ON STABILITY IN CONTROL SYSTEMS

By Emilio Roxin†

Brown University and University of Buenos Aires

[redacted]

†This report was supported in part by the National Aeronautics and Space Administration under Contract No. NGR 40-002-015 and in part by the United States Air Force through the Air Force Office of Scientific Research under Contract No. AF-AFOSR-693-64.

ON STABILITY IN CONTROL SYSTEMS

by

Emilio Roxin

Brown University and University of Buenos Aires

1. Introduction

An axiomatic foundation of the theory of control systems was developed recently, based upon the notion of attainable set (Barbashin [1], Roxin [6], [7], [8]). Starting from a set of basic axioms, one proves that the properties of the so defined systems (called sometimes "generalized dynamical systems" or "generalized control systems") are in accordance with those of commonly known control systems. The main advantage of this approach lies in the fact that concepts like invariance, recurrence, stability, etc., are introduced in its greatest generality, showing their intrinsic nature.

The relation of these systems with those defined by contingent equations were studied in [9]. A way of defining generalized control systems locally, on a closed subset of the phase space, was given in [11].

In the present paper, definitions of different kinds of stability for generalized control systems are given, similar to those known for classical dynamical systems (see, for example, Massera [5]). Practically every kind of stability for dynamical systems, correspond to a strong and a weak similar property in the case of control systems. This was already mentioned in a communication of the author [10].

It should be noted that the relationship of different kinds of stability of control systems with some "Liapunov functions" was already studied, in a few cases, by Zubov [13]; here it is not treated, but it is, obviously, a good subject for further investigations.

[redacted]

2. Definition of general control systems

Consider as phase space X a complete, locally compact metric space. Elements of X will be denoted by small letters $(x, y, \ldots)$, subsets of X by capitals $(Y, F, A, \ldots)$. Let also denote:

i) $\rho(x, y)$ the distance between the points x, $y \in X$.

ii) $\rho(A, x) = \rho(x, A) = \inf\{\rho(x, y);\ y \in A\}$, (distance between the point x and the set A).

iii) $\beta(A, B) = \sup\{\rho(x, B);\ x \in A\}$, ("deviation" of the set A from the set B).

iv) $\alpha(A, B) = \alpha(B, A) = \max\{\beta(A, B),\ \beta(B, A)\}$, (distance between the sets A, B in the Hausdorff pseudo-metric).

v) $\gamma(A, B) = \inf\{\rho(x, B);\ x \in A\} =$

$$\inf\{\rho(x, y);\ x \in A,\ y \in B\}.$$

vi) $S_\varepsilon(A) = \{x \in X;\ \rho(x, A) < \varepsilon\}$, ($\varepsilon$-neighborhood of the set A).

The independent variable t (which will be called time) may be assumed to take all real values or all non-negative values ($t \in R$ or $t \in R^+$ respectively). Generally, only $t \in R^+$ will be considered, but in most cases the difference is irrelevant.

A control system will be assumed given by its "attainability function" $F(x_o, t_o, t)$, which corresponds to the set of all points attainable, at time t, from x_o at time t_o. It is sometimes also called "integral funnel".

The following axioms are assumed to hold:

I. $F(x_o, t_o, t)$ is a closed non-empty subset of X, defined for every $x_o \in X$, $t_o \leq t$.

II. $F(x_o, t_o, t_o) = \{x_o\}$ for every $x_o \in X$, $t_o \in R$.

III. For any $t_o \leqq t_1 \leqq t_2$:

$$F(x_o, t_o, t_2) = \bigcup_{x_1 \in F(x_o, t_o, t_1)} F(x_1, t_1, t_2).$$

IV. For any $x_1 \in X$, $t_o \leqq t_1$, there exists some $x_o \in X$ such that $x_o \in F(x_o, t_o, t_1)$.

V. For each $x_o \in X$, $t_o \leqq t_1$, $\varepsilon > 0$, there is $\delta > 0$ such that $|t - t_1| < \delta$ implies

$$\alpha(F(x_o, t_o, t), \ F(x_o, t_o, t_1)) < \varepsilon.$$

VI. For each $x_o \in X$, $t \leqq \tau$, $\varepsilon > 0$, there is $\delta > 0$ such that

$$\rho(x_o, y_o) < \delta, \ |t - t'| < \delta, \ |\tau - \tau'| < \delta, \ t' \leqq \tau'$$

imply

$$\beta(F(y_o, t', \tau'), \ F(x_o, t, \tau)) < \varepsilon.$$

It was shown in [8] how the behaviour of the control system can be satisfactorily derived from these axioms. In the case when the control system is only defined on a closed subset of the space X, the axioms have to be modified as pointed out in [11].

The following properties proved in [8], will be needed.

The attainability function $F(x, t, \tau)$ can be extended backwards, i.e., for $\tau < t$ (in [8] this extension was denoted by G). The properties of this backward extension are almost the same as for the forward part, the main exception being that the continuity of $F(x, t, \tau)$ in τ (axiom V) may fail and F become unbounded (finite escape time backwards).

Definition 2.1: A mapping $u: I \to X$, defined in some interval $I = [t_o, t_1]$ and such that

$$t_o \leqq \tau_o \leqq \tau_1 \leqq t_1$$

implies

$$u(\tau_1) \in F(u(\tau_o), \tau_o, \tau_1),$$

is called a motion of the control system F; the corresponding curve in X-space, a trajectory.

The continuity of a motion follows from its definition and axioms I - VI.

A motion $u_1: [t_a, t_b] \to X$ is a prolongation of the motion $u_2:[t_c, t_d] \to X$, if $[t_a, t_b] \supset [t_c, t_d]$ and $u_1:(t) = u_2(t)$ for $t \in [t_c, t_d]$.

In [8] the following properties are proved.

Theorem 2.1: if $x_1 \in F(x_o, t_o, t_1)$, there exists a motion $u(t)$ of the control system, such that $u(t_o) = x_o$, $u(t_1) = x_1$.

Theorem 2.2: if the motions $u_i(t)$, $(i = 1, 2, 3, \ldots)$ of a control system are all defined in an interval $[t_o, t_1]$ (or $[t_o, +\infty)$), and if $\lim_{i \to \infty} u_i(t_o) = x_o$, then some subsequence $u_{i_k}(t)$ converges to a certain motion $u_o(t)$ and the convergence is uniform in any finite interval.

Finally, the notation

$$F(A, t_o, t) = \bigcup_{x \in A} F(x, t_o, t)$$

will be used. If A is compact, then $F(A, t_o, t)$ is also compact for every $t \geq t_o$.

3. Strong stability

Definition 3.1: The set $A \subset X$ is called strongly positively invariant with respect to a certain control system, if for any $x_o \in A$, $t_o \leq t$, the relation

$$F(x_o, t_o, t) \subset A$$

holds. If A consists of a single point, it will also be called a strong point of rest.

Note: If the control system is defined only in the closed subset $Y \subset X$, then A must be assumed to belong to the interior of Y, at positive distance from its boundary.

Definition 3.2: The strongly positively invariant set $A \subset X$ is called strongly stable, if for every $\varepsilon > 0$ and $t_o \geq 0$, there is $\delta = \delta(\varepsilon, t_o) > 0$ such that $\rho(x_o, A) < \delta$ implies

$$F(x_o, t_o, t) \subset S_\varepsilon(A)$$

for all $t \geq t_o$.

This stability will also be called *strong Liapunov stability*. Now, as it is done in classical dynamical systems (see, for example, Yoshizawa [12] and especially Massera [5]), it is possible to define the following stability-type properties, which for simplicity are denoted by numbers followed by "s" (in order to indicate that it is a stability of the "strong" type). The properties are:

1s) The strong Liapunov stability according to definition 3.2.

