SMITHSONIAN INSTITUTION
ASTROPHYSICAL OBSERVATORY

Research in Space Science

SPECIAL REPORT

Number 194

ON THE ACCURACY OF THE GRAVITATIONAL POTENTIAL AS DERIVED FROM CAMERA OBSERVATIONS OF ARTIFICIAL SATELLITES

by

Walter J. Köhnlein

November 15, 1965

CAMBRIDGE, MASSACHUSETTS 02138

SAO Special Report No. 194

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Abstract.--The standard deviations of the gravitational potential and of the geocentric radius of an equipotential surface are derived from the numerical results obtained by I. Izsak and Y. Kozai in their latest determinations of the tesseral and zonal coefficients, respectively. The error influences of the zonal and tesseral terms and their combined influence are given for different sections of outer space.

Author

The geopotential in free space is usually written

$$U = \frac{\mu}{r}\left\{1 + \sum_{n=2}^{\infty}\sum_{m=0}^{n}\left(\frac{a}{r}\right)^n (C_{nm}\cos m\lambda + S_{nm}\sin m\lambda)\, P_{nm}(\sin\varphi)\right\} \tag{1}$$

$$+ \frac{\omega^2 r^2}{2}\cos^2\varphi\ , \qquad \text{(in standard notation)}$$

where the Legendre associated functions are defined as

$$P_{nm}(\sin\varphi) = \cos^m\varphi\, \frac{d^m P_n(\sin\varphi)}{d(\sin\varphi)^m}\ .$$

We assume that only the harmonic coefficients C_{nm}, S_{nm} are affected by errors and compute their influence on the potential U or, what is equivalent, the accuracy with which the undulations and/or the geocentric radius r of an equipotential surface can be determined.

[1] This work was supported in part by grant No. NsG 87-60 from the National Aeronautics and Space Administration.

[2] Geodesist, Smithsonian Astrophysical Observatory, Cambridge, Massachusetts.

If we write equation (1) in the general form

$$F(U, r, C_{nm}, S_{nm}) = 0 \,, \qquad n = 2, 3, \ldots; \qquad 0 \le m \le n, \tag{2}$$

we obtain by differentiation along an equipotential surface (U = const)

$$\frac{\partial F}{\partial r}\, dr + \sum_{n=2} \sum_{m=0}^{n} \frac{\partial F}{\partial C_{nm}}\, dC_{nm} + \sum_{n=2} \sum_{m=1}^{n} \frac{\partial F}{\partial S_{nm}}\, dS_{nm} = 0 \,, \tag{3}$$

or, after splitting up the function into an even zonal, an odd zonal, and a tesseral (sectorial) part:

$$\begin{aligned}
-\frac{\partial F}{\partial r}\, dr &= \sum_{\nu=2} \frac{\partial F}{\partial C_{2\nu-2\;0}}\, dC_{2\nu-2\;0} && \text{(even zonal)} \\
&+ \sum_{\nu=2} \frac{\partial F}{\partial C_{2\nu-1\;0}}\, dC_{2\nu-1\;0} && \text{(odd zonal)} \\
&+ \sum_{n=2} \sum_{m=1}^{n} \left(\frac{\partial F}{\partial C_{nm}}\, dC_{nm} + \frac{\partial F}{\partial S_{nm}}\, dS_{nm} \right), && \left(\begin{matrix}\text{tesseral,} \\ \text{sectorial}\end{matrix}\right)
\end{aligned} \tag{4}$$

$$\nu = 2, 3, \ldots \,.$$

The upper limit in the summation over n and ν, respectively, depends on the degree of the harmonic coefficients under consideration.

The zonal coefficients C_{n0} in the above equations are computed from the secular motions of the perigee and the longitude of the ascending node for even zonal harmonics, and from the amplitudes of the long-periodic variations of the orbital elements for odd zonal harmonics, while the tesseral coefficients are usually determined from the residuals as obtained by a differential orbit improvement. By using a least-squares adjustment procedure, Kozai (1964) gets a normal matrix 7 x 7 for the 7 even zonal coefficients and a matrix 6 x 6 for the 6 odd zonal coefficients, while Izsak (1965) obtains the tesseral (sectorial) coefficients up to n,m = 6,6 from a normal matrix 38 x 38.

If we call the observations used in the adjustment computations L_i, $\bar{L}_j$, $\bar{\bar{L}}_k$, respectively, we can express the harmonic coefficients in the following way:

$$
\begin{aligned}
C_{2\nu-2\,0} &= C_{2\nu-2\,0}(L_i)\ , && \text{(even zonal)} \\
C_{2\nu-1\,0} &= C_{2\nu-1\,0}(\bar{L}_j)\ , && \text{(odd zonal)} \\
C_{nm} &= C_{nm}(\bar{\bar{L}}_k)\ , && \left(\begin{array}{c}\text{tesseral,}\\ \text{sectorial}\end{array}\right) \\
S_{nm} &= S_{nm}(\bar{\bar{L}}_k)\ , &&
\end{aligned}
\tag{5}
$$

and their differentials as

$$
\begin{aligned}
dC_{2\nu-2\,0} &= \sum_i \alpha_i \, dL_i\ , \\
dC_{2\nu-1\,0} &= \sum_j \bar{\alpha}_j \, d\bar{L}_j\ , \\
dC_{nm} &= \sum_k \bar{\bar{\alpha}}_k \, d\bar{\bar{L}}_k\ , \\
dS_{nm} &= \sum_k \bar{\bar{\beta}}_k \, d\bar{\bar{L}}_k\ .
\end{aligned}
\tag{6}
$$

