu\Big] + A_2(1+\nu)\cos\alpha\Big\} \tag{4.26}$$

Thus the general solution of the sixth order system of equations in the free region has been written in terms of the known functions ψ, β_r, and six unknown coefficients which will be determined later.

C. SOLUTION IN THE CONTACT REGION

Based on (4.4), (4.5), and (4.6) the dimensionless form of the system of equations which will be used to solve the spherical cap problem in the contact region is

$$\dot{\bar{N}}_\alpha + (\bar{N}_\alpha - \bar{N}_\theta)\,\mathrm{ctn}\,\alpha + C_1\bar{V} = 0 \tag{4.27a}$$

$$\dot{\bar{M}}_\alpha + (\bar{M}_\alpha - \bar{M}_\theta)\,\mathrm{ctn}\,\alpha - C_2\bar{V} = 0 \tag{4.27b}$$

$$\bar{N}_\alpha = \dot{\bar{u}} + \nu\bar{u}\,\mathrm{ctn}\,\alpha + (1+\nu)\bar{w} \tag{4.27c}$$

$$\bar{N}_\theta = \nu \dot{\bar{u}} + \bar{u} \operatorname{ctn}\alpha + (1+\nu)\,\bar{w} \tag{4.27d}$$

$$\bar{M}_\alpha = \dot{\beta} + \nu\beta \operatorname{ctn}\alpha \tag{4.27e}$$

$$\bar{M}_\theta = \nu\dot{\beta} + \beta \operatorname{ctn}\alpha \tag{4.27f}$$

$$\bar{V} = \dot{\bar{w}} - \bar{u} + \beta \tag{4.27g}$$

$$\bar{w}_o = (1-\cos\alpha) - \bar{w}\cos\alpha + \bar{u}\sin\alpha \tag{4.27h}$$

where

$$C_1 = \frac{5(1-\nu)}{12}\,, \qquad C_2 = \frac{5R^2(1-\nu)}{h^2} \tag{4.28}$$

The dimensionless stress resultants and displacements are defined in the same manner as in the previous chapter on the half-ring problem. The normal stress q^+ is found from the second of the equilibrium Equations (4.4).

A fourth order differential equation in $\bar{w}$ will now be derived and solved. Substitution of (4.27e), (4.27f) and (4.27g) into (4.27b) yields the result

$$\ddot{\beta} + \dot{\beta}\operatorname{ctn}\alpha - (\operatorname{ctn}^2\alpha + C_3)\beta = C_2(\dot{\bar{w}} - \bar{u}) \tag{4.29}$$

where

$$C_3 = C_2 + \nu \tag{4.30}$$

The result of substituting (4.27c), (4.27d), and (4.27g) into (4.27a) and using (4.27h) to eliminate $\overline{u}$ is given by

$$\ddot{\overline{w}} + \left[-\sec\alpha \csc\alpha + (C_1 + \nu)\tan\alpha\right] \dot{\overline{w}} - C_4 \overline{w}$$
$$= C_4 (\overline{w}_o \sec\alpha + 1 - \sec\alpha) - C_1 \beta \tan\alpha \tag{4.31}$$

The Equation (4.31) may now be solved for β and this result along with the restraint Equation (4.27h) can be substituted into (4.29) yielding

$$\ddot{\ddot{\overline{w}}} - (2\csc\alpha \sec\alpha + \tan\alpha)\dddot{\overline{w}} + (3\csc^2\alpha + 2 - C_3)\ddot{\overline{w}}$$
$$-(3\csc^4\alpha \tan\alpha - C_3 \csc^2\alpha \tan\alpha + C_7 \tan\alpha)\dot{\overline{w}} - C_6 \overline{w}$$
$$= C_6 (1 - \sec\alpha + \overline{w}_o \sec\alpha) \tag{4.32}$$

where

$$C_6 = C_4 + C_5 \quad ; \quad C_4 = C_1 + \nu - 1$$
$$C_5 = C_1 C_2 - C_3 C_4 \tag{4.33}$$
$$C_7 = -2C_1 + 1 - 2\nu + \nu C_2 + \nu C_1 + \nu^2$$

Because (4.32) governs only the small contact region of the spherical cap, approximations such as

$$\csc\alpha \simeq \frac{1}{\alpha} \; ; \; \operatorname{ctn}\alpha \simeq \frac{1}{\alpha} \; ; \; \tan\alpha \simeq \alpha \; ; \; \sec\alpha \simeq 1 \tag{4.34}$$

will be introduced. The quantity C_6 may be written

$$C_6 = (1-\nu)C_2 - (1+\nu)C_1 + 2\nu - \nu^2 \tag{4.35}$$

The ratio $(R/h)^2$ is considered to be a large number. Hence, the approximations

$$C_6 \simeq (1-\nu)C_2 \quad , \quad C_3 \simeq C_2 \tag{4.36}$$

will also be introduced. The Equation (4.32) may now be approximated by

$$\begin{aligned} &\alpha^4 \ddddot{\bar{w}} - 2\alpha^3 \dddot{\bar{w}} + \alpha^2 (3 - Kb^4\alpha^2)\ddot{\bar{w}} \\ &\quad - \alpha(3 - Kb^4\alpha^2)\dot{\bar{w}} - b^4\alpha^4 \bar{w} \\ &\quad = b^4\alpha^4\left[\bar{w}_0 \sec\alpha + (1 - \sec\alpha)\right] \end{aligned} \tag{4.37}$$

with the use of (4.34), (4.36) and introduction of the definitions

$$b^4 = \frac{5R^2(1-\nu)^2}{h^2} \quad , \quad Kb^4 = \frac{5R^2(1-\nu)}{h^2} \tag{4.38}$$

It should be noted that only one term has been kept in the series expansion of the coefficient of $\dddot{\bar{w}}$ whereas two terms have been kept in the coefficients of $\ddot{\bar{w}}$ and $\dot{\bar{w}}$. This is done in order that the quantity $Kb^4\alpha^2$ may assume values of the same order of magnitude as three.

The homogeneous solution of (4.37) may be found by power series techniques. For computational reasons it has been found convenient to rewrite the homogeneous part of (4.37) in the form

$$\begin{aligned} &z^4 \bar{w}'''' - 2z^3 \bar{w}''' + z^2(3 - Kb^2z^2)\bar{w}'' \\ &\quad - z(3 - Kb^2z^2)\bar{w}' - z^4\bar{w} = 0 \end{aligned} \tag{4.39}$$

where

$$z = b\alpha \tag{4.40}$$

and "prime" represents differentiation with respect to z. Substitution of the series

$$\overline{w} = \sum_{n=0}^{\infty} b_n \, z^{n+c} \tag{4.41}$$

into (4.39) gives the indicial equation

$$c(c^3 - 8c^2 + 20c - 16) = 0 \tag{4.42}$$

which has the four roots

$$c = 4, 2, 2, 0. \tag{4.43}$$

The nonlogarithmic solution which may be found when C = 4 is given by

$$\overline{w}_1 = z^4 + \frac{Kb^2}{24} z^6 + k_1(1+k_3) z^8 + \frac{Kb^2 k_1}{80}(2+k_3) z^{10} + k_1 k_2 (1+3k_3+k_4) z^{12} + \ldots \tag{4.44}$$

where

$$k_1 = \frac{1}{1152}, \quad k_2 = \frac{1}{9600}, \quad k_3 = K^2 b^4, \quad k_4 = k_3^2 \tag{4.45}$$

The other solutions of (4.39) may be found by repeated use of substitutions of the type

$$\overline{w}_2(z) = \overline{w}_1(z) \int v_1(z)\, dz \tag{4.46}$$

which serve to reduce the order of the equation by introduction of a new dependent variable. The other three solutions of the homogeneous equation may be stated as

$$
\begin{aligned}
\bar{w}_2 &= z^2 + \frac{Kb^2}{24} z^4 + k_1(6+2k_3) z^6 \\
&\quad + \frac{Kb^2 k_1}{24}(4+k_3) z^8 + 5k_1k_2(3+5k_3+k_4) z^{10} + \cdots \\
\bar{w}_3 &= z^2 \log z + \frac{Kb^2}{8} z^4 \log z + \cdots \\
&\quad + \frac{z^2}{2} + \frac{Kb^2}{48} z^4 + \cdots \\
\bar{w}_4 &= 1 + \frac{5Kb^2}{24} z^2 + k_1(1+k_3) z^4 + \cdots \\
&\quad + \frac{Kb^2}{3} z^2 \log z + \cdots
\end{aligned}
\tag{4.47}
$$

Expansion of the right hand side of (4.37) results in the expression

$$
\begin{aligned}
& b^4 \alpha^4 \left(w_0 \sec\alpha + 1 - \sec\alpha \right) \\
&\quad = b^4 \alpha^4 \left(w_0 - \frac{\alpha^2}{2} + \frac{w_0 \alpha^2}{2} + \cdots \right)
\end{aligned}
\tag{4.48}
$$

This series will be truncated after two terms since $\bar{w}_0$ and $\alpha^2/2$ both are small quantities. A particular solution of the resulting differential equation

$$
\begin{aligned}
& \alpha^4 \overset{....}{\bar{w}} - 2\alpha^3 \overset{...}{\bar{w}} + \alpha^2(3-Kb^4\alpha^2)\ddot{\bar{w}} - \alpha(3-Kb^4\alpha^2)\dot{\bar{w}} \\
&\quad - b^4\alpha^4 \bar{w} = b^4 \alpha^4 \left(\bar{w}_0 - \frac{\alpha^2}{2} \right)
\end{aligned}
\tag{4.49}
$$

is given by

$$
\bar{w}_p = -\bar{w}_0 + \frac{\alpha^2}{2} \tag{4.50}
$$

Finally the general solution of (4.50) may be written

$$\bar{w} = B_1 \bar{w}_1 + B_2 \bar{w}_2 + B_3 \bar{w}_3 + B_4 \bar{w}_4 + \bar{w}_p \tag{4.51}$$

where B_1, B_2, B_3, and B_4 are unknown constants of integration.

The remaining displacements may be expressed in terms of $\bar{w}$ and its derivatives as

$$\begin{aligned} \bar{u} &= \bar{w} \operatorname{ctn}\alpha + \bar{w}_o \csc\alpha + \operatorname{ctn}\alpha - \csc\alpha \\ \beta &= \frac{1}{C_1}\Big\{ - \ddot{\bar{w}} \operatorname{ctn}\alpha + (\csc^2\alpha - C_1 - \nu)\dot{\bar{w}} \\ &\quad + C_4 \bar{w} \operatorname{ctn}\alpha + C_4\left[\operatorname{ctn}\alpha - (1 - \bar{w}_o)\csc\alpha\right]\Big\} \end{aligned} \tag{4.52}$$

The stress resultants are given in terms of $\bar{u}$, $\bar{w}$, and β by Equations (4.27c) through (4.27g) while the normal stress q^+ may be given by the relation

$$q^+ = -\frac{1}{R}\left[\dot{V} + V \operatorname{ctn}\alpha - N_\theta - N_\alpha\right] \tag{4.53}$$

Certain symmetry and boundary conditions must be satisfied by the displacements and stress resultants at the origin $\alpha = 0$. The conditions

$$\bar{w}(0) = -\bar{w}_o \ , \quad \bar{u}(0) = 0 \ , \quad \left.\frac{\partial \bar{w}}{\partial \alpha}\right|_{\alpha=0} = 0 \tag{4.54}$$

are all automatically satisfied. The condition

$$\beta(0) = 0 \tag{4.55}$$

is satisfied only if $B_3 = 0$ and $B_4 = 0$ since the substitution of $\bar{w}_3$ and $\bar{w}_4$ into (4.52) yields a nonzero result at the origin due to the effect of the terms $Z^2 \log Z$ which appears in both of these solutions. When the condition on β is satisfied, the condition on the shear resultant is automatically satisfied. The normal stress resultants and the stress couples may have nonzero values at the origin.