2s) The same definition 3.2, but with $\delta(\varepsilon, t_o) = \delta(\varepsilon)$ independent of t_o (uniform strong stability).

3s) For every $t_o \geq 0$ there exists a $\delta_o(t_o) > 0$ such that for any motion with $u(t_o) = x_o \in S_{\delta_o}(A)$,

$$\lim_{t \to +\infty} \rho(u(t), A) = 0$$

holds (quasi-asymptotic strong stability).

4s) Property (3s) with δ_o independent of $t_o \geq 0$.

5s) For every $t_o \geq 0$ there is $\delta_o(t_o) > 0$ such that $\rho(x_o, A) < \delta_o$ implies

$$\lim_{t \to +\infty} \beta(F(x_o, t_o, t), A) = 0$$

(i.e., property (3s) uniformly for all motions $u(t)$ starting at (x_o, t_o)).

6s) Property (5s) with δ_o independent of $t_o \geq 0$.

7s) For every $t_o \geq 0$ there is $\delta_o(t_o) > 0$ such that

$$\lim_{t \to +\infty} \beta(F(S_{\delta_o}(A), t_o, t), A) = 0$$

(i.e., property (5s) uniformly in $x \in S_{\delta_o}(A)$: quasi-equi-asymptotic strong stability).

8s) Property (7s) with δ_o independent of $t_o \geqq 0$.

9s) There is $\delta_o > 0$ such that

$$\lim_{\tau \to +\infty} \beta(F(S_{\delta_o}(A), t_o, t_o + \tau), A) = 0$$

uniformly for all $t_o \geqq 0$ (uniform quasi-equi-asymptotic strong stability).

The relations between these properties are indicated in Fig. 1. Both groups of properties 1-2 and 3-9 are independent, as the following example shows.

Example 3.1: Let $X = R$ and the control system be defined in Fig. 2, where the motions $u(t)$ are given graphically (this characterizes them sufficiently well, the decrease for $t \to +\infty$ may for instance be taken exponentially). It should be noted that through $x_o = 0$ there are infinitely many different motions for every t_o. Axioms I-VI are satisfied, as it is easy to verify.

The set $A = \{x: x < 0\}$ is positively strongly invariant and satisfies property (9s), but it does not satisfy (1s). Therefore, both groups of properties in Fig. 1 are independent.

It may be noted that if the definitions are not restricted to $t_o \geqq 0$, but taken for all $t_o \in R$, then property (9s) is not satisfied any more, but property (8s) is.

In this example, the set A is not closed. Indeed, for a compact A we can prove:

Theorem 3.1: For a compact, positively strongly invariant set, property (7s) implies (1s).

Proof: let A be compact, positively strongly invariant and satisfy property (7s). Then, for every $t_o \geqq 0$ and $\varepsilon > 0$, there are $\delta_o > 0$ and $t_1 \geqq t_o$ such that

$$\beta(F(S_{\delta_o}(A), t_o, t), A) < \varepsilon$$

for all $t \geqq t_1$.

If A is a single point, it follows from axiom VI that there is $\delta_1 > 0$ such that for all t in the interval $[t_o, t_1]$,

$$(3.1) \qquad \beta(F(S_{\delta_1}(A), t_o, t), A) < \varepsilon.$$

Taking $\delta = \min(\delta_1, \delta_2)$, this value satisfies property (1s).

If A is not a single point, the existence of δ_1 satisfying (3.1) can be proved as follows. Take for every $x \in A$ a value $\delta_x > 0$ such that $\beta(F(S_{\delta_x}(x), t_o, t), A) < \varepsilon$ uniformly in $t_o \leqq t \leqq t_1$. A is covered by a finite collection: $A \subset \bigcup_i S_{\delta_{x_i}}(x_i)(i = 1, 2, \ldots, p)$. Then $\bigcup S_{\delta_{x_i}}$ is a neighborhood of A and there is some δ_1 satisfying $S_{\delta_1}(A) \subset \bigcup_i S_{\delta_{x_i}}(x_i)$, and therefore

$$F(S_{\delta_1}(A), t_o, t) \subset S_\varepsilon(A)$$

for all $t_o \leqq t \leqq t_1$.

Theorem 3.2: Properties (2s) and (3s) together imply (5s).

Proof: let $A \subset X$ be positively strongly invariant and satisfy properties (2s) and (3s). Let $t_o \geq 0$ be given and $\delta_o = \delta_o(t_o)$ be the same as in the definition of property (3s). It will be proved that the same δ_o satisfies (5s).

Assuming, indeed, the contrary, there is some $x_o \in S_{\delta_o}(A)$ and a sequence $t_i \to +\infty$ such that

$$(3.2) \qquad \beta(F(x_o, t_o, t_i), A) > a > 0 \quad (i = 1, 2, 3, \ldots).$$

As A satisfies (2s), there is $\delta > 0$ such that $\rho(x, A) < \delta$ implies $\beta(F(x, t, \tau), A) < a$ for all $\tau \geq t \geq t_o$. According to (3.2) there is a motion $u_1(t)$ through (x_o, t_o) such that

$$\rho(u_1(t_1), A) > a$$

and, therefore,

$$\rho(u_1(t), A) > \delta$$

for all $t \in [t_o, t_1]$. In the same way there is, for each $i = 2, 3, \ldots$, a motion $u_i(t)$ such that $u_i(t_o) = x_o$ and $\rho(u_i(t_i), A) > a$, and, therefore,

$$\rho(u_i(t), A) > \delta$$

for all $t \in [t_o, t_i]$. By theorem 2.2 some subsequence of $u_i(t)$ converges to a limit motion $u_o(t)$ for all $t \geq t_o$, which therefore satisfies

$$\rho(u_o(t), A) > \delta$$

for all $t \geq t_o$, contrary to property (3s).

The same proof applies to the following:

<u>Theorem 3.3</u>: Properties (2s) and (4s) together imply (6s).

For compact sets the following stronger results are valid.

<u>Theorem 3.4</u>: If $A \subset X$ is conditionally compact (i.e. the closure of A is compact), positively strongly invariant and satisfies properties (2s) and (3s), then A also satisfies (7s).

Proof: let $t_o \geq 0$, $\delta_o(t_o)$ defined according to property (3s) and $\delta_o > \eta > 0$. It will be proved that η satisfies the requirement of property (7s).

Assuming the contrary, there are $\varepsilon > 0$, $x_i \in S_\eta(A)$ and $t_i \to +\infty$ $(i = 1, 2, 3, \ldots)$ such that

$$\beta(F(x_i, t_o, t_i), A) > \varepsilon > 0 \quad (i = 1, 2, 3, \ldots).$$

As the closure of $S_\eta(A)$ may be assumed compact, the proof coincides essentially with the preceding one, taking

$$u_i(t_o) = x_i$$

and

$$\rho(u_i(t_i), A) > \varepsilon.$$

Therefore

$$\rho(u_i(t), A) > \delta > 0$$

for all $t \in [t_o, t_i]$, δ being related to ε by property (2s). By compactness, $x_i \to x_o \in S_{\delta_o}(A)$ may be assumed, so that there is some limit motion $u_o(t)$, for which

$$\rho(u_o(t), A) > \delta$$

for all $t \geqq t_o$, contradicting property (3s).

In the same way one proves the following.

Theorem 3.5 : If $A \subset X$ is conditionally compact, positively strongly invariant and satisfies properties (2s) and (4s), then A also satisfies property (8s).

The definitions (1s) to (9s) should, of course, be such that no two of them turn out to be identical (to imply each other). This is obvious in many cases, because it is known for classical dynamical systems (which are a special case of control systems, the strong stability being for them the common stability). For less obvious typical cases, two examples are given here.

