LONG-TERM MOTION OF A LUNAR SATELLITE

Frances A. Frost

October 1965

Goddard Space Flight Center
Greenbelt, Maryland

CONTENTS

LONG-TERM MOTION OF A LUNAR SATELLITE

I. INTRODUCTION

The purpose of this report is to examine two special methods of studying the long-term motion of a lunar orbiter. The Hamiltonian used is the long-period Hamiltonian (in which the argument of pericenter and inclination of the satellite appear but other angular variables have been eliminated) as derived by Kozai and Giacaglia, et. al., (References 2, 3). It includes the perturbation of the earth as a point mass and the principal part of the oblateness of the moon.

This long-period Hamiltonian is

$$F = \frac{1}{6} K_1 L (5 - 3\eta^2)\left(-1 + \frac{3H^2}{G^2}\right) + \frac{1}{3} K_2 \eta^{-3}\left(-1 + \frac{3H^2}{G^2}\right) + \frac{5}{2} K_1 L (1 - \eta^2)\left(1 - \frac{H^2}{G^2}\right)\cos 2g$$
$$= \frac{1}{6} C, \tag{1}$$

where

$\eta^2 = 1 - e^2$

$e =$ eccentricity of the satellite's orbit

$L = \sqrt{\mu a}$

$G = L\sqrt{1 - e^2}$

$H = G \cos i$

$\mu =$ gravitational constant times the mass of the moon $= 3.6601891 \times 10^{-3}$ decamegameters^3/centiday^2

$a =$ semimajor axis of the satellite's orbit (decamegameters)

$i =$ inclination of satellite orbit plane to the moon's equatorial plane

$g =$ argument of pericenter of the satellite

$$K_1 = \frac{3n_{☾}^2}{8\epsilon n} \quad (\text{centiday}^{-1})$$

$n_{☾}$ = mean motion of the moon
= 2.2802713×10^{-3} (radians/centiday)

$n = \mu^2 L^{-3}$ = mean motion of the satellite

ϵ = ratio of sum of masses of earth and moon to the mass of the earth
= 1.0123001

$$K_2 = \frac{3}{4} J_2 b^2 n^2 \quad (\text{decamegameters}^2/\text{centiday}^2)$$

J_2 = principal part of the oblateness of the moon
= 2.41×10^{-4} (Reference 2)

b = mean radius of the moon
= 0.1738 decamegameters

Since the Hamiltonian (1) is not an explicit function of time, it has a constant value, $F = 1/6\ C$.

Special methods are needed for the problem of the long-term motion since the standard procedures – von Zeipel's method and the method of successive approximations – break down at this point for the lunar satellite (e.g. see Reference 2). The special methods, which involve harmonic analysis and elliptic integrals, to be considered in this report were suggested many years ago by E.W. Brown in his paper "On the Stellar Problem of Three Bodies" (Reference 1).

In the method involving harmonic analysis it is assumed that η^2 and dt/dg may be represented by cosine series with the argument of pericenter g as the variable. Integrating dt/dg gives g as a sum of a linear expression of time and a sine series in g. On reversing this expression we have g as a function of time. Then η^2 is approximated as a cosine series in time. In the second of these methods proposed by Brown,

$$\dot{\eta} = \frac{1}{L} \frac{\partial F}{\partial g}, \quad \dot{g} = -\frac{\partial F}{\partial G},$$

and an approximation of cos 2g found by neglecting the oblateness of the moon are used to give $d\eta^2/dt$ and $dg/d\eta^2$ as functions of η^2 alone. These result in elliptic integrals of the first and third kinds as the solutions for η^2 as a function of time and g as a function of η^2 (and therefore of time).

For the harmonic analysis method it is necessary to solve the Hamiltonian for η^2 corresponding to chosen values of the argument of pericenter. To be able to compare the values of η^2 and g as obtained from harmonic analysis and elliptic integrals the portion of the Hamiltonian containing the oblateness of the moon was neglected, both in solving for η^2 and evaluating dt/dg. Setting $J_2 = 0$ results in the following quadratic equation in $x = \eta^2$:

$$3(1 - 5\cos 2g)x^2 - [5 + 9\nu^2 - 15\cos 2g(1 + \nu^2) + C']\, x + 15\,\nu^2\,(1 - \cos 2g) = 0$$

where

$$C' = 2(5 - 6\,\eta_1^2 + 3\,\nu^2)$$

and $\nu = H/L$ are constants. η_1^2 is the value of η^2 when $g = 0$. If the oblateness of the moon is included in the Hamiltonian, however, an approximation used in this report is to represent $(1 - e^2)^{-3/2}$ by its Maclaurin series terminating the series to give the quadratic equation in e^2 (for small to moderate values of e)

$$\alpha e^4 + \beta e^2 + \gamma = 0,$$

where

$$\alpha = 3K_1 L + \frac{3}{4} K_2 (-1 + 15\nu^2) - 15K_1 L\cos 2g$$

$$\beta = (K_1 L + K_2)(-1 + 9\nu^2) + C + 15\,K_1 L(1 - \nu^2)\cos 2g$$

$$\gamma = 2(K_1 L + K_2)(-1 + 3\nu^2) - C.$$

This quadratic is then solved for $\eta^2 = 1 - e^2$, given specific values of cos 2g:

$$\eta^2 = 1 - e^2 = 1 + \frac{\beta}{2\alpha} \mp \sqrt{\frac{\beta^2}{4\alpha^2} - \frac{\gamma}{\alpha}}$$

(the sign is chosen so that η^2 lies between zero and one).