The convergence properties of the series for $\overline{w}_1$ and $\overline{w}_2$ and their first three derivatives may be examined by means of a comparison of successive terms in the various expansions. If the n-th term in one of these series is called a_n, an examination of the recurrence relations which determine the various terms shows that

$$\left| a_{n+1} \right| \leq \left| K b^2 Z^2 a_n \right| \tag{4.56}$$

The values of R/h and $\overline{\alpha}$ which were chosen for the numerical portion of this analysis were such that $Kb^2Z^2 < .003$. Rapid convergence of the series for $\overline{w}_1$, $\overline{w}_2$ and their derivatives was thus assured.

D. BOUNDARY MATCHING OF THE TWO SOLUTIONS AND NUMERICAL RESULTS

As the general solutions have now been found in both the contact and free regions, the only remaining steps are the matching of the stress resultant and deformation quantities at the common boundary $\alpha = \overline{\alpha}$ and the satisfying of displacement conditions at the fixed edge $\alpha = \alpha_e$. The same procedure will be used for the spherical cap as was used in the previous chapter on the half-ring.

Consisting of Equations (4.14), (4.20), (4.21), (4.25), and (4.26) the solutions in the free region have the six unknown coefficients A_1,

A_2, A_3, A_4, A_5, and A_6. In the contact region the unknown coefficients are B_1 and B_2. As before, neither w_o nor $\bar{\alpha}$ have been specified up to this point. Thus nine boundary conditions and either of w_o or $\bar{\alpha}$ must be specified for the solution of the unknown quantities.

At the boundary $\alpha = \alpha_e$ the requirements that

$$u(\alpha_e) = 0 \; , \; \beta_r(\alpha_e) = 0 \; , \; w_r(\alpha_e) = 0 \qquad (4.57)$$

will satisfy the condition for a fixed boundary. Thus the six remaining conditions which may be applied at the common boundary $\bar{\alpha}$ are

$$\begin{aligned} u_c(\bar{\alpha}) &= u_f(\bar{\alpha}) \; , \; N_{\alpha c}(\bar{\alpha}) = N_{\alpha f}(\bar{\alpha}) \\ w_c(\bar{\alpha}) &= -w_{fr}(\bar{\alpha}) \; , \; M_{\alpha c}(\bar{\alpha}) = -M_{\alpha fr}(\bar{\alpha}) \\ \beta_c(\bar{\alpha}) &= -\beta_{fr}(\bar{\alpha}) \; , \; V_c(\bar{\alpha}) = -V_f(\bar{\alpha}) \end{aligned} \qquad (4.58)$$

where the subscript "f" indicates quantities defined in the free region, the subscript "c" indicates quantities defined in the contact region, and the subscript "r" indicates the reverse sign convention. Because of the continuity requirements on N_α and M_α it follows that N_θ and M_θ will automatically be made continuous as a result.

Again, because of the complexity of the system of equations, digital techniques were used to find the unknown quantities w_o, B_1, B_2, A_1, A_2, A_3, A_4, A_5, and A_6 in terms of the specified contact length $\bar{\alpha}$. These results were then used to compute the dimensionless results shown in Figures 8-12 and Tables 11-15. The definitions for the dimensionless quantities remain unchanged.

Figure 8 and 9 show typical curves for the displacements $\overline{u}$, $\overline{w}$, and β. The principal deformation is the normal displacement $\overline{w}$. The values of β shown in Figure 9 are much smaller than would be predicted by an analysis using the Kirchhoff hypothesis indicating that shear deformation is apparently important. In Table 11 the quantity R/h is chosen to be 10, while in Table 12 the ratio has the smaller value of 5. In the case of the thinner shell the quantity β is larger as would be expected.

The normal stress resultants and the stress couples are shown in Figure 10. The quantity $\overline{M}_\alpha$ is negative in most of the contact region indicating the flattening of the shell. The moment is positive for most of the free region indicating the sharp curvature near the edge of the contact region. The normal stress resultant $\overline{N}_\alpha$ is negative and nearly constant as would be expected for a cap.

Typical shear resultant and normal pressure curves are shown in Figure 11. An interesting aspect of this curve is the indication that $\overline{q}^+$ approaches a nonzero limit as the contact length $\overline{\alpha}$ approaches zero. This fact has been demonstrated numerically for the spherical cap problem and analytically for the half-ring problem. Even though $\overline{q}^+$ is nonzero for $\overline{\alpha} = 0$, the limit value of applied load is zero as would be expected for the shear deformation theory. This might be viewed as the result of a finite nonzero pressure acting over an area with zero magnitude. Finally Figure 12 shows a plot of the normal

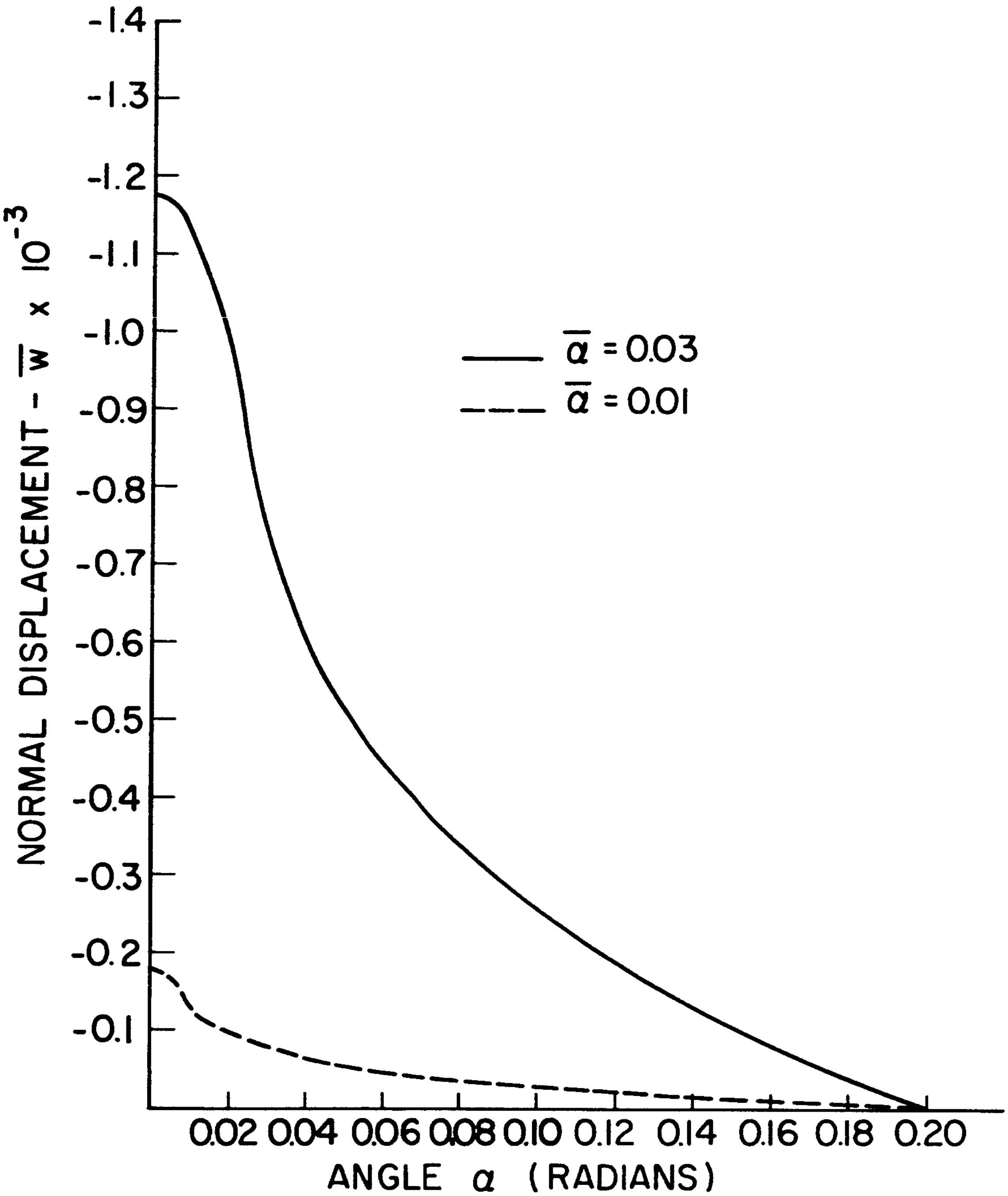

Figure 8. Normal displacement versus position for cap. $\overline{\alpha}$ = 0.01, 0.03, α_e = 0.2, R/h = 10, R = 5 in., E = 222 lb/in.^{2}, ν = 0.5.