Example 3.2: $X = R$ and the control system is an ordinary dynamical system whose motions are given in Fig. 3. The set $\{0\}$ satisfies properties (2s) and (7s), but not (4s).

Example 3.3: $X = R^2$ and polar coordinates ρ, θ are used. With the auxiliary function $h(s)$ given in Fig. 4a, the equation of the motions are given by:

$$\dot{\theta} = \begin{cases} \text{sign } \theta \quad \text{for} \quad -\pi < \theta < 0 \quad \text{and} \quad 0 < \theta < \pi , \\ 0 \quad \text{for} \quad \theta = 0 \quad \text{and} \quad \theta = \pm \pi , \end{cases}$$

(so that $\theta = \theta(t)$ are given in Fig. 4b), and

$$\rho(t) = [e^{-t} + h(\theta)] \cdot \text{const} ,$$

this constant being defined by the initial conditions:

$$\text{const} = \frac{\rho(t_o)}{e^{-t_o} + h(\theta(t_o))} .$$

The motions starting at $\rho(0) = \rho_o$, $\theta(0) = 0$ lie on the funnel-shaped surface of equation

$$\rho = \rho_o[e^{-t} + h(\theta)]$$

drawn in Fig. 4c.

For every motion, $\rho(t) \to 0$, so that the solution $\rho \equiv 0$ satisfies property (3s). In spite of this, the attainable set $F[\rho(0) = \rho_o, \theta(0) = 0, t]$, which is the cross-section of the above mentioned surface, does not tend to zero because for $\theta = \theta^* = \frac{\pi}{2}$,

$$\rho^* = \rho_o[e^{-t} + 1] \to \rho_o \qquad \text{for} \quad t \to +\infty$$

Therefore property (3s) does not imply (5s).

Example 3.4: Let $X = R^2$ and the motions defined by

$$x = k \cos \alpha$$
$$y = k e^{-t} \sin \alpha$$
$$\alpha = \text{arc tg}(t + c) \text{ or } \alpha = \pm \frac{\pi}{2} .$$

Here α is taken mod 2π and k and c are constants determined by the initial conditions. This system satisfies property (5s) but not (7s) (see Fig. 5).

4. Weak stability

Definition 4.1: The set $A \subset X$ is called weakly positively invariant with respect to a certain control system, if for every $x_o \in A$, $t_o \geq 0$, there exists some motion $u(t)$ such that $u(t_o) = x_o$ and $u(t) \in A$ for all $t \geq t_o$. If A consists of a single point, it also will be called a weak point of rest.

Note: If the control system is defined only on the closed subset $Y \subset A$, then the motion $u(t)$ should be defined (not empty) for all $t \geq t_o$. For the stability properties defined below, A are assumed to belong to the interior of Y, at a finite distance from ∂Y.

Theorem 4.1: (Barbashin [1]): necessary and sufficient for the weak positive invariance of a closet set A, is the condition

$$F(x_o, t_o, t) \cap A \neq \Phi$$

for every $x_o \in A$, $t \geq t_o$ (Φ is the empty set).

Definition 4.1: the weakly positively invariant set $A \subset X$ is called weakly stable, if for every $\varepsilon > 0$ and $t_o \geq 0$, there is $\delta = \delta(\varepsilon, t_o) > 0$ such that $\rho(x_o, A) < \delta$ implies the existence of some motion $u(t)$ with $u(t_o) = x_o$ and $\rho(u(t), A) < \varepsilon$ for all $t \geq t_o$.

This kind of stability will be called also weak Liapunov stability.

Now, as in the preceding section, the following stability properties are defined; the "w" indicates that they correspond to the weak type.

1w) The weak Liapunov stability according to definition 4.1.

2w) The same definition 4.1, but with $\delta(\varepsilon, t_o)$ independent of $t_o \geq 0$ (uniform weak stability).

3w) For every $t_o \geq 0$ there is $\delta_o(t_o) > 0$ such that $\rho(x_o, A) < \delta_o$ implies

$$\lim_{t \to +\infty} \gamma(F(x_o, t_o, t), A) = 0.$$

(where $\gamma(A, B) = \inf \{\rho(a, b);\ a \in A,\ b \in B\}$)

4w) Property (3w) with δ_o independent of $t_o \geq 0$.

5w) For every $t_o \geq 0$ there is $\delta_o(t_o) > 0$ such that if $\rho(x_o, A) < \delta_o$, there is some motion $u(t)$ with $u(t_o) = x_o$ and

$$\lim_{t \to +\infty} \rho(u(t), A) = 0$$

(quasi-asymptotic weak stability).

6w) Property (5w) with δ_o independent of $t_o \geq 0$.

7w) For every $t_o \geq 0$ there is $\delta_o(t_o) > 0$ and for every $\varepsilon > 0$ there is $T = T(t_o, \varepsilon)$ such that $\rho(x_o, A) < \delta_o$ implies the existence of a motion $u(t)$ with $u(t_o) = x_o$ and $\lim \rho(u(t), A) = 0$ for $t \to +\infty$, in such a way that $\rho(u(t), A) < \varepsilon$ for all $t \geq t_o + T$ (quasi-equi-asymptotic weak stability).

8w) Property (7w) with δ_o independent of $t_o \geqq 0$.

9w) Property (8w) with $T = T(\varepsilon)$ independent of $t_o \geqq 0$ (uniform quasi-equi-asymptotic weak stability).

<u>Note</u>: For a strongly stable compact set A and any finite interval $[t_1, t_2]$, it was proved in [8] that a value $\delta(\varepsilon)$ can be taken such that the stability condition of definition 3.2 is satisfied for all $t_o \in [t_1, t_2]$. This is similar to the classical dynamical systems. The following example shows, however, that this is not true for the weak stability.

<u>Example 4.1</u>: Let $X = R^2$ and the motions of the control system defined by:

a) In the solid pyramidal cone $t > 0$, $|x| < t - y$, $|x| < 2y - t$ the motions are given by

$$\frac{dx}{dt} = \frac{x}{t} \quad ; \quad \frac{dy}{dt} = \frac{y}{t} \; .$$

b) Outside that cone: $\frac{dx}{dt} = \frac{dy}{dt} = 0$.

c) On the boundary of that cone, the tanget to the motion $(\frac{dx}{dt}, \frac{dy}{dt})$ at any point is required to belong to the convex hull of the set of tangents at infinitely nearby points, plus the vector $\dot{x} = \pm 1$, $\dot{y} = 0$.

This way, the motions are really defined by a contingent equation (see Roxin [9]) and are shown in Fig. 6. At the points of the boundary of the pyramidal cone, the solutions are not unique. It is easy to verify that the origin $x = y = 0$ is weakly positively invariant and satisfies property (1w). On the other hand, there is no $\delta_o(\varepsilon, t_o)$ valid for all $0 < t_o < T_1$ for any $T > 0$, $\varepsilon > 0$.

This example can be easily modified in such a way that it applies to properties (3w), (5w) and (7w) (the only thing to do is to change conveniently the motions outside the pyramidal cone). Therefore, it makes sense to define the properties:

1*w) For every finite interval $[t_1, t_2] \in R^+$, there is $\delta_o > 0$ such that the condition of property (1w) is satisfied for all $t_o \in [t_1, t_2]$.

3*w) Similarly for property (3w).

5*w) Similarly for property (5w).

7*w) Similarly for property (7w), for both $\delta_o(t_o)$ and $T(\varepsilon, t_o)$.

The relations between all these properties are given in Fig. 7.