II. THE HARMONIC ANALYSIS METHOD

Given the initial conditions

$$\begin{aligned} L &= 0.06 \\ G &= 0.055 \\ H &= 0.05 \\ g &= 0 \end{aligned}$$

we find

$$\begin{aligned} e &= 0.39965264 \\ a &= 0.98\,355573 \text{ decamegameters} \\ i &= 24.619974^\circ \\ K_1 &= 3.105573 \times 10^{-5} \\ K_2 &= 2.1003149 \times 10^{-8} \\ C &= 7.6893300 \times 10^{-6}. \end{aligned}$$

We assume

$$\eta^2 = \sum_{k \geq 0} a_k \cos 2kg \quad \text{and} \quad \frac{dt}{dg} = \sum_{k \geq 0} b_k \cos 2kg,$$

terminating the series with cos 8g so that

$$\eta^2 = a_0 + a_1 \cos 2g + a_2 \cos 4g + a_3 \cos 6g + a_4 \cos 8g$$

$$\frac{dt}{dg} = b_0 + b_1 \cos 2g + b_2 \cos 4g + b_3 \cos 6g + b_4 \cos 8g.$$

Solving the Hamiltonian for η^2 given the five values $2g = 0, \pi, \pi/2, \pi/3, 2\pi/3$, we have the following results:

$2g$	0	π	$\frac{\pi}{2}$	$\frac{\pi}{3}$	$\frac{2\pi}{3}$
η^2, $K_2 = 0$	.84027778	.77996317	.80665828	.82299779	.79219924
η^2, $K_2 \neq 0$	.84027778	.78055119	.80702653	.82317074	.79270692

Corresponding to each of the five special values of $2g$ with the associated values of η^2, dt/dg is evaluated using

$$\frac{dg}{dt} = -\frac{\partial F}{\partial G} = K_1\eta\left[\left(-1+\frac{5\nu^2}{\eta^4}\right)+5\left(1-\frac{\nu^2}{\eta^4}\right)\cos 2g\right]-\frac{K_2}{L\eta^4}\left(1-\frac{5\nu^2}{\eta^2}\right)$$

Then the following table may be used to solve for the coefficients in the series for dt/dg:

$2g$	0	π	$\frac{\pi}{2}$	$\frac{\pi}{3}$	$\frac{2\pi}{3}$
$\frac{dt}{dg}$, $K_2 = 0$	8781.8656	6732.7808	8268.1582	8735.5882	7538.4678
$\frac{dt}{dg}$, $K_2 \neq 0$	8663.7170	6662.5543	8155.7627	8611.8994	7446.8692

Letting f_k denote the k^{th} value of η^2 in the table, the equations

$$f_1 = a_0 + a_1 + a_2 + a_3 + a_4$$

$$f_2 = a_0 - a_1 + a_2 - a_3 + a_4$$

$$f_3 = a_0 \qquad - a_2 \qquad + a_4$$

$$f_4 = a_0 + \frac{1}{2}a_1 - \frac{1}{2}a_2 - a_3 - \frac{1}{2}a_4$$

$$f_5 = a_0 - \frac{1}{2}a_1 - \frac{1}{2}a_2 + a_3 - \frac{1}{2}a_4$$

may be solved for the coefficients a_k in the terminated series expansion for η^2. For $K_2 = 0$ we then have

$$a_0 = \frac{1}{6}(f_1 + f_2) + \frac{1}{3}(f_4 + f_5) = .80843917$$

$$a_1 = \frac{1}{3}(f_1 - f_2 + f_4 - f_5) = .03637106$$

$$a_2 = \frac{1}{4}(f_1 + f_2) - \frac{1}{2} f_3 = .00173110$$

$$a_3 = \frac{1}{6}(f_1 - f_2) - \frac{1}{3}(f_4 - f_5) = -.00021374$$

$$a_4 = f_3 - a_0 + a_2 = -.00004979.$$

Thus, if $K_2 = 0$,

$$\eta^2 = .80843917 + .03037106 \cos 2g + .00173110 \cos 4g - .00021374 \cos 6g - .00004979 \cos 8g \tag{2}$$

while, if $K_2 \neq 0$,

$$\eta^2 = .80876405 + .03006347 \cos 2g + .00169398 \cos 4g - .00020018 \cos 6g - .00004354 \cos 8g. \tag{2'}$$

