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TECHNICAL PAPER proposed for presentation at Third International Heat Transfer Conference
Chicago, Illinois, August 8-12, 1966

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AN ANALYSIS OF THE TRANSIENT SOLIDIFICATION OF A FLOWING WARM LIQUID ON A CONVECTIVELY COOLED WALL

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ABSTRACT

A study is made of the frozen layer that forms when a warm liquid flows over a flat plate that is cooled below the freezing temperature of the liquid by a coolant flowing along the other side of the plate. Three analytical procedures are employed and compared for accuracy and convenience in application. One is an iterative procedure, yielding three analytical closed-form solutions, each successive solution being a more accurate, higher order approximation. The two other analytical procedures require numerical integration for evaluation. Numerical results are presented graphically so that rapid estimates of frozen layer growth can be made.

AUSZUG

Die Studie befasst sich mit der erstarrten Schicht die entsteht wenn eine warme Flüssigkeit über eine flache Platte fliesst und diese Platte durch ein Kühlmittel an der Unterseite auf den Gefrierpunkt der betreffenden Flüssigkeit gebracht wird. Die drei verwendeten analytischen Methoden werden in Bezug auf Genauigkeit und Bequemlichkeit der Anwendung verglichen. Eine von diesen drei Methoden ist iterativ und liefert drei analytische Lösungen in geschlossener Form, in denen jede Folgende eine genauere Annäherung einer höheren Grössenordnung ist. Die beiden anderen analytischen Methoden erfordern numerische Integration zur Auswertung. Numerische Ergebnisse werden graphisch dargestellt und machen daher rasches Abschätzen des Wachstums der erstarrten Schicht möglich.

АННОТАЦИЯ

Исследуется замороженный слой образующийся при обтекании плоской пластинки теплой жидкостью с одной стороны и охладителем с другой стороны. Используются три аналитических приема которые сравниваются по точности и удобству приложения. Один является итеративным приемом дающим три замкнутых решений каждое из которых представляет более точное приближение высшего порядка. Остающиеся два аналитических приема должны оценивается численным интегрированием. Численные результаты представленн графически для быстрой оценки роста замороженного слоя.

INTRODUCTION

This paper is concerned with the transient solidification of a warm flowing liquid that is in contact with a wall that is cooled below the freezing point of the liquid by a coolant flowing on the opposite side. This type of solidification problem arises in important and familiar applications such as continuous casting of metals and freezing of rivers. The application that motivated the present study is concerned with a number of advanced propulsion devices that utilize liquid-liquid heat exchangers in which the coolant is at a temperature below the freezing point of the warm liquid. If the temperature or flow rate of the warm liquid is low enough, a frozen layer will form on the exchanger walls. The prevention of serious flow blockage by excessive solidification requires an understanding of the transient formation of the solid layer on the exchanger walls.

The phenomenon of freezing (or melting) of a material has been the subject of considerable mathematical analysis for two reasons. The first is because of its practical importance in applications such as mentioned above. The second is that the problem offers a challenge for mathematical study because the liquid-solid interface is a moving boundary, which provides a nonlinear mathematical condition. Exact analytical solutions for the transient frozen layer thicknesses and heat transfer are usually difficult to obtain except for boundary conditions of certain types. This has led to the situation where in some instances physically unrealistic boundary conditions have been imposed in order to achieve a mathematical solution. Some exact solutions for conditions of practical importance are summarized by Carslaw and Jaeger [1][1]. No attempt will be made here to review the mathematical bibliography which is not directly pertinent to the present study. A few references which provide various mathematical methods and have lists of references are given by [2-5].

When an exact analytical solution cannot be found, numerical or approximate methods must be employed. Murray and Landis [6] devised a numerical finite difference technique and demonstrated it for two sample problems. Cochran [7] developed a lumped-parameter method, wherein all the heat capacity is lumped at the center of the frozen material. A heat balance integral approach was demonstrated on a variety of freezing and melting problems by Goodman [8] where the temperature distribution in the material was approximated by a second or third order polynomial to evaluate the integrals of the conduction equation. Adams [9] employed an iterative technique to find the frozen layer thickness and heat transfer in metal castings with and without accounting for the conduction into the mold. This method does not involve any approximations in principle, but provides an approximate result because in some instances only the first iteration can be conveniently carried out in analytical form.

[1]Number in brackets denote references.

The model chosen for study here is a one-dimensional solidification of a liquid that flows over one side of a thin flat wall (Fig. 1). The other side of the wall is being convectively cooled by a fluid that is at a temperature below the freezing point of the liquid. The wall on which the liquid solidifies is assumed to be sufficiently thin so that the heat flow needed to subcool the wall during the transient is negligible compared with the heat flow through the wall. This heat flow is comprised of the latent heat of fusion, the subcooling heat capacity of the frozen layer, and the convection at the solidified interface.

Three analytical methods will be employed. For two of these the transient heat conduction equation which governs the energy balance within the solidified layer has been integrated in a general way to provide an expression for the frozen layer growth as a function of time. The general integrated equation is evaluated for the present problem by using a method of successive analytically iterated approximations and also by using an approximate temperature profile of the type proposed by Goodman [8]. The former technique has been utilized in a different manner by Adams [9] for some other problems. The analytical approximations are carried out here in an improved manner so that higher order solutions can be found, and an analytical expression very close to an exact solution is obtained. The third analytical method is one proposed by Goodman [8] where a heat balance at the frozen layer-liquid interface is used. The methods are compared with regard to accuracy and ease of application. One of the closed form solutions represents for all practical purposes the exact solution and is used to evaluate the final results that are presented in graphical form.

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ANALYSIS

The model chosen for this study is the one-dimensional configuration shown in Figure 1. A warm liquid at a fixed temperature T_l flows over one side of a thin plane wall providing a constant convective heat transfer coefficient h_l. A coolant at a fixed temperature T_c flows over the opposite side of the plate providing a constant convective heat transfer coefficient h_c. A transient solidification process can then be initiated in a number of ways such as by reducing the temperature or the flow rate of the warm liquid, introducing the coolant if it had not been flowing, or lowering T_c of the flowing coolant. These changes will cause the wall to further cool until the freezing temperature is reached on the surface of the plate exposed to the warm liquid. At this instant $(\tau = 0, X(\tau) = 0)$, since the specific heat of the wall has been neglected, a linear temperature distribution will exist in the plate as shown in Figure 1, and solidification is assumed to begin.

As the solid layer on the wall increases in thickness, heat is being extracted from the boundary layer of the warm liquid and is transferred by convection to the liquid-solid interface. There it combines with the latent heat released by the solidification process and is conducted through both the frozen layer and cold wall to the wall surface in contact with the coolant, where it is convectively removed. An additional amount of heat removal is needed to subcool the frozen layer, and this is also transferred to the coolant (the heat removal necessary to subcool the wall is neglected in comparison with the other heat flows). The solid layer continues to grow until it achieves a steady-state thickness X_s. Constant properties are assumed throughout the analysis.

Steady-State Thickness of Frozen Layer

In many solidification problems the frozen layer never approaches a steady-state thickness; rather it continues to propagate with time into the liquid. When, however, the liquid flowing over the frozen layer is at a temperature above the freezing point, the frozen layer will achieve a steady-state thickness. This thickness will be used as a reference length in the later analyses of growing layers and hence will be separately derived here.