Here, α_i, $\bar{\alpha}_j$, $\bar{\bar{\alpha}}_k$, $\bar{\bar{\beta}}_k$ are coefficients depending on the observation equations and the elements of the inverses of their normal matrices. Equation (6) introduced into (4) leads finally to a relation:

$$
dr = dr \quad (dL_i,\ d\bar{L}_j,\ d\bar{\bar{L}}_k)\ , \qquad i,j,k = 1,2\ldots \tag{7}
$$

from which the standard deviation σ_r of the geocentric radius r can be obtained (under the assumption of a normally distributed error system):

$$
\sigma_r = \pm \left(\sigma_0^2 \sum_i x_i + \bar{\sigma}_0^2 \sum_j \bar{x}_j + \bar{\bar{\sigma}}_0^2 \sum_k \bar{\bar{x}}_k \right)^{\frac{1}{2}}\ , \tag{8}
$$

where the σ_0, $\bar{\sigma}_0$, $\bar{\bar{\sigma}}_0$ denote the standard deviations (of weight unit) of L_i, $\bar{L}_j$, $\bar{\bar{L}}_k$, respectively, and

$$\sum_i x_i = \left(\frac{\mu}{r}\right)^2 \underset{\sim}{f}^* \underset{\sim}{Q}\, \underset{\sim}{f} \Big/ \left(\frac{\partial F}{\partial r}\right)^2 ,$$

$$\sum_j \bar{x}_j = \left(\frac{\mu}{r}\right)^2 \bar{\underset{\sim}{f}}^* \bar{\underset{\sim}{Q}}\, \bar{\underset{\sim}{f}} \Big/ \left(\frac{\partial F}{\partial r}\right)^2 , \qquad (9)$$

$$\sum_k \bar{\bar{x}}_k = \left(\frac{\mu}{r}\right)^2 \bar{\bar{\underset{\sim}{f}}}^* \bar{\bar{\underset{\sim}{Q}}}\, \bar{\bar{\underset{\sim}{f}}} \Big/ \left(\frac{\partial F}{\partial r}\right)^2 ,$$

with the transposed vectors

$$\underset{\sim}{f}^* = \left[\left(\frac{a}{r}\right)^2 P_{20}, \left(\frac{a}{r}\right)^4 P_{40}, \cdots, \qquad \left(\frac{a}{r}\right)^{2\nu-2} P_{2\nu-2\,0}\right],$$

$$\bar{\underset{\sim}{f}}^* = \left[\left(\frac{a}{r}\right)^3 P_{30}, \left(\frac{a}{r}\right)^5 P_{50}, \cdots, \qquad \left(\frac{a}{r}\right)^{2\nu-1} P_{2\nu-1\,0}\right],$$

$$\begin{aligned}\bar{\bar{\underset{\sim}{f}}}^* = \Bigg[&\left(\frac{a}{r}\right)^2 P_{22} \cos 2\lambda, \left(\frac{a}{r}\right)^2 P_{22} \sin 2\lambda, \left(\frac{a}{r}\right)^3 P_{31} \cos \lambda, \left(\frac{a}{r}\right)^3 P_{31} \sin \lambda, \\ &\left(\frac{a}{r}\right)^3 P_{32} \cos 2\lambda, \left(\frac{a}{r}\right)^3 P_{32} \sin 2\lambda, \left(\frac{a}{r}\right)^3 P_{33} \cos 3\lambda, \left(\frac{a}{r}\right)^3 P_{33} \sin 3\lambda, \\ &\left(\frac{a}{r}\right)^4 P_{41} \cos \lambda, \left(\frac{a}{r}\right)^4 P_{41} \sin \lambda, \cdots, \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ &\left(\frac{a}{r}\right)^n P_{nm} \cos m\lambda, \left(\frac{a}{r}\right)^n P_{nm} \sin m\lambda\Bigg] \cdot 1 \le m \le n\end{aligned} \qquad (10)$$

The square matrices $\underset{\sim}{Q}, \bar{\underset{\sim}{Q}}, \bar{\bar{\underset{\sim}{Q}}}$ are the inverses of the normal matrices in Kozai's and Izsak's least-squares analyses and represent the correlation in their solutions between the harmonic coefficients. From equation (8) we also get the separate influence of the zonal terms:

$$\sigma_{r(\text{zonal even})} = \sigma_0 \sqrt{\sum_i x_i} , \qquad (11)$$

$$\sigma_{r(\text{zonal odd})} = \bar{\sigma}_0 \sqrt{\sum_j \bar{x}_j} , \qquad (12)$$

or their combination

$$\sigma_{r(\text{even + odd})} = \pm\left(\sigma_0^2 \sum_i x_i + \bar{\sigma}_0^2 \sum_j \bar{x}_j\right)^{\frac{1}{2}} . \tag{13}$$

The transformation to the standard deviation of the geopotential is obtained by multiplying equations (8) and (11) through (13) with $\frac{\partial F}{\partial r}$.

The above analytical expressions are developed on the assumption of a normal distribution of the observational errors. This may be verified to a certain extent by a careful selection of the numerical data available and the mathematical procedure devised for the analysis of the observations, etc. Although the absolute magnitudes of the standard deviations may not express the actual but rather some internal accuracy, we can generalize the results to a certain extent by computing a normalized standard deviation system, making the greatest value unity and diminishing the rest in a proportional manner. This new system represents, then, more generally the "optical" method used in Kozai's and Izsak's determinations of the gravitational potential.

The normalized standard deviations (shown in Tables 1 and 2, and Figures 1 and 2) refer in the case of the geocentric radius to an equipotential surface, while for the gravitational potential the accuracy is referred to a sphere with a radius[3] depending on the space section under consideration. The computations are performed at sea level (with $\omega = 0$ and $\omega = 0.72921 \times 10^{-4}$ radians/sec) and at heights 1,000 km, 10,000 km, and 100,000 km above sea level (with $\omega = 0$); at infinity the standard deviations are zero according to equation (10). The numerical results for the northern and southern hemispheres were so close that only one-half of the diagrams is shown in Figures 1 and 2.

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[3]Earth's equatorial radius (6 378 165 m) plus height above sea level; however, the results are practically unchanged if used for an equipotential surface instead.