Similarly, the five coefficients b_k in the terminated cosine series for dt/dg may be evaluated, so that if $K_2 = 0$,

$$\frac{dt}{dg} = 8010.4597 + 1082.0684 \cos 2g - 255.4175 \cos 4g - 57.5260 \cos 6g + 2.2810 \cos 8g$$

and, if $K_2 \neq 0$,

$$\frac{dt}{dg} = 7907.3015 + 1055.3976 \cos 2g - 246.3136 \cos 4g$$

$$- 54.8163 \cos 6g + 2.1476 \cos 8g.$$

Integrating this expression for dt/dg gives

$$t = b_0 g + \frac{b_1}{2} \sin 2g + \frac{b_2}{4} \sin 4g + \frac{b_3}{6} \sin 6g + \frac{b_4}{8} \sin 8g$$

(the constant of integration is zero in this example because of the initial condition $g = 0$). Then

$$g = \frac{1}{b_0} t - \frac{b_1}{2b_0} \sin 2g - \frac{b_2}{4b_0} \sin 4g - \frac{b_3}{6b_0} \sin 6g - \frac{b_4}{8b_0} \sin 8g.$$

In particular,

$$(K_2 = 0) \quad g = .12483678 \times 10^{-3}\, t - .06754097 \sin 2g$$

$$+ .00797137 \sin 4g + .00119689 \sin 6g - .00003559 \sin 8g$$

$$(K_2 \neq 0) \quad g = .12646539 \times 10^{-3}\, t - .06673563 \sin 2g + .00778754 \sin 4g$$

$$+ .00115539 \sin 6g - .00003395 \sin 8g.$$

Since we may write $g = \tau + \varphi(g)$, where τ is a linear function of time and $\varphi(g)$ is a trigonometric function in g, Lagrange's expansion theorem may be applied to reverse the equation and give g as a function in time. By this theorem, if

$$y = x + \varphi(y),$$

then a series may be used to express y as a function of x:

$$y = x + \varphi(x) + \frac{1}{2!}\ \frac{\partial}{\partial x}(\varphi^2(x)) + \frac{1}{3!}\ \frac{\partial^2}{\partial x^2}(\varphi^3(x)) + \cdots$$

In this case, only the first four terms of this series were taken and all terms after $\sin 8g$ were eliminated.

Hence,

$$g = \left[\tau + \left(a + \frac{A}{2} + \frac{E}{6}\right) \sin 2\tau + \left(b + \frac{B}{2} + \frac{P}{6}\right) \sin 4\tau \right.$$
$$\left. + \left(c + \frac{J}{2} + \frac{M}{6}\right) \sin 6\tau + \left(d + \frac{D}{2} + \frac{N}{6}\right) \sin 6\tau\right]$$

where

	$k_2 = 0$	$k_2 \neq 0$
a	= -.067540965	-.06673563
b	= .0079713748	.00778754
c	.0011968934	.00115539
d	-.000035595648	-.00003395

$$A = -2(ab + bc + cd)$$

$$B = 2(a^2 - 2ac - 2bd)$$

$$J = 6a(b - d)$$

$$D = 4(2ac + b^2)$$

$$E = 6\left\{-\frac{a^3}{2} - b^2 a - c^2 a - d^2 a - \frac{b^2 c}{2} - bcd + \frac{a^2 c}{2} + bad\right\}$$

$$P = 24\left\{-bac - adc - a^2 b - bc^2 - bd^2 + \frac{a^2 d}{2} - \frac{b^3}{2} - \frac{c^2 d}{2}\right\}$$

$$M = 54\left\{-b^2c - bcd - \frac{ab^2}{c} - d^2c - adb + \frac{a^3}{6} - a^2c - \frac{c^3}{2}\right\}$$

$$N = 48\left\{ba^2 - 2abc - 2b^2d - bc^2 - 2a^2d - d^3 - 2dc^2\right\}$$

Thus, for $K_2 = 0$,

$$g = \tau - .06685092 \sin 2\tau + .01255112 \sin 4\tau - .00091832 \sin 6\tau + .00007192 \sin 8\tau \tag{3}$$

where $\tau = 0.12483678 \times 10^{-3}\, t$ (t in centidays), while if $K_2 \neq 0$,

$$g = \tau - .06606959 \sin 2\tau + .01225829 \sin 4\tau - .00088524 \sin 6\tau + .00006836 \sin 8\tau. \tag{3'}$$

($\tau = .12646539 \times 10^{-3}\, t$, t in centidays).

Letting 2τ take on the five special values $0, \pi, \pi/2, \pi/3, 2\pi/3$, these equations for g give corresponding values of 2g; from these we may evaluate η^2 using the terminated cosine series (2) and (2'). This gives the following results:

2τ	0	π	$\frac{\pi}{2}$	$\frac{\pi}{3}$	$\frac{2\pi}{3}$
g, $K_2 = 0$	0	$\frac{\pi}{2}$	.71946556	.47651148	.97849565
η^2, $K_2 = 0$	.84027778	.77996317	.81080056	.82570548	.79556239
g, $K_2 \neq 0$	0	$\frac{\pi}{2}$	.72021381	.47693763	.97942282
η^2, $K_2 \neq 0$	.84027778	.78055119	.81107413	.82582292	.79599495

Assuming η^2 may be written as a cosine series in τ and taking the first five terms of the series, we may evaluate the coefficients of these terms as above. Thus, we have for $K_2 = 0$,

$$\eta^2 = .81046278 + .03015257 \cos 2\tau - .00034004 \cos 4\tau$$

$$+ .00000474 \cos 6\tau - .00000226 \cos 8\tau$$

$$(\tau = .12483678 \times 10^{-3}\, t) \tag{4}$$

and for $K_2 \neq 0$,

$$\eta^2 = .81074412 + .02985152 \cos 2\tau - .00032982 \cos 4\tau$$

$$+ .00001177 \cos 6\tau + .00000019 \cos 8\tau. \tag{4'}$$

$$(\tau = .12646539 \times 10^{-3}\, t)$$

In this way both g and η^2 may be found as functions of time.