It is assumed that the liquid-solid interface at the boundary of the frozen layer is at the freezing temperature T_f. If the heat flow is taken as positive in the positive x direction, the convected flux from the liquid to the liquid-solid interface is

$$-q = h_l(T_l - T_f) \equiv \text{constant}$$

Using the overall heat transfer resistance between the liquid-solid interface and the coolant gives the relation

$$-q = h_l(T_l - T_f) = \frac{T_f - T_c}{\frac{X_s}{k} + \frac{a}{k_w} + \frac{1}{h_c}}$$

This is solved for the steady-state thickness

$$X_s = \frac{k}{h_l}\frac{T_f - T_c}{T_l - T_f} - k\left(\frac{a}{k_w} + \frac{1}{h_c}\right) \tag{1}$$

For $X_s = 0$, Equation (1) gives the relation between variables required to just avoid freezing. For example, solving for T_l gives

$$T_l\Big|_{X_s = 0} = T_f + \frac{T_f - T_c}{h_l\left(\frac{a}{k_w} + \frac{1}{h_c}\right)} \tag{2}$$

In order to prevent freezing the liquid temperature T_l must be equal to or greater than the value given by Equation (2).

General Equations for Frozen Layer Growth and Temperature Distribution

The heat flow within the frozen layer is governed by the transient heat conduction equation,

$$k\frac{\partial^2 T}{\partial x^2} = \rho c_p \frac{\partial T}{\partial \tau} \tag{3}$$

Equation (3) is integrated from any position x within the layer to the solid-liquid interface X

$$k\frac{\partial T}{\partial x}\Big|_X - k\frac{\partial T}{\partial x}\Big|_x = \rho c_p \int_x^X \frac{\partial T}{\partial \tau}\,dx \tag{4}$$

At the interface the heat conducted into the solidified layer is equal to that supplied by the latent heat of fusion and the convection from the flowing liquid,

$$k\frac{\partial T}{\partial x}\Big|_X = \rho L\frac{dX}{d\tau} + h_l(T_l - T_f) \tag{5}$$

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Equation (5) is substituted into Equation (4) to give

$$-k \left.\frac{\partial T}{\partial x}\right|_x = -\rho L \frac{dX}{d\tau} - h_l(T_l - T_f) + \rho c_p \int_x^X \frac{\partial T}{\partial \tau}\, dx \tag{6}$$

The term on the left side is the heat flow crossing any position x at any time τ. The last term on the right is the heat removed to subcool the portion of the solidified layer between x and X. The integration of Equation (6) from the wall ($x = 0$) to any position x results in an expression for the instantaneous temperature distribution

$$T(x,\tau) - T_2 = \frac{\rho L}{k}\frac{dX}{d\tau}x + \frac{h_l}{k}(T_l - T_f)x - \frac{\rho c_p}{k}\int_0^x\left[\int_x^X \frac{\partial T}{\partial \tau}\, dx\right]dx \tag{7}$$

To eliminate the variable T_2 which is a function of time, Equation (6) is written at $x = 0$ to give the heat flow at the wall, and this is equated to the heat flow through the wall and into the coolant,

$$q(x = 0) = -\rho L \frac{dX}{d\tau} - h_l(T_l - T_f) + \rho c_p \int_0^X \frac{\partial T}{\partial \tau}\, dx = -\frac{T_2 - T_c}{\frac{1}{h_c} + \frac{a}{k_w}} \tag{8}$$

Then Equation (8) is solved for T_2 and substituted in Equation (7) to give

$$T(x,\tau) = T_c + \left(\frac{1}{h_c} + \frac{a}{k_w}\right)\left[\rho L\frac{dX}{d\tau} + h_l(T_l - T_f) - \rho c_p\int_0^X \frac{\partial T}{\partial \tau}\, dx\right] + \frac{\rho L}{k}\frac{dX}{d\tau}x + \frac{h_l}{k}(T_l - T_f)x - \frac{\rho c_p}{k}\int_0^x\left(\int_x^X \frac{\partial T}{\partial \tau}\, dx\right)dx \tag{9}$$

Equation (9) is then placed in dimensionless form by letting $x' = x/X_s$, $X' = X/X_s$, $\tau' = h_l(T_l - T_f)\tau/\rho L X_s$, and $T' = (T - T_f)/(T_c - T_f)$ and using Equation (1) in the form $(X_s h_l/k)(T_l - T_f)/(T_f - T_c) = R/(1 + R)$ to give

$$1 - T' = \frac{1 + Rx'}{1 + R}\left(\frac{dX'}{d\tau'} + 1\right) + \frac{RS}{1 + R}\left[\frac{1}{R}\int_0^{X'} \frac{\partial T'}{\partial \tau'}\, dx' + \int_0^{x'}\left(\int_{x'}^{X'} \frac{\partial T'}{\partial \tau'}\, dx'\right)dx'\right] \tag{10}$$

As a consequence of the nondimensionalization it is noted that two parameters S and R have appeared. Their physical significance is discussed in the section RESULTS AND DISCUSSION.

By applying the rules for differentiating under an integral, the integral terms of Equation (10) can be transformed so that the derivative is taken outside the integral signs, and Equation (10) becomes

$$T' = 1 - \frac{1 + Rx'}{1 + R}\left(\frac{dX'}{d\tau'} + 1\right) - \frac{RS}{1 + R}\frac{\partial}{\partial \tau'}\left(\frac{1}{R}\int_0^{X'} T'\, dx' + \int_0^{x'}\int_{x'}^{X'} T'\, dx'\, dx'\right) \tag{11}$$

By reversing the order of integration, the double integral in Equation (11) can be transformed into equivalent single integrals. This is accomplished by first writing the double integral in terms of dummy variables of integration η and β as follows:

$$\int_0^{x'}\int_{x'}^{X'} T'(x',X')\, dx'\, dx' \equiv \int_0^{x'}\int_\eta^{X'} T'(\beta,X')\, d\beta\, d\eta$$

where X' is treated as a constant. Reversing the order of integration leads to

$$\int_0^{x'}\int_\eta^{X'} T'(\beta,X')\, d\beta\, d\eta = \int_0^{x'}\int_0^{\beta} T'(\beta,X')\, d\eta\, d\beta + \int_{x'}^{X'}\int_0^{x'} T'(\beta,X')\, d\eta\, d\beta$$

$$= \int_0^{x'} \beta T'(\beta,X')\, d\beta + \int_{x'}^{X'} x'T'(\beta,X')\, d\beta$$

so that

$$\int_0^{x'}\int_{x'}^{X'} T'(x',X')\, dx'\, dx' = \int_0^{x'} x'T'(x',X')\, dx' + x'\int_{x'}^{X'} T'(x',X')\, dx' \tag{12}$$

Equation (11) then takes the form

$$T'(x',X') = 1 - \frac{1 + Rx'}{1 + R}\left(\frac{dX'}{d\tau'} + 1\right) - \frac{RS}{1 + R}\frac{\partial}{\partial \tau'}\left(\frac{1}{R}\int_0^{X'} T'\, dx' + x'\int_{x'}^{X'} T'\, dx' + \int_0^{x'} x'T'\, dx'\right) \tag{13}$$

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For convenience let $I(x',X') \equiv \frac{1}{R}\int_0^{X'} T'\,dx' + x'\int_{x'}^{X'} T'\,dx' + \int_0^{x'} x'T'\,dx'$ and note that $\partial I/\partial\tau' = (\partial I/\partial X')(dX'/d\tau')$. As a result, Equation (13) takes the form

$$T' = 1 - \frac{1 + Rx'}{1 + R}\left(\frac{dX'}{d\tau'} + 1\right) - \frac{RS}{1 + R}\frac{\partial I}{\partial X'}\frac{dX'}{d\tau'} \tag{14}$$

When Equation (14) is evaluated at the solid-liquid interface $x' = X'$, there results

$$1 = \frac{1 + RX'}{1 + R}\left(\frac{dX'}{d\tau'} + 1\right) + \frac{RS}{1 + R}\frac{\partial G(X')}{\partial X'}\frac{dX'}{d\tau'} \tag{15}$$

where

$$G(X') \equiv I(x' = X', X') = \frac{1}{R}\int_0^{X'} T'\,dx' + \int_0^{X'} x'T'\,dx'$$