Remarks to Tables 1 and 2

Geocentric latitude φ: first column on the left of each table; northern hemisphere "+," southern hemisphere "-."

Geocentric longitude λ: first line on the top of each table (to be multiplied by 10); east of Greenwich "+," west of Greenwich "-."

The normalized standard deviation is multiplied by 10 in the tables. If the actual standard deviation is computed with the help of k_U and k_r this scaling must be canceled out in the final result.

Table 1. Normalized standard deviation. Tesseral (sectorial) part. All elements are multiplied by 10.

	-18	-17	-16	-15	-14	-13	-12	-11	-10	-9	-8	-7	-6	-5	-4	-3	-2	-1	0+	1+	2+	3+	4+	5+	6+	7+	8+	9+	10+	11+	12+	13+	14+	15+	16+	17+	18
90	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
80	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
70	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3
60	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4
50	5	5	6	6	5	5	5	5	6	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5
40	6	6	7	7	6	6	6	7	7	6	6	6	6	6	6	6	6	6	6	6	6	6	6	6	6	6	6	6	6	6	6	6	6	6	7	6	6
30	7	7	8	8	8	7	7	8	8	7	7	7	7	7	7	7	7	7	7	7	7	7	7	7	7	7	7	7	6	7	7	7	7	7	8	8	7
20	9	8	9	9	9	8	9	9	9	8	8	8	8	8	8	8	8	8	8	8	8	8	8	8	7	7	8	7	7	7	8	8	8	8	9	9	9
10	9	9	9	10	9	9	10	10	9	9	9	9	9	9	8	8	8	9	9	8	8	9	8	8	8	8	8	8	7	8	8	8	9	9	9	9	9
-0	10	9	10	10	9	9	10	10	9	10	10	9	9	9	9	9	9	9	9	8	9	9	9	9	8	8	8	8	8	8	8	8	9	9	9	10	10
-10	9	9	9	10	9	9	9	9	9	9	9	9	9	9	8	8	9	9	8	8	9	9	8	8	8	8	8	8	8	8	8	8	8	9	9	9	9
-20	8	8	8	8	8	8	8	8	8	8	8	8	8	8	8	7	8	8	7	7	8	8	8	7	8	7	7	7	7	7	7	7	7	8	8	8	8
-30	7	7	7	7	7	7	7	7	7	7	7	7	7	7	7	7	7	7	6	7	7	7	7	7	7	7	6	6	6	6	6	6	6	7	7	7	7
-40	6	6	6	6	6	6	6	6	6	6	6	6	6	6	6	6	6	5	6	6	6	6	6	6	6	6	5	5	5	5	5	5	5	6	6	6	6
-50	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5
-60	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4
-70	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3
-80	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
-90	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

sea level ($\omega = 0$ and $\omega = 0.72921 \times 10^{-4}$ radians/sec)

	-18	-17	-16	-15	-14	-13	-12	-11	-10	-9	-8	-7	-6	-5	-4	-3	-2	-1	0+	1+	2+	3+	4+	5+	6+	7+	8+	9+	10+	11+	12+	13+	14+	15+	16+	17+	18
90	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
80	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
70	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3
60	4	5	5	5	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4
50	5	6	6	6	5	5	5	6	6	6	5	5	5	6	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	6	5	5
40	6	6	7	7	7	6	6	7	7	7	6	6	6	7	7	6	6	6	6	6	6	6	6	6	6	6	6	6	6	6	6	6	6	7	7	7	6
30	7	7	8	8	8	7	7	8	8	8	7	7	7	7	7	7	7	7	7	7	7	7	7	7	7	7	7	7	7	7	7	7	7	8	8	8	7
20	8	8	9	9	9	8	9	9	9	9	8	8	8	8	8	8	8	8	8	8	8	8	8	8	8	8	8	8	7	7	8	8	8	9	9	9	8
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-0	9	9	10	10	9	9	10	10	10	10	9	9	9	9	8	8	9	9	9	8	9	9	9	9	8	8	8	8	8	8	8	8	9	9	10	10	9
-10	9	9	9	10	9	9	9	9	9	9	9	9	9	9	8	8	9	9	8	8	9	9	9	8	8	8	8	8	8	8	8	8	8	9	9	9	9
-20	8	8	9	9	8	8	9	8	8	9	8	8	8	8	8	8	8	8	7	8	8	8	8	7	8	7	7	7	7	7	7	7	8	8	8	8	8
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-60	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4
-70	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3
-80	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
-90	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

1,000 km elevation ($\omega = 0$)

	-18	-17	-16	-15	-14	-13	-12	-11	-10	-9	-8	-7	-6	-5	-4	-3	-2	-1	0+	1+	2+	3+	4+	5+	6+	7+	8+	9+	10+	11+	12+	13+	14+	15+	16+	17+	18
90	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
80	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
70	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3
60	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4
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40	6	7	7	7	6	6	6	6	6	6	6	6	6	6	6	6	6	6	6	6	6	7	7	6	6	6	6	6	6	6	6	5	6	6	6	6	6
30	7	7	8	8	7	7	7	7	8	7	7	7	7	7	7	7	7	7	7	7	8	8	8	7	7	7	7	7	7	7	7	7	7	7	7	7	7
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-0	9	9	9	9	9	9	9	9	9	9	9	8	8	8	8	8	8	8	8	9	10	10	10	9	8	8	8	8	8	8	8	9	9	9	9	9	9
-10	9	9	9	9	9	9	9	9	9	9	9	8	8	8	8	8	8	8	8	9	9	10	9	8	7	7	8	8	8	8	8	9	9	9	9	9	9
-20	8	8	8	8	8	8	8	9	8	9	8	8	8	8	8	8	7	7	7	8	[illegible]	9	9	8	7	7	7	8	8	8	8	8	8	8	9	8	8
-30	8	8	8	8	8	8	8	7	7	7	7	7	7	8	8	7	7	6	7	7	8	8	8	7	6	7	7	7	7	7	7	7	7	7	8	8	8
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-50	5	5	6	6	6	5	5	5	5	5	5	5	5	6	6	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5
-60	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4
-70	3	3	3	3	3	3	3	3	2	2	3	3	3	3	3	3	3	3	2	2	2	2	2	2	3	3	3	3	3	3	3	3	2	2	2	3	3
-80	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
-90	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

