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TENSILE FRACTURE ALONG A BIMATERIAL INTERFACE

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ABSTRACT

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The conditions for the existence of tensile fracture along a bimaterial interface have been determined by an extension of Yoffe's method. The velocity of such a fracture is specified by the combination of elastic properties of the two media. The fracture can move only at the specific velocity, without acceleration or deceleration. The maximum possible velocity of a fracture along an interface is the lower of the two Rayleigh wave velocities. The relative displacements on either side of the fracture surface depend~~s~~ on the velocity of the fracture as well as the combination of elastic properties. Several curves are presented showing the relationship between the elastic constants of the two media and the corresponding fracture velocity. A critical relation between the Poisson's ratio and shear moduli of the two media has been given. If this relation is satisfied, catastrophic failure cannot occur along the interface.

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1. INTRODUCTION

A natural extension of the study of fracture* propagation in infinite homogeneous media would be the study of fracture in anisotropic media. As a first step in this direction, it is proposed to investigate the possibility of a fracture along a plane boundary separating two isotropic media of differing elastic properties. Such a study would be of interest in geology, in the study of faulting in stratified rocks, and in engineering, in the study of failure along welded or cemented joints.

The corresponding static case of the determination of stresses around a crack at a bimaterial interface has been studied by WILLIAMS (1959) and his group. Williams found that stresses in the neighbourhood of the crack tip possess a sharp oscillatory character, of the type $r^{-\frac{1}{2}} \sin(b \log r)$, where r is the distance from the tip and b is a constant. Motion of dislocations along a bimaterial interface has been investigated by WEERTMAN (1964). He determined that the limiting velocity of a dislocation was the lowest sound velocity (i.e. the transverse wave velocity) in either media.

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* A fracture is here defined as the dynamic counterpart of a crack. A crack is defined as a static cavity with longitudinal dimensions much larger than the lateral one. The 'Yoffe fracture' is a fracture which maintains a constant length during propagation.

In this paper the method of YOFFE (1951) is used to study the conditions under which a fracture may propagate along a bimaterial interface. Yoffe's method was originally applied to fractures of constant length propagating at constant velocity. However, one may construct a growing fracture by considering two coincident Yoffe fractures propagating in opposite directions with the same velocity. The relative stress distribution is the same as that obtained by unilateral motion of a Yoffe fracture. Yoffe's method has been considerably simplified by the use of standard solutions of dual-integral equations (BILBY AND BULLOUGH, 1954).

The radius of the tip of the Yoffe fracture increases with velocity (COTTERELL, 1964). This may or may not be true for actual fractures. Hence it has been suggested (MANSINHA, in press) that the tip radius of a fracture is a constant which depends only on the properties of the medium, and not on the velocity of propagation. In the sequel we shall consider both the Yoffe fracture and the fracture of constant shape.

2. THE HOMOGENEOUS CASE

For completeness we will briefly recount here the method of solution of YOFFE (1951) and BILBY AND BULLOUGH (1954). Let a fracture of length 2a move in the positive x direction with velocity c in an isotropic homogeneous elastic medium under the action of a stress $p_{yy} = T$ applied at ∞. Let p_{ij} be the stresses in the vicinity of the fracture and let u, v represent the displacement in the x and y directions respectively. The boundary conditions in the half space $y > o$ are

$$p_{ij} = 0 \quad \text{at} \quad \infty$$

and on y=o

$$\begin{aligned} p_{yy} &= -T & &\text{for } |\bar{x}| < a \\ v &= 0 & &\text{for } |\bar{x}| > a \\ p_{xy} &= 0 & &\text{for all } |\bar{x}| \end{aligned} \tag{1}$$

where $\bar{x} = x - ct$,

Let us assume a set of surface disturbance of the type

$$u = \int_0^\infty A(s)\left[\exp(-\gamma s y) - \frac{2\beta\gamma}{(1+\beta^2)}\exp(-\beta s y)\right]\sin(s\bar{x})\,ds \tag{2}$$

$$v = \int_0^\infty A(s)\left[\exp(-\gamma s y) - \frac{2}{(1+\beta^2)}\exp(-\beta s y)\right]\cos(s\bar{x})\,ds \tag{3}$$

where

$$\gamma^2 = 1 - \frac{c^2}{c_1^2}$$

$$\beta^2 = 1 - \frac{c^2}{c_2^2}$$

and c_1 and c_2 are the longitudinal and transverse wave velocities respectively.

The above expressions satisfy the third of boundary conditions (1). Substituting (2) and (3) into the other two boundary conditions one obtains a set of dual integral equations. Solution of these equations will give A(s). Substituting the expression for A(s) in expressions (2) and (3) one obtains the solution for displacement and then the stresses. The solution is considerably simplified by the use of complex functions of the type $f(\bar{x} + i\Upsilon y)$ and $g(\bar{x} + i\beta y)$. An alternative method is by the generalised solution of RADOK (1956).

For our purpose we need the stresses and the displacements on y= 0. They are

$$p_{yy} = \begin{cases} -T & |\bar{x}| < a \\ T\left(\dfrac{\bar{x}}{(\bar{x}^2 - a^2)^{1/2}} - 1\right) & |\bar{x}| > a \end{cases} \quad \ldots (4)$$

$$u = -\frac{T}{H}\left(\frac{1}{\Upsilon} - \frac{2\beta}{(1+\beta^2)}\right) \cdot \begin{cases} \bar{x} & |\bar{x}| < a \\ \bar{x} - \sqrt{\bar{x}^2 - a^2} & |\bar{x}| > a \end{cases} \quad \ldots (5)$$

$$v = \frac{T}{H}\left(\frac{2}{(1+\beta^2)} - 1\right) \cdot \begin{cases} \sqrt{a^2 - \bar{x}^2} & |\bar{x}| < a \\ 0 & |\bar{x}| > a \end{cases} \quad \ldots (6)$$

where

$$H = -\mu \left[\frac{(\beta^2+1)^2 - 4\beta\tau}{\tau\,(1+\beta^2)} \right]$$

and μ is the shear modulus.