Tables 1 and 2 give the values of η^2 and g. Equations (4) and (4') are plotted against time in Figure 1. This shows the shift that occurs when the oblateness of the moon is considered in the long-period Hamiltonian. Figure 2 shows similar results for g as given by Equations (3) and (3').

As may be seen from these figures, the effect of the moon's oblateness on a satellite at this distance from the moon (5.659 moon radii) is not great. For a satellite near the surface of the moon, the effect of the moon's oblateness on the eccentricity and argument of pericenter of the satellite would be much more pronounced.

Table 1
η^2, g Harmonic Analysis (K_2 = 0)

t (days)	η^2	g Radians	g Degrees	F $\times 10^{-5}$	F - C $\times 10^{-10}$
0	.84027782	0	0	.76086524	0
20	.83675070	.2273639	13.02699	.76086623	.089812
40	.82692477	.4543801	26.03406	.76086372	-.160298
60	.81298783	.6850423	39.25003	.76086474	-.059685
80	.79215000	.9287134	53.21135	.76086799	.266027
100	.78627455	1.1952296	68.48161	.76085606	-.927684
120	.78029692	1.4843190	85.04521	.76086390	-.143245
140	.78191710	1.7799106	101.98136	.76085884	-.649720
160	.79068695	2.0606739	118.06792	.76086051	-.481463
180	.80424327	2.3168499	134.33726	.76086867	.334239
200	.81911010	2.5537270	146.31778	.76086276	-.2575
220	.83167979	2.7820046	159.39712	.76086533	0
240	.83906558	3.0090020	172.40312	.76086577	.043769
260	.83965478	3.2365252	185.43923	.76086555	.022168
280	.83332190	3.4636120	198.45035	.76086267	.051159
300	.82144059	3.6913262	211.49741	.76086784	-.266027
320	.80671501	3.9260582	224.94656	.76086301	.250679
340	.79268607	4.1783800	239.40354	.76086301	-.232489
360	.78291847	4.4552836	255.26894	.76085703	-.829345
380	.78002546	4.7497567	272.14101	.76086506	-.027284
400	.78480573	5.0413928	288.85053	.76085565	-.967474
420	.79594596	5.3120934	304.36053	.76086641	.107434
440	.81050099	5.5589943	318.50691	.76086596	.062527
460	.82482343	5.7910522	331.80284	.76086308	-.225099
480	.83551665	6.0182398	344.81974	.76086616	.082991
500	.84017939	6.2454964	357.84058	.76086530	-.003410
520	.83780875	.1897622	10.87257	.76086611	.077875
540	.82891087	.4167136	23.87593	.76086439	-.093791

Table 2

η^2, g: Harmonic Analysis ($K_2 \neq 0$)

t (days)	η^2	g Radians	g Degrees	F $\times 10^{-5}$	ΔF $\times 10^{-8}$
0	.84027778	0	0	.76393302	0
20	.83668256	.23048543	13.205842	.76898250	.0495106
40	.82670981	.46072723	26.397726	.76906286	.1298758
60	.81265809	.69497792	39.819302	.76911933	.1863440
80	.79790366	.94291820	54.025233	.76917259	.2395950
100	.78620654	1.2142525	69.571544	.76921679	.2837964
120	.78072811	1.5074170	86.383185	.76926319	.3201989
140	.78300817	1.8058868	103.46969	.76923517	.3021796
160	.79240119	2.0877218	119.61765	.76919183	.2588421
180	.80631778	2.3446295	134.33738	.76914220	.2092065
200	.82111800	2.5832362	148.00853	.76908830	.1553075
220	.83316153	2.8143461	161.25016	.76901856	.0855607
240	.83963119	3.0445164	174.43794	.76894303	.0100385
260	.83905608	3.2751774	187.65384	.76895149	.0184968
280	.83156597	3.5053372	200.84103	.76903174	.0987483
300	.81886887	3.7369943	214.11400	.76909695	.1639591
320	.80395451	3.9777790	227.90995	.76915085	.2178580
340	.79050779	4.2384158	242.84334	.76919868	.2656861
360	.78212960	4.5239208	259.20157	.76924155	.3085574
380	.78109380	4.8229269	276.33336	.76924995	.3169589
400	.78771252	5.1136422	292.99012	.76920982	.2768274
420	.80013334	5.3809400	308.30515	.76916465	.2316596
440	.81502776	5.625948	322.34308	.76911089	.1778971
460	.82865910	5.8590324	335.69783	.76905185	.1188595
480	.83771752	6.0893168	348.89215	.76896972	.0367322
500	.84018579	.03656091	2.094786	.76893447	.0014779
520	.83548432	.26695049	15.295136	.76899601	.0630166
540	.82467794	.49738946	28.498317	.76907304	.1400508