Solving Equation (15) for $dX'/d\tau'$ gives

$$\frac{dX'}{d\tau'} = \frac{R(1 - X')}{1 + RX' + RS\frac{dG}{dX'}} \tag{16}$$

Separating the variables and integrating yields

$$\tau' = \int_0^{X'} \frac{1 + RX'}{R(1 - X')}\,dX' + S\int_0^{X'} \frac{1}{1 - X'}\frac{dG}{dX'}\,dX' \tag{17}$$

When the first integral on the right is evaluated and the second integral is integrated by parts, the final expression for $\tau' = \tau'(X')$ is

$$\tau' = \left[-X' - \frac{1 + R}{R}\ln(1 - X')\right] + S\left[\frac{G(X')}{1 - X'} - \int_0^{X'}\frac{G(X')}{(1 - x')^2}\,dX'\right] \tag{18}$$

This is the expression that will be used to compute the frozen layer thickness as a function of time. Since $G(X')$ contains the temperature distribution, this distribution is obtained from Equation (14) with Equation (16) used to eliminate $dX'/d\tau'$. The final relation for the temperature distribution in the frozen layer becomes

$$T' = \frac{R(1-x')}{1+R} - \left(\frac{1+Rx'+RS\frac{\partial I}{\partial X'}}{1+R}\right)\left[\frac{R(1-X')}{1+RX'+RS\frac{dG}{dX'}}\right] \tag{19}$$

Equation (19) is an integral equation for the temperature $T'(x',X')$. To solve Equation (18), Equation (19) must first be solved so that the integrals that comprise G in Equation (18) can be evaluated. Two methods for carrying this out will be examined here.

The first method, and ultimately the more accurate of the two, is an iterative technique whereby the integrals in I and G are evaluated using temperature distributions that are obtained from successively better approximations of Equations (19). It appears that this procedure will ultimately lead to an exact solution. In the second method Equation (19) is not used. Instead the temperature distribution to be used in $G(X)$ of Equation (18) is approximated by a second order polynomial with coefficients that are evaluated using the physical conditions at the boundaries of the frozen layer. Since this approximate temperature profile does not arise from the energy equation, the second method cannot yield an exact solution. The two methods are outlined in the next sections.

Solution by Analytical Iterations[2]

An iterative procedure has been utilized by Adams [9] for some solidification problems, and before discussing the present method, a few comments are in order. When Adams' procedure was applied to the present problem, the first iteration (second approximation) resulted in a cumbersome quadratic equation for $dX'/d\tau'$. After solving the quadratic equation, a numerical integration was required to determine $X'(\tau')$. A second iteration would have been extremely difficult. Hence an alternate approach was devised in which higher order iterations could be analytically obtained thereby leading to more accurate results. These will now be presented.

A first order approximation to the temperature distribution and growth times is found by neglecting the effect of heat capacity within the frozen layer. When the c_p becomes zero, the parameter $S = 0$, and Equations (18) and (19) reduce to

$$\tau_I' = -X' - \frac{1 + R}{R}\ln(1 - X') \tag{20}$$

$$T_I' = \frac{R(X' - x')}{1 + RX'} = \frac{RX'}{1 + RX'}(1 - \xi) \tag{21}$$

A second approximation for T' and $\tau' = \tau'(X')$ was obtained by substituting Equation (21) for T_I' in Equations (18) and (19) with the S terms now retained. After the integrations in $I(x',X')$ and $G(X')$ are performed, the results can be simplified to

$$\tau_{II}' = \tau_I' - S\left\{\frac{RX'}{3(RX' + 1)}\left[X' + \frac{R + 2}{R(R + 1)}\right] + \frac{3 + 3R + R^2}{3(1 + R)^2}\ln(1 - X') + \frac{\ln(RX' + 1)}{3R(1 + R)^2}\right\} \tag{22}$$

and

$$T_{II}' = \frac{R(1 - X'\xi)}{1 + R} - \left[\frac{1 + RX'\xi + RS\left.\frac{\partial I}{\partial X'}\right|_I}{1 + R}\right]\left[\frac{R(1 - X')}{1 + RX' + RS\left.\frac{dG}{dX'}\right|_I}\right] \tag{23}$$

[2] This will be referred to later as method "AI" (Analytical Iteration).

where

$$\left.\frac{dG(X')}{dX'}\right|_{I} = \frac{R^2}{(1 + RX')^2}\left(\frac{X'}{R^2} + \frac{X'^2}{R} + \frac{X'^3}{3}\right) \tag{24}$$

$$\left.\frac{\partial I}{\partial X'}\right|_{I} = \frac{R^2}{(1 + RX')^2}\left[\frac{1}{R}\left(\frac{X'}{R} + \frac{X'^2}{2}\right) + \left(\frac{X'^2\xi}{R} + \frac{X'^3\xi}{2}\right) - \left(\frac{X'^2\xi^2}{2R} + \frac{X'^3\xi^3}{6}\right)\right] \tag{25}$$

When the procedure is repeated, that is, when Equation (23) is substituted into Equations (18) and (19), the third approximation results, which is the final approximation evaluated here, that is,

$$\tau'_{III} = \tau'_{I} + S\left[\frac{G_{II}(X')}{1 - X'} - \int_0^{X'} \frac{G_{II}(X')}{(1 - X')^2}\, dX'\right] \tag{26}$$

$$T'_{III} = \frac{R(1 - X'\xi)}{1 + R} - \left[\frac{1 + RX'\xi + RS\left.\frac{\partial I}{\partial X'}\right|_{II}}{1 + R}\right]\left[\frac{R(1 - X')}{1 + RX' + RS\left.\frac{dG}{dX'}\right|_{II}}\right] \tag{27}$$

where

$$G_{II}(X') = \frac{X' + \frac{R - 1}{2}X'^2 - \frac{RX'^3}{3}}{1 + R} + \left(\left.\frac{dX'}{d\tau'}\right|_{II}\right)\left[\frac{-\frac{X'}{R} - X'^2 - \frac{RX'^3}{3}}{1 + R} - \frac{R^3S}{(1+R)(1+RX')^2}\left(\frac{X'^2}{R^3} + \frac{4X'^3}{3R^2} + \frac{2X'^4}{3R} + \frac{2X'^5}{15}\right)\right] \tag{28}$$

$$\begin{aligned}
\left.\frac{\partial I}{\partial X'}(X',\xi)\right|_{II} = {} & \frac{(1 + RX'\xi)(1 - X')}{1 + R} + \left(\left.\frac{dX'}{d\tau'}\right|_{II}\right)\left\{-\frac{(1 + RX')(1 + RX'\xi)}{R(1 + R)}\right. \\
& + \frac{R^3S}{(1 + R)(1 + RX')^2}\left[\left(\frac{1 + RX'\xi}{R}\right)\left(-\frac{2X'}{R^2} - \frac{5X'}{2R} - \frac{5X'^3}{6}\right) + \left(\frac{1 + RX'}{R}\right)\left(\frac{X'^2\xi^2}{2R} + \frac{X'^3\xi^3}{6}\right)\right] \\
& - \frac{2R^4S}{(1 + R)(1 + RX')^3}\left[\left(\frac{1 + RX'\xi}{R}\right)\left(-\frac{X'^2}{R^2} - \frac{5X'^3}{6R} - \frac{5X'^4}{24}\right) + \left(\frac{X'}{R} + \frac{X'^2}{2}\right)\left(\frac{X'^2\xi^2}{2R} + \frac{X'^3\xi^3}{6}\right)\right. \\
& \left.\left. - \frac{X'^4\xi^4}{24R} - \frac{X'^5\xi^5}{120}\right]\right\} + \frac{d}{dX'}\left(\left.\frac{dX'}{d\tau'}\right|_{II}\right)\left\{-\frac{\left(X' + \frac{RX'^2}{2}\right)(1 + RX'\xi)}{R(1 + R)}\right. \\
& + \frac{X'^2\xi^2 + \frac{RX'^3\xi^3}{3}}{2(1 + R)} + \frac{R^3S}{(1 + R)(1 + RX')^2}\left[\left(\frac{1 + RX'\xi}{R}\right)\left(-\frac{X'^2}{R^2} - \frac{5X'^3}{6R} - \frac{5X'^4}{24}\right)\right. \\
& \left.\left. + \left(\frac{X'}{R} + \frac{X'^2}{2}\right)\left(\frac{X'^2\xi^2}{2R} + \frac{X'^3\xi^3}{6}\right) - \frac{X'^4\xi^4}{24R} - \frac{X'^5\xi^5}{120}\right]\right\}
\end{aligned} \tag{29}$$