3. THE BIMATERIAL CASE

Let us assume that the boundary between the two media M' and M" is plane and coincides with $y = 0$ (Fig. 1). In the sequel a single prime will refer to the elastic constants, displacements and stresses in the medium M', and the double prime will refer to like terms in the lower medium M". Terms without prime(s) are general and refer to both media. For a fracture to propagate along the plane $y = 0$ some additional boundary conditions have to be satisfied. The conditions are a) the displacements should be continuous across the interface and b) the stresses acting across the interface should be continuous. Or, on $y = 0$

$$\begin{aligned} u' &= u'' \\ v' &= v'' \\ p'_{yy} &= p''_{yy} \\ p'_{xy} &= p''_{xy} \end{aligned} \tag{7}$$

Since a free surface exists for $|\bar{x}| < a$, the displacements need not be continuous for this range of $|\bar{x}|$. The stresses p_{ij} also need not be continuous. From boundary conditions (1) it is seen that $v' = v'' = 0$ for $|\bar{x}| > a$. Further the stress $p'_{xy} = p''_{xy} = 0$ for all $|\bar{x}|$. Hence the additional boundary conditions to be satisfied reduce to

$$\begin{aligned} u' &= u'' \quad \text{for } |\bar{x}| > a \\ p'_{yy} &= p''_{yy} \quad \text{for } |\bar{x}| > a \end{aligned} \tag{8}$$

From (4) one obtains

$$P_{yy} = T\left(\frac{\bar{x}}{(\bar{x}^2 - a^2)^{1/2}} - 1\right) \qquad |\bar{x}| > a$$

As the above expression is a function of the applied stress and the geometry only, and is not dependent on elastic properties of the medium, therefore the following expression always holds true

$$P'_{yy} = P''_{yy} = T\left(\frac{\bar{x}}{(\bar{x}^2 - a^2)^{1/2}} - 1\right) \qquad |\bar{x}| > a, \; y = 0$$

The last remaining displacement condition gives us

$$\frac{T}{H'}\left(\frac{1}{\gamma'} - \frac{2\beta'}{(1+\beta'^2)}\right) = \frac{T}{H''}\left(\frac{1}{\gamma''} - \frac{2\beta''}{(1+\beta''^2)}\right)$$

or

$$\frac{[2\beta'\gamma' - (1+\beta'^2)]}{\mu'[4\beta'\gamma' - (1+\beta'^2)^2]} = \frac{[2\beta''\gamma'' - (1+\beta''^2)]}{\mu''[4\beta''\gamma'' - (1+\beta''^2)^2]} \tag{9}$$

Let

$$c_f = c/c'_2 \;,\; K'_c = c'_1/c'_2 \;,\; K''_c = c''_1/c''_2 \;,\; K_2 = c'_2/c''_2 \;,\; K_\mu = \mu'/\mu''$$

Rewriting equation (9) we have

$$\frac{[2\{(1 - c_f^2)(1 - c_f^2/K_c'^2)\}^{1/2} - 2 + c_f^2]}{[4\{(1 - c_f^2)(1 - c_f^2/K_c'^2)\}^{1/2} - (2 - c_f^2)^2]}$$

$$- \frac{K_\mu[2\{(1 - c_f^2 \cdot K_2^2)(1 - c_f^2 \cdot K_2^2/K_c''^2)\}^{1/2} - 2 + c_f^2 \cdot K_2^2]}{[4\{(1 - c_f^2 \cdot K_2^2)(1 - c_f^2 \cdot K_2^2/K_c''^2)\}^{1/2} - (2 - c_f^2 \cdot K_2^2)^2]} = 0$$

(10)

When $c \rightarrow 0$ the limiting form of the above equation may be seen to be

$$\lim_{c \to 0} K_\mu = \frac{(1 - K_c''^2)}{(1 - K_c'^2)} \qquad (11)$$

This expression may be obtained by noting that as the velocity c approaches zero we have

$$\gamma = 1 - \frac{c^2}{2c_1^2}$$

and

$$\beta = 1 - \frac{c^2}{2c_2^2}$$

4. FRACTURE OF CONSTANT SHAPE

The tip radius of the Yoffe fracture increases with increase of velocity (see Fig. 1 of COTTERELL, 1964). This may not be true for actual fractures. Therefore, it is possible that the tip radius of a fracture in an isotropic homogeneous elastic medium is a constant (MANSINHA, in press). The boundary conditions for such a fracture are

$$p_{ij} = 0 \text{ at } \infty$$

and on $y = 0$

$$\begin{aligned} v &= 0 && \text{for } \lvert \bar{x} \rvert > a \\ v &= n(a^2 - x^2)^{\frac{1}{2}} && \text{for } \lvert \bar{x} \rvert < a \\ p_{xy} &= 0 && \text{for all } \lvert \bar{x} \rvert \end{aligned} \qquad (12)$$

where n is a small arbitrary constant, with the ellipticity of the fracture being given by (1-n).

Let us, as before, discuss the continuity of displacements and stresses at the interface. For $\lvert \bar{x} \rvert < a$, a free surface exists and the stresses and displacements need not be continuous over that range of $\lvert \bar{x} \rvert$. We also note that $v = p_{xy} = 0$ for $\lvert \bar{x} \rvert > a$. Thus, boundary conditions (12) reduce to (8). However, the expressions for u and p_{yy} are different from those for the Yoffe fracture. From MANSINHA (in press) we obtain the expression for the displacement u and stresses p_{yy} on $y = 0$:

$$u = n \frac{(1+\beta^2 - 2\beta\gamma)}{(\beta^2 - 1)\gamma} \left[\bar{x} - (\bar{x}^2 - a^2)^{\frac{1}{2}} \right] \qquad \lvert \bar{x} \rvert > a$$

$$\ldots (13)$$

and

$$p_{y\,y} = n\mu \; \frac{[(1+\beta^2)^2 - 4\beta\gamma]}{(1-\beta^2)\,\gamma} \cdot \left(\frac{\bar{x}}{(\bar{x}^2 - a^2)^{1/2}} - 1\right) \qquad |\bar{x}| > a \quad (14)$$