10,000 km elevation ($\omega = 0$)

	-18	-17	-16	-15	-14	-13	-12	-11	-10	-9	-8	-7	-6	-5	-4	-3	-2	-1	0+	1+	2+	3+	4+	5+	6+	7+	8+	9+	10+	11+	12+	13+	14+	15+	16+	17+	18
90	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
80	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
70	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
60	2	2	3	3	3	2	2	2	2	2	2	3	3	3	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	3	3	2	2	2	2	2	2
50	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4
40	5	6	6	6	6	6	5	5	5	5	5	6	6	6	6	5	5	5	5	5	6	6	6	5	5	5	5	5	6	6	6	6	5	5	5	5	5
30	7	7	7	7	7	7	7	6	6	7	7	7	7	7	7	7	6	6	7	7	7	7	7	7	7	6	6	7	7	7	7	7	7	7	6	7	7
20	8	8	9	9	9	9	8	8	8	8	8	9	9	9	8	8	7	7	8	8	9	9	9	8	8	7	8	8	8	9	9	9	8	8	8	8	8
10	9	9	9	10	9	9	9	8	8	9	9	9	10	9	9	9	8	8	9	9	10	10	9	9	8	8	8	9	9	10	10	9	9	9	8	8	9
-0	9	9	10	10	10	9	9	9	9	9	9	10	10	10	9	9	8	8	9	9	10	10	10	9	9	8	9	9	9	10	10	10	9	9	9	9	9
-10	9	9	9	10	9	9	9	8	9	9	9	9	10	9	9	9	8	8	9	9	10	10	9	9	9	8	8	9	9	10	10	9	9	9	8	8	9
-20	8	8	9	9	9	9	8	9	8	8	8	9	9	9	8	8	8	8	8	8	9	9	9	8	8	8	8	8	8	9	9	9	8	8	8	8	8
-30	7	7	7	7	7	7	7	7	7	7	7	7	7	7	7	7	6	6	7	7	7	7	7	7	7	6	7	7	7	7	7	7	7	7	7	7	7
-40	5	6	6	6	6	6	5	5	5	5	6	6	6	6	6	5	5	5	5	6	6	6	6	6	5	5	5	5	6	6	6	6	5	5	5	5	5
-50	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4
-60	2	2	3	3	3	2	2	2	2	2	2	3	3	3	3	2	2	2	2	2	2	2	2	2	2	2	2	2	2	3	3	2	2	2	2	2	2
-70	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
-80	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
-90	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

100,000 km elevation ($\omega = 0$)

Table 2. Normalized standard deviation. Combined effect (zonal and tesseral, sectorial part). All elements are multiplied by 10.

	-18	-17	-16	-15	-14	-13	-12	-11	-10	-9	-8	-7	-6	-5	-4	-3	-2	-1	0	+1	+2	+3	+4	+5	+6	+7	+8	+9	+10	+11	+12	+13	+14	+15	+16	+17	+18
90	6	6	6	6	6	6	6	6	6	6	6	6	6	6	6	6	6	6	6	6	6	6	6	6	6	6	6	6	6	6	6	6	6	6	6	6	6
80	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2
70	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	3	3	3	3	3	3	4	4	4	4	4	4	3	3	3	3	4	4
60	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4
50	5	5	6	6	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5
40	6	6	7	7	6	6	6	6	7	6	6	6	6	6	6	6	6	6	6	6	6	6	6	6	6	6	6	6	5	6	6	6	6	6	6	6	6
30	7	7	8	8	7	7	7	8	8	7	7	7	7	7	7	7	7	7	7	7	7	7	7	7	7	7	7	6	6	6	7	7	7	7	8	7	7
20	8	8	8	9	8	8	8	9	9	8	8	8	8	8	8	8	8	8	8	7	8	8	8	8	7	7	8	7	7	7	8	7	8	8	9	9	8
10	9	9	9	10	9	9	10	10	9	9	9	9	9	9	8	8	8	9	9	8	8	9	8	8	8	8	8	8	8	8	8	8	9	9	9	9	9
-0	10	9	10	10	10	9	10	10	10	10	10	9	9	9	9	9	9	9	9	9	9	9	9	9	9	8	8	8	8	8	9	9	9	9	9	10	10
-10	9	9	9	9	9	9	9	9	9	9	9	9	9	9	8	8	9	9	8	8	9	9	8	8	8	8	8	8	8	8	8	8	8	9	9	9	9
-20	8	8	8	8	8	8	8	8	8	8	8	8	8	8	7	7	8	8	7	7	8	8	7	7	7	7	7	7	7	7	7	7	7	8	8	8	8
-30	7	7	7	7	7	7	7	7	7	7	7	6	6	7	7	7	6	6	6	7	7	7	6	6	7	6	6	6	6	6	6	6	6	6	7	7	7
-40	6	6	6	6	6	6	6	6	6	6	6	5	5	6	6	6	5	5	5	6	6	6	5	5	6	5	5	5	5	5	5	5	5	5	6	6	6
-50	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	4	5	5	5	5	5	5	5	5	5	4	4	5	5	5	5	5	5	5	5
-60	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4
-70	3	3	4	4	4	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3
-80	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2
-90	6	6	6	6	6	6	6	6	6	6	6	6	6	6	6	6	6	6	6	6	6	6	6	6	6	6	6	6	6	6	6	6	6	6	6	6	6

sea level ($\omega = 0$ and $\omega = 0.72921 \times 10^{-4}$ radians/sec)