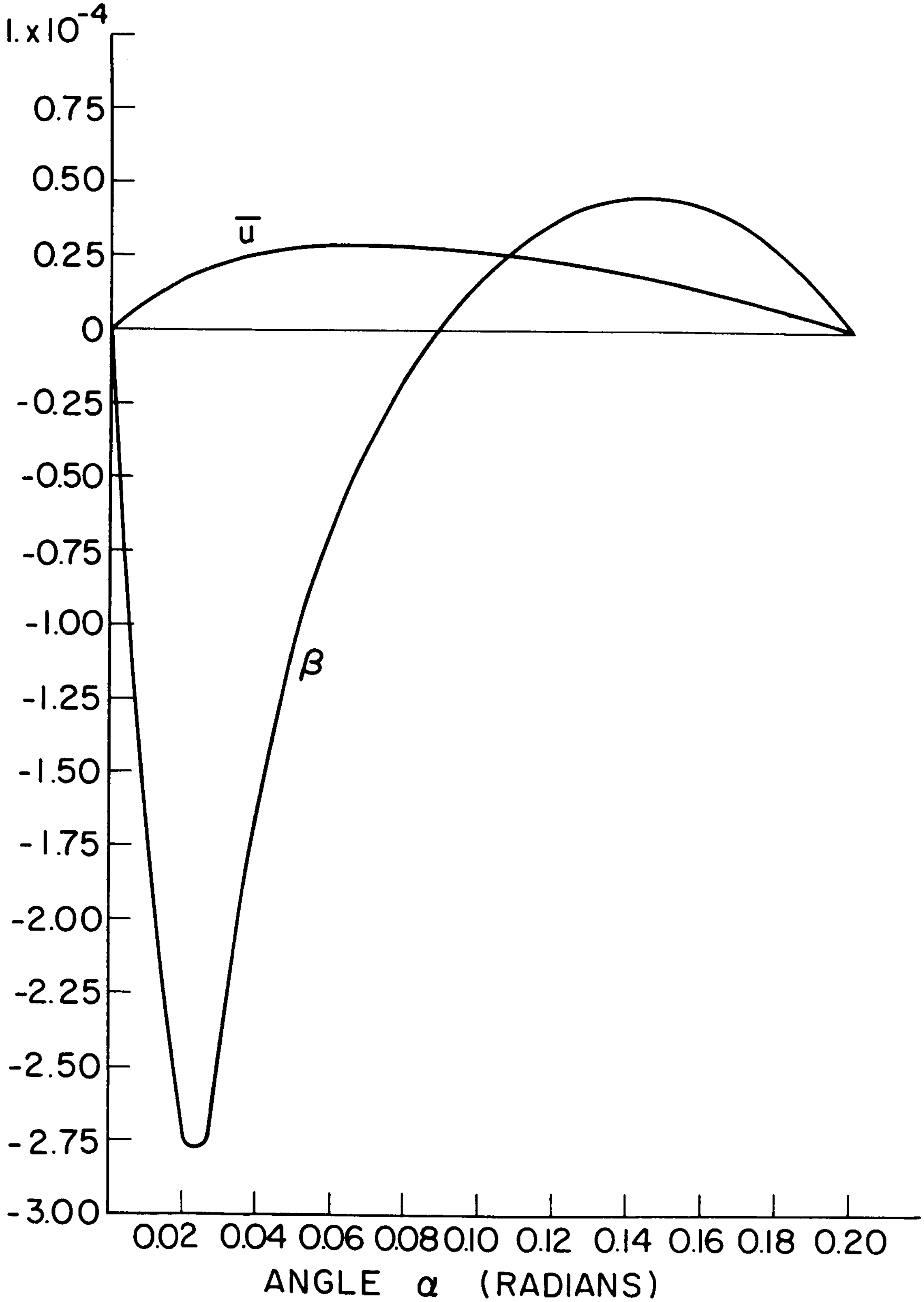

Figure 9. Displacement $\overline{u}$ and rotation β versus position for cap. $\overline{\alpha} = 0.03$, $\alpha_e = 0.2$, $R/h = 10$, $R = 5$ in., $E = 222$ lb/in., $\nu = 0.5$.

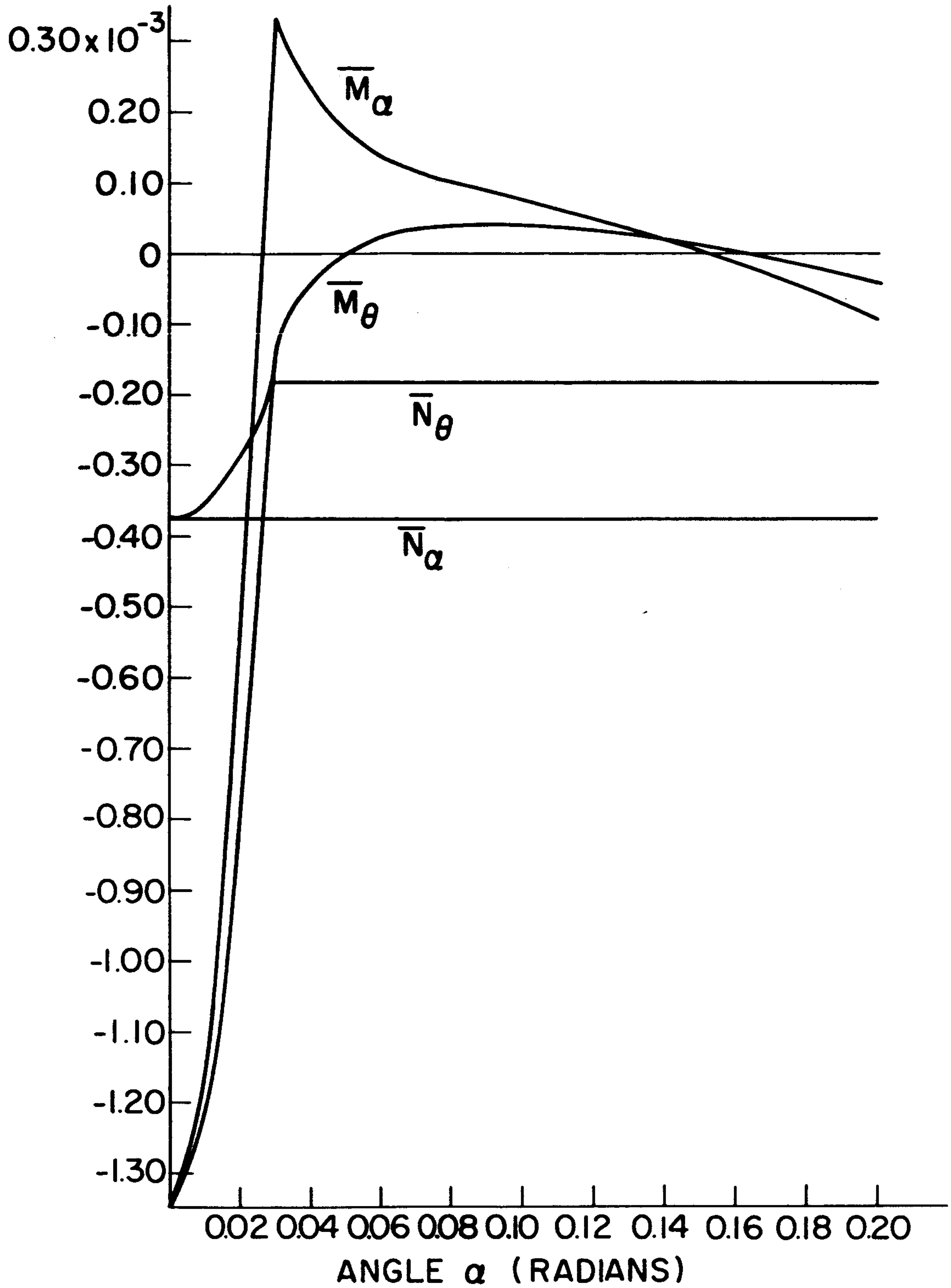

Figure 10. Normal stress resultants and stress couples versus position for cap. $\bar{\alpha} = 0.03$, $\alpha_e = 0.2$, $R/h = 10$. $R = 5$ in., $E = 222$ lb/in^2. $\nu = 0.5$.

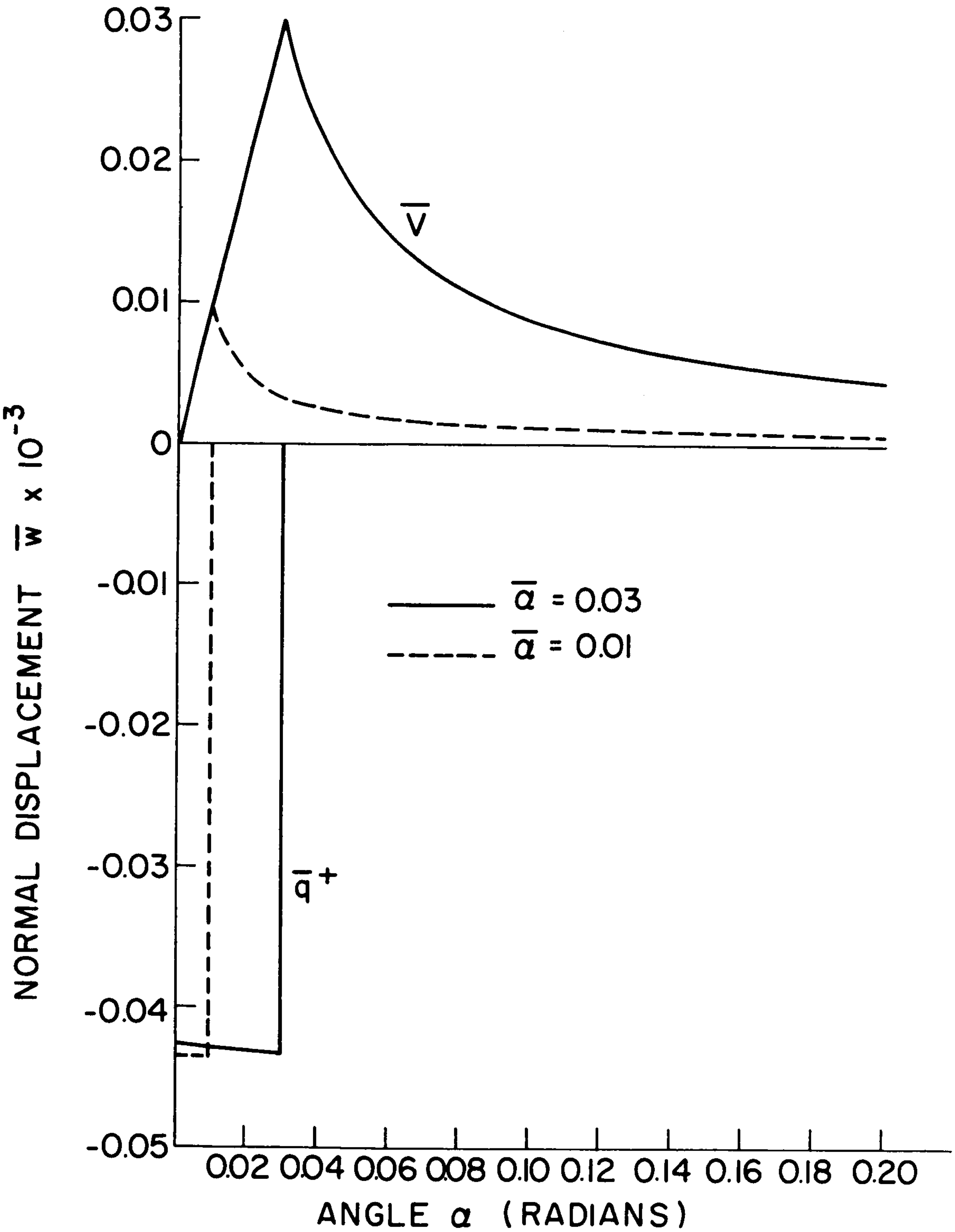

Figure 11. Shear resultant $\overline{V}$ and normal pressure $\overline{q}^{+}$ versus position for cap. $\overline{\alpha} = 0.01, 0.03$, $\alpha_e = 0.2$, $R/h = 10$, $R = 5$ in., $E = 222$ lb/in.^{2}, $\nu = 0.5$.