Neglecting the geometrical factors, the two conditions are given by

$$n' \; \frac{(1+\beta'^2 - 2\beta'\gamma')}{(\beta'^2 - 1)\,\gamma'} = n'' \; \frac{(1+\beta''^2 - 2\beta''\gamma'')}{(\beta''^2 - 1)\,\gamma''} \qquad (15)$$

or

$$\frac{n'}{n''} = \frac{(1+\beta''^2 - 2\beta''\gamma'')(\beta'^2 - 1)\,\gamma'}{(1+\beta'^2 - 2\beta'\gamma')\,(\beta'^2 - 1)\,\gamma''} \qquad (16)$$

and

$$n'\mu' \; \frac{[(1+\beta'^2)^2 - 4\beta'\gamma']}{(1-\beta'^2)\,\gamma'} = n''\mu'' \; \frac{[(1+\beta''^2)^2 - 4\beta''\gamma'']}{(1-\beta''^2)\,\gamma''} \qquad (17)$$

If (15) is substituted in (17) it is immediately seen that the conditions for the propagation of a fracture of constant shape along a bimaterial interface are the same as those for a Yoffe fracture (expression (9).) Expression (16) makes it possible for us to determine the relative shapes of the fracture on both sides of the interface.

As the velocity c $\rightarrow$ 0 the limiting form of expression (16) is given by

$$\lim_{c \to 0} \frac{n'}{n''} = \frac{K_c'^2}{K_c''^2}$$

5. RESULTS

Equations (9) and (16) have been numerically solved for various values of the elastic properties of media M' and M" with the Rice Computer. It should be noted that the Poisson's ratio σ completely determines the ratio of dilatational and transverse wave velocities in a medium. The expression is given by

$$c_1^2/c_2^2 = 2(1-\sigma)/(1-2\sigma)$$

We assumed values of σ' and σ'', K_2 and c_f and determined the ratio K_μ. The ratio of the densities can be easily determined from K_μ and K_2.

Figures 2 through 6 give the curves K_μ versus c_f for various combinations of σ' and σ'', and for different values of K_2. For every combination of σ' and σ'', K_2 and K_μ there exists a c_f. Thus the velocity of a tensile fracture along a bimaterial interface is determined by the elastic constants of the two media. It can be seen that for low values of K_μ, c_f is almost independent of K_2.

The expressions within the brackets in equation (8) in the denominators is nothing but the Rayleigh wave equations for both media M' and M" (EWING, JARDETZKY AND PRESS, 1957). Hence the condition for fracture along an interface is tied in with the Rayleigh wave velocity c_R in either media. For $K_2 \leqslant 1$, $K_\mu \rightarrow \infty$ as $c_f \rightarrow c_R'$. For $K_2 > 1$ the limiting velocity value of c_f is given by

$$c_f = \frac{c_R''/c_2''}{c_2'/c_2''}$$

Figure 8 shows the curves n'/n'' versus c_f for various K_2 and $\sigma' = 0$, $\sigma'' = 0.45$. The ratio n'/n'' will give at any point $|\bar{x}| < a$, $y = 0$, the ratio of the normal displacement v'/v''. For $\sigma' = \sigma''$ and $K_2 = 1$, the media are essentially identical. Therefore, $n'/n'' = 1$ for all c_f. For $c_f = 0$, n'/n'' is independent of K_2 and depends only on σ' and σ''.

6. DISCUSSION

A 'welded' interface merely implies that there is no slippage at the interface. Nothing is said about the 'strength' of the bonding. Therefore, one cannot definitely state whether or not a fracture will propagate along an interface. The results so far only permit us to state that if such a fracture exists it will move at a velocity determined by the combination of elastic constants of the two media. It should be noted that the results do not allow for acceleration or deceleration. Hence, the fracture must start and stop suddenly.

In Fig. 6 it can be seen that for $K_2 = 0.8$ and 1.0 two velocities c_f exist for same values of K_μ. The question then arises as to at which of these two velocities the fracture will propagate. Since both these velocities are lower than c_R the choice will probably depend on energy considerations.

A somewhat unexpected result is that the limiting velocity of fractures at an interface is the lower of the two Rayleigh wave velocities, and not the Stoneley wave velocity. It will be recalled that Stoneley waves are a type of wave propagating along and near a bimaterial interface.

The limiting value of K_μ at $c_f \to 0$ depends only on σ' and σ''. The $K_2 - c_f$ curves are almost horizontal near $c_f \to 0$. Therefore when K_μ differs slightly from K_μ at $c=0$ creeping fractures can exist for all K_2 and σ' and σ''. When two media are being joined, catastrophic failure along the interface may be prevented by choosing the materials such that

$$K_{\mu}(0) = \frac{\mu'}{\mu''}\bigg|_{c=0} = \frac{[1 - 2(1-\sigma'')/(1-2\sigma'')]}{[1 - 2(1-\sigma')/(1-2\sigma')]}$$

This would insure that only very very slow fractures may propagate along the interface. Precise satisfaction of the above relation will mean that no fracture can propagate along the interface.

ACKNOWLEDGEMENT

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FIGURES

Figure 1. Coordinate system. The fracture of length 2a moves in the positive X direction along the surface separating mediums M' and M".

Figures 2, 3 and 4. Curves of K_μ versus c/c'_2 for two media having the same Poisson's ratio but different transverse wave velocities.

Figures 5, 6 and 7. Curves of K_μ versus c/c'_2 for two media having different Poisson's ratio and different transverse wave velocities.

Figure 8. Curves of n'/n" versus c/c'_2 for $\sigma' = 0$ and $\sigma'' = 0.45$, for various values of K_2.