	-18	-17	-16	-15	-14	-13	-12	-11	-10	-9	-8	-7	-6	-5	-4	-3	-2	-1	0	+1	+2	+3	+4	+5	+6	+7	+8	+9	+10	+11	+12	+13	+14	+15	+16	+17	+18
90	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2
80	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2
70	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3
60	4	4	5	5	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4
50	5	6	6	6	5	5	5	5	6	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	6	5	5
40	6	6	7	7	7	6	6	7	7	6	6	6	6	7	6	6	6	6	6	6	6	6	6	6	6	6	6	6	6	6	6	6	6	6	7	7	6
30	7	7	8	8	8	7	7	8	8	7	7	7	7	7	7	7	7	7	7	7	7	7	7	7	7	7	7	7	7	7	7	7	7	7	8	8	7
20	8	8	9	9	9	8	9	9	9	9	8	8	8	8	8	8	8	8	8	8	8	8	8	8	8	8	8	8	7	7	8	8	8	8	9	9	8
10	9	9	9	10	9	9	10	10	10	9	9	9	9	9	8	8	9	9	9	8	9	9	9	8	8	8	8	8	7	8	8	8	9	9	10	10	9
-0	9	9	10	10	9	9	10	10	10	10	10	9	9	9	8	8	9	9	9	8	9	9	9	9	8	8	8	8	8	8	8	8	9	9	10	10	9
-10	9	9	9	10	9	9	9	9	9	9	9	9	9	9	8	8	9	9	8	8	9	9	9	8	8	8	8	8	8	8	8	8	8	9	9	9	9
-20	8	8	8	8	8	8	8	8	8	8	8	8	8	8	8	8	8	8	7	8	8	8	8	7	8	7	7	7	7	7	7	7	8	8	8	8	8
-30	7	7	7	7	7	7	7	7	7	7	7	7	7	7	7	7	7	7	6	7	7	7	7	7	7	7	6	6	6	6	6	6	7	7	7	7	7
-40	6	6	6	6	6	6	6	6	6	6	6	6	6	6	6	6	6	5	6	6	6	6	6	6	6	6	6	6	6	5	5	6	6	6	6	6	6
-50	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5
-60	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4
-70	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3
-80	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2
-90	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2

1,000 km elevation ($\omega = 0$)

	-18	-17	-16	-15	-14	-13	-12	-11	-10	-9	-8	-7	-6	-5	-4	-3	-2	-1	0	+1	+2	+3	+4	+5	+6	+7	+8	+9	+10	+11	+12	+13	+14	+15	+16	+17	+18
90	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2
80	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2
70	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3
60	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4
50	5	5	6	6	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5
40	6	7	7	7	6	6	6	6	6	6	6	6	6	6	6	6	6	6	6	6	6	7	7	6	6	6	6	6	6	6	6	5	6	6	6	6	6
30	7	7	8	8	7	7	7	7	8	7	7	7	7	7	7	7	7	7	7	7	8	8	8	7	7	7	7	7	7	7	7	7	7	7	7	7	7
20	8	8	9	9	8	9	8	8	9	8	8	8	8	8	8	8	7	7	7	8	9	9	9	8	7	7	7	8	8	8	8	8	8	8	8	8	8
10	8	8	9	9	9	9	8	9	9	9	9	8	8	8	8	8	8	8	8	9	9	10	9	8	8	8	8	8	8	8	8	9	9	9	9	9	8
-0	9	9	9	9	9	9	9	9	9	9	9	8	8	8	8	8	8	8	8	9	10	10	10	9	8	8	8	8	8	8	8	9	9	9	9	9	9
-10	9	9	9	9	9	9	9	9	9	9	9	8	8	8	8	8	8	8	8	9	9	10	9	8	8	7	8	8	8	8	8	8	9	9	9	9	9
-20	8	8	8	8	8	8	8	8	8	8	8	8	8	8	8	8	7	7	7	8	9	9	9	8	7	7	8	8	8	8	8	8	8	8	9	8	8
-30	8	8	8	8	8	8	8	7	7	7	7	7	7	8	8	7	7	6	7	7	8	8	8	7	6	7	7	7	7	7	7	7	7	7	8	8	8
-40	7	7	7	7	7	7	7	6	6	6	6	6	7	7	7	6	6	6	6	6	7	7	6	6	6	6	6	6	6	6	6	5	6	6	6	6	7
-50	5	5	6	6	6	5	5	5	5	5	5	5	5	6	6	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5
-60	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4
-70	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3
-80	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2
-90	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2

10,000 km elevation ($\omega = 0$)

	-18	-17	-16	-15	-14	-13	-12	-11	-10	-9	-8	-7	-6	-5	-4	-3	-2	-1	0	+1	+2	+3	+4	+5	+6	+7	+8	+9	+10	+11	+12	+13	+14	+15	+16	+17	+18
90	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2
80	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2
70	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2
60	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3
50	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4
40	5	6	6	6	5	5	5	5	5	5	5	6	6	6	6	5	5	5	5	5	6	6	6	5	5	5	5	5	6	6	6	6	5	5	5	5	5
30	7	7	7	7	7	7	7	6	6	7	7	7	7	7	7	7	6	6	7	7	7	7	7	7	7	6	6	7	7	7	7	7	7	7	6	6	7
20	8	8	9	9	9	8	8	8	8	8	8	9	9	9	8	8	7	7	8	8	9	9	9	8	8	7	8	8	8	9	9	9	8	8	8	8	8
10	9	9	9	10	9	9	9	8	8	9	9	9	10	9	9	9	8	8	9	9	10	10	9	9	8	8	8	9	9	10	10	9	9	9	8	8	9
-0	9	9	10	10	10	9	9	9	9	9	9	10	10	10	9	9	8	8	9	9	10	10	10	9	9	8	9	9	9	10	10	10	9	9	9	9	9
-10	9	9	9	10	9	9	9	8	8	9	9	9	10	9	9	9	8	8	9	9	10	10	9	9	9	8	8	9	9	10	10	9	9	9	8	8	9
-20	8	8	9	9	9	8	8	8	8	8	8	9	9	9	8	8	8	8	8	8	9	9	9	8	8	8	8	8	8	9	9	9	8	8	8	8	8
-30	7	7	7	7	7	7	7	7	7	7	7	7	7	7	7	7	6	6	7	7	7	7	7	7	7	6	6	7	7	7	7	7	7	7	7	7	7
-40	5	6	6	6	6	5	5	5	5	5	6	6	6	6	6	5	5	5	5	6	6	6	6	5	5	5	5	5	6	6	6	5	5	5	5	5	5
-50	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4
-60	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3
-70	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2
-80	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2
-90	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2