Example 3.1 (Fig. 2) is valid also for the weak stability; $x = 0$ is a weak point of rest which satisfies property (9w) but not (1w). This proves the independence of both groups of properties in Fig. 7.

Similarly, example 3.2 shows that property (7w) does not imply (4w).

The following example shows that property (3w) does not imply (5w).

<u>Example 4.2:</u> Let $X = R^2$ and

$$x = k \cos \alpha(t)$$
$$y = k\, e^{-t} \sin \alpha(t).$$

Here, k is a constant determined by the initial conditions, and the functions $\alpha(t)$ are given (mod 2π) in Fig. 8a. It is to be noted that $\alpha(t) \equiv 0$ is an admissible function, from which other curves branch off.

The motions lie on tubes which become more flat as $t \to + \infty$, but the attainable set from any point of the tube-surface is, for sufficiently large t, the whole cross section of the tube-surface; therefore, its minimal distance to the origin tends to zero (property (3w)).

5. Finite stability

In the preceding two sections, the properties number 1 and 2 correspond to the common (Liapunov) stability, and those numbered 3 to 9, to the quasi-asymptotic stabilities. Assuming both to hold, one obtains the very important asymptotic stabilities. As in control systems there is no assumption about uniqueness of motion $x(t)$ through each point (x_o, t_o) (which restricts so much the classical dynamical systems), there can be defined even stronger stabilities than the asymptotic ones, by requiring that the motions $x(t)$ not only tend to, but actually reach the invariant set A in finite time. This type of stability will be called finite stability ; it can be defined for the strong and for the weak stability, and like the asymptotic one, it will be split up into the quasi-finite plus the Liapunov stability.

Once the main idea is established, the development is quire straightforward. Even some examples given above can be slightly changed so that they apply to the finite stabilities.

Finite stabilities of the strong type (here A is a strongly positively invariant set).

10s) For every $t_o \geq 0$ there exists a $\delta_o(t_o) > 0$ such that for every motion $u(t)$ with $u(t_o) = x_o \in S_{\delta_o}(A)$, there is a finite value $\tau_f > 0$ such that $u(t_o + \tau_f) \in A$ (and therefore $u(t) \in A$ for all $t > t_o + \tau_f$). In general, τ_f depends on the motion $u(t)$. (This is the quasi-finite-strong stability.)

11s) Property (10s) with δ_o independent of $t_o \geqq 0$.

12s) For every $t_o \geqq 0$, there is $\delta_o(t_o) > 0$ such that $x_o \in S_{\delta_o}(A)$ implies the existence of $\tau_f = \tau_f(x_o, t_o) > 0$ such that

$$F(x_o, t_o, t_o + \tau_f) \subset A,$$

and therefore $F(x_o, t_o, t)$ A for all $t > t_o + \tau_f$.

13s) Property (12s) with δ_o independent of t_o.

14s) For every $t_o \geqq 0$ there is $\delta_o(t_o) > 0$ and a finite $\tau_f(t_o) > 0$ such that

$$F(S_{\delta_o}(A), t_o, t_o + \tau_f) \subset A.$$

This is the quasi-equi-finite strong stability.

15s) Property (14s) with δ_o independent of t_o.

16s) Property (15s) with τ_f independent of t_o (uniform quasi-equi-finite strong stability).

Obviously the following implications hold:

$10s \rightarrow 3s$; $11s \rightarrow 4s$; $12s \rightarrow 5s$; $13s \rightarrow 6s$;

$14s \rightarrow 7s$; $15s \rightarrow 8s$; $16s \rightarrow 9s$.

Fig. 9 shows the implications between the stabilities of this last group.

<u>Finite stabilities of the weak type</u> (here A is a weakly positively invariant set).

10w) For every $t_o \geqq 0$ there is $\delta_o(t_o) > 0$ such that if $x_o \in S_{\delta_o}(A)$, there is a motion $u(t)$ with $u(t_o) = x_o$ and $u(t_o + \tau_f) \in A$ for some finite $\tau_f > 0$ (and, therefore, this motion can be prolonged indefinitely in A). This is the quasi-finite weak stability.

10*w) Property (10w), and for any finite interval $[t_1, t_2] \in R^+$, $\delta_o(t_o)$ can be taken to hold uniformly for all $t_o \in [t_1, t_2]$.

11w) Property (10w), with δ_o independent of $t_o \geq 0$.

12w) For every $t_o \geq 0$ there is $\delta_o(t_o) > 0$ and some value τ_f, $0 < \tau_f = \tau_f(t_o)$ such that $x_o \in S_{\delta_o}(A)$ implies the existence of a motion $u(t)$ with $u(t_o) = x_o$ and $u(t_o, \tau_f) \in A$ (quasi-equi-finite weak stability).

12*w) Property (12w), and for any finite interval $[t_1, t_2] \in R^+$, $\delta_o(t_o)$ can be taken to hold uniformly for all $t_o \in [t_1, t_2]$.

13w) Property (12w), with δ_o independent of $t_o \geq 0$.

14w) Property (13w), and $\tau_f = \tau_f(\delta_o)$ independent of $t_o \geq 0$ (uniform quasi-equi-finite weak stability).

Obviously, the following implications hold:

$10w \rightarrow 5w$; $10^*w \rightarrow 5^*w$; $11\,w \rightarrow 6w$;

$12w \rightarrow 7w$; $12^*w \rightarrow 7^*w$; $13\,w \rightarrow 8w$; $14w \rightarrow 9w$.

Figure 10 shows the implications between the stabilities of this last group.

<u>Remarks about the finite stabilities:</u> The importance of motions arriving at the origin (supposed to be a positively weakly invariant set) in a finite time, plays an important role in control theory. Therefore, the stabilities of the finite type were already used, without special denomination by numerous authors (for example, Kalman [2], Markus and Lee [4], LaSalle [3]). LaSalle

even pointed out the importance of considering stability in finite time intervals. The region of attraction for the finite weak stability corresponds to what is known as the domain of controllability ([4]). It may be noted that most asymptotically stable systems of the real physical world are, indeed, finitely stable.

The strong type of finite stability has not been used, apparently, but a rather trivial example shows that it can appear even in the simple case of:

$$\dot{x} = -2\sqrt{|x|}\,.\,(2+u)\,.\,\text{sign}\;x$$

with the control $u(t)$ restricted by $|u| \leq 1$. In this equation, the extreme values of $u(t)$ correspond to the motions

for $u = s$:

$$|x(t)| = \begin{cases} [\sqrt{|x_o|} + 3t_o - 3t]^2 & \text{for} \quad t \leq t_o + \dfrac{\sqrt{|x_o|}}{3} \\ 0 \quad \text{for} \quad t \geq t_o + \dfrac{\sqrt{|x_o|}}{3} \;; \end{cases}$$

for $u = -s$:

$$|x(t)| = \begin{cases} [\sqrt{|x_o|} + t_o - t]^2 & \text{for} \quad t \leq t_o + \sqrt{|x_o|} \\ 0 \quad \text{for} \quad t \geq t_o + \sqrt{|x_o|}\,. \end{cases}$$

The attainable set is indicated in Fig. 11.

Of course, the definitions given above do not solve any specific problem, but they may help to treat systematically cases which appear frequently in applications.

REFERENCES

[1] Barbashin, E. A., "On the theory of generalized dynamical systems", Uch. Zap. M.G.U. No. 135 (1949), pp. 110-133.

[2] Kalman, R. E., "Contributions to the theory of control", Boletin Soc. Mat. Mexicana (1960), pp. 102-119.

[3] LaSalle, J. P., "The time optimal control problem", Contrib. Theory of Non-linear Oscillations, Vol. 5, pp. 1-24, Princeton Univ. Press (1960).