$$\begin{aligned}
\left.\frac{dG(X')}{dX'}\right|_{II} = {} & \frac{1 + (R - 1)X' - RX'^2}{1 + R} + \frac{d}{dX'}\left(\left.\frac{dX'}{d\tau'}\right|_{II}\right)\left\{\frac{-\frac{X'}{R} - X'^2 - \frac{RX'^3}{3}}{1 + R} - \frac{R^3S}{(1+R)(1+RX')^2}\left[\frac{X'^2}{R^3} + \frac{4}{3}\frac{X'^3}{R^2} + \frac{2X'^4}{3R} + \frac{2X'^5}{15}\right]\right\} \\
& + \left(\left.\frac{dX'}{d\tau'}\right|_{II}\right)\left\{\frac{-\frac{1}{R} - 2X' - RX'^2}{1 + R} - \frac{R^3S}{(1 + R)(1 + RX')^2}\left(\frac{2X'}{R^3} + \frac{4X'^2}{R^2} + \frac{8X'^3}{3R} + \frac{2X'^4}{3}\right)\right. \\
& \left. + \frac{2R^4S}{(1 + R)(1 + RX')^3}\left(\frac{X'^2}{R^3} + \frac{4X'^3}{3R^2} + \frac{2X'^4}{3R} + \frac{2X'^5}{15}\right)\right\}
\end{aligned} \tag{30}$$

$$\left.\frac{dX'}{d\tau'}\right|_{II} = \frac{R(1 - X')}{1 + RX' + RS\left.\frac{dG}{dX'}\right|_{I}}$$

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$$\frac{d}{dX'}\left(\frac{dX'}{d\tau'}\bigg|_{II}\right) = \left[(1 + RX')^3 + R^3S\left(\frac{X'}{R^2} + \frac{X'^2}{R} + \frac{X'^3}{3}\right)\right]^{-2}\left\{R\left[(1 + RX')^3 + R^3S\left(\frac{X'}{R^2} + \frac{X'^2}{R} + \frac{X'^3}{3}\right)\right]\right.$$

$$\left.\times\left[2R(1 - X')(1 + RX') - (1 + RX')^2\right] - R(1 - X')(1 + RX')^2\left[3R(1 + RX')^2 + R^3S\left(\frac{1}{R^2} + \frac{2X'}{R} + X'^2\right)\right]\right\} \tag{31}$$

No attempt was made to analytically evaluate the integral in Equation (26) as there did not seem to be any advantage to doing so because of its algebraic complexity. Since the entire equation was to be evaluated on a digital computer, the integration was performed numerically.

It is obvious from the complexity of the temperature distribution Equations (27) to (31) that a fourth analytically determined approximation would be almost impossible to carry out. As will be shown, a fourth approximate solution is unnecessary because the AI method converged rapidly with the first three approximate solutions.

Approximate Solution by Prescribed Temperature Method[3]

The basic expression (Eq. (18)) that predicts the dimensionless growth times τ' for a layer of thickness X' contains integrals, in the function G, of the instantaneous temperature distributions in the frozen layer. Because the temperature distributions appear as integrands, the growth time τ' is not very sensitive to the exact shape of the distribution. Therefore, using a reasonable approximation for the temperature profile should result in fairly accurate predictions for $\tau' = \tau'(X')$. This idea was first applied to problems of freezing and melting by Goodman [8].

For the present problem, the temperature profile within the frozen layer is approximated by a second order polynomial of the form

$$T' = A(X' - x') + B(X' - x')^2 \tag{32}$$

Equation (32) already satisfies the boundary condition that the temperature is equal to T_f at the liquid-solid interface. Two additional boundary conditions are used to evaluate A and B. The boundary condition at the wall is that the following heat balance be satisfied:

$$k\left.\frac{\partial T}{\partial x}\right|_{x=0} = \frac{T_2 - T_c}{\frac{1}{h_c} + \frac{a}{k_w}}$$

which has the dimensionless form

$$\left.\frac{\partial T'}{\partial x'}\right|_{x'=0} = R[T'(x' = 0) - 1] \tag{33}$$

Substituting Equation (32) into Equation (33) provides the condition

$$-A - 2X'B = R(AX' + BX'^2 - 1) \tag{34}$$

The second boundary condition is at the liquid-solid interface and is given by Equation (5).

[3]This will be referred to later as method "PT" (Prescribed Temperature).

Equation (5) is modified in order to express it entirely in terms of T, without $dX/d\tau$ appearing, by using the following relation which can be written for any location in the frozen layer:

$$dT = \left.\frac{\partial T}{\partial x}\right|_\tau dx + \left.\frac{\partial T}{\partial \tau}\right|_x d\tau$$

At the instant when the interface moves through position x, x = X, $dx/d\tau$ becomes $dX/d\tau$, and the derivative $dT/d\tau = 0$ since the interface is always at T_f. Then

$$\left\{\left.\frac{\partial T}{\partial x}\right|_{\substack{\text{at interface}\\ \text{at given time}}}\right\}\frac{\partial X}{\partial \tau} = -\left.\frac{\partial T}{\partial \tau}\right|_{\substack{\text{at fixed } x \text{ as}\\ \text{interface moves}\\ \text{across that } x}}$$

This is used to eliminate $dX/d\tau$ from Equation (5)

$$\left.\frac{\partial T}{\partial x}\right|_X = -\frac{\rho L}{k}\frac{\left.\frac{\partial T}{\partial \tau}\right|_X}{\left.\frac{\partial T}{\partial x}\right|_X} + \frac{h_l}{k}(T_l - T_f)$$

If the relation $\partial T/\partial\tau = (k/\rho c_p)(\partial^2 T/\partial x^2)$ from the conduction equation is used, this boundary condition can be rearranged into

$$\left(\left.\frac{\partial T}{\partial x}\right|_X\right)^2 = -\frac{L}{c_p}\left.\frac{\partial^2 T}{\partial x^2}\right|_X + \frac{h_l}{k}(T_l - T_f)\left.\frac{\partial T}{\partial x}\right|_X$$

In dimensionless form it becomes

$$\left(\left.\frac{\partial T'}{\partial x'}\right|_{X'}\right)^2 = \frac{1}{S}\left.\frac{\partial^2 T'}{\partial x'^2}\right|_{X'} - \frac{R}{1 + R}\left.\frac{\partial T'}{\partial x'}\right|_{X'} \tag{35}$$

The temperature profile (Eq. (32)) is substituted into Equation (35), and the result is rearranged into

$$B = \frac{S}{2}\left(A^2 - \frac{AR}{1 + R}\right) \tag{36}$$

which gives B when A is known. Equations (34) and (36) are solved simultaneously to yield A

$$A = \frac{R}{2(1 + R)} - \frac{1 + RX'}{S(2X' + RX'^2)} + \left\{\left[-\frac{R}{2(1 + R)} + \frac{1 + RX'}{S(2X' + RX'^2)}\right]^2 + \frac{2}{S(2X' + RX'^2)}\right\}^{1/2} \tag{37}$$

The coefficient A is the root of a quadratic equation, and the positive sign in front of the square root was chosen so that A would approach the proper limit as steady state is achieved. The limit is found by realizing that the steady-state temperature profile in the frozen layer must be linear and hence from Equation (32), $B(X' \to 1) = 0$. Then from Equation (36), $A(X' \to 1) = R/(1 + R)$, which is also the limit of Equation (37).