100,000 km elevation ($\omega = 0$)

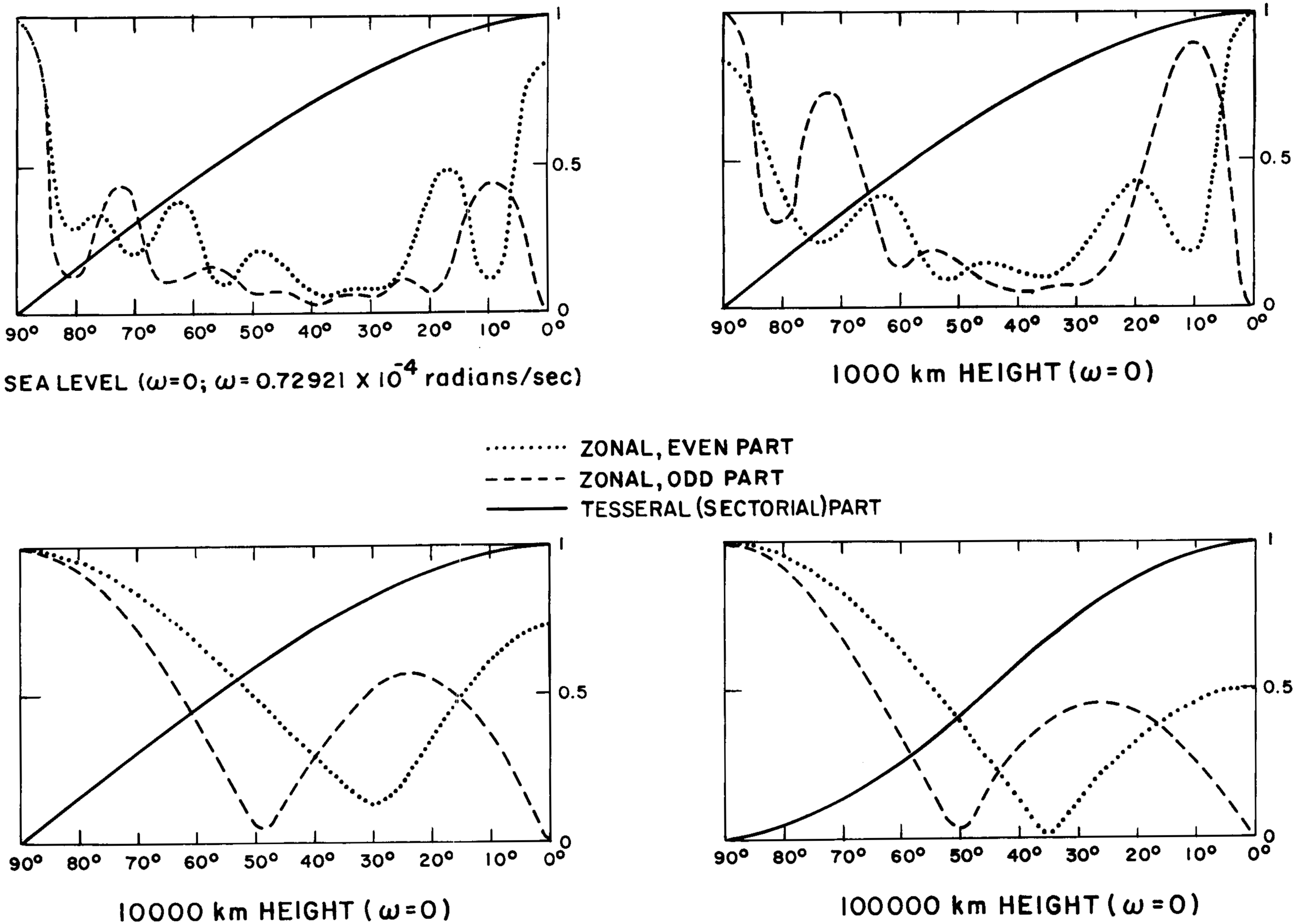

Figure 1.--Normalized standard deviations (individual effects).