[4] Lee, E. B., and Markus, L., "Optimal control for non-linear processes", Archiv Rat. Mech. Anal. 8 (1961), pp. 36-58.

[5] Massera, J. L., "Contributions to stability theory", Ann. of Math. 64 (1956), pp. 182-205.

[6] Roxin, E., "Axiomatic theory of control systems", RIAS Tech. Rep. 62-12 (1962).

[7] -----, "Axiomatic foundation of the theory of control systems", Second Int. Conf. of IFAC, Basel, 1963.

[8] -----, "Stability in general control systems". Journ. of Diff. Equ., Vol. Vol. 1, No. 2, (in press), Acad. Press.

[9] -----, 'On generalized dynamical systems defined by contingent equations", Journ. of Diff. Equ., Vol. 1, No. 2, (in press)Acad.Press.

[10] -----, "Stabilität in allgemeinen Regelungssystemen", Third Conf. on Non-linear Oscill., Berlin, 1964.

[11] -----, "Local definition of generalized control systems", to be published.

[12] Yoshizawa, T., "Asymptotic behaviour of solutions of ordinary differential equations near sets", RIAS Tech. Rep. 61-5 (1961).

[13] Zubov, V. I., "Methods of A. M. Liapunov and their application", Izdat. Leningrad Univ. (1957), English translation publ. by Noordhoff, Groningen, 1964.

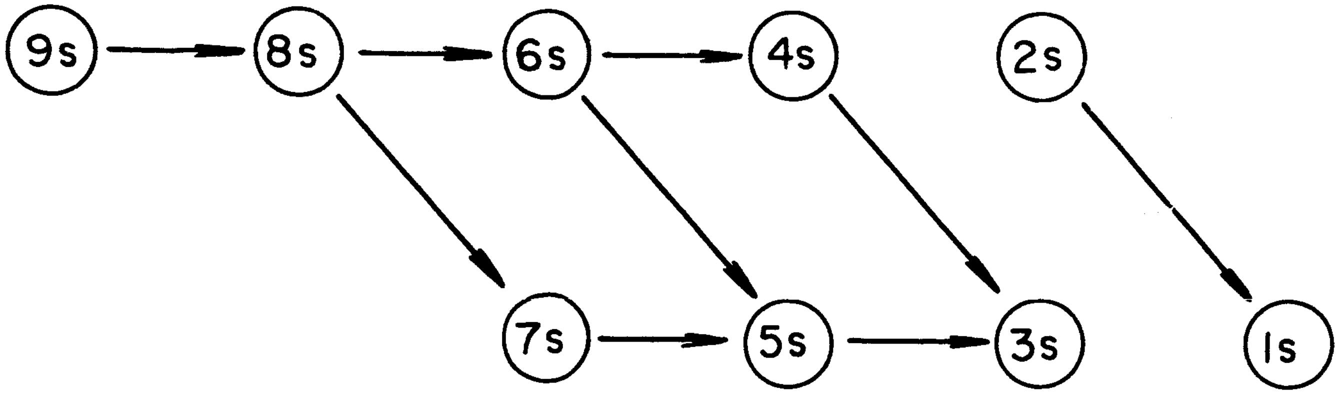


FIG. 1

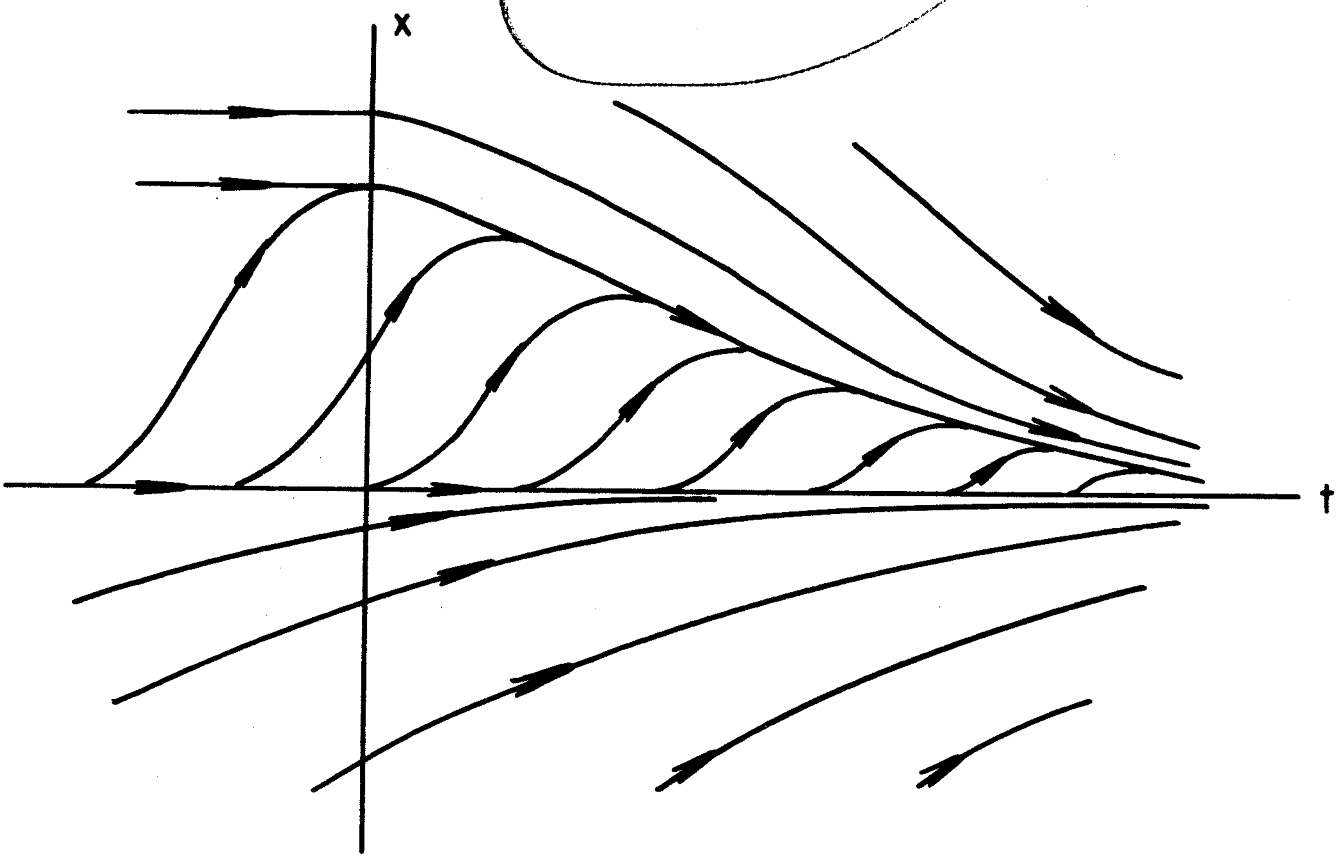
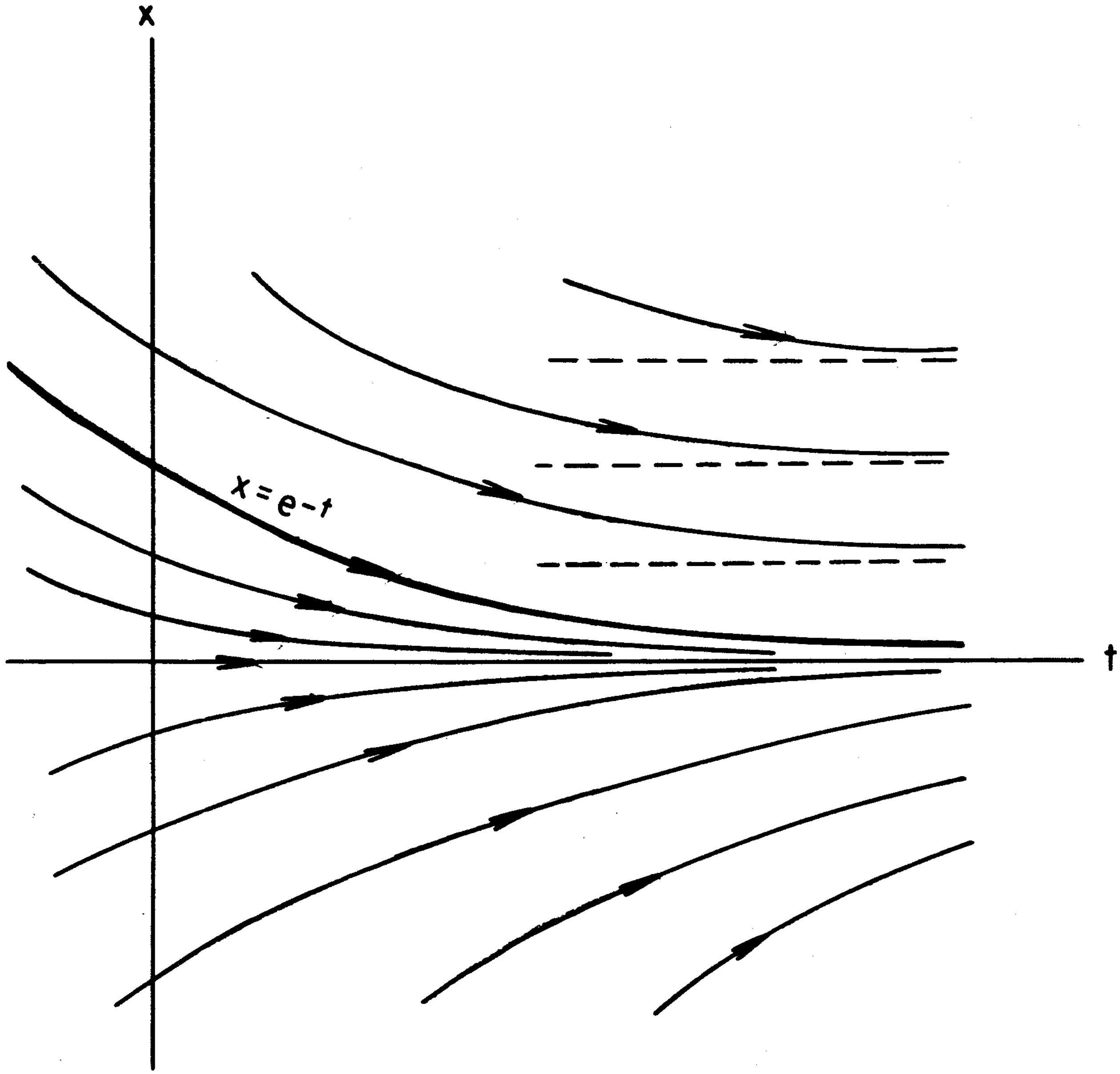