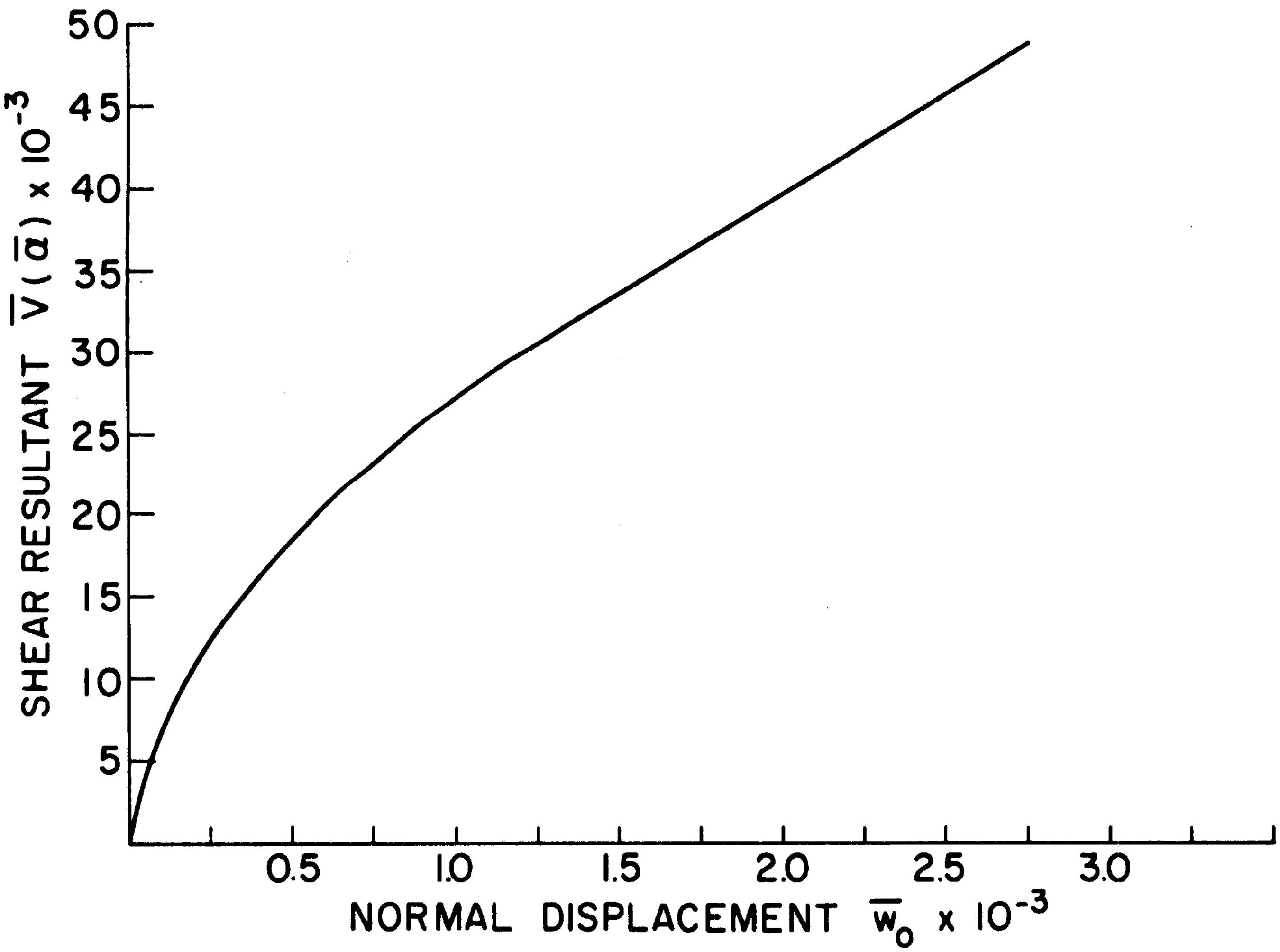

Figure 12. Normal displacement $\overline{w}_o$ versus applied load as given by $\overline{V}(\overline{\alpha})$ for cap. $R/h = 10$, $R = 5$ in., $E = 222$ lb/in.^{2}

load, as given by the shear resultant $\overline{V}$ evaluated at $\alpha = \overline{\alpha}$, versus the central normal deflection $\overline{w}_o$.

The Tables 11 and 12 are computed for two values of R/h with other parameters held constant. The deformations shown in the two tables are similar, as forced by the boundary conditions, but the stress quantities are greater for the thicker ring as would be expected. In Tables 12 and 13 the length of the cap, as given by α_e, is varied. As would be expected, the deformations of the longer cap are larger while the stress quantities are smaller. In Tables 13 and 14 the contact length $\overline{\alpha}$ is varied. Smaller stress and deflections are noted for the smaller value of $\overline{\alpha}$.

In Tables 14 and 15 the material properties of the cap are varied. Since the magnitudes of the dimensionless quantities are similar for the two tables, the deformations $\overline{u}$ and $\overline{w}$ of the caps are similar while the stresses carried by the stiff cap are much greater as would be expected. It should be noted in Table 15 that the rotation β and the stress couples M_α and M_θ are positive in the contact region. This would not be expected in view of the moment equilibrium equation from (4.4). This may point out a shortcoming of some aspect of the theoretical or numerical analysis. This behavior was noted only for caps which were stiff and thick at the same time and seemed to affect the stress couples and rotation only in the small region in the vicinity of the contact area.

When the numerical results were substituted into the original equilibrium Equations (4.4), it was found that the first two equations were well satisfied. However, the moment equation was not well satisfied, especially for caps with large α_e and R/h. At the present time the source of this difficulty is unknown. Future studies of the approximations which have been made in the analytical and numerical work should lead to the source of the problem.

CHAPTER V

EXPERIMENTS ON HALF-RINGS

Figure 13 is a sketch of the model used to test the shear deformation theory. The network of lines was inked on the surface so that

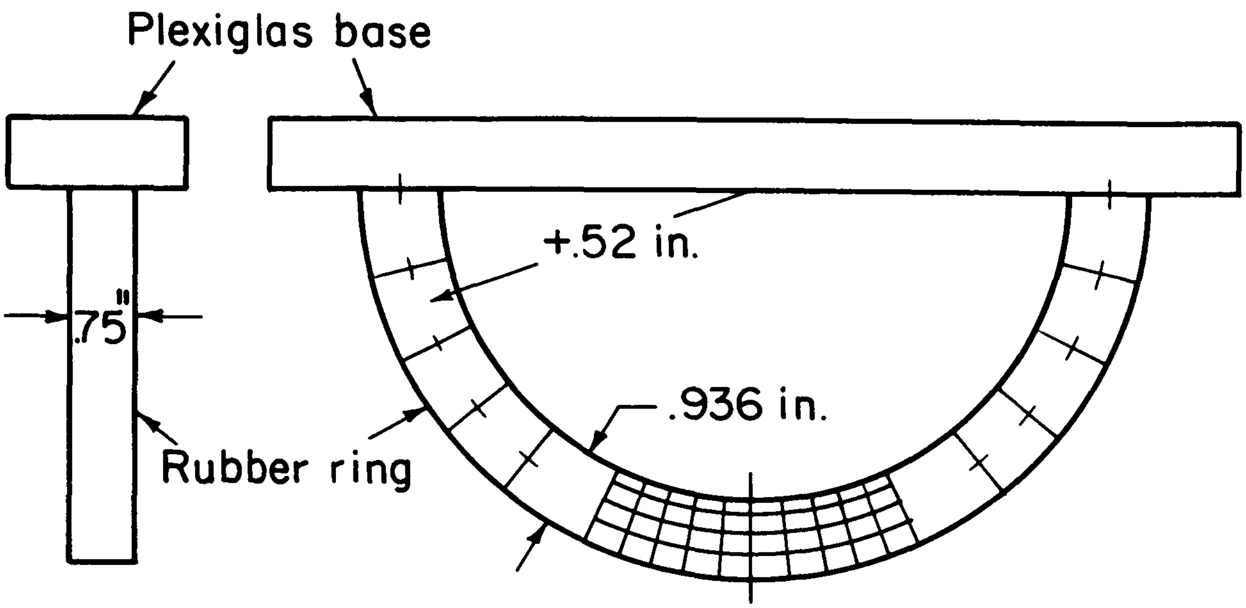

Figure 13. Sketch of rubber test ring with E = 227.6 lb/in.^{2}, ν = .464, and G = 77.6 lb/in.^{2}.

photographs of the deformation field could be taken. Dow-Corning Silastic RTV room-curing rubber was chosen as the material for two reasons. First, it is easy to cast into any desired shape. Second, it can be easily bonded onto most hard metallic and nonmetallic surfaces during the casting process. Thus the model was cast and bonded to hard Plexiglas at the fixed edge $\alpha = \alpha_e$ in one operation.

Because of the ease of molding this material it was also easy to cast strips for modulus tests at the same time the half-ring was being cast. The modulus remained nearly constant over a large range of deflections and loading rates and so, for the purposes of the current experiment, it was assumed that the material was Hookean.

The simplest tests to carry out were the load versus central deflection tests shown in Figure 14. The half-ring was mounted in a table model Instron testing machine and traveling head deflection versus applied load graphs were produced directly. Pictures of the ring were also made simultaneously to record the deformation pattern. The Instron data was easily reproducible and showed excellent agreement with shear deformation theory. The negative of the deformed ring for $\bar{N}$ (90) = .007 was enlarged and superimposed on a negative of the undeformed ring. From this picture the experimental points shown in Figure 15 were determined. Agreement with shear theory was again good.

In Figure 16 is shown a plot of applied load versus length of the contact region. The experimental data was obtained by inking the outer edge $\zeta = h/2$ of the ring and pressing it against a piece of paper. An independent check of this data was accomplished when tests with a pressure sensing device mounted in the rigid contacting surface yielded about the same results. A possible reason for the discrepancy is the neglecting of all dependence on the coordinate direction normal to the α - ζ plane. It is well-known that the cross-section of a rectangular beam does not remain rectangular during bending due to the Poisson's ratio effect. The result in this experiment was that the

