

**NASA TECHNICAL
MEMORANDUM**



NASA TM X-52182

NASA TM X-52182

N66-17577

FACILITY FORM 802

(ACCESSION NUMBER)	(THRU)
<u>18</u>	<u>1</u>
(PAGES)	(CODE)
<u>TM-X-52182</u>	<u>33</u>
(NASA CR OR TMX OR AD NUMBER)	(CATEGORY)

RAREFIED-GAS HEAT TRANSFER BETWEEN PARALLEL

PLATES BY A MONTE CARLO METHOD

by Morris Perlmutter
Lewis Research Center
Cleveland, Ohio

GPO PRICE \$ _____

CFSTI PRICE(S) \$ _____

Hard copy (HC) 1.00

Microfiche (MF) .50

ff 653 July 65

TECHNICAL PAPER proposed for presentation at
Third International Heat Transfer Conference
Chicago, Illinois, August 8-12, 1966

NATIONAL AERONAUTICS AND SPACE ADMINISTRATION • WASHINGTON, D.C. • 1966

RAREFIED-GAS HEAT TRANSFER BETWEEN PARALLEL PLATES
BY A MONTE CARLO METHOD

by Morris Perlmutter

Lewis Research Center
Cleveland, Ohio

TECHNICAL PAPER proposed for presentation at
Third International Heat Transfer Conference
Chicago, Illinois, August 8-12, 1966

NATIONAL AERONAUTICS AND SPACE ADMINISTRATION

RAREFIED-GAS HEAT TRANSFER BETWEEN PARALLEL PLATES BY A MONTE CARLO METHOD

by Morris Perlmutter

Lewis Research Center
National Aeronautics and Space Administration
Cleveland, Ohio

ABSTRACT

The problem of rarefied-gas heat transfer between parallel plates is treated by a Monte Carlo procedure. Hard-sphere molecule collisions are assumed. The distributions of target molecules are assumed to be locally Maxwellian, with thermal velocity and density varying across the channel. The temperature, density, and heat transfer are found for flow conditions ranging from Knudsen numbers of 0.05 to the free-molecule regime for a wall-temperature ratio of 1 to 4. The present results were compared with the results of another Monte Carlo solution obtained by using a different method and were found to agree with those results; they were also compared with other analytical solutions. The present results also agreed with the results obtained from a continuum solution with slip boundaries except that the continuum conductivity was too high for the larger Knudsen number cases.

AUSZUG

Die Frage der Wärmeübertragung in verdünntem Gas zwischen parallelen Platten wird mit Hilfe einer Monte-Carlo Methode behandelt. Zusammenstöße von Hartsphärenmolekülen werden angenommen. Es wird vorausgesetzt dass die Beschussmoleküle örtlich eine Maxwellverteilung haben und dass die thermische Geschwindigkeit und Dichte im Querschnitt veränderlich sind. Die Methode liefert die Temperatur, Dichte und Wärmeübertragung für Strömungsbedingungen, die zwischen Knudsenzahlen von 0.05 und dem freien Molekülfluss für ein Verhältnis von Wand zu Temperatur bei 1 : 4 liegen. Die vorliegenden Ergebnisse werden mit den Resultaten einer anderen Monte-Carlo Lösung unter Verwendung einer unterschiedlichen Methode verglichen und stimmten mit diesen Resultaten überein; sie wurden auch noch mit anderen analytischen Lösungen verglichen. Die gegenwärtigen Resultate waren auch im Gleichklang mit durch eine Kontinuumlösung mit Gleitlinien erhaltenen Ergebnisse aber die Kontinuumsleitfähigkeit war zu hoch in Fällen mit grösseren Knudsenzahlen.

АННОТАЦИЯ

Исследуется теплопередача разреженным газом между параллельными пластинами способом Монте-Карло. Предполагается соударение жестких сферических молекул. Принято местное максвелловское распределение молекул мишени с переменной скоростью теплового движения и переменной плотностью поперек канала. Температура, плотность и теплопередача найдены для режима обтекания с числами Кнудсена начиная от 0.05 до режима обтекания со свободными молекулами при значениях коэффициента "стена-температура" от 1 до 4. Результаты были сравнены с результатами решения полученного другим способом Монте-Карло и оказались с ними в согласии. Результаты также были сравнены с другими аналитическими решениями. Они также в согласии с результатами полученными для сплошной среды с границами скольжения; но проводимость сплошной среды оказалась чрезмерно высокой для случаев с более высокими числами Кнудсена.

INTRODUCTION

In rarefied-gas transport problems, the Navier-Stokes' equation for momentum transfer or the Fourier equation for conduction are no longer applicable because these equations assume local isotropy and small gradients compared with the path lengths of the molecules. Therefore, the more fundamental Boltzmann equation must be resorted to. This equation is difficult to treat by the usual analytical procedures because of its complexity.