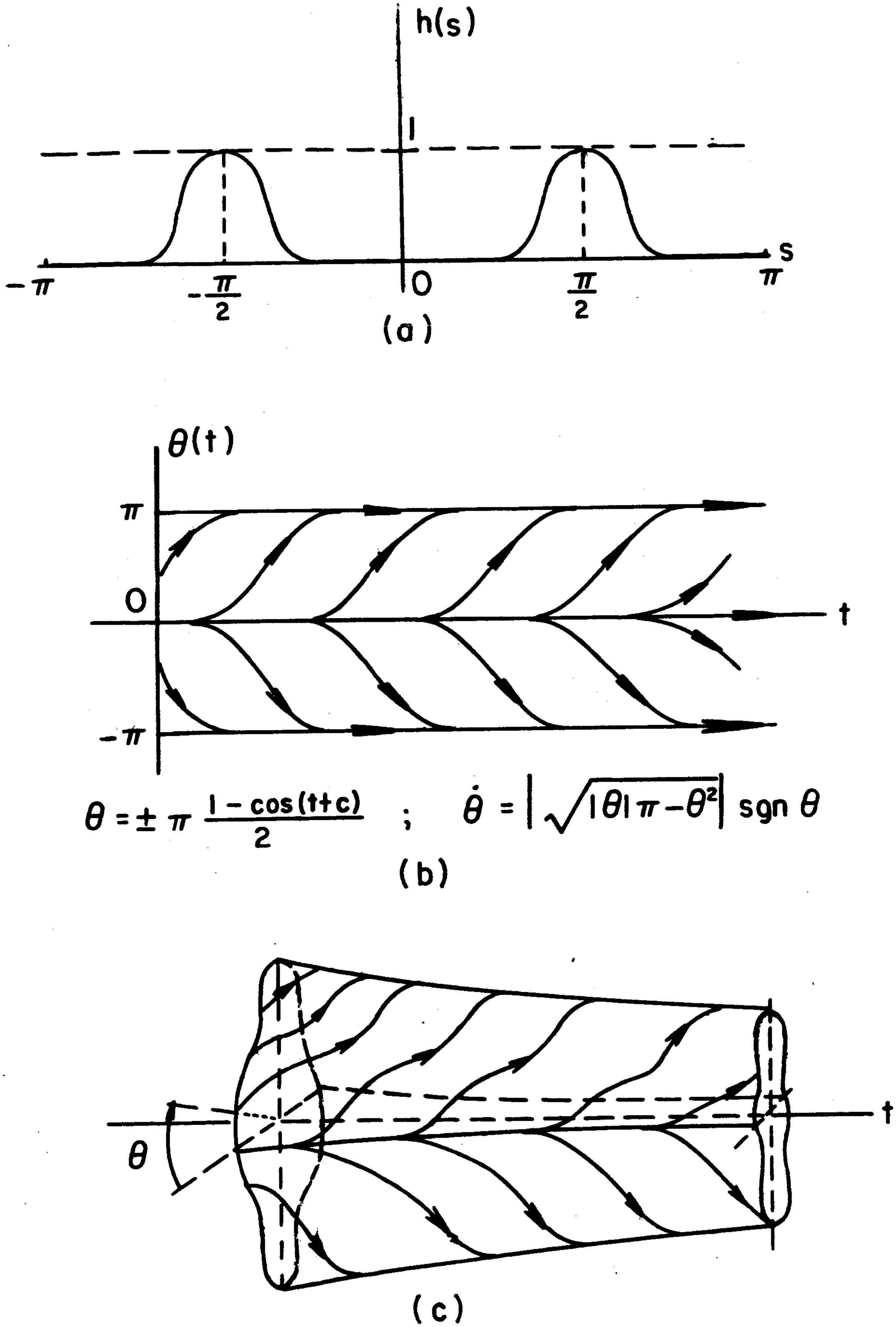
FIG. 2



For $x \le e^{-t}$: $x(t) = x_o e^{-t}$

for $x \ge e^{-t}$: $x(t) = x_o - 1 + e^{-t}$

FIG. 3



h(s)
1
s
−π
−π/2
0
π/2
π
(a)
θ(t)
π
0
t
−π
$\theta = \pm \pi \frac{1-\cos(t+c)}{2}$; $\dot{\theta} = \left| \sqrt{|\theta|\pi - \theta^2} \right| \operatorname{sgn} \theta$
(b)
θ
t
(c)