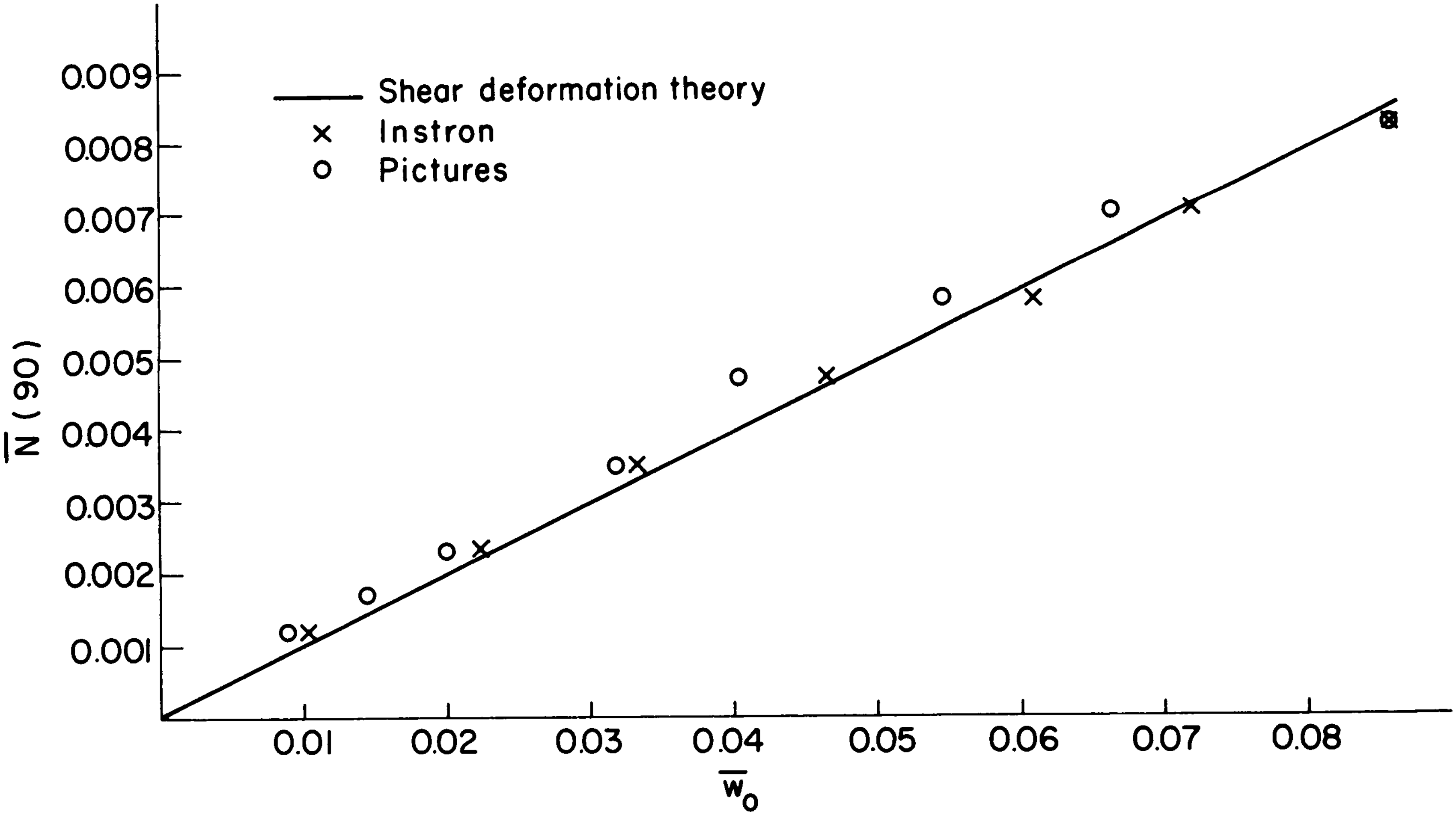

Figure 14. Applied load versus central deflection for experiment.

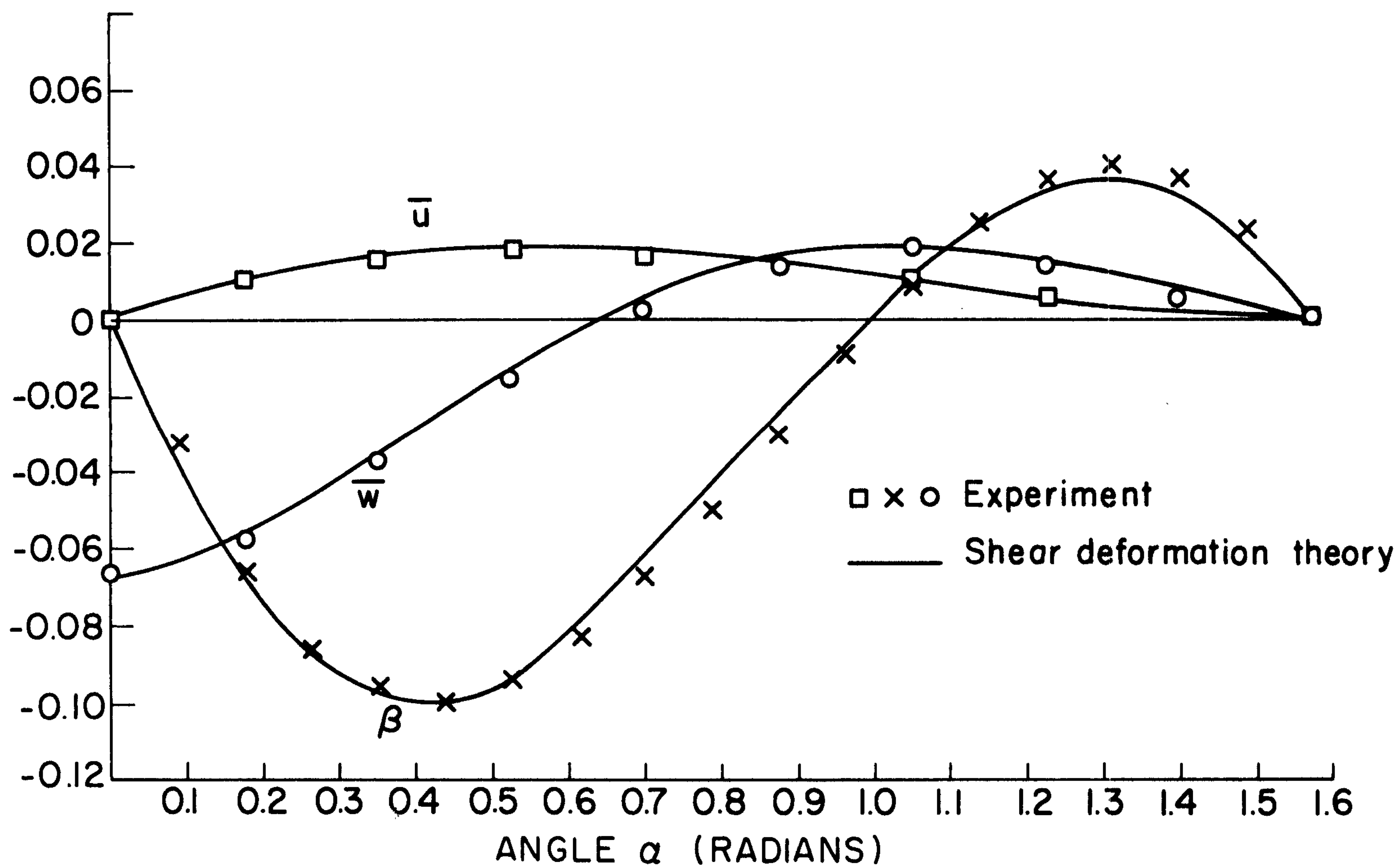

Figure 15. Deformation for experimental ring. $\overline{\alpha} = 0.525$, $R = 4.52$ in., $R/h = 4.58$, $E = 227.6$ lb/in^2. $\nu = .464$.

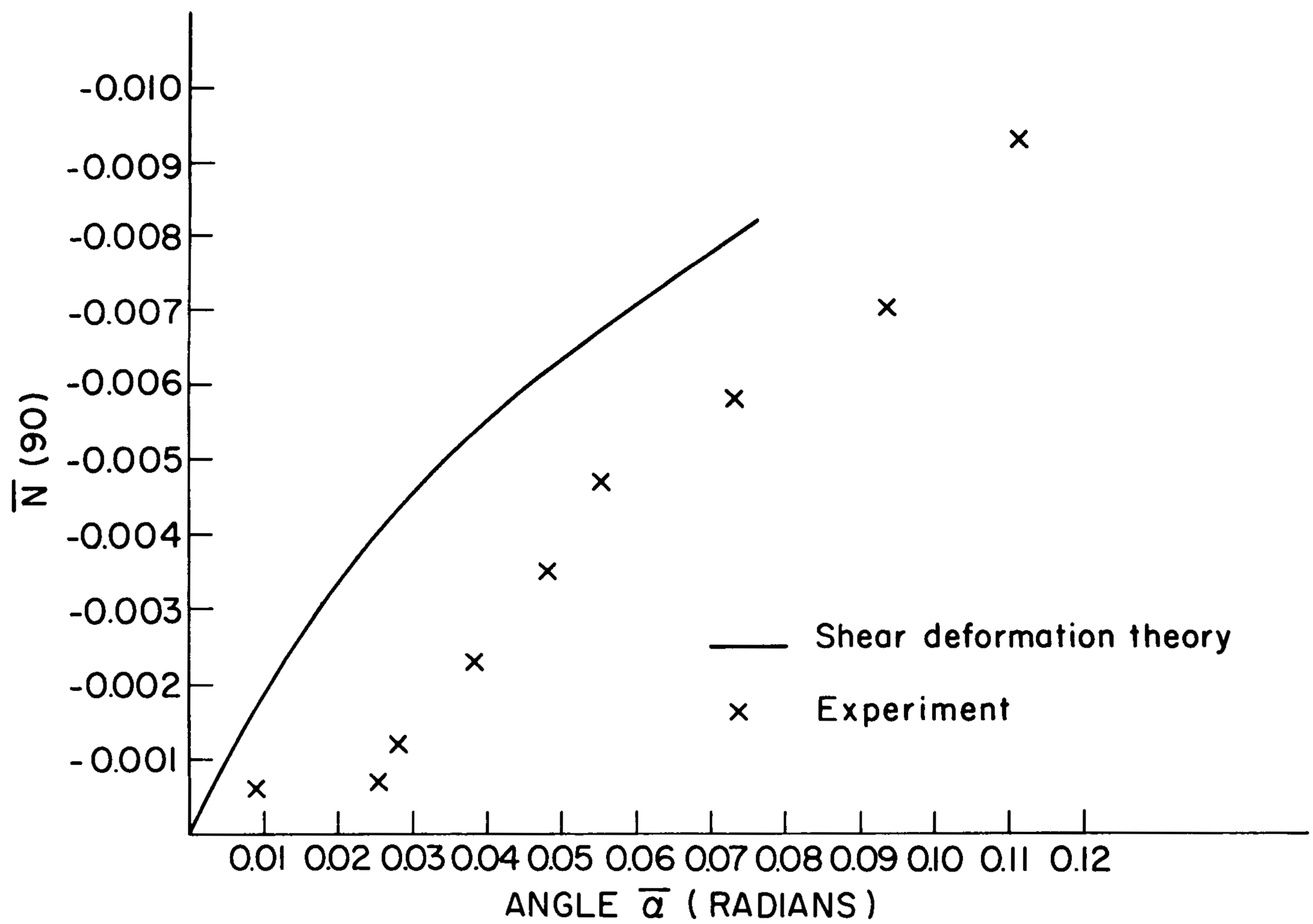

Figure 16. Applied load versus contact length for experiment.

ink prints of the contact region were not rectangular thus making it difficult to determine a true value for $\bar{\alpha}$.

Figure 17 shows a plot of pressure distribution versus location. There is no indication of peak values of pressure near the edge of the contact region in the experimental data. However these peak values have been observed for thin rings in other research involving thin ring analogies for pneumatic tires. The fact that a nonzero value of $\bar{q}^{+}$ is predicted at the edge of the contact region is a shortcoming of the shear deformation theory as applied in this thesis. It is presumed necessary to include the effects of transverse normal deformation to overcome this difficulty.