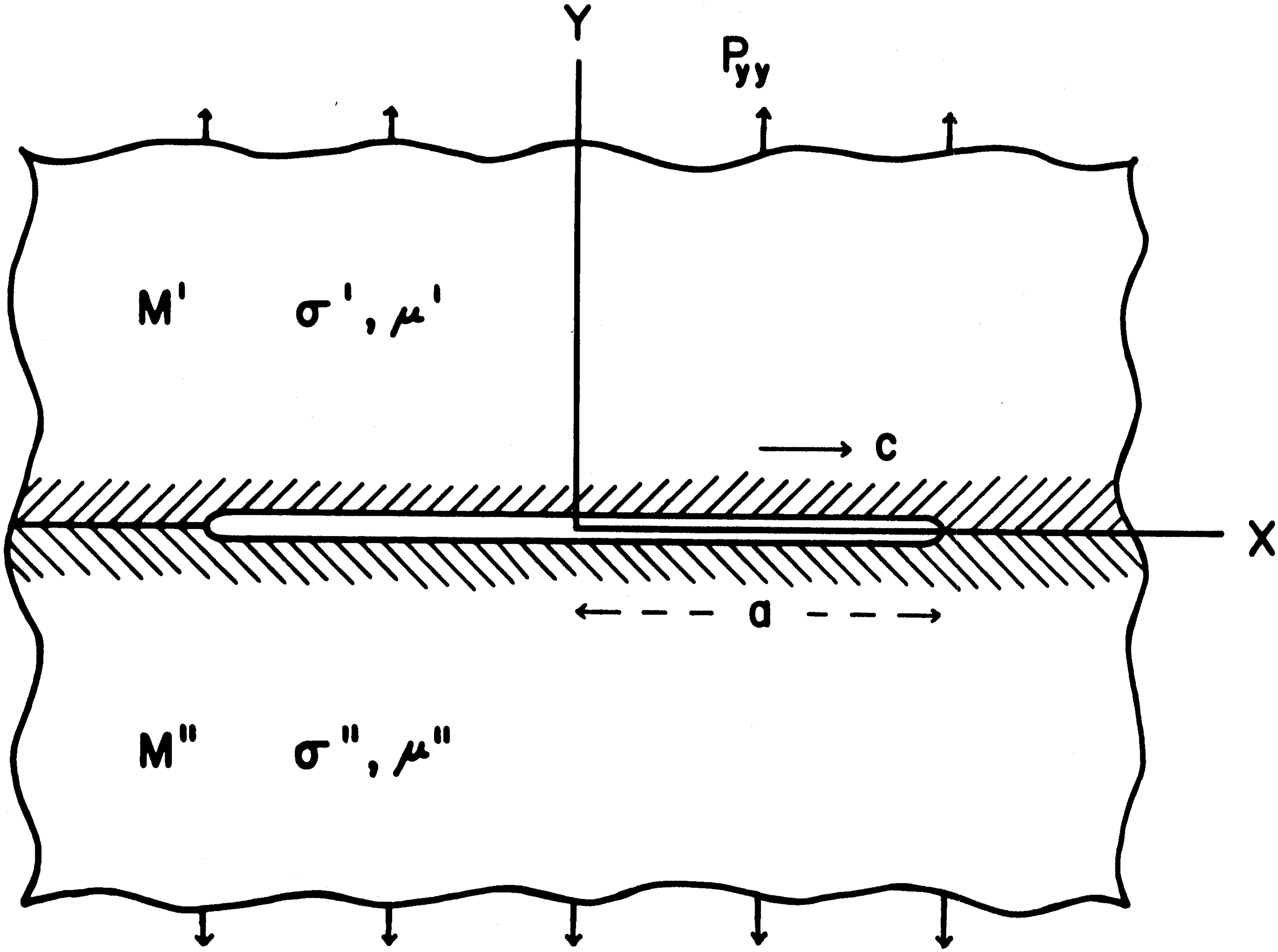


Fig 1

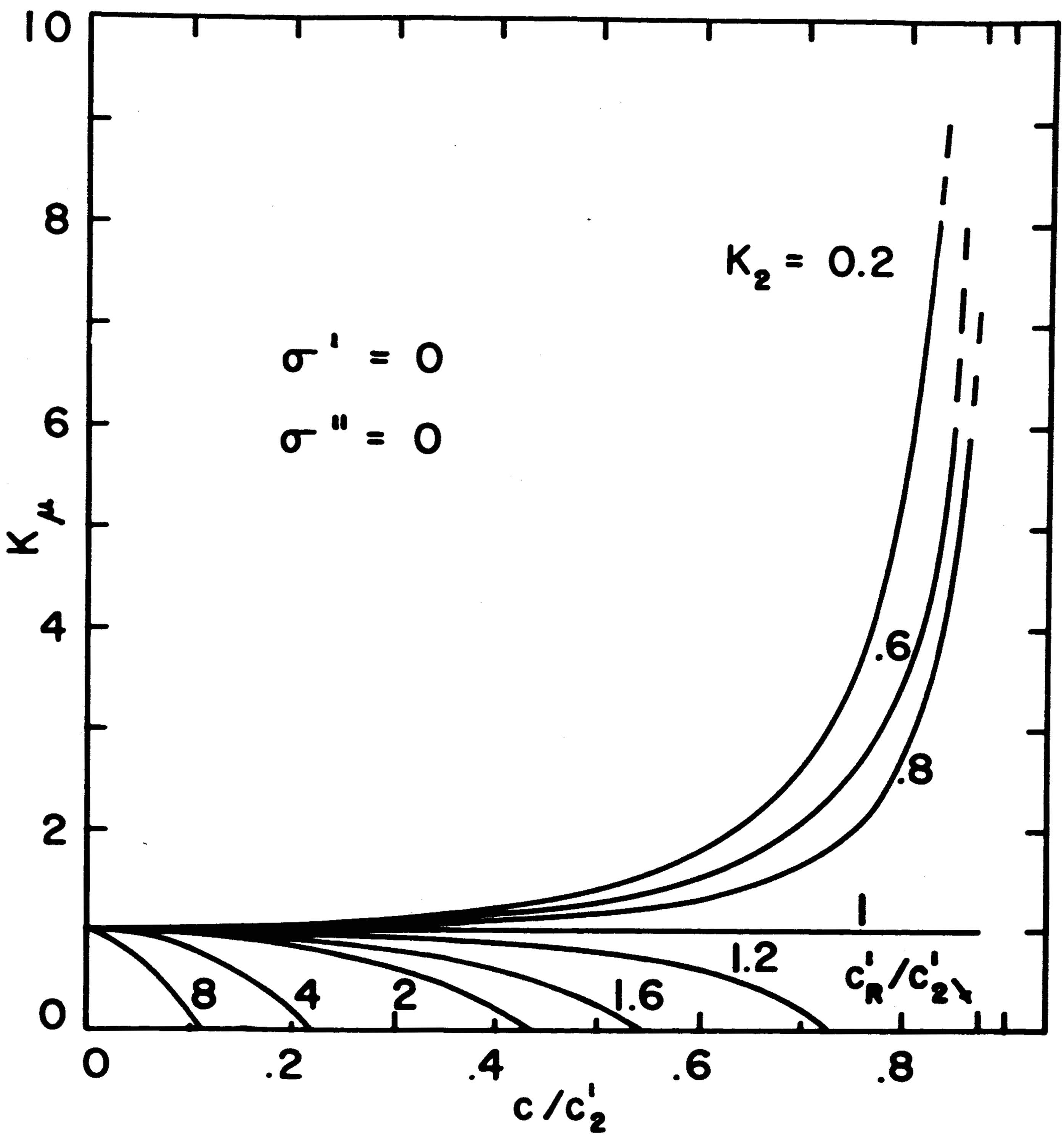