FIG. 4

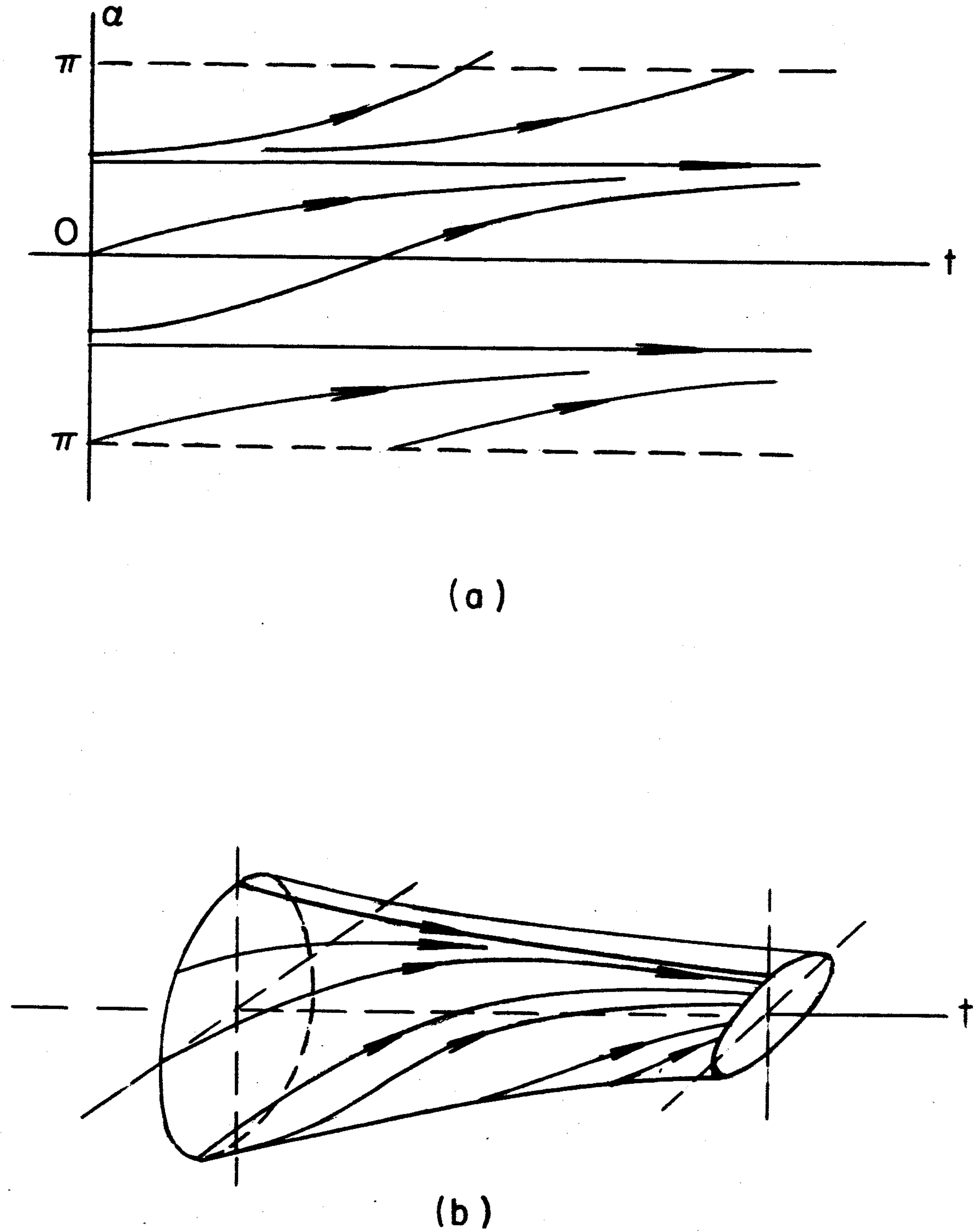


FIG. 5

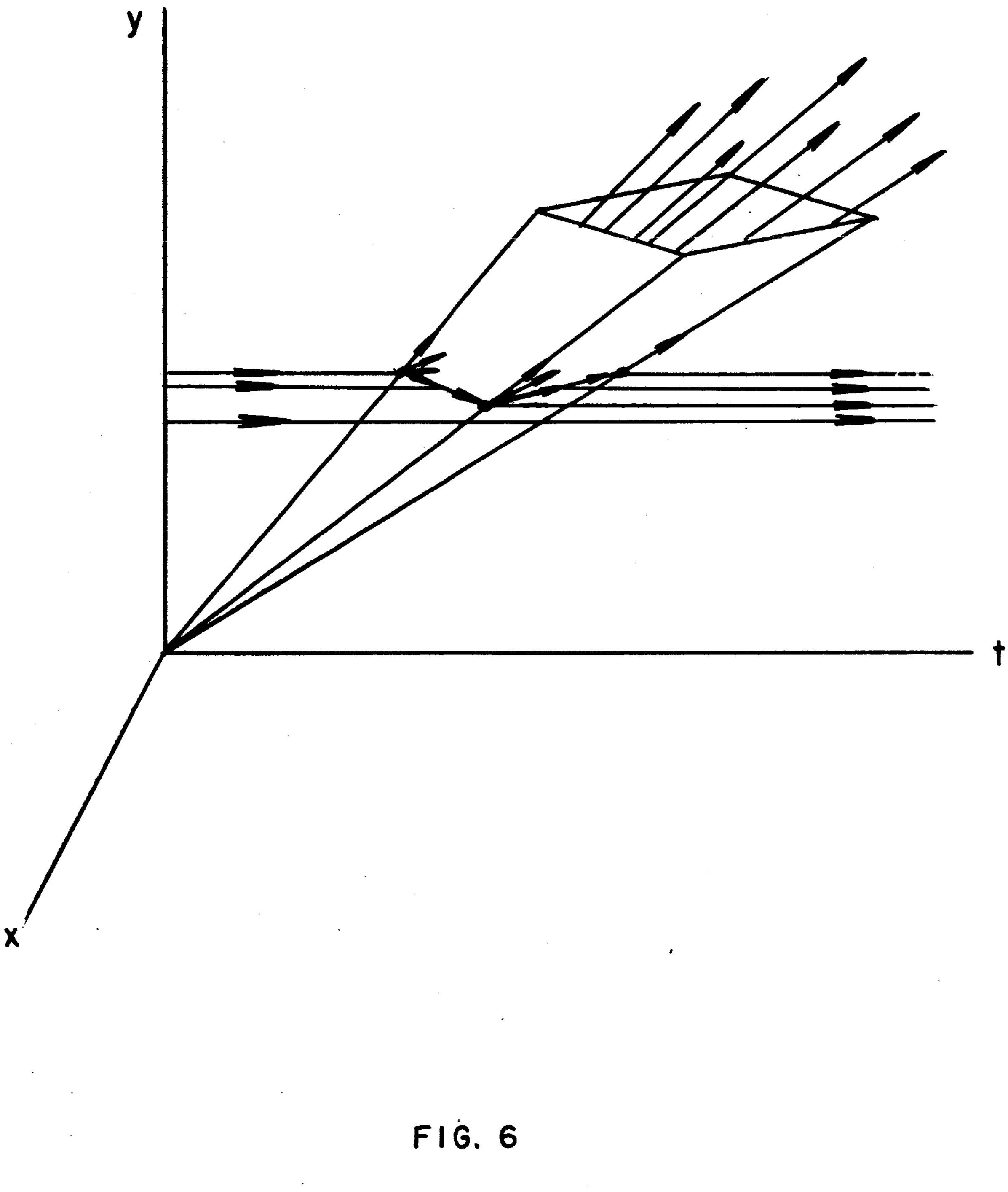


FIG. 6

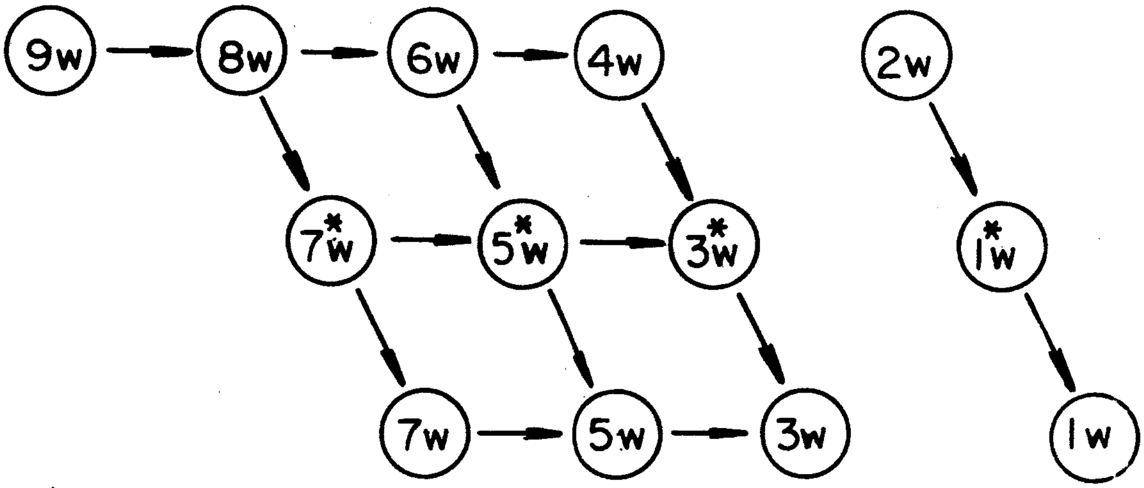
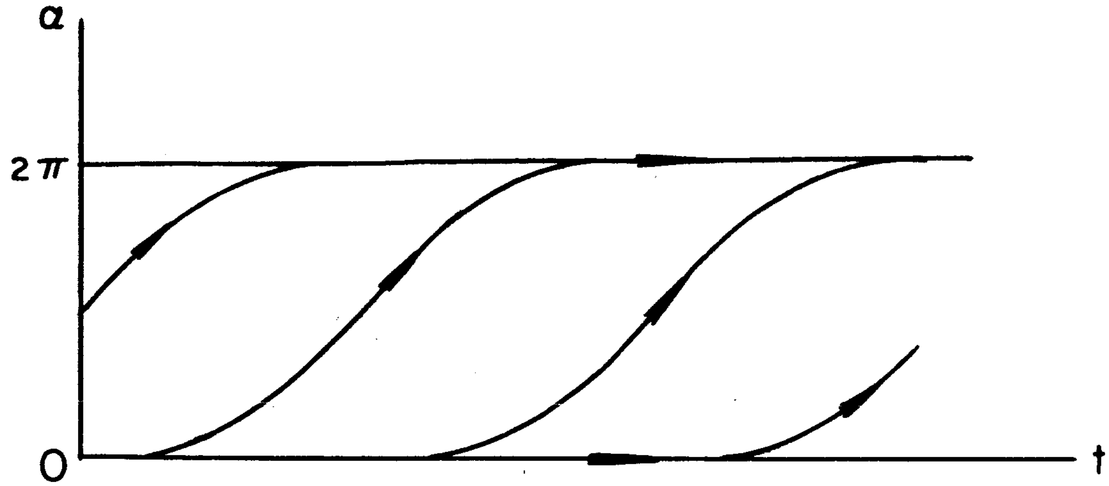