Equations (36) and (37) give A and B as explicit functions of X'; hence, the temperature distribution, Equation (32), is given as a function of X' and x'. The temperature distribution is then used to evaluate the integrals in Equation (18)

The quantity $G(X')$ is

$$G(X') = \frac{1}{R}\left(\frac{A}{2} X'^2 + \frac{B}{3} X'^3\right) + \frac{AX'^3}{6} + \frac{BX'^4}{12} \tag{38}$$

while the quantity

$$\int_0^{X'} \frac{G(X')}{(1 - X')^2}\, dX' \tag{39}$$

has to be integrated numerically because the A and B contained in G are very complicated functions of X'. Equation (18) is then evaluated to give the time required to form a thickness of frozen layer, $\tau' = \tau'(X')$, for any values of the parameters R and S.

With A and B known as a function of X' and hence as a function of τ', the temperature profile in the frozen layer can be evaluated at any time during the layer growth from Equation (32), which can also be placed in the form

$$T' = AX'(1 - \xi) + BX'^2(1 - \xi)^2 \tag{40}$$

Determination of Frozen Layer Growth by Integrating Heat Balance at Liquid-Solid Interface[4]

This alternate approach suggested by Goodman [8] begins with the interface condition that follows Equation (34)

$$\frac{dX}{d\tau} = -\frac{\left.\frac{\partial T}{\partial \tau}\right|_X}{\left.\frac{\partial T}{\partial x}\right|_X} = -\left.\frac{\frac{k}{\rho c_p}\frac{\partial^2 T}{\partial x^2}}{\frac{\partial T}{\partial x}}\right|_X$$

This has the dimensionless form

$$\frac{dX'}{d\tau'} = -\frac{1 + R}{RS}\left.\frac{\frac{\partial^2 T'}{\partial x'^2}}{\frac{\partial T'}{\partial x'}}\right|_{X'} \tag{41}$$

The approximate temperature profile, Equation (32), is used to evaluate the derivatives at the interface to give

$$\frac{dX'}{d\tau'} = \frac{1 + R}{RS}\frac{2B}{A}$$

The quantity B is eliminated by using Equation (36), and the variables are then separated. The result is integrated to yield

$$\tau' = \int_0^{X'} \frac{dX'}{A\frac{1 + R}{R} - 1} \tag{42}$$

This was integrated numerically for various R and S using A as a function of X' from Equation (37). As in the previous method, the temperature distribution is found from Equation (40) with B obtained from Equation (36).

RESULTS AND DISCUSSION

Significance and Ranges of Parameters and Dimensionless Variables

The subcooling parameter $S \equiv c_p(T_f - T_c)/L$ appears because of the manner in which specific heat term was nondimensionalized in the basic equations. Although the numerator $c_p(T_f - T_c)$ is the maximum internal energy that could be removed if a unit mass of the solidified material were subcooled from the freezing temperature to the lowest temperature in the system T_c, the parameter does not give any indication of how much subcooling is actually experienced by the frozen layer. As the coolant temperature approaches absolute zero, S increases to a maximum value. For many materials employed in engineering practice such as molten metals and water, the maximum S is approximately 3.

The parameter $R \equiv (X_s/k)\Big/\left[(1/h_c) + (a/k_w)\right]$ is a ratio of heat flow resistances. The X_s/k is the resistance offered by the steady-state thickness of the frozen layer, while $(1/h_c) + (a/k_w)$ is the resistance offered by the wall and the convective coefficient on the coolant side of the wall. Since the thickness X_s can be very thin, the value of R can approach zero. For large X_s or if h_c is large and a/k_w small, R can be very large; hence, R can range from zero to infinity. To aid the clarity of the discussion and the physical understanding of the reader, it is convenient to regard the parameter R as a measure of the steady-state frozen layer thickness X_s for fixed values of h_c, a, k, and k_w.

The steady-state frozen layer thickness X_s in Equation (1) depends on the many independent variables in the system. For example increasing the liquid temperature T_l, the liquid heat transfer coefficient h_l, or the coolant temperature T_c; or decreasing the coolant side convective coefficient h_c all decrease the steady ice layer thickness. The dimensionless instantaneous solidified thickness $X' \equiv X/X_s$ varies by definition from zero to unity.

The dimensionless time given by

$$\tau' = \frac{\tau h_l(T_l - T_f)}{\rho L X_s}$$

[4] This will be referred to later as method "IF" (Interface).

is the ratio of two heat flow quantities. The $\tau h_l(T_l - T_f)$ is the total heat convected from the warm liquid to the surface of the frozen layer during the time τ. The $\rho L X_s$ is the amount of latent heat that is removed to form the steady-state frozen layer. A large τ' indicates that the heat convected to the frozen layer by the warm liquid has been large compared with the heat of solidification.

Comparison of Solutions

Figure 2 shows the growth of the solidified layer as predicted by four different equations presented in the analysis for several values of the parameters R and $S > 0$. Also included are results for the case where the subcooling in the frozen layer is neglected, $S = 0$ as computed from Equation (20). First consider the lower set of curves where $R = 0.01$ which corresponds to thin frozen layers. In the limit when heat capacity is neglected, $S = 0$, all the solutions obtained from Equation (18) are in agreement, by definition, with that given by Equation (20). When S is increased to its maximum practical value of 3, all of the solution methods are in agreement and are only slightly removed from the $S = 0$ curve.

The upper set of curves are for a thick frozen layer ($R = 100$). The curves for $S = 3$ represent the extreme case of large heat capacity effects coupled with a thick frozen layer and provide the maximum deviation between the calculation methods. One of the most significant findings is the very close agreement between the second and third analytical iterations (curves "AI_{II}" and "AI_{III}"). This indicates that the third iteration has converged very close to the final solution. This will also be shown by the temperature distributions which will be given in Figure 3. Hence for practical purposes the third iteration can be regarded as the correct solution. The method using a prescribed temperature profile (PT curves) is in very good agreement with the analytical iterations, and hence the prescribed temperature method can also be used with good accuracy for the present problem. The interface method (IF curves) deviates somewhat from the other methods, but at a given τ' still provides X' values within 5 percent of the correct solution. The reason for this deviation stems from the fact that the growth times are related to the first and second partial derivatives of the temperature at the interface (Eq. (41)). For the IF method these derivatives are obtained from the approximate temperature profile (Eq. (32)) rather than from a solution of the energy equation as in the AI method.

Temperature Distributions Within Freezing Layer

A few temperature distributions within the freezing layer are shown in Figure 3. The dimensionless temperature is shown as a function of position within the layer for four different dimensionless layer thicknesses. An interesting characteristic is that all the temperature profiles are fairly close to being linear. The heat convected by the warm liquid to the interface combined with the latent heat of fusion that also arises at the interface, is conducted through the frozen layer and tends to establish a linear temperature profile. The heat capacity of the layer however tends to produce a curved profile. The fact that most of the temperature profiles are nearly linear suggests that the subcooling energy is small compared with the convective and latent heats at the frozen layer interface.

Figure 3(a) shows the temperature distributions as computed by various methods for a large value of both R and S. The rapid convergence of the successive analytical iteration solutions (AI) is demonstrated. The difference between the temperature distributions for the second and third iterations is very small. It appears that a fourth approximation would be indistinguishable from the third approximation and would have an insignificant effect on the frozen layer growth curves in Figure 2. The profiles evaluated from the prescribed temperature (PT) and interface integration (IF) methods are not too far removed from the analytical iteration profiles. Figures 3(b) and (c) show how the agreement of all the methods is improved as either S or R is made smaller.

Solidification Times as a Function of R and S Parameters

While all the solutions agree well with the third analytical iteration which is taken to be the correct solution, they vary considerably in their form and difficulty of evaluation. The simplest solution is Equation (20) for the $S = 0$ condition, but its range of application is restricted to small R values as indicated by Figure 2. The second analytical iteration, Equation (22), is quite accurate for all ranges of R and S and has a form that can be easily and quickly evaluated even on a desk calculator if necessary. The third AI, PT, and IF solutions all require a digital computer for their evaluation. For these reasons the second AI solution is the most convenient analytical form given here for use in engineering applications. The graphical results which will now be presented were evaluated from the third AI solution (Eq. (26)).