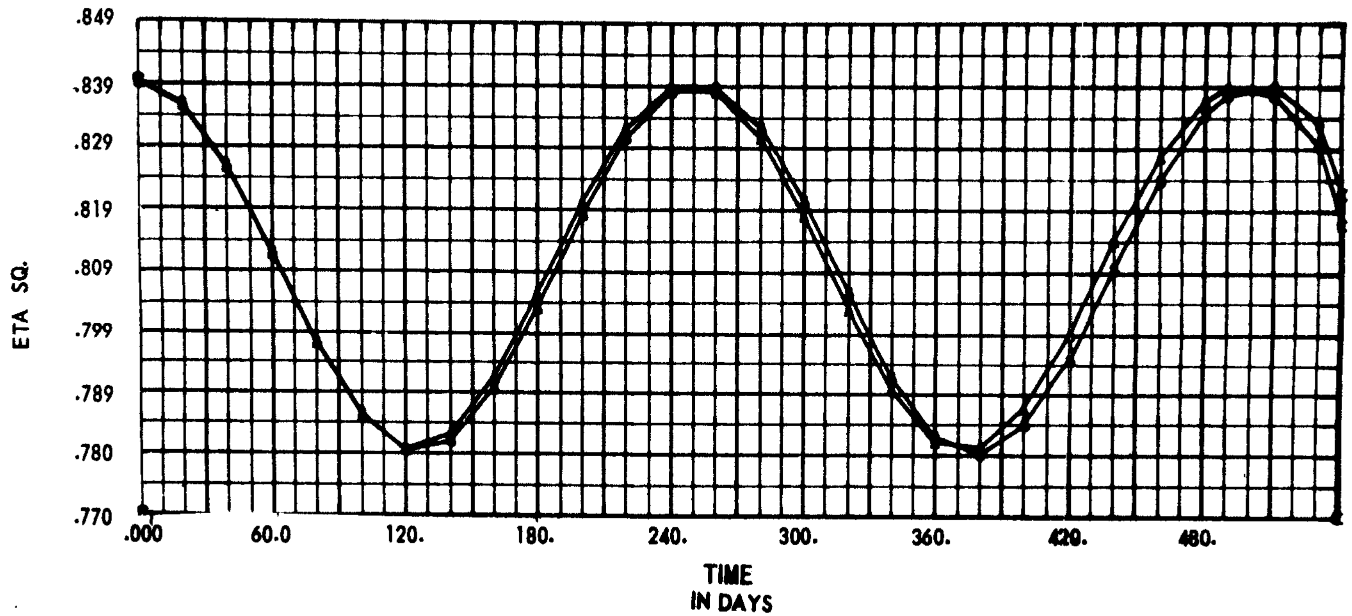

Figure 1

K_2 is zero, S; K_2 is not zero, A.

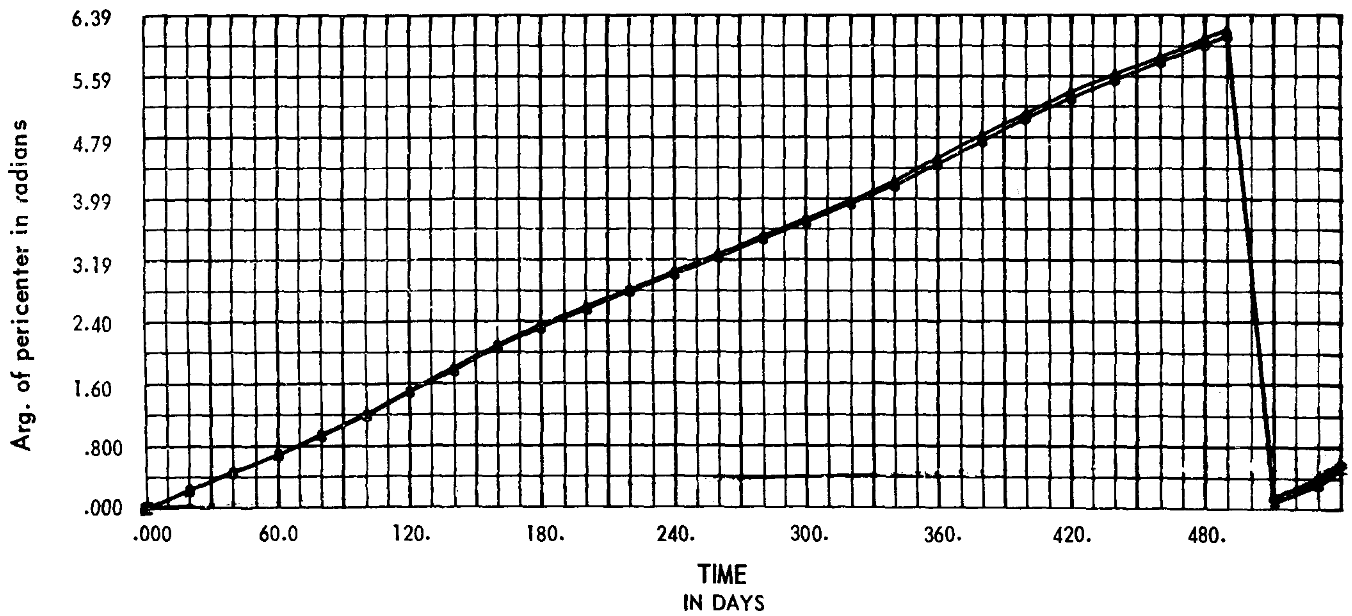

Figure 2

K_2 is zero, S; K_2 is not zero, A.