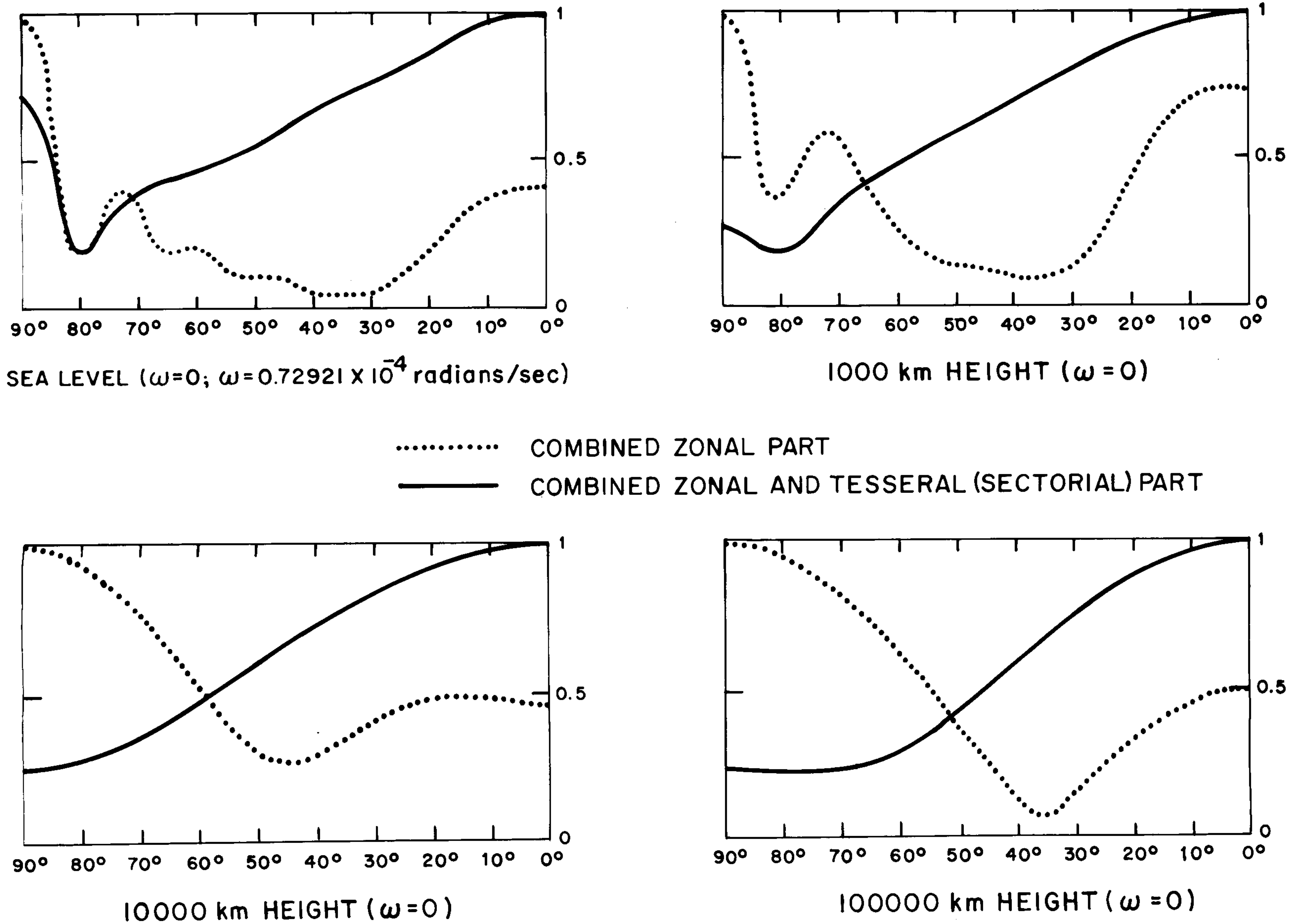

Figure 2.--Normalized standard deviations (combined effects).

Numerical results

1. Constants and coefficients

a) Constants used:

$\mu = 3.986032 \times 10^{20}$ cm^3 sec^{-2} (mass of the Earth $\times$ gravitational constant)

$a = 6\ 378\ 165$ m (equatorial radius)

$\omega = 0.72921 \times 10^{-4}$ radians/sec (Earth's angular velocity).

b) Harmonic coefficients obtained by least-squares methods (Izsak, Kozai)

Zonal Coefficients (Kozai, 1964)

C_{20}	$-1\ 082.645 \times 10^{-6}$	C_{30}	2.546×10^{-6}
C_{40}	1.649	C_{50}	0.210
C_{60}	-0.646	C_{70}	0.333
C_{80}	0.270	C_{90}	0.053
$C_{10\,0}$	0.054	$C_{11\,0}$	-0.302
$C_{12\,0}$	0.357	$C_{12\,0}$	0.114
$C_{14\,0}$	-0.179		

$C_{n0} = -J_n$ as used by Kozai (1964).

Orbital parameters of the satellites used for the determination of the zonal coefficients.

Satellite	i	e	a x 10^3 km
1959 α1	32°.8	0.16	8.29
1959 η1	33.4	0.19	8.48
1960 ι2	47.2	0.01	7.97
1961 ν	28.8	0.09	7.53
1961 o1,2	66.8	0.01	7.33
1961 $\alpha\delta$1	95.9	0.01	10.01
1962 $\alpha\epsilon$	44.8	0.24	9.69
1962 $\beta\mu$1	50.1	0.01	7.53
1962 $\beta\upsilon$	47.5	0.28	10.78

Tesseral (sectorial) coefficients (Izsak, 1965)

$\bar{C}_{22}$	2.08×10^{-6}	$\bar{S}_{22}$	-1.25×10^{-6}
$\bar{C}_{31}$	1.60	$\bar{S}_{31}$	-0.04
$\bar{C}_{32}$	0.38	$\bar{S}_{32}$	-0.80
$\bar{C}_{33}$	-0.17	$\bar{S}_{33}$	1.40
$\bar{C}_{41}$	-0.38	$\bar{S}_{41}$	-0.40
$\bar{C}_{42}$	0.20	$\bar{S}_{42}$	0.58
$\bar{C}_{43}$	0.69	$\bar{S}_{43}$	-0.10
$\bar{C}_{44}$	-0.11	$\bar{S}_{44}$	0.43
$\bar{C}_{51}$	-0.14	$\bar{S}_{51}$	-0.04
$\bar{C}_{52}$	0.24	$\bar{S}_{52}$	-0.27
$\bar{C}_{53}$	-0.67	$\bar{S}_{53}$	0.05
$\bar{C}_{54}$	-0.13	$\bar{S}_{54}$	0.16
$\bar{C}_{55}$	0.08	$\bar{S}_{55}$	-0.41
$\bar{C}_{61}$	-0.02	$\bar{S}_{61}$	0.12
$\bar{C}_{62}$	0.05	$\bar{S}_{62}$	-0.23
$\bar{C}_{63}$	0.05	$\bar{S}_{63}$	0.00
$\bar{C}_{64}$	0.07	$\bar{S}_{64}$	-0.39
$\bar{C}_{65}$	-0.28	$\bar{S}_{65}$	-0.38
$\bar{C}_{66}$	-0.12	$\bar{S}_{66}$	-0.59

Izsak (1964, p. 2630) gives the normalization coefficients for the transformation of the normalized values $\bar{C}_{nm}$ into the conventional values C_{nm}.