In Figure 4 is presented the dimensionless time τ' against R with S as a parameter. Each group of curves gives the τ' required to form a given frozen layer thickness X'.

At each X' it is interesting to note the behavior of the group of curves as R becomes small. All the curves for $S \neq 0$ approach the curve for $S = 0$, which means that the effect of heat capacity tends to become negligible as R diminishes. A small R means that the temperature drop across the frozen layer is small compared with the total drop across the wall resistance and convective resistance on the coolant side. This means that for small R the actual frozen layer subcooling is a small part of the maximum possible subcooling $T_f - T_c$. Hence the effect of the S parameter dies away at small R.

As R becomes large it is seen that for any X', the τ' is greater as S is increased from 0 to 3. From this fact two important observations can be made concerning the subcooling of the frozen layer. The first observation is that when the heat capacity is not included in an analysis of a growing frozen layer, serious errors can result in the predicted growth times. This conclusion will be shown as follows by comparing, for a fixed R, two cases, one where the specific heat is neglected so that $S = 0$, and another where $S > 0$. Under these conditions all the conditions such as the h_l, h_c, T_l, T_c, k, k_w, X_s, remain the same for both cases. The ratios of the times τ' and τ for $S = 0$

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and $S > 0$ are then equal; that is,

$$\frac{\tau'_{c_p>0}}{\tau'_{c_p=0}} = \frac{\tau_{c_p>0}}{\tau_{c_p=0}}$$

From Figure 4 it is found that the τ' ratio is always greater than unity and hence $\tau_{c_p>0} > \tau_{c_p=0}$. Thus, a simplified analysis neglecting heat capacity will always lead to predicting growth times τ that are too short.

The second observation is that, when the heat capacity is properly accounted for, the dimensional time τ does not increase as τ' increases when the S parameter of the frozen layer is increased. In fact the opposite is generally true, τ decreases. For a given solidifying material S is increased by lowering the coolant temperature T_c. A lower coolant temperature, however, increases the solidification rate which more than compensates for the increased heat capacity that must be removed. This will now be discussed in more detail.

Consider a fixed R and a fixed steady-state frozen layer thickness X_s. For a given liquid where L, c_p, and T_f are assumed constant, an increase in S is accomplished by lowering the coolant temperature T_c. Then since R depends on X_s, and X_s contains the ratio $(T_f - T_c)/h_l(T_l - T_f)$, the quantity $h_l(T_l - T_f)$ must be proportionately increased to keep X_s unchanged. If S is increased from S_1 to S_2, then

$$\frac{\left[h_l(T_l - T_f)\right]_2}{\left[h_l(T_l - T_f)\right]_1} = \frac{S_2}{S_1}$$

Using the definition of the dimensionless time results in

$$\frac{\tau_2\left[h_l(T_l - T_f)\right]_2}{\tau_1\left[h_l(T_l - T_f)\right]_1} = \frac{\tau'_2}{\tau'_1} = \frac{\tau_2}{\tau_1}\frac{S_2}{S_1}$$

From Figure 4, it is found that for any X/X_s the ratio S_2/S_1 is always larger than τ'_2/τ'_1 so that $\tau_2 < \tau_1$. The conclusion is that when a frozen layer of a specified steady-state thickness is being formed in a given liquid, the layer will grow faster as the coolant temperature T_c is decreased even though the subcooling of the frozen layer is thereby increased.

Another feature of Figure 4 is that for any given X' the τ' is essentially no longer dependent on R when $R > 100$. This implies that for large R's, the combined thermal resistance of the wall on which the layer is formed and the coolant convective resistance is negligible when compared to that of the frozen layer. As a consequence, the instantaneous frozen layer thickness is governed solely by the degree of subcooling imposed on the layer. This is shown by the following simple relation derived from the second approximate iterative solution (Eq. (22)) by taking a limit as $R \to \infty$

$$\tau'_{II}\Big|_{R\to\infty} = \frac{3 + S}{3}\left(-X' - \ln|1 - X'|\right) \qquad (43)$$

where the expression in parentheses is the $S = 0$ solution, Equation (20), for large R. A comparison of Equation (43) with Figure 4 shows that the equation fits the curves quite well for $R > 100$. As a consequence, the factor $(3 + S)/3$ represents a correction factor that can be used to get rapid estimates from the $S = 0$ solution for $R > 100$ and $0 \leq S \leq 3$.

One important aspect of Figure 4 is its usefulness in making rapid estimates of the thickness X against time τ for any given R and S. To illustrate, consider a given liquid, coolant, and wall material where the properties are known and the T_l, T_c, h_l, h_c are specified. With this information given, the quantities R and S can be calculated. Then from Figure 4, the τ' values can be found by simple interpolation for each X' shown. Having obtained values of τ' against X', the relation between dimensional τ and X can easily be calculated from the definitions of τ' and X'.

CONCLUDING REMARKS

The goal of the analysis was to develop a means for predicting the transient growth of the frozen layer that forms when a flowing warm liquid is in contact with a cold flat plate that is convectively cooled on the opposite side. This goal was achieved by employing two basic approaches to the problem: (1) a growth relation obtained by integrating the transient conduction equation over the entire frozen layer thickness and (2) a heat balance at the frozen layer-liquid interface.

By use of a proper dimensionless layer thickness and time variable, the frozen layer growth can be expressed as a function of two parameters. One parameter provides a measure of the maximum possible subcooling energy of the frozen layer as compared with the latent heat of fusion. The second is the ratio of the heat flow resistance of the steady-state layer, to the combined resistance of the wall and convection coefficient on the coolant side.

From the two analytical approaches five solutions were developed, four from the first approach and one from the second. All solutions except one gave accurate predictions of thickness against time for all values of the parameters. The one exception was valid for only thin frozen layers where subcooling was unimportant, as this solution had neglected the heat capacity of the frozen layer. Of the accurate solutions, two that were analytically derived successive iterations were the most convenient to use and the most precise. The form of one of these iterative approximations was sufficiently simple that it could easily be evaluated on a desk calculator. The other, from which the graphical results were prepared, appeared well converged to the correct solution.

The results of this analysis led to several conclusions. The first is that the subcooling of a frozen layer during its formation slows the growth rate substantially for some conditions. This is particularly true of thick layers. The second is that the temperature profiles in the frozen layer were found to be almost linear at all times. This suggests that the energy required to subcool the frozen layer is much less than the combined convective and latent heat passing through the layer. Lastly, the analytical iterative technique developed here resulted in a rapidly converging means for solving the nonlinear transient freezing problem.