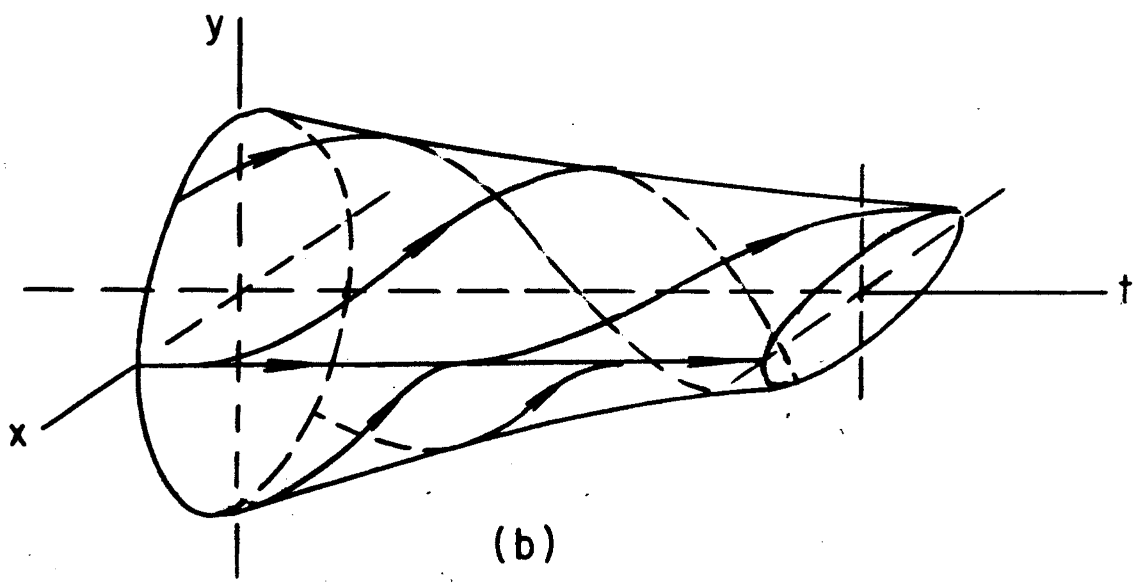


FIG. 7



(a)



(b)

FIG. 8

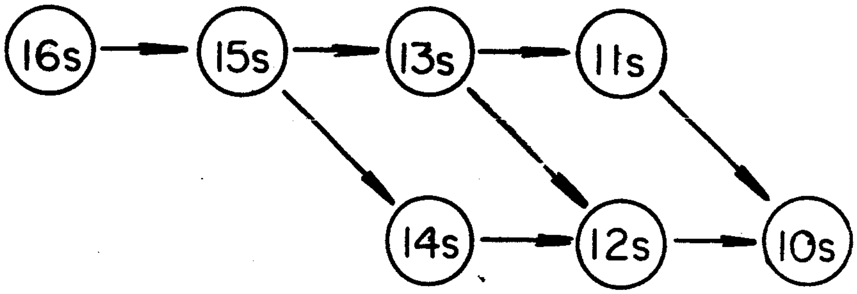


FIG. 9

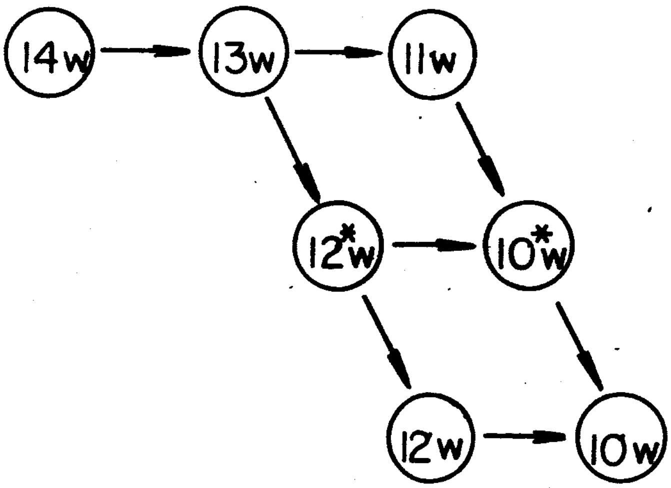


FIG. 10