Fig 2

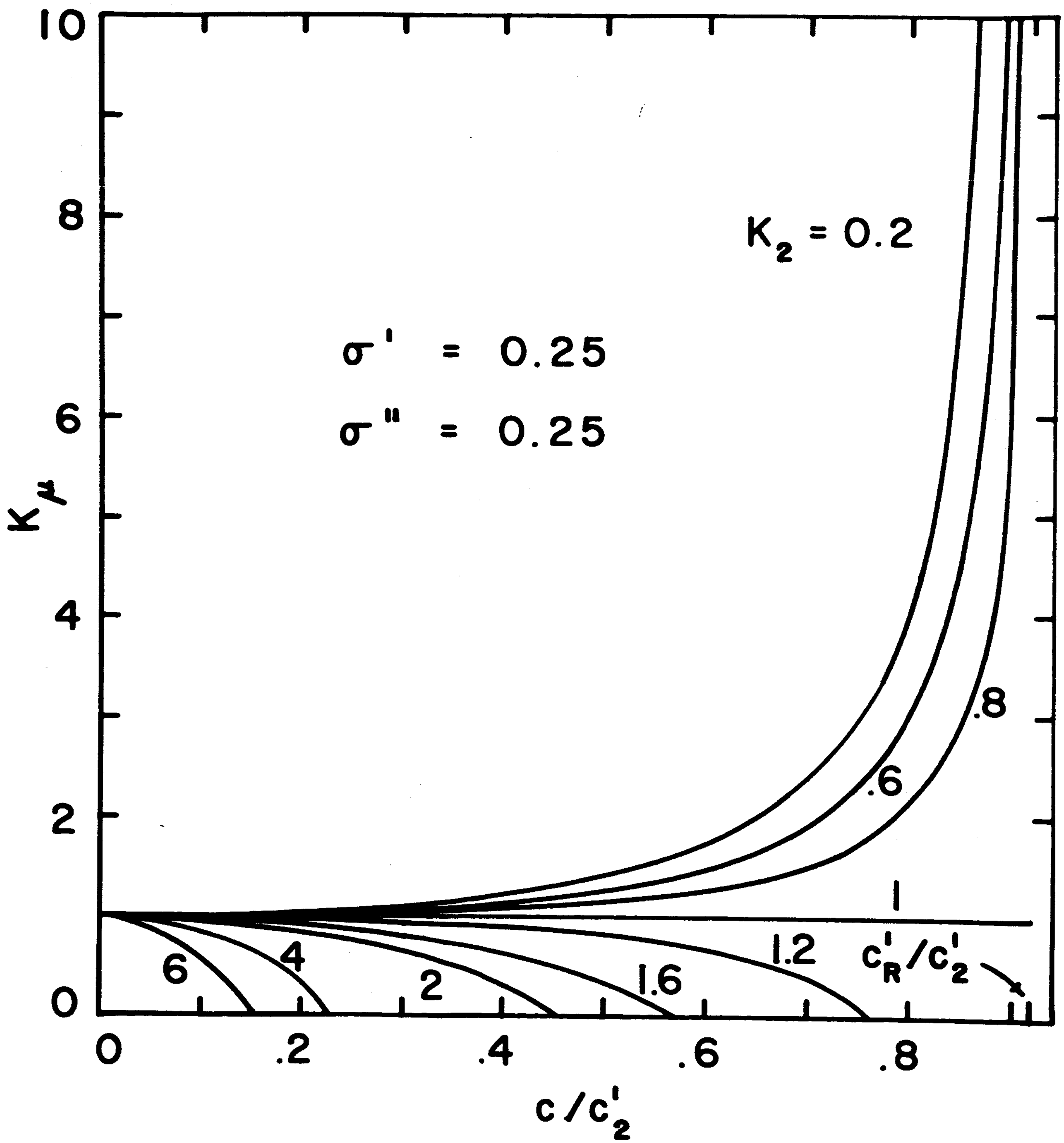