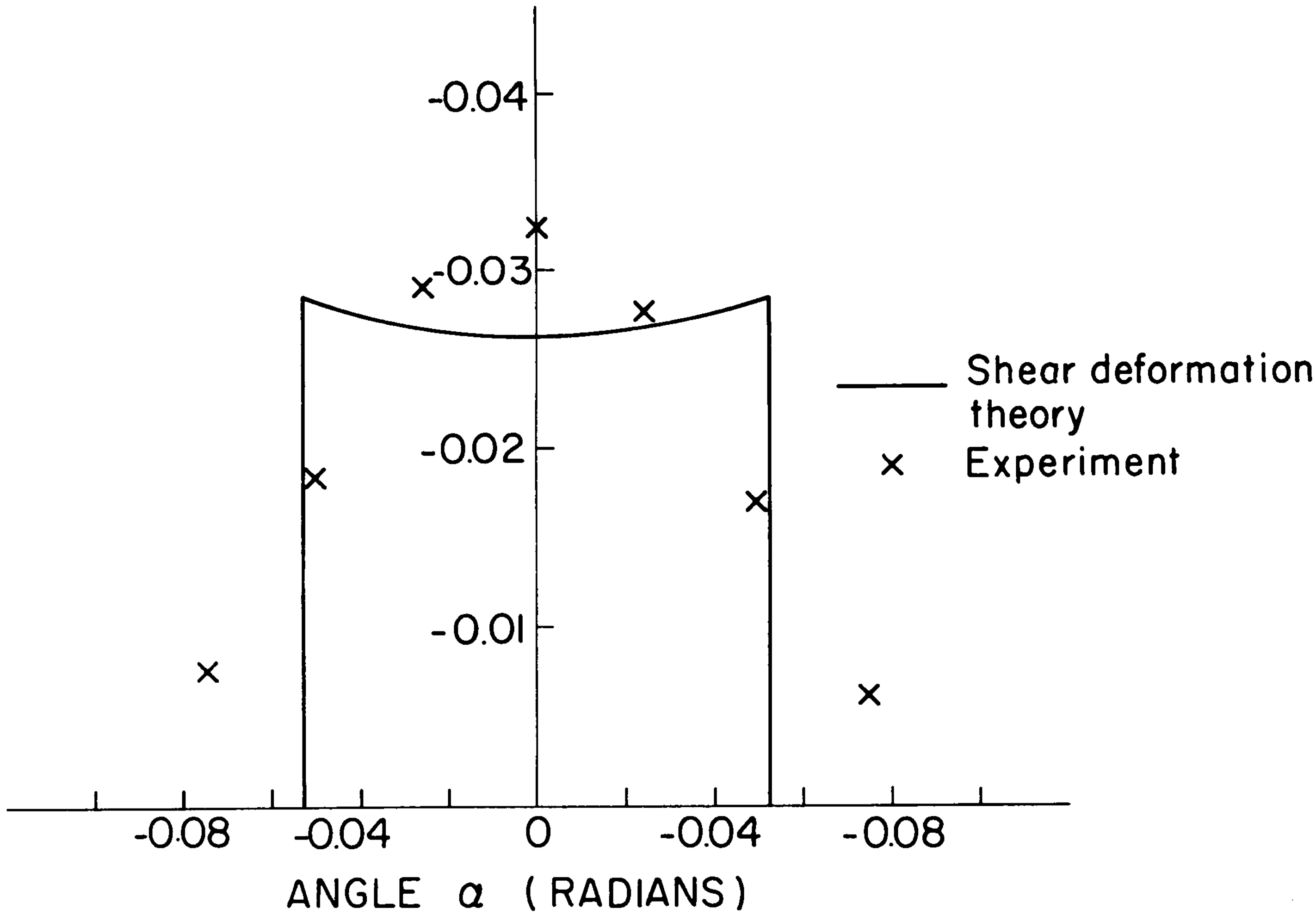

Figure 17. Pressure distribution $\overline{q}^{+}$ versus α for experiment.

CHAPTER VI

CONCLUSIONS

A class of elastic shell problems involving the contact of certain symmetric shells with rigid flat smooth surfaces has been considered. Shear deformation has been included in the governing equations. It has been shown that a possible means for analyzing such problems involves the solving of the system equations separately in the free region and in the contact region. The two solutions are then matched at the common boundary at the edge of the contact region.

Numerical computations of the deformations and stress resultants for half-rings and spherical caps with radius-to-thickness ratios which are 10 or less have shown that the effect of shear deformation should be included to avoid serious error. Experiments performed on half-rings tend to verify the shear-deformation analysis. An exception is the prediction of the normal stress in the contact region. Although the shear-deformation analysis offers a better approximation than the classical bending theory, it may be surmised that transverse normal stresses and strains should not be neglected for accurate predictions of this quantity.

LIST OF REFERENCES

1. Wu, C. H. and Plunkett, R., "On the Contact Problem of Thin Circular Rings," J. Appl. Mech., 32 (1965), 11-21.

2. Gudramovich, V. S. and Mossakovskii, V. I., "A Contact Problem for a Circular Ring which is Reinforced by a Cylindrical Shell," [In Russian], Izv. Akad. N. SSSR, OTN, Mech. i Mashinstro., No. 2 (1961), 153-156.

3. Timoshenko, S., Strength of Materials Vol. II, 3rd ed., D. Van Nostrand, New York (1956).

4. Timoshenko, S. and Woinowsky-Krieger, S., Theory of Plates and Shells, 2nd ed., McGraw-Hill, New York (1959).

5. Clark, S. K., Tielking, J. T., Robbins, D. H., and Schreyer, H. L., "Analysis of a Pneumatic Tire under Load," ORA Report 02957-17-T, Univ. of Mich., Ann Arbor (1963).

6. Clark, S. K., "An Analog for the Static Loading of a Pneumatic Tire," ORA Report 02857-19-T, Univ. of Mich., Ann Arbor (1964).

7. Clark, S. K., "An Analog for the Rolling Pneumatic Tire under Load," ORA Report 02957-20-T, Univ. of Mich., Ann Arbor (1965).

8. Essenburg, F., "On a Class of Nonlinear Axisymmetric Plate Problems," J. Appl. Mech., 27 (1960), 677-680.

9. Naghdi, P. M., "On the Theory of Thin Elastic Shells," Q. Appl. Math., 14 (1957), 369-380.

10. Naghdi, P. M., "The Effect of Transverse Shear Deformation on the Bending of Elastic Shells of Revolution," Q. Appl. Math., 15 (1957), 41-52.

11. Reissner, E., "On a Variational Theorem in Elasticity," J. Math. Phys., 29 (1950), 90-95.

12. DeSilva, C. N. and Cohen, H., "Some Remarks on Spherical Shells," J. Appl. Mech., 32 (1965), 121-128.

LIST OF REFERENCES (Concluded)

13. Reissner, E., "On the Theory of Thin Elastic Shells," H. Reissner Anniversary Volume, Edwards Press, Ann Arbor, Mich., (1949), 231-241.

14. Langer, R., "On the Asymptotic Solution of Ordinary Differential Equations, with Reference to the Stokes' Phenomenon about a Singular Point," Trans. Am. Math. Soc., 37 (1935), 397-416.

APPENDIX

NUMERICAL VALUES OF LOADS AND DISPLACEMENTS FOR HALF-RINGS AND SPHERICAL CAPS WITH VARIOUS GEOMETRICAL AND MATERIAL PROPERTIES

TABLE 1

SHEAR THEORY

Loads and deformations of half-ring for $\bar{\alpha} = 0.08$, R = 5 in., E = 222 lb/in.^{2}, R/h = 5, $\nu = 0.5$

α	β	$\bar{u}$	$\bar{w}$	$\bar{N}$	$\bar{M}$	$\bar{V}$	$\bar{q}^+$
0.02	-0.012	0.0015	-0.0844	-6.79×10^{-3}	-0.0610	0.0074	-0.0176
0.04	-0.025	0.0030	-0.0837	-6.84	-0.0596	0.0150	-0.0183
0.06	-0.037	0.0045	-0.0827	-6.92	-0.0571	0.0230	-0.0195
0.08	-0.048	0.0060	-0.0812	-7.04	-0.0536	0.0316	-0.0220
0.228	-0.117	0.0179	-0.0560	-8.24	-0.0175	0.0248	
0.470	-0.126	0.0239	-0.0232	-9.13	+0.0092	0.0170	
0.665	-0.092	0.0237	-0.0057	-9.68	0.0255	0.0085	
0.860	-0.037	0.0189	+0.0234	-9.85	0.0308	-0.0002	
1.055	-0.018	0.0121	0.0272	-9.65	0.0249	-0.0090	
1.250	+0.050	0.0055	0.0192	-9.09	0.0081	-0.0174	
1.445	+0.039	0.0015	0.0061	-8.19	-0.0192	-0.0252	
1.571	0.000	0.0000	0.0000				

TABLE 2

SHEAR THEORY

Loads and deformations of half-ring for $\bar{\alpha} = 0.16$, R = 5 in., R/h = 5, E = 222 lb/in.^{2}, $\nu = 0.5$

α	β	$\bar{u}$	$\bar{w}$	$\bar{N}$	$\bar{M}$	$\bar{V}$	$\bar{q}^+$
0.01	-0.008	0.001	-0.120	-0.0105	-0.0774	0.0020	-0.0108
0.06	-0.047	0.0064	-0.118	-0.0106	-0.0750	0.0125	-0.0120
0.11	-0.084	0.0115	-0.113	-0.0108	-0.0689	0.0254	-0.0151
0.16	-0.116	0.0163	-0.106	-0.0114	-0.0579	0.0431	-0.0206
0.355	-0.182	0.0305	-0.063	-0.0128	-0.0097	0.0323	
0.55	-0.169	0.0356	-0.016	-0.0139	+0.0241	0.0204	
0.745	-0.104	0.0323	+0.021	-0.0145	+0.0420	0.0077	
0.94	-0.020	0.0237	+0.038	-0.0145	+0.0435	-0.0053	
1.135	+0.050	0.0136	+0.036	-0.0140	0.0285	-0.0181	
1.33	0.075	0.0053	0.020	-0.0130	-0.0025	-0.0303	
1.525	0.026	0.0006	0.003	-0.0115	-0.0482	-0.0412	
1.571	0.000	0.0000	0.000				

TABLE 3

SHEAR THEORY

Loads and deformations of half-ring for $\bar{\alpha} = 0.1$, $R = 5$ in., $R/h = 10$, $E = 222$ lb/in.^{2}, $\nu = 0.5$

α	β	$\bar{u}$	$\bar{w}$	$\bar{N}$	$\bar{M}$	$\bar{V}$	$\bar{q}^+$
0.02	-0.0183	0.0018	-0.0958	-2.90×10^{-3}	-0.0454	1.66×10^{-3}	-2.17×10^{-3}
0.04	-0.0364	0.0037	-0.0951	-2.92	-0.0447	3.51	-2.49
0.06	-0.0541	0.0055	-0.0941	-2.94	-0.0435	5.76	-3.05
0.08	-0.0712	0.0073	-0.0925	-2.97	-0.0416	8.67	-3.93
0.10	-0.0874	0.0091	-0.0906	-3.01	-0.0389	12.56	-5.21
0.295	-0.1819	0.0231	-0.0571	-3.48	-0.0105	9.64	
0.49	-0.1809	0.0293	-0.0136	-3.83	+0.0100	6.36	
0.685	-0.1168	0.0274	+0.0227	-4.02	0.0217	2.84	
0.88	-0.0246	0.0201	0.0415	-4.07	0.0244	- 0.79	
1.075	+0.0597	0.0111	0.0402	-3.96	0.0177	- 4.39	
1.270	0.1005	0.0040	0.0239	-3.70	0.0021	- 7.83	
1.465	0.0636	0.0006	0.0049	-3.29	-0.0219	-10.97	
1.571	0.0000	0.0000					