III. THE SOLUTION WITH ELLIPTIC INTEGRALS

The second method suggested by Brown, starting with $\dot{\eta}$ and $\dot{g}$, results in elliptic integral expressions for η^2 and g, but for this solution the oblateness portion of the Hamiltonian is neglected.

From the long-period Hamiltonian (1) and the definition of the Delaunay variables, we have

$$\frac{d\eta}{dt} = \frac{1}{L}\frac{dG}{dt} = \frac{1}{L}\frac{\partial F}{\partial g} = -5K_1(1-\eta^2)\left(1-\frac{\nu^2}{\eta^2}\right)\sin 2g. \tag{5}$$

Setting $K_2 = 0$ in the Hamiltonian gives

$$W = (5-3\eta^2)\left(-1+\frac{3\nu^2}{\eta^2}\right) + 15(1-\eta^2)\left(1-\frac{\nu^2}{\eta^2}\right)\cos 2g, \tag{6}$$

where the constant W is evaluated for $g = 0$. Then, $W = 2(5-6\eta_1^2+3\nu^2)$ where η_1^2 is the value of η^2 when $g = 0$. On solving the Hamiltonian (6) for $\cos 2g$, we have

$$\cos 2g = \left[15(1-\eta^2)\left(1-\frac{\nu^2}{\eta^2}\right)\right]^{-1}\left[(5-3\eta^2)\left(1-\frac{3\nu^2}{\eta^2}\right) + 2(5-6\eta_1^2+3\nu^2)\right]$$

which may be substituted in (6) for $\sin 2g = [1-\cos^2 2g]^{1/2}$ to give

$$\frac{dx}{dt} = 2\eta\dot{\eta} = \mp 4\sqrt{6}\,K_1\,[(x-x_1)(x-x_2)(x-x_3)]^{1/2},$$

where $x = \eta^2$, $x_1 = \eta_1^2$ and the roots x_2, x_3 of the polynomial are defined by the equations

$$x_2 x_3 = \frac{5}{3}\nu^2$$

$$x_2 + x_3 = \frac{1}{3}(5+5\nu^2-2x_1).$$

In this expression for dx/dt the minus sign is used when $\sin 2g$ is positive and the plus sign when $\sin 2g$ is negative.

If the roots of the polynomial are relabeled to give a strict ordering $x_1' < x_2' < x_3'$ with the relationship $x_1' \leq x \leq x_2'$, we have

$$\frac{dx}{\sqrt{(x - x_1')(x - x_2')(x - x_3')}} = \mp 4 K_1 \sqrt{6}\ dt.$$

Integrating both sides gives

$$\int_{x_0}^{x} \frac{dx}{\sqrt{(x - x_1')(x - x_2')(x - x_3')}} = \mp 4 K_1 \sqrt{6} \int_{t_0}^{t} dt$$

where x_0 is the initial value of x and t_0 may be taken as zero.

Set

$$x = \frac{x_3' - x_2' y^2}{1 - y^2}$$

$$dx = \frac{2(x_3' - x_2') y\, dy}{(1 - y^2)^2}$$

$$k = \sqrt{\frac{x_2' - x_1'}{x_3' - x_1'}}$$

and we have

$$\mp 4 K_1\ t \sqrt{6} = \frac{2}{\sqrt{x_3' - x_1'}} \int_{y_0}^{y} \frac{dy}{\sqrt{(1 - k^2 y^2)(1 - y^2)}}$$

or

$$\int_{y_0}^{y} \frac{dy}{\sqrt{(1 - k^2 y^2)(1 - y^2)}} = \mp 2 K_1 \, t \sqrt{6(x_3' - x_1')}$$

Let us define

$$v = 2K_1 \, t \sqrt{6(x_3' - x_1')}$$

$$u = \int_{y_1}^{y_0} \frac{dy}{\sqrt{(1 - k^2 y^2)(1 - y^2)}}$$

Then

$$v = \int_{x_0}^{x} \frac{\sqrt{x_3' - x_1'}\, dx}{2\sqrt{(x - x_1')(x - x_2')(x - x_3')}}$$

$$= \left\{ \int_{x_1}^{x} \frac{dx}{\sqrt{(x - x_1')(x - x_2')(x - x_3')}} \pm \int_{x_1}^{x_0} \frac{dx}{\sqrt{(x - x_1')(x - x_2')(x - x_3')}} \right\} \frac{\sqrt{x_3' - x_1'}}{2}$$

$$= \left[\int_{y_1}^{y} \frac{dy}{\sqrt{(1 - y^2)(1 - k^2 y^2)}} \pm \int_{y_1}^{y_0} \frac{dy}{\sqrt{(1 - y^2)(1 - k^2 y^2)}} \right] \frac{\sqrt{x_3' - x_1'}}{2}$$

where

$$y = \pm \sqrt{\frac{x - x_3'}{x - x_2'}}$$

so that

$$u \mp v = \int_0^{\sqrt{\frac{x - x_1'}{x_2' - x_1'}}} \frac{dy}{\sqrt{(1 - y^2)(1 - k^2 y^2)}} .$$