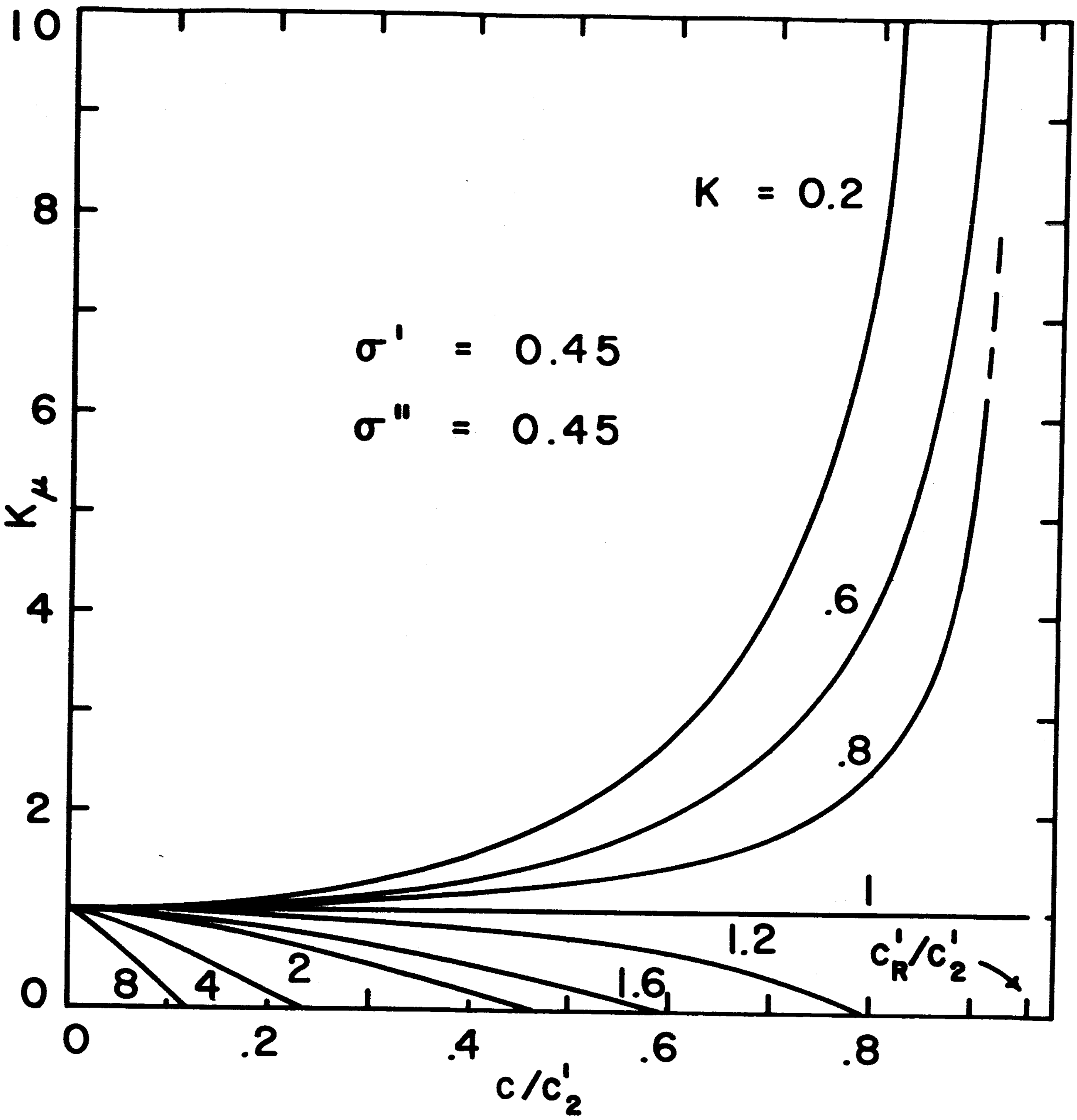
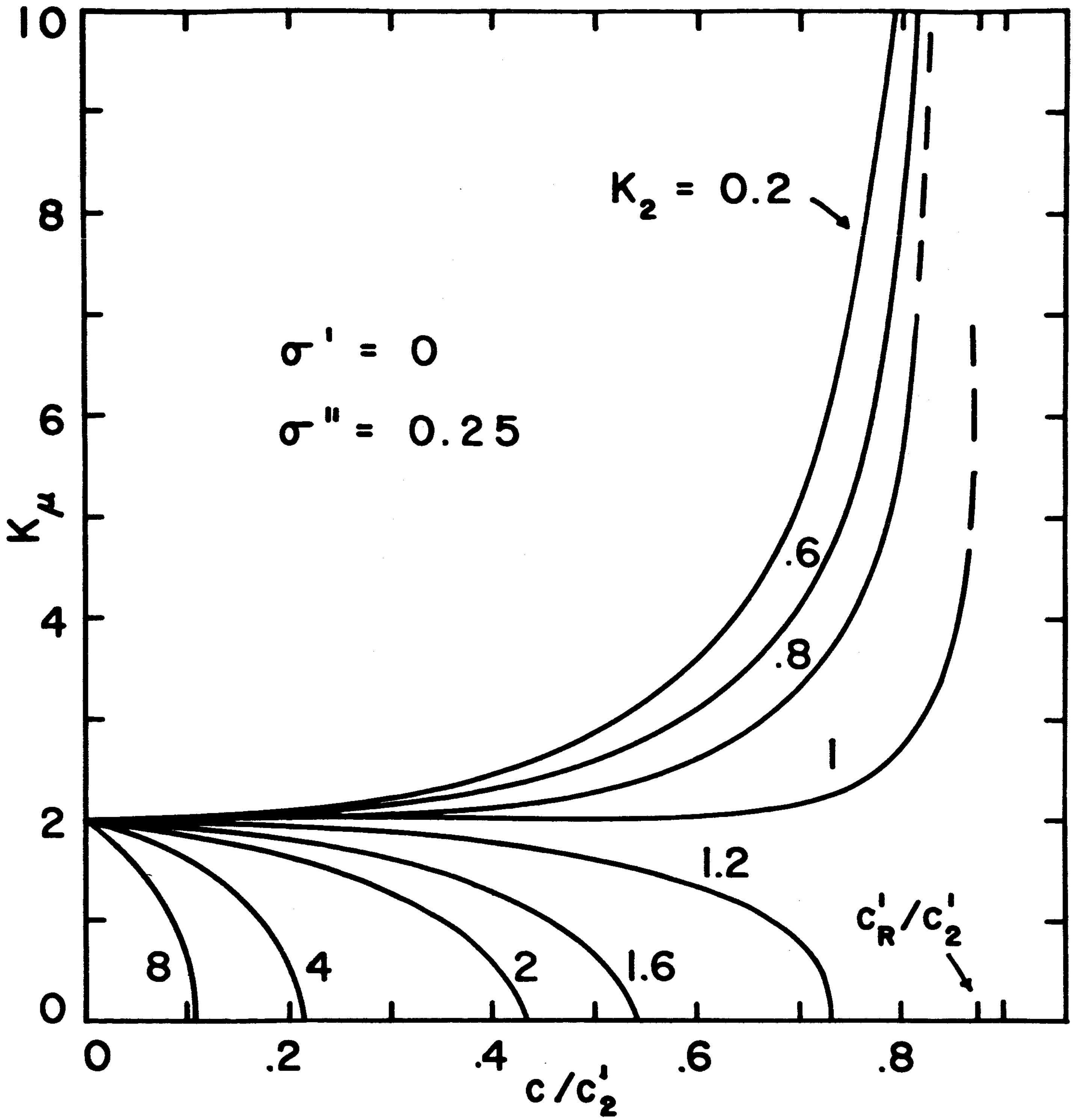