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NOMENCLATURE

A	coefficient of linear term in Eq. (32)
a	thickness of cooled wall
B	coefficient of quadratic term in Eq. (32)
c_p	specific heat of solidified material
h	convective heat transfer coefficient
k	thermal conductivity of solidified material
k_w	thermal conductivity of wall
L	latent heat of fusion
q	heat flux in x direction
R	dimensionless parameter: $(X_s/k)\big/\left[(1/h_c) + (a/k_w)\right]$
S	dimensionless parameter: $c_p(T_f - T_c)/L$
T	temperature
T'	dimensionless temperature, $(T - T_f)/(T_c - T_f)$
X	thickness of frozen layer
X'	dimensionless thickness of frozen layer, X/X_s
X_s	thickness of frozen layer at steady state
x	position coordinate in frozen layer
x'	dimensionless coordinate, x/X_s
ξ	dimensionless coordinate, x/X
ρ	density of solidified material
τ	time
τ'	dimensionless time, $\tau h_l(T_l - T_f)/\rho L X_s$

Subscripts:

c	refers to coolant
f	at freezing point
l	liquid phase of solidifying substance
2	interface between frozen layer and wall
I,II, III	successive iterative approximations

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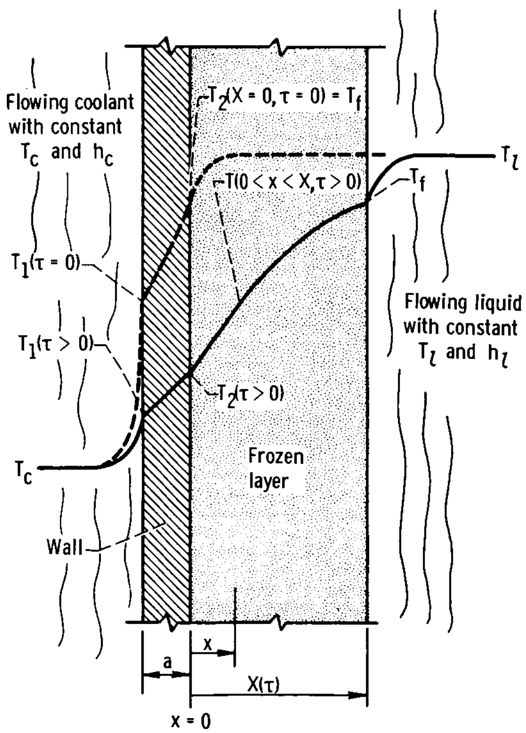

Figure 1. - One-dimensional model of transient solidification of a warm liquid on a convectively cooled plate.

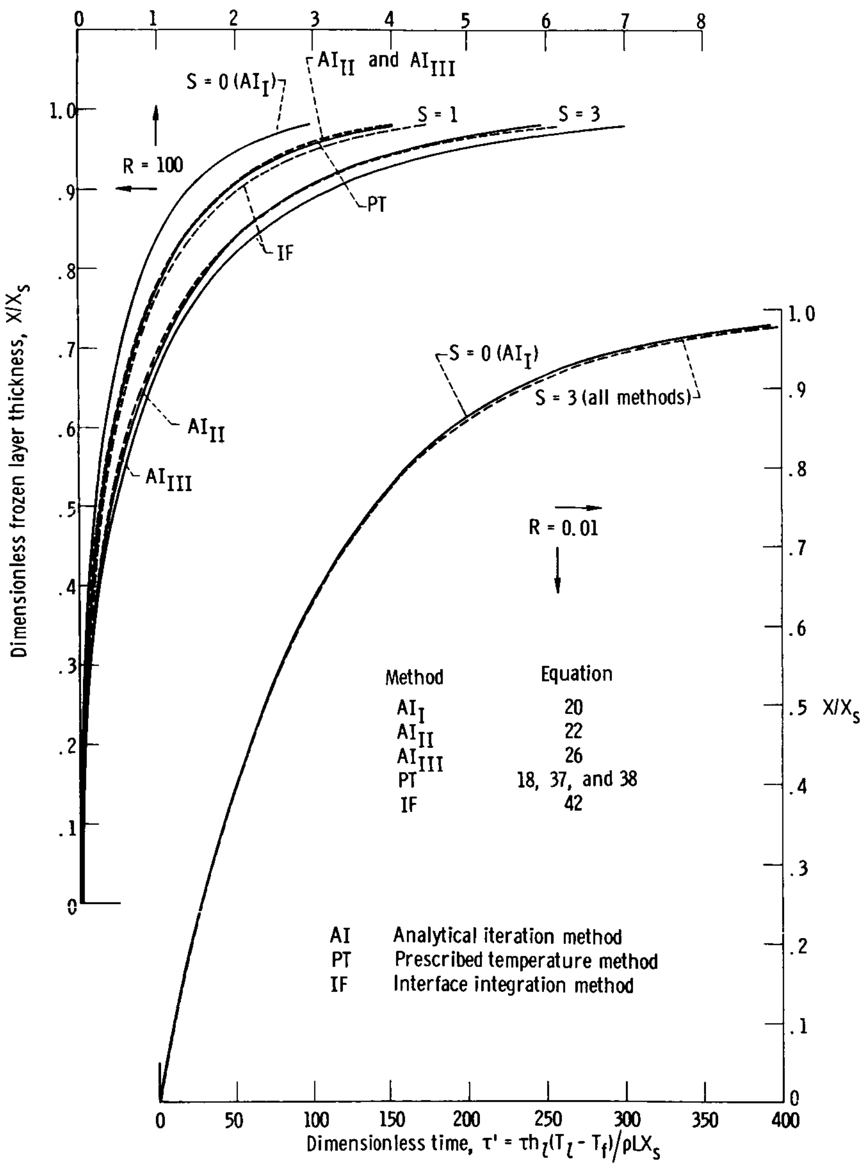

Figure 2. - Comparison of methods for predicting the instantaneous thickness of the frozen layer. (Some curves dotted for clarity only.)

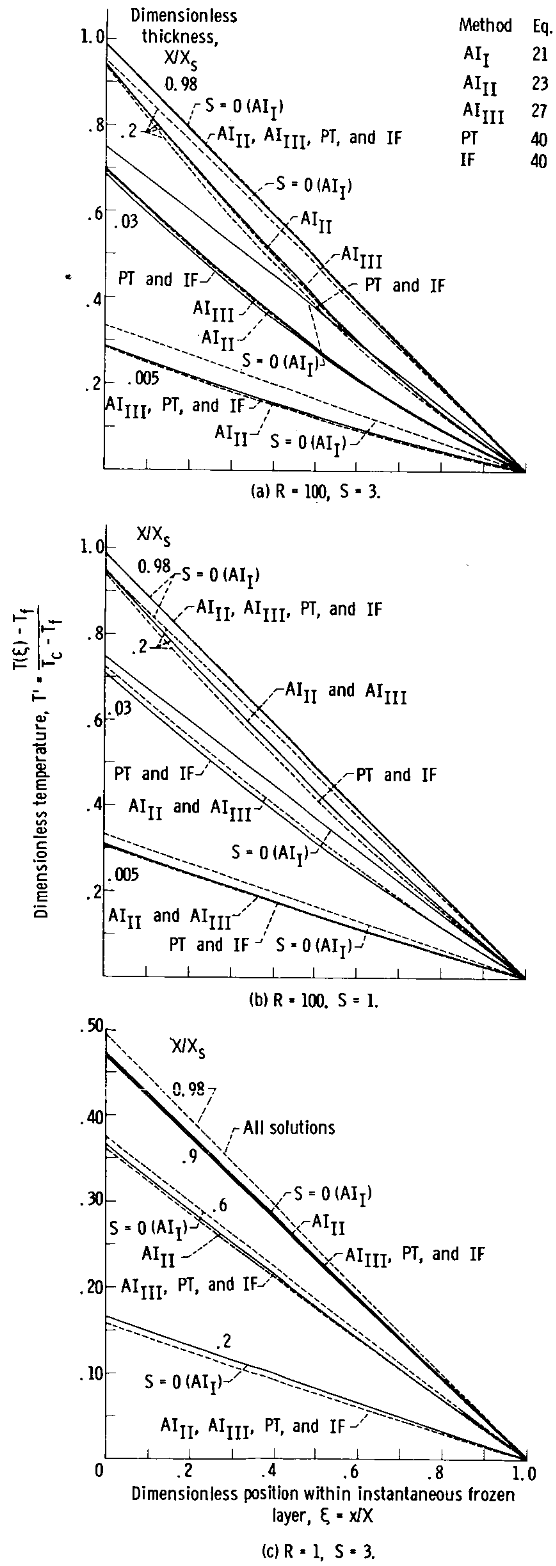

Figure 3. - Temperature distributions in frozen layer computed by various methods. (Some curves dotted for clarity only.)