TABLE 4

SHEAR THEORY

Loads and deformations of half-ring for $\bar{\alpha} = 0.1$, $R = 5$ in., $R/h = 50$, $E = 222$ lb/in.^{2}, $\nu = 0.5$

α	β	$\bar{u}$	$\bar{w}$	$\bar{N}$	$\bar{M}$	$\bar{V}$	$\bar{q}^+$
0.02	-0.020	0.0020	-0.1002	-0.148×10^{-3}	-0.999×10^{-2}	0.008×10^{-3}	-0.007×10^{-3}
0.04	-0.040	0.0040	-0.0996	-0.148	-0.996	0.059	-0.027
0.06	-0.060	0.0060	-0.0985	-0.149	-0.977	0.302	-0.115
0.08	-0.079	0.0079	-0.0969	-0.151	-0.914	1.108	-0.403
0.10	-0.100	0.098	-0.0949	-0.150	-0.956	0.620	-0.238
0.295	-0.215	0.0252	-0.0584	-0.173	-0.257	0.474	
0.49	-0.213	0.0318	-0.0093	-0.190	+0.246	0.311	
0.685	-0.133	0.0293	+0.0314	-0.199	0.532	0.137	
0.88	-0.020	0.0208	0.0515	-0.201	0.592	0.043	
1.075	+0.083	0.0107	0.0479	-0.196	0.423	-0.221	
1.270	0.130	0.0032	0.0271	-0.183	0.031	-0.391	
1.465	0.081	0.0002	0.0047	-0.163	-0.569	-0.546	
1.571	0.000	0.0000	0.0000				

TABLE 5

SHEAR THEORY

Loads and deformations of half-ring for $\bar{\alpha} = 0.1$, R = 5 in., R/h = 50, E = 30×10^{6} lb/in.^{2}, $\nu = 0.25$

α	β	$\bar{u}$	$\bar{w}$	$\bar{N}$	$\bar{M}$	$\bar{V}$	$\bar{q}^{+}$
0.02	-0.020	0.0020	-0.1002	-0.149×10^{-3}	-0.997×10^{-2}	$+0.03\times10^{-3}$	-0.017×10^{-3}
0.04	-0.040	0.0040	-0.0996	-0.150	-0.981	+0.20	-0.105
0.06	-0.059	0.0060	-0.0985	-0.154	-0.876	+1.30	-0.638
0.08	-0.074	0.0079	-0.0969	-0.169	-0.409	+6.14	-2.987
0.10	-0.100	0.0098	-0.0949	-0.151	-0.962	+0.44	-0.242
0.295	-0.216	0.0252	-0.0584	-0.174	-0.259	0.33	
0.49	-0.214	0.0318	-0.0093	-0.191	+0.247	0.22	
0.685	-0.134	0.0293	+0.0314	-0.201	0.535	0.10	
0.88	-0.020	0.0208	0.0515	-0.203	0.595	-0.03	
1.075	+0.083	0.0107	0.0479	-0.197	0.425	-0.16	
1.27	0.131	0.0032	0.0271	-0.184	0.031	-0.27	
1.465	0.082	0.0002	0.0047	-0.164	-0.570	-0.38	
1.571	0.000	0.0000	0.0000				

TABLE 6

BENDING THEORY

Loads and displacements of half-ring for $\bar{\alpha} = 0.08$, R = 5 in., R/h = 5, E = 222 lb/in.^{2}, ν = 0.5

α	β	$\bar{u}$	$\bar{w}$	$\bar{N}$	$\bar{M}$	$\bar{V}$	$\bar{q}^+$
0.00	-0.000	0.000×10^{-3}	-0.124	-0.0132	-0.0967	-0.000×10^{-3}	-1.21×10^{-3}
0.02	-0.020	2.2	-0.124	-0.0132	-0.0967	-0.630×10^{-3}	-1.21
0.04	-0.039	4.3	-0.123	-0.0132	-0.0969	-1.26×10^{-3}	-1.20
0.06	-0.059	6.4	-0.122	-0.0132	-0.0971	-1.89×10^{-3}	-1.19
0.08	-0.079	8.5	-0.121	-0.0132	-0.0973	-2.54×10^{-3}	-1.17
0.08	-0.079	8.5	-0.121	-0.0132	-0.0973	+0.0610	
0.275	-0.204	26.0	-0.087	-0.0154	-0.0309	0.0476	
0.470	-0.217	35.2	-0.039	-0.0170	+0.0179	0.0324	
0.665	-0.154	35.0	+0.005	-0.0180	0.0475	0.0159	
0.86	-0.052	27.8	0.032	-0.0183	0.0565	-0.0011	
1.055	+0.047	17.5	0.037	-0.0179	0.0447	-0.0182	
1.250	0.103	8.3	0.023	-0.0169	0.0125	-0.0345	
1.445	0.077	2.3	0.005	-0.0151	-0.0388	-0.0495	

TABLE 7

BENDING THEORY

Loads and displacements of half-ring for $\bar{\alpha} = 0.16$, R = 5 in., R/h = 5, E = 222 lb/in.^{2}, $\nu = 0.5$

α	β	$\bar{u}$	$\bar{w}$	$\bar{N}$	$\bar{M}$	$\bar{V}$	$\bar{q}^+$
0.00	-0.000	0.0000	-0.1534	-0.0186	-0.0956	-0.00×10^{-3}	-2.20×10^{-3}
0.04	-0.039	0.0053	-0.1525	-0.0186	-0.0958	-1.25	-2.19
0.08	-0.078	0.0104	-0.1499	-0.0186	-0.0963	-2.53	-2.16
0.12	-L.118	0.0155	-0.1454	-0.0185	-0.0971	-3.83	-2.12
0.16	-0.158	0.0203	-0.1392	-0.0185	-0.0982	-5.18	-2.05
0.16	-0.158	0.0203	-0.1392	-0.0185	-0.0982	+0.0747	
0.355	-0.272	0.0387	-0.0888	-0.0212	-0.0182	0.0561	
0.55	-0.253	0.0457	-0.0275	-0.0230	+0.0377	0.0353	
0.745	-0.151	0.0417	+0.0215	-0.0240	0.0674	0.0132	
0.94	-0.017	0.0305	0.0450	-0.0241	0.0697	-0.0094	
1.135	+0.095	0.0174	0.0412	-0.0233	0.0447	-0.0316	
1.33	0.132	0.0071	0.0198	-0.0216	-0.0069	-0.0527	
1.525	0.044	0.0010	0.0010	-0.0190	-0.0829	-0.0717	

TABLE 8

BENDING THEORY

Loads and displacements of half-ring for $\bar{\alpha} = 0.1$, R = 5 in., R/h = 10, E = 222 lb/in.^{2}, ν = 0.5

α	β	$\bar{u}$	$\bar{w}$	$\bar{N}$	$\bar{M}$	$\bar{V}$	$\bar{q}^+$
0.00	-0.0000	0.0000	-0.1108	-3.84×10^{-3}	-0.0495	0.00×10^{-4}	-2.08×10^{-4}
0.02	-0.0199	0.0021	-0.1105	-3.84	-0.0496	-1.59	-2.08
0.04	-0.0398	0.0042	-0.1099	-3.84	-0.0496	-3.19	-2.07
0.06	-0.0598	0.0063	-0.1087	-3.84	-0.0497	-4.80	-2.05
0.08	-0.0798	0.0084	-0.1072	-3.84	-0.0499	-6.42	-2.03
0.10	-0.0999	0.0104	-0.1053	-3.83	-0.0500	-8.07	-2.01
0.10	-0.0999	0.0104	-0.1053	-3.83	-0.0500	+0.0167	
0.295	-0.2223	0.0268	-0.0678	-4.44	-0.0138	0.0129	
0.49	-0.2228	0.0342	-0.0166	-4.87	+0.0123	0.0085	
0.685	-0.1429	0.0320	+0.0266	-5.12	0.0273	0.0038	
0.88	-0.0271	0.0234	0.0488	-5.18	0.0306	-0.0010	
1.075	+0.0789	0.0128	0.0467	-5.03	0.0222	-0.0058	
1.27	0.1296	0.0045	0.0268	-4.71	0.0224	-0.0104	
1.465	0.0816	0.0007	0.0047	-4.20	-0.0282	-0.0146	

TABLE 9

BENDING THEORY

Loads and displacements of half-ring for $\bar{\alpha}$ = 0.1, R = 5 in., R/h = 50, E = 222 lb/in.^{2}, ν = 0.5

α	β	$\bar{u}$	$\bar{w}$	$\bar{N}$	$\bar{M}$	$\bar{V}$	$\bar{q}^+$
0.00	-0.000	0.0000	-0.1037	-0.157×10^{-3}	-0.0100	-0.00×10^{-5}	-1.79×10^{-6}
0.02	-0.020	0.0021	-0.1035	-0.157	-0.0100	- 0.44	-1.79
0.04	-0.040	0.0041	-0.1029	-0.157	-0.0100	- 0.89	-1.78
0.06	-0.060	0.0062	-0.1018	-0.157	-0.0100	- 1.34	-1.77
0.08	-0.080	0.0082	-0.1002	-0.157	-0.0101	- 1.79	-1.75
0.10	-0.100	0.0102	-0.982	-0.157	-0.0101	- 2.25	-1.73
0.10	-0.100	0.0102	-0.0982	-0.157	-0.0101	47.2	
0.295	-0.223	0.0261	-0.0608	-0.181	-0.0028	36.2	
0.49	-0.221	0.0329	-0.0099	-0.199	+0.0025	23.8	
0.685	-0.139	0.0304	+0.0325	-0.209	0.0055	10.5	
0.88	-0.021	0.0216	0.0535	-0.211	0.0062	- 3.2	
1.075	+0.086	0.0111	0.0498	-0.205	0.0044	-16.8	
1.27	0.136	0.0033	0.0281	-0.192	0.0003	-29.7	
1.465	0.085	0.0002	0.0049	-0.171	-0.0059	-41.5	

TABLE 10

BENDING THEORY

Loads and displacements of half-ring for $\bar{\alpha} = 0.1$, R = 5 in., R/h = 50, E = $30\mathrm{x}10^6$ lb/in.^{2}, ν = 0.25