Thus we may solve for x as a function of time, using the Jacobi elliptic function of u and v:

$$\sqrt{\frac{x - x_1'}{x_2' - x_1'}} = \text{sn } (u \pm v)$$

$$x = x_1' + (x_2' - x_1') \text{ sn}^2 (u \pm v)$$

$$= x_1' + (x_2' - x_1') \left[\frac{\text{sn u cn v dn v} \pm \text{sn v cn u dn u}}{1 - k^2 \text{ sn}^2 \text{ u sn}^2 \text{ v}}\right]^2 .$$

For the given initial conditions we have

$$x_1' = .77996317$$

$$x_2' = .84027778$$

$$x_3' = 1.48392569$$

$$k^2 = .08567872$$

$$\text{sn u} = 1$$

$$\text{cn u} = 0$$

$$x = .77996317 + .06031461 \left[\frac{\text{cn v dn v}}{1 - .08567872 \text{ sn}^2 \text{ v}}\right]^2 .$$

To evaluate g , the Hamiltonian is solved for cos 2g, giving

$$\cos 2g = \left[15(1-x)\left(1-\frac{\nu^2}{x}\right)\right]^{-1}\left[\frac{C}{K_1 L} + \frac{2K_2}{K_1 L}\;\frac{1-3\nu^2 x^{-1}}{x^{3/2}} + (5-3x)\left(1-\frac{3\nu^2}{x}\right)\right].$$

This expression is substituted for cos 2g in

$$\dot{g} = -\frac{K_1}{\sqrt{x}}\left[\left(x-\frac{5\nu^2}{x}\right) - 5\left(x-\frac{\nu^2}{x}\right)\cos 2g\right] - \frac{K_2}{Lx^2}\left(1-\frac{5\nu^2}{x}\right).$$

Since

$$\dot{g} = \frac{dg}{dx}\cdot\frac{dx}{dt} = \mp 4\sqrt{6}\;K_1\;\sqrt{Q(x)}\;\frac{dg}{dx}\;,$$

where $Q(x) = (x-x_1)(x-x_2)(x-x_3)$, this substitution gives

$$\frac{dg}{dx} = -\frac{K_1}{\sqrt{xQ(x)}}\left(x-\frac{5\nu^2}{x}\right) + \frac{5K_1}{\sqrt{xQ(x)}}\left(x-\frac{\nu^2}{x}\right)\left[\frac{C}{K_1 L} + \frac{2K_2}{K_1 L}\;\frac{1-3\nu^2 x^{-1}}{x^{3/2}}\right.$$

$$\left. + (5-3x)\left(1-\frac{3\nu^2}{x}\right)\right]\left[15(1-x)\left(1-\frac{\nu^2}{x}\right)\right]^{-1} - \frac{K_2}{Lx^2\sqrt{Q(x)}}\left(1-\frac{5\nu^2}{x}\right).$$

Integrating this expression, we have

$$\mp 4\sqrt{6}\;K_1 g = \frac{1}{3}\left[\frac{C}{L} + 2(1-3\nu^2)K_1\right]\int_{x_0}^{x}\frac{dx}{(1-x)\sqrt{xQ(x)}}$$

$$-\frac{\nu^2}{3}\left[\frac{C}{L} + 2(3\nu^2-5)K_1\right]\int_{x_0}^{x}\frac{dx}{(x-\nu^2)\sqrt{xQ(x)}}$$

$$
-\frac{1}{3}\left[\frac{C}{L} + 2(1 + 3\nu^2)\, K_1\right] \int_{x_0}^{x} \frac{dx}{\sqrt{xQ(x)}}
$$

$$
+\frac{2}{3}\,\frac{K_2}{L}\,(1 - 3\nu^2) \int_{x_0}^{x} \frac{dx}{(1 - x)\sqrt{Q(x)}} - \frac{4}{3\nu^2}\,\frac{K_2}{L} \int_{x_0}^{x} \frac{dx}{(x - \nu^2)\sqrt{Q(x)}}
$$

$$
+\frac{2}{3}\,\frac{K_2}{\nu^2 L}\,(\nu^2 - 3\nu^4 - 2) \int_{x_0}^{x} \frac{dx}{x\sqrt{Q(x)}}
$$

$$
-\frac{1}{3}\,\frac{K_2}{L}\,(7 + 6\nu^2) \int_{x_0}^{x} \frac{dx}{x^2\sqrt{Q(x)}} + 3\nu^2\,\frac{K_2}{L} \int_{x_0}^{x} \frac{dx}{x^3\sqrt{Q(x)}}.
$$

In actually evaluating g only the first three summands of the preceding were used since the oblateness of the moon was omitted from the Hamiltonian. The elliptic functions were evaluated using Reference 6.