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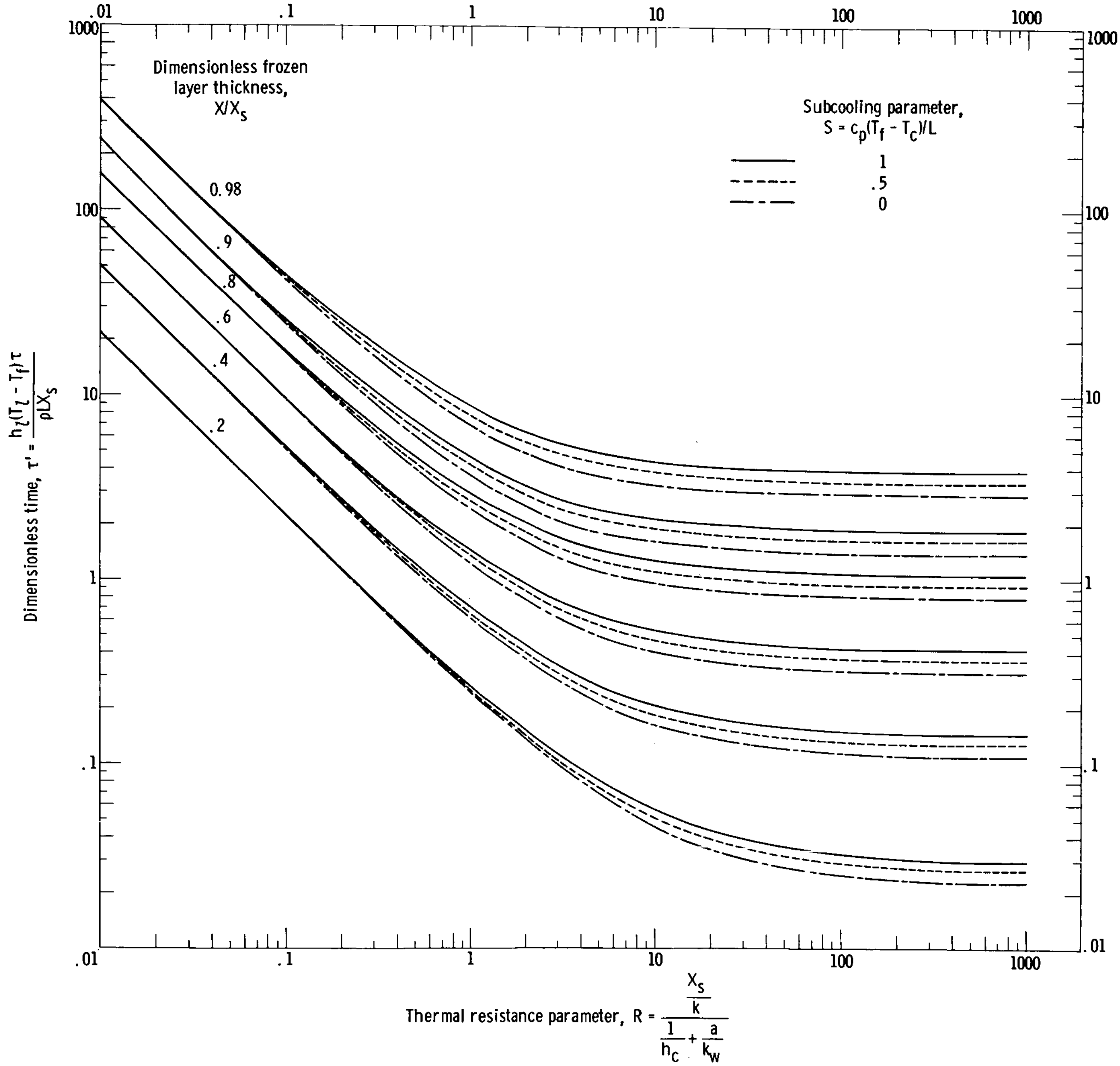
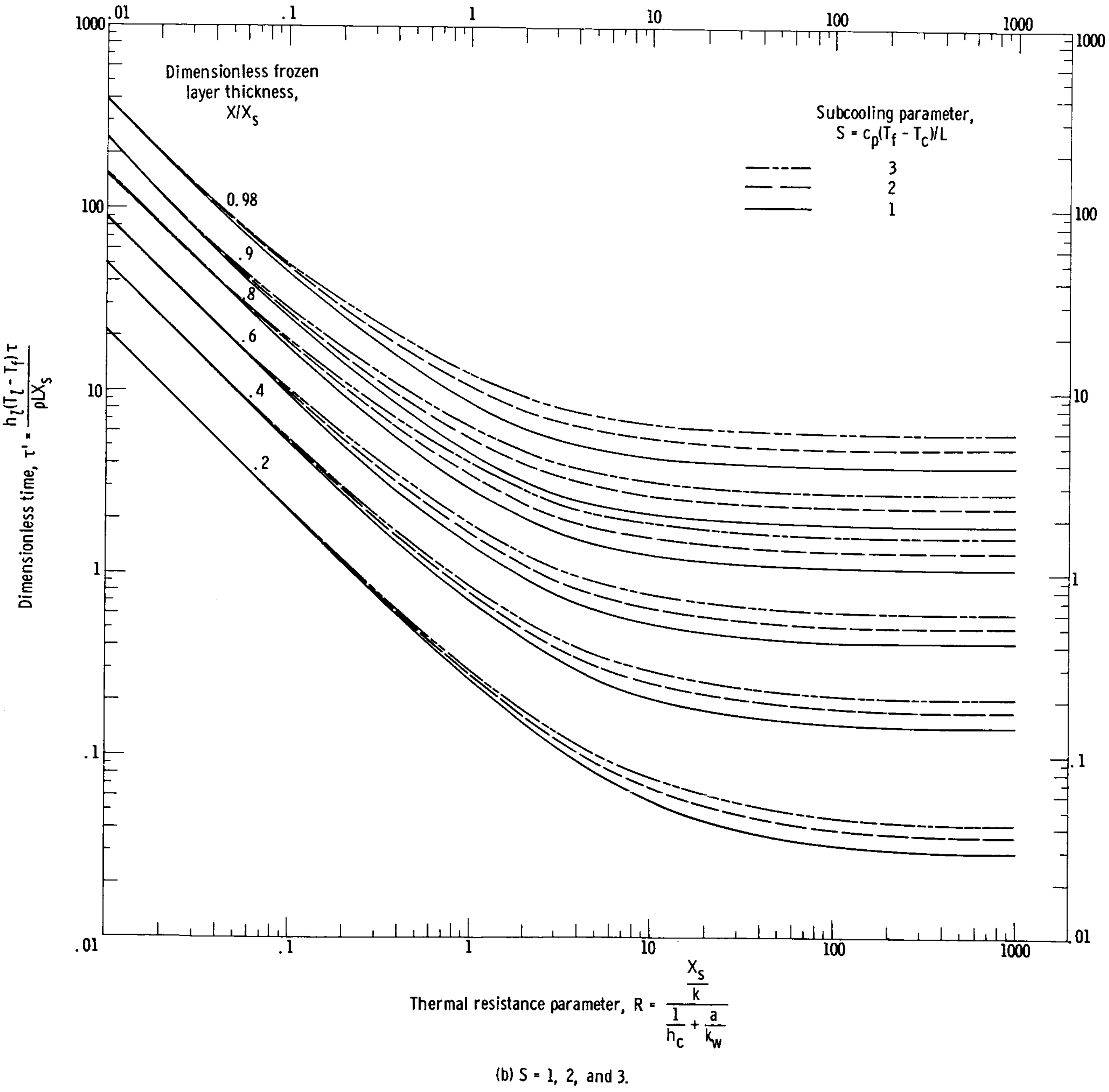


(a) S = 0, 0.5, and 1.

Figure 4. - Dimensionless growth times against the thermal resistance parameter for various dimensionless thicknesses and subcooling parameters.



(b) S = 1, 2, and 3.

Figure 4. - Concluded.