Fig 3



10
8
6
4
2
0
K_μ
K = 0.2
.6
.8
1
1.2
1.6
2
4
8
$\sigma' = 0.45$
$\sigma'' = 0.45$
c'_R/c'_2
0
.2
.4
.6
.8
c/c'_2

Fig 4



$K_2 = 0.2$
$\sigma' = 0$
$\sigma'' = 0.25$
.6
.8
1
1.2
8
4
2
1.6
c'_R/c'_2
K_μ
c/c'_2
0
2
4
6
8
10
0
.2
.4
.6
.8

Fig 5

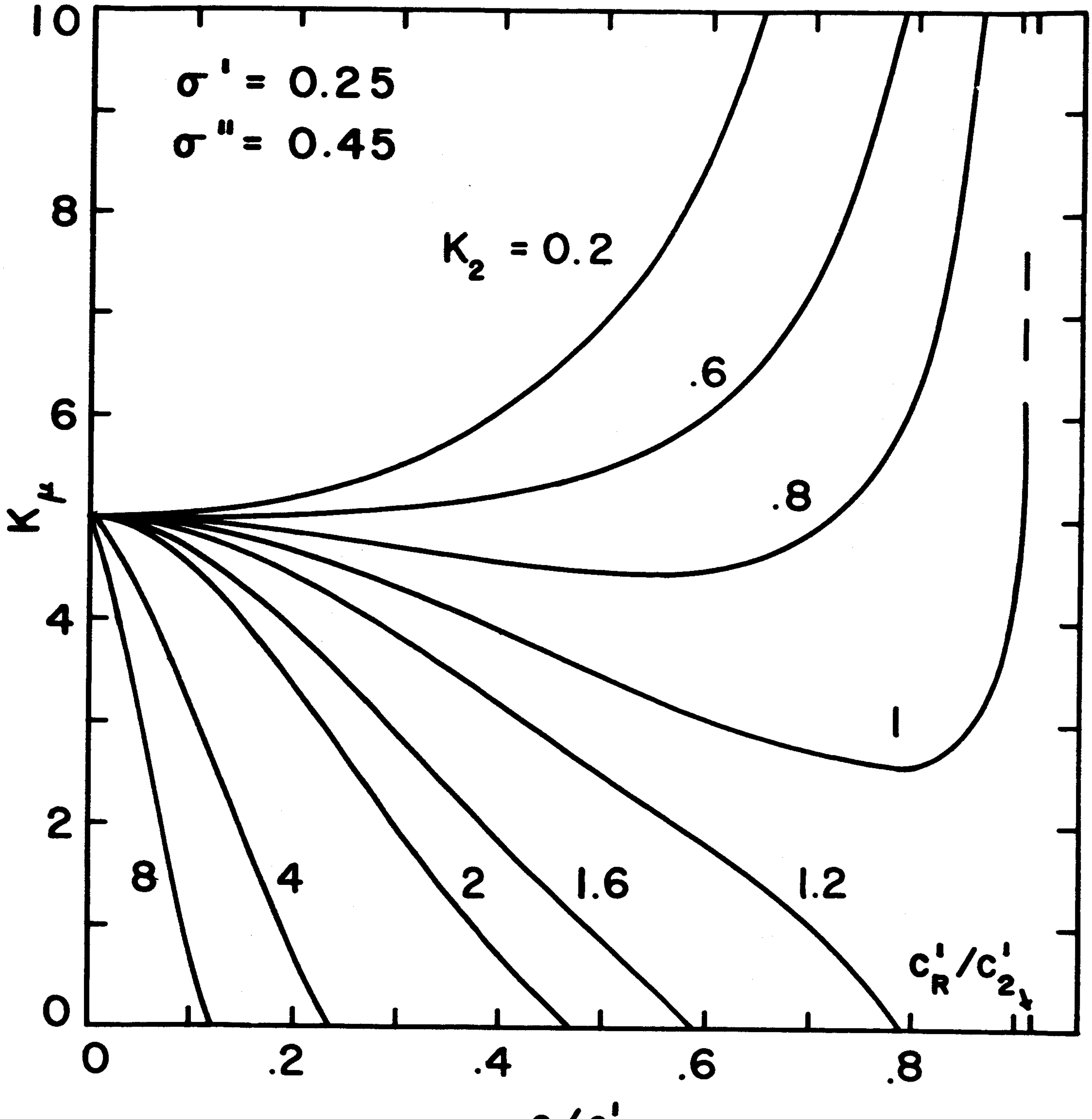


Fig 6

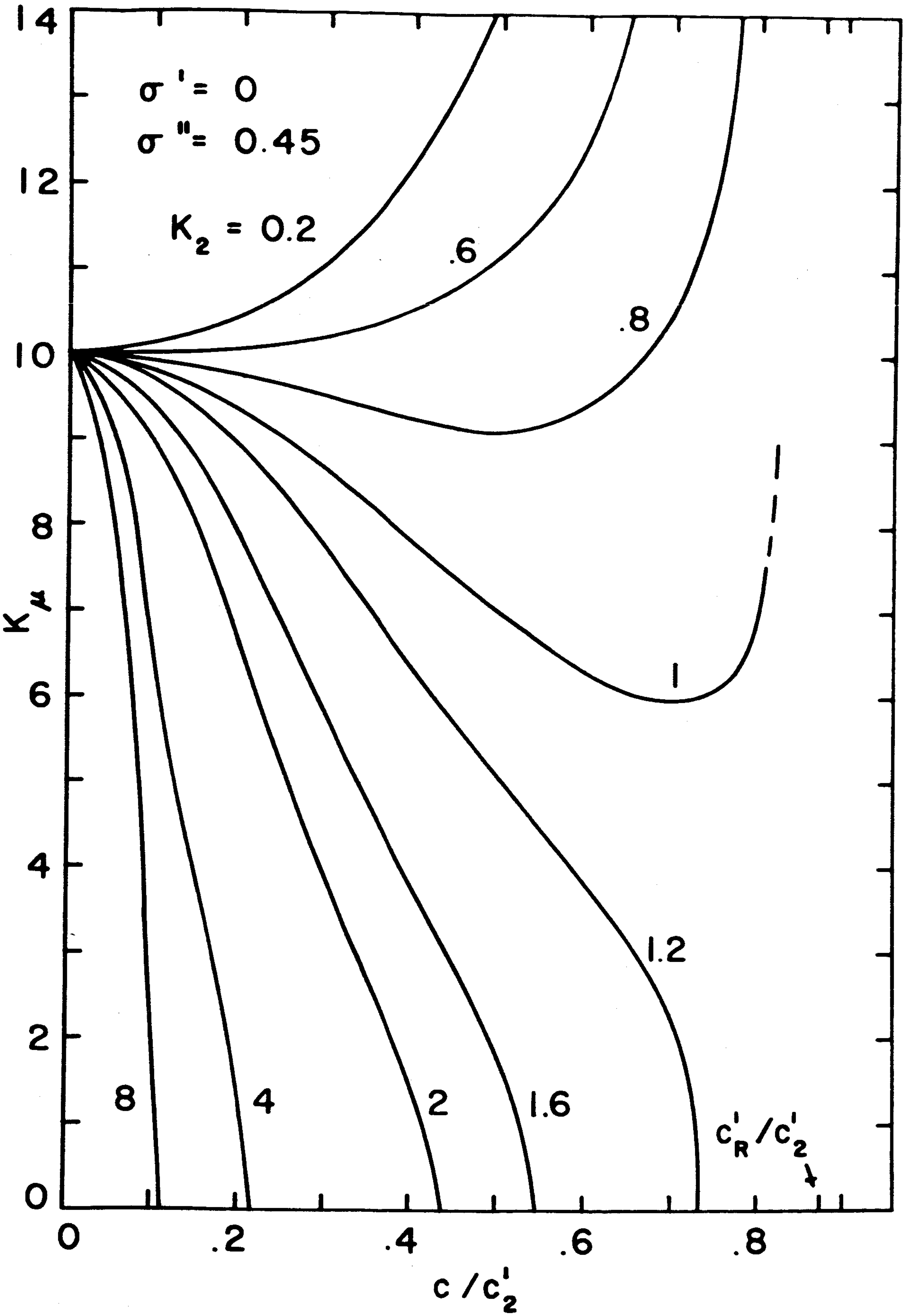


Fig 7

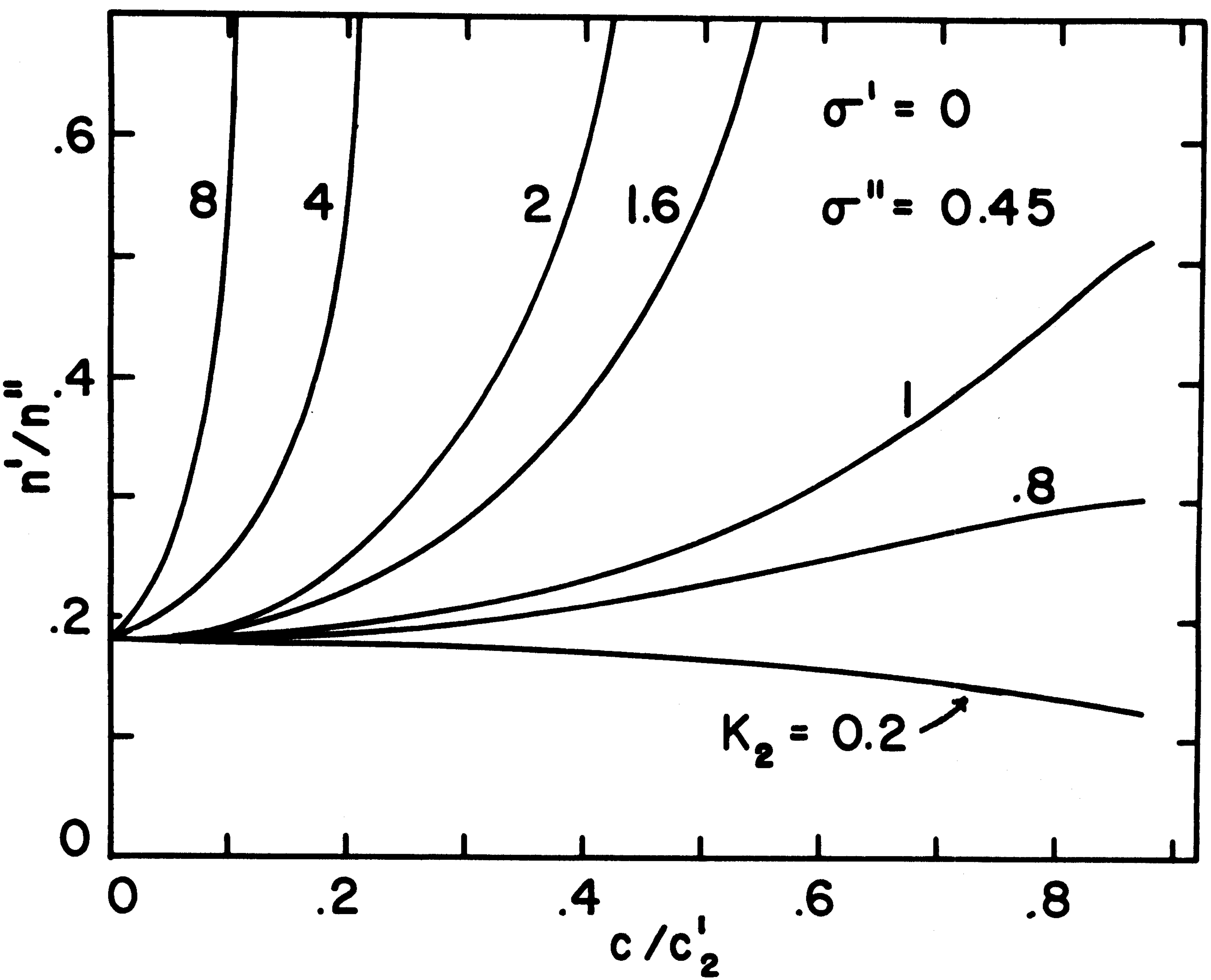


Fig 8