IV. CONCLUSIONS

The results of these two special methods involving harmonic analysis and elliptic integrals are compared using the given initial conditions and the harmonic analysis solutions (3) and (4) for η^2 and g when $K_2 = 0$. These results are tabulated in table 3 together with the differences between the values obtained by the two methods.

In table 2 are the values of η^2 and g when the oblateness of the moon is retained in the Hamiltonian and a Maclaurin series is used to give a quadratic equation in e^2. The value of the Hamiltonian (1) was computed for η^2 and g (at 20 day intervals) and compared to the initial value ($\eta^2 = .84027778$, $g = 0$) $C = .76893300 \times 10^{-5}$.

Although g is periodic in this example and all values are admissible, there is the possibility that a lunar satellite may have a stable orbit but that for some value g_0 of the argument of pericenter, dg/dt will be zero so that dt/dg will be undefined. In such an event the special values of 2g used to evaluate the coefficients in the cosine series representation of η^2 and dt/dg must be restricted to the region of admissible values, i.e., $g_0 < g < (\pi - g_0)$.

Table 3
η^2, g; Elliptic Integrals and Harmonic Analysis (J_2 = 0)

t (days)	η^2 Elliptic Integrals	η^2 Harmonic Analysis	$\lvert\Delta \eta^2\rvert$	g Elliptic Integrals (radians)	g Harmonic Analysis (radians)	$\lvert\Delta g\rvert$
0	.84027778	.84027778	0×10^{-5}	0	0	0×10^{-4}
20	.83674774	.83675070	.296	.22747356	.2273639	1.0966
40	.82691973	.82692477	.504	.45445976	.4543801	.7966
60	.81298180	.81298783	.603	.68513324	.6850423	.9094
80	.79820373	.79821500	1.127	.92893740	.9287134	2.240
100	.78626473	.78627455	.982	1.1954244	1.1952296	1.946
120	.78029580	.78029692	.112	1.4844038	1.4843190	.848
140	.78191519	.78191710	.191	1.7799258	1.7799106	.152
160	.79068037	.79068695	.658	2.0605636	2.0606739	1.103
180	.80424132	.80424327	.195	2.3167928	2.3168499	.571
200	.81911161	.81911010	.151	2.5537699	2.5537270	.429
220	.83167986	.83167979	.007	2.7820064	2.7820046	.018
240	.83906625	.83906558	.067	3.0090287	3.0090020	.267
260	.83965270	.83965478	.208	3.2366909	3.2365252	1.657
280	.83331311	.83332190	.879	3.4638279	3.4636120	2.159
300	.82142913	.82144059	1.146	3.6914969	3.6913262	1.707
320	.80670000	.80671501	1.501	3.9263168	3.9260582	2.586
340	.79266806	.79268607	1.801	4.1787443	4.1783300	4.143
360	.78290935	.78291847	.912	4.4555575	4.4552326	3.249
380	.78002606	.78002546	.060	4.7499516	4.7497667	1.909
400	.78480504	.78480573	.069	5.0414831	5.0413928	.903
420	.79594658	.79594596	.062	5.3120974	5.3120934	.040
440	.81050828	.81050099	.729	5.5591054	5.5589943	1.111
460	.82483048	.82482343	.705	5.7911936	5.7910822	1.314
480	.83552045	.83551665	.380	6.0183377	6.0182398	.979
500	.84018034	.84017939	.895	6.2456836	6.2454964	1.872
520	.83780104	.83780875	.771	.19007482	.1897622	3.1262
540	.82889504	.82891087	1.583	.41701462	.4167136	3.0102

The harmonic analysis method makes it possible to include the oblateness of the moon when finding η^2 and g as functions of time. Greater accuracy may be achieved by the retention of terms of the series after 8g or 8ψ. An advantage of this method is the ability to include the oblateness of the moon. This is especially advantageous if the Orbiter is to be near the moon where the effect of the moon's oblateness will be greater.

REFERENCES

1. Brown, E. W. "The Stellar Problem of Three Bodies," Monthly Notices of the Royal Astronomical Society, Vol 97, 56-66, 116-127, 388-395 (1936).

2. Giacaglia, Georgio E. O. et al., The Motion of a Satellite of the Moon. GSFC X-547-65-218, June 1965.

3. Kozai, Y. "Motion of a Lunar Orbiter," Journal of the Astronomical Society of Japan, Vol 15, 301-312 (1963).

4. Byrd, P., and Friedman, M., Handbook of Elliptic Integrals for Engineers and Physicists, Springer-Verlag, 1954.

5. Burington, R.S., Handbook of Mathematical Tables and Formulas, Handbook Publishers, Inc., Sandusky, Ohio.

6. Velez, Carmelo E., Subroutines for Elliptic Functions, F 178, 179, 180. June 1965.