Most of the analyses of rarefied-gas heat transfer have been for the linearized problem. As yet, few nonlinear results have been published. The present analysis of heat transfer between parallel plates at different temperatures enclosing a rarefied gas with hard-sphere molecule collisions is by a Monte Carlo procedure. This method reduces the complexity of the analysis thereby avoiding many of the simplifying assumptions generally made in the usual analytical procedures but at the expense of added numerical computations.

A linearized solution to the problem of heat transfer between parallel plates enclosing a hard-sphere gas was analyzed by Gross and Ziering [1].¹ They assume the difference in wall temperature was small compared to the magnitude of the wall temperatures so that the problem can be described by the linearized Boltzmann equation. The distribution function was then approximated by half-range polynomials in velocity space and the space-dependent coefficient were determined by forming half-range moment equations.

The nonlinear problem was treated by Levin and Haviland [2] for hard-sphere molecule collisions by using Lee's moment method. This method consists of choosing a distribution of molecular velocities in the form of half Maxwellians that contain a number of functions related to the local density and thermal velocity. A sufficient number of moment equations are then solved to determine these functions. Haviland and Levin have also treated this problem by a Monte Carlo procedure

¹Numbers in brackets denote references.

[3], [4], and [5]. The major difference between the present analysis and that of Haviland and Levin is in the methods of choosing target-molecule collision partners and evaluating the local mean properties.

The Monte Carlo Procedure

The Monte Carlo procedure is a model sampling technique. A model is created and histories of sample molecules are followed through this model by making choices at points of decision from the appropriate probability distribution. By averaging certain properties of the sample molecules at various positions, macroscopic quantities of interest can be obtained.

The model is shown in Figure 1. The wall at x_2 of 0 is considered to be at temperature $T_{w,0}$ and the other wall at x_2 of D is at temperature $T_{w,1}$. A rarefied, hard-sphere molecule gas is contained between the walls.

The sample molecule history is started at wall 0 by picking its velocity components from the appropriate distribution of velocities of the absorbed and reemitted molecules. The space between the walls is divided into zones as shown in Figure 1. The sample molecule, after leaving the wall, will either pass through the first zone or will have a collision with a target molecule in this zone. This will depend on whether the path length to collision for the sample molecule is longer or shorter than the distance the sample molecule must travel to pass through the zone. If the sample molecule passes through the first zone with no collision, it is started again at the beginning of the next zone with its velocity components unchanged.

If there is a collision in the zone, however, the point of collision is found at the end of the path length to collision. The distribution of target-molecule velocities is assumed to have the form of a Maxwellian distribution throughout each zone. The distribution of target-molecule collision partners is derived from this distribution of target molecules. A target molecule is picked from the distribution of target-molecule collision partners in the zone. A collision calculation is carried out for the sample- and target-molecule collision partners, and new velocity components are found for the sample molecules after collision. The history of the sample molecule is then continued from the point of collision with the new sample-molecule velocity components. The collision partner is ignored after collision for reasons discussed in [4].

A new path length to collision is found for the sample molecule, and this is compared in length with the distance from the point of collision to the zone boundary. If the path length to collision is greater, the sample molecule is started at the beginning of the next zone with its velocity components unchanged. In the other case, there is a collision in the zone and the collision calculations are repeated as before.

If the sample molecule strikes the wall at $x_2 = D$, it is absorbed and reemitted with new velocity components picked from the appropriate distribution of velocities based on the temperature of the wall. After leaving the upper wall it is followed as before.

The transport and fluid characteristics of interest are density distribution, temperature distribution, and heat transfer across the channel.

These can be obtained [6] by locating scoring positions at various locations across the channel (see Fig. 1). By scoring the various characteristics of the sample molecules as they pass the scoring cross section, the transport properties and fluid characteristics of interest can be obtained at these scoring positions.

In the present analysis it was assumed that the target molecules in each zone were in a Maxwellian velocity distribution based on the local temperature and density of each zone. From the Monte Carlo solution based on this assumed distribution of target molecules, the temperature and density at each scoring position are found. The Monte Carlo solution is then rerun with the new local temperature and density in the Maxwellian distribution of target molecules. The problem was iterated in this manner until the density and temperature distribution found from the sample molecule histories agreed with the density and temperature distribution used for the target molecules.

In [3], [4], and [5], a Monte Carlo solution is used to treat the present problem. That solution is carried out with an assumed distribution of target molecules in a tabular form. By scoring the velocity components of many sample molecules as they pass through each zone, the velocity distribution of the molecules in each zone can be obtained in tabular form. These velocity distributions are then used as the target-molecule distributions in the next iteration. This process is continued until the sample-molecule distribution found agrees with the target-molecule distribution assumed. Then, from the distributions in each zone, the macroscopic quantities of interest can be found by numerical integration of the moments of the distribution. This was carried out for one hard-sphere-molecule case with a Knudsen number of 2 and a wall-temperature ratio of 1 to 4. In the present method, it is not necessary to find and store the entire distributions of target molecules in each increment since the form of the target molecule distribution is assumed.