α	β	$\bar{u}$	$\bar{w}$	$\bar{N}$	$\bar{M}$	$\bar{V}$	$\bar{q}^+$
0.00	-0.000	0.0000	-0.1037	$-0.157\mathrm{x}10^{-3}$	-0.0100	$-0.000\mathrm{x}10^{-5}$	$-1.79\mathrm{x}10^{-6}$
0.02	-0.020	0.0021	-0.1035	-0.157	-0.0100	- 0.64	-1.79
0.04	-0.040	0.0041	-0.1029	-0.157	-0.0100	- 1.28	-1.78
0.06	-0.060	0.0062	-0.1018	-0.157	-0.0100	- 1.93	-1.77
0.08	-0.080	0.0082	-0.1002	-0.157	-0.0101	- 2.58	-1.75
0.10	-0.100	0.0102	-0.0982	-0.157	-0.0101	- 3.24	-1.73
0.10	-0.100	0.0102	-0.0982	-0.157	-0.0101	68.0	
0.295	-0.223	0.0261	-0.0608	-0.181	-0.0028	52.1	
0.49	-0.222	0.0329	-0.0099	-0.199	+0.0025	34.2	
0.685	-0.139	0.0304	+0.0325	-0.209	0.0055	15.1	
0.88	-0.021	0.0216	0.0535	-0.211	0.0062	- 4.6	
1.075	+0.086	0.0111	0.0498	-0.205	0.0044	-24.1	
1.27	0.136	0.0033	0.0281	-0.192	0.0003	-42.8	
1.465	0.085	0.0002	0.0049	-0.171	-0.0059	-59.8	

TABLE 11

LOADS AND DEFORMATIONS OF CAP

$\bar{\alpha} = 0.03$, $\alpha_e = 0.1$, $R/h = 10$, $R = 5$ in., $E = 222$ lb/in.^{2}, $\nu = 0.5$

α	$\bar{u}$	$\bar{w}$	β	$\bar{N}_\alpha$	$\bar{N}_\theta$	$\bar{M}_\alpha$	$\bar{M}_\theta$	$\bar{V}$	$\bar{q}^+$
0.0010	0.07×10^{-5}	-0.920×10^{-3}	-0.151×10^{-4}	-0.391×10^{-3}	-0.391×10^{-3}	-1.14×10^{-3}	-1.14×10^{-3}	1.0×10^{-3}	-0.0430
0.0068	0.44	-0.898	-0.100×10^{-3}	-0.391	-0.382	-1.06	-1.09	6.7	-0.0430
0.0126	0.79	-0.842	-0.173	-0.391	-0.357	-0.88	-0.96	12.4	-0.0431
0.0184	1.08	-0.752	-0.223	-0.391	-0.318	-0.61	-0.75	18.2	-0.0432
0.0242	1.30	-0.628	-0.240	-0.389	-0.264	-0.22	-0.48	24.0	-0.0434
0.0300	1.40	-0.472	-0.213	-0.389	-0.197	+0.28	-0.13	30.0	
0.0440	1.41	-0.320	-0.123	-0.389	-0.195	0.16	-0.02	20.3	
0.0580	1.23	-0.212	-0.072	-0.389	-0.193	0.11	+0.01	15.4	
0.0720	0.92	-0.126	-0.038	-0.389	-0.193	0.09	0.02	12.3	
0.0860	0.50	-0.058	-0.016	-0.389	-0.194	0.06	0.02	10.3	
0.1000	0.00	0.000	0.000	-0.388	-0.194	0.04	0.02	8.8	

TABLE 12

LOADS AND DEFORMATIONS OF CAP

$\bar{\alpha} = 0.03$, $\alpha_e = 0.1$, $R/h = 5$, $R = 5$ in., $E = 222$ lb/in.^{2}, $\nu = 0.5$

α	$\bar{u}$	$\bar{w}$	β	$\bar{N}_\alpha$	$\bar{N}_\theta$	$\bar{M}_\alpha$	$\bar{M}_\theta$	$\bar{V}$	$\bar{q}^+$
0.006	0.39×10^{-5}	-0.902×10^{-3}	-0.234×10^{-4}	-0.388×10^{-3}	-0.382×10^{-3}	-0.570×10^{-3}	-0.578×10^{-3}	5.97×10^{-3}	-86.8×10^{-3}
0.012	0.76	-0.848	-0.438	-0.388	-0.358	-0.480	-0.513	11.95	-87.2
0.018	1.06	-0.758	-0.581	-0.388	-0.321	-0.330	-0.407	17.94	-87.2
0.024	1.29	-0.632	-0.632	-0.388	-0.267	-0.124	-0.259	23.94	-87.2
0.030	1.40	-0.470	-0.561	-0.388	-0.196	+0.144	-0.069	29.94	-87.2
0.044	1.41	-0.318	-0.325	-0.388	-0.193	0.083	-0.012	20.39	
0.058	1.23	-0.212	-0.191	-0.388	-0.193	0.059	+0.004	15.44	
0.072	0.92	-0.126	-0.103	-0.388	-0.193	0.045	0.012	12.40	
0.086	0.50	-0.058	-0.042	-0.388	-0.193	0.034	0.014	10.36	
0.100	0.00	-0.000	-0.000	-0.388	-0.196	0.026	0.012	8.87	

TABLE 13

LOADS AND DEFORMATIONS OF CAP

$\bar{\alpha} = 0.03$, $\alpha_e = 0.2$, $R/h = 10$, $R = 5$ in., $E = 222$ lb/in.^{2}, $\nu = 0.5$

α	$\bar{u}$	$\bar{w}$	β	$\ddot{N}_\alpha$	$\bar{N}_\theta$	$\bar{M}_\alpha$	$\bar{M}_\theta$	$\bar{V}$	$\bar{q}^+$
0.006	0.055×10^{-4}	-0.00117	-1.05×10^{-4}	-0.378×10^{-3}	-0.371×10^{-3}	-0.00126	-0.00129	0.0059	-0.0429
0.012	0.107	-0.00111	-1.95	-0.378	-0.347	-0.00106	-0.00114	0.0118	-0.0430
0.018	0.154	-0.00102	-2.59	-0.378	-0.309	-0.00074	-0.00091	0.0177	-0.0432
0.024	0.193	-0.00089	-2.81	-0.376	-0.254	-0.00027	-0.00058	0.0237	-0.0434
0.030	0.220	-0.00073	-2.48	-0.376	-0.184	+0.00033	-0.00015	0.0297	-0.0437
0.040	0.249	-0.00061	-1.65	-0.376	-0.182	0.00023	-0.00004	0.0223	
0.050	0.268	-0.00053	-1.10	-0.376	-0.181	0.00017	+0.00000	0.0178	
0.060	0.279	-0.00045	-0.71	-0.376	-0.180	0.00014	0.00002	0.0148	
0.080	0.282	-0.00034	-0.17	-0.375	-0.180	0.00010	0.00004	0.0111	
0.110	0.251	-0.00022	+0.29	-0.375	-0.182	0.00006	0.00004	0.0080	
0.140	0.189	-0.00013	0.46	-0.375	-0.184	0.00002	0.00002	0.0062	
0.170	0.104	-0.00006	0.37	-0.375	-0.186	-0.00003	-0.00001	0.0051	
0.200	0.000	-0.00000	0.00	-0.375	-0.187	-0.00009	-0.00004	0.0042	

TABLE 14

LOADS AND DEFORMATIONS OF CAP

$\bar{\alpha} = 0.01,\ \alpha_e = 0.2,\ R/h = 10,\ R = 5$ in., $E = 222$ lb/in.^{2}, $\nu = 0.5$

α	$\bar{u}$	$\bar{w}$	β	$\bar{N}_\alpha$	$\bar{N}_\theta$	$\bar{M}_\alpha$	$\bar{M}_\theta$	$\bar{V}$	$\bar{q}^+$
0.002	0.030×10^{-5}	-0.176×10^{-3}	-0.28×10^{-4}	-0.416×10^{-4}	-0.407×10^{-4}	-0.1005×10^{-3}	-0.1014×10^{-3}	2.0×10^{-3}	-0.0435
0.004	0.059	-0.170	-0.51	-0.415	-0.381	-0.0843	-0.0900	4.0	-0.0435
0.006	0.086	-0.160	-0.68	-0.415	-0.338	-0.0576	-0.0714	6.0	-0.0435
0.008	0.109	-0.146	-0.73	-0.414	-0.277	-0.0203	-0.0446	8.0	-0.0436
0.010	0.129	-0.128	-0.64	-0.413	-0.199	+0.0284	-0.0097	10.0	-0.0436
0.014	0.160	-0.113	-0.38	-0.414	-0.198	0.0186	-0.0008	7.1	
0.020	0.198	-0.098	-0.14	-0.414	-0.197	0.0138	+0.0041	5.0	
0.030	0.245	-0.080	+0.10	-0.415	-0.197	0.0105	0.0065	3.3	
0.064	0.311	-0.047	0.58	-0.415	-0.198	0.0073	0.0073	1.6	
0.098	0.297	-0.029	0.83	-0.416	-0.201	0.0041	0.0057	1.0	
0.132	0.231	-0.017	0.86	-0.416	-0.203	0.0000	0.0024	0.7	
0.166	0.129	-0.007	0.61	-0.416	-0.206	-0.0005	-0.0016	0.6	
0.200	0.000	0.000	0.00	-0.416	-0.208	-0.0012	-0.0057	0.5	

TABLE 15

LOADS AND DEFORMATIONS OF CAP

$\bar{\alpha} = 0.01$, $\alpha_e = 0.2$, $R/h = 10$, $R = 5$ in., $E = 10^5$ lb/in.^{2}, $\nu = 0.4$

α	$\bar{u}$	$\bar{w}$	β	$\bar{N}_\alpha$	$\bar{N}_\theta$	$\bar{M}_\alpha$	$\bar{M}_\theta$	$\bar{V}$	$\bar{q}^+$
0.002	0.029×10^{-5}	-0.171×10^{-3}	0.06×10^{-3}	-4.00×10^{-5}	-3.90×10^{-5}	1.92×10^{-5}	1.94×10^{-5}	2.00×10^{-3}	-0.050
0.004	0.056	-0.165	0.11	-4.00	-3.60	1.71	1.80	4.00	-0.050
0.006	0.082	-0.155	0.15	-4.00	-3.11	1.39	1.59	6.00	-0.050
0.008	0.105	-0.141	0.18	-4.00	-2.42	0.95	1.31	8.00	-0.050
0.010	0.124	-0.123	0.19	-4.00	-1.53	0.36	0.93	10.00	-0.050
0.015	0.161	-0.106	0.20	-4.02	-1.53	0.50	0.77	6.7	
0.020	0.191	-0.094	0.23	-4.02	-1.53	0.56	0.71	5.0	
0.030	0.235	-0.077	0.30	-4.02	-1.53	0.59	0.65	3.4	
0.064	0.300	-0.046	0.54	-4.02	-1.53	0.48	0.54	1.5	
0.098	0.286	-0.028	0.70	-4.03	-1.55	0.26	0.40	1.0	
0.132	0.223	-0.016	0.69	-4.03	-1.58	0.04	0.20	0.7	
0.166	0.125	-0.007	0.48	-4.03	-1.60	-0.44	-0.06	0.6	
0.200	0.000	0.000	0.00	-4.03	-1.62	-0.93	-0.38	0.5	