Orbital parameters of the satellites used for the determination of the tesseral (sectorial) coefficients.

Satellite	i	e	a × 10^3 km	No. of observations
1959 α1	32°9	0.16	8.30	3552
1959 α2	32.9	0.18	8.49	226
1959 η	33.4	0.19	8.50	2644
1960 ι2	47.2	0.01	7.97	6809
1961 $\alpha\delta$1	95.9	0.01	10.01	3435
1961 δ1	38.9	0.12	7.97	5088
1961 o1	66.8	0.01	7.32	972
1961 o2	66.8	0.01	7.32	432
1962 $\alpha\epsilon$1	44.8	0.24	9.67	1625
1962 $\beta\mu$1	50.1	0.01	7.51	1239
1962 $\beta\upsilon$1	47.5	0.28	10.76	222

2. Inverses of the normal matrices.--The matrices $\underset{\sim}{Q}$ and $\underset{\sim}{\bar{Q}}$ can be derived from the equations of condition published by Kozai (1964). Izsak's matrix $\underset{\sim}{\bar{\bar{Q}}}$ was taken directly from a computer tape used in his tesseral harmonics program.

3. Standard deviation of the gravitational potential and of the geocentric radius

a) Individual effects

In Figure 1 the normalized standard deviations are plotted as functions of the geocentric latitude. If we multiply the values with the coefficients k_U or k_r we obtain the actual standard deviations of the gravitational potential or of the geocentric radius originating from the zonal (even, odd) and tesseral influences (see Table 3), respectively. The results at sea level can be used both for ($\omega = 0.72921 \times 10^{-4}$ radians/sec) and ($\omega = 0$) because the differences are practically negligible.

Table 3

Coeff. \ Standard dev.	Even zonal part	Odd zonal part	Tesseral (sect.) part	Height above sea level (km)
$k_U [cm^2/sec^2]$	0.45×10^5	0.81×10^5	0.14×10^6	0
$k_r [cm]$	46	82	133	
$k_U [cm^2/sec^2]$	0.10×10^5	0.11×10^5	0.57×10^5	1,000
$k_r [cm]$	14	15	77	
$k_U [cm^2/sec^2]$	0.20×10^3	0.25×10^3	0.15×10^4	10,000
$k_r [cm]$	1.4	1.7	9.9	
$k_U [cm^2/sec^2]$	0.85	0.16	4.1	100,000
$k_r [cm]$	0.24	0.05	1.2	

The even zonal part produces the highest inaccuracy at the poles and at the equator while the standard deviation becomes a minimum in the middle latitudes. This also holds true for the odd zonal part except at the equator where the standard deviation is zero for theoretical reasons. If we average the tesseral influence (Table 1, p. 16) along a latitude curve we obtain for comparison a sinusoidal-like line which originates at the poles and reaches its maximum at the equator. Of particular interest is the almost independence of the standard deviation as a function of the longitude (Table 1). Hence the maxima and minima of the equipotential surfaces are as well determined as, for example, are the flatter areas, and the standard deviation seems primarily to be a function of the latitude.[4]

b) Combined effects

In Table 2 we show the combined effect of all three influences: the even and the odd zonal, and the tesseral parts. These values are normalized again and assigned with the proper coefficients k_U, k_r for the computation of the standard deviations in the gravitational potential and geocentric radius, respectively (see Table 4). At sea level the accuracy is again minimal at the poles and at the equator, while at higher elevations the most accurate areas are around the poles because of the dominating influence of the nonzonal part. Taking the mean along a latitude curve of Table 2 we compare the values with the normalized standard deviation of the combined zonal part (even and odd) in Figure 2.

[4] The values at 100,000 km elevation reflect the results expected for a three-axial ellipsoid of revolution.

Table 4

Standard dev. / Coeff.	Even and odd zonal parts	Zonal (even, odd) and tesseral (sectorial) parts	Height above sea level (km)
$k_U[cm^2/sec^2]$	0.92×10^5	0.14×10^6	0
$k_r[cm]$	94	144	
$k_U[cm^2/sec^2]$	0.14×10^5	0.58×10^5	1,000
$k_r[cm]$	19	79	
$k_U[cm^2/sec^2]$	0.32×10^3	0.15×10^4	10,000
$k_r[cm]$	2.2	9.9	
$k_U[cm^2/sec^2]$	0.86	4.1	100,000
$k_r[cm]$	0.24	1.2	

The coefficients k_U and k_r have little meaning because they strongly depend on the number of observations, the computational process used by the authors (Izsak, 1965), etc. This is actually the reason why it is so hypothetical to combine different statistical systems (as in this section), except for some very general statements such as those mentioned above.

Conclusions

The standard deviations in Izsak's and Kozai's least-squares procedures reflect the spread between the camera observations and the mathematical models devised for the determination of the harmonic coefficients. Using these values as internal statistical indicators, we find that the accuracy (combined effect) of the geopotential and the geocentric radius is greatest in the middle latitudes and decreases toward the poles and the equator. For geodetic purposes this means that one may refer in the gravity field (derived from these harmonic coefficients) and its geometric structure with some advantage to parameters (i.e., Earth's radius, gravity, etc.) of middle latitudes rather than those of the equator.

Acknowledgments

I. Izsak and Y. Kozai kindly allowed me to use some of their numerical results prior to publication. All the computer programs are the skillful work of Miss L. Rich, to whom I am much indebted.

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