In the present Monte Carlo solution, since the collision partners are ignored, only conservation of mass is satisfied exactly since molecules are not allowed to "disappear" in transit between the walls. Conservation of momentum and energy will only be satisfied if the correct target molecule distribution is assumed.

ANALYSIS

Velocity Components of Emitted Sample Molecule

The sample molecule history begins at the surface at temperature $T_{w,0}$. It is assumed that the molecules incident on the surface are perfectly accommodated; that is, they are in a Maxwellian distribution at the wall, based on the wall temperature. This assumption is discussed in [7]. In this case, the normalized Maxwellian velocity distribution of the molecules moving away from the wall $V_2 > 0$ is given as

$$\frac{\rho_{+,0}}{M} f_{+,0} d^3V = \frac{2\rho_{+,0}}{M\pi^{3/2}C_{w,0}^3} \left[\exp\left(-\frac{V^2}{C_{w,0}^2}\right) dV_1 dV_2 dV_3 \right]_{V_2 > 0} \quad (1)$$

The $\rho_{+,0}/M$ is the number density of molecules moving away from the wall, and $C_{w,0}$ is the ther-

mal velocity based on the wall temperature $[(2k/M)T_{w,0}]^{1/2}$.

If we consider the positive x_2 direction normal to the wall, the distribution of velocities of the molecules leaving the wall per unit time per unit area is $V_2 f_{+,0}$ ([4] and [6]). This distribution can be transformed to cylindrical coordinates $V_1 = V_r \cos \theta$, $V_2 = V_2$, $V_3 = V_r \sin \theta$ and can be normalized by $\langle V_2 \rangle_{+,0} = C_{w,0}/\sqrt{\pi}$ [4] to give

$$f(\theta, V_2, V_r) = \frac{V_2 f_{+,0} d^3 V}{\langle V_2 \rangle_{+,0}} = \frac{2V_2}{\pi C_{w,0}^4} \left[\exp\left(-\frac{V_2^2}{C_{w,0}^2}\right) \right] V_r d\theta dV_2 dV_r = f_\theta f_{V_2} f_{V_r} d\theta dV_2 dV_r \quad (2)$$

The distributions for $f_\theta f_{V_2} f_{V_r}$ can be written separately as

$$f_\theta d\theta = \frac{d\theta}{2\pi} \quad (3a)$$

$$f_{V_2} dV_2 = \frac{2V_2}{C_{w,0}^2} \left[\exp\left(-\frac{V_2^2}{C_{w,0}^2}\right) \right] dV_2 \quad (3b)$$

$$f_{V_r} dV_r = \frac{2V_r}{C_{w,0}^2} \left[\exp\left(-\frac{V_r^2}{C_{w,0}^2}\right) \right] dV_r \quad (3c)$$

The velocity components of our sample molecule leaving the surface must be picked from these distributions. A convenient way of picking from a distribution for the high-speed computer is to transform the distribution to a uniform distribution in R by setting the random number R equal to the cumulative distribution function. For example,

$$R = \int_0^\theta f_\theta d\theta = \frac{\theta}{2\pi} \quad (4)$$

Then, by using a high-speed computer to generate a random number R between 0 and 1, θ can be obtained from Equation (4) such that for a large number of samples picked in this manner, the distribution in Equation (3a) will be satisfied.

Similarly, V_2 can be picked from

$$R_{V_2} = \int_0^{V_2} \frac{2V_2}{C_{w,0}^2} \left[\exp\left(-\frac{V_2^2}{C_{w,0}^2}\right) \right] dV_2 \quad (5)$$

Or, since picking R is equivalent to picking $1 - R$,

$$V_2 = \left(-C_{w,0}^2 \ln R_{V_2} \right)^{1/2} \quad (6a)$$

so that by picking R_{V_2} V_2 can be obtained from Equation (6).

The velocity V_r is obtained in the same manner and is given by

$$V_r = \left(-C_{w,0}^2 \ln R_{V_r} \right)^{1/2} \quad (6b)$$

The V_2 , V_r , and θ picked give the direction and velocity of the sample molecule as it leaves the wall. This same result was used in [4] and [7]. Finally, V_1 and V_3 can be obtained from

$$V_3 = V_r \sin \theta; \quad V_1 = V_r \cos \theta \quad (7)$$

Calculation of the Path Length λ to Collision

The probability that a sample molecule will collide in the incremental path length λ to $\lambda + d\lambda$ is given in [8] as

$$f_\lambda = \frac{\exp(-\lambda/\lambda_s)}{\lambda_s} \quad (8)$$

where λ_s is the mean free path to collision of the sample molecule moving at velocity V_s in that zone.

A path length to collision for the sample molecule from this distribution can then be chosen by the same procedure described earlier.

$$\lambda = -\lambda_s \ln R_\lambda \quad (9)$$

To use this relation, the mean free path λ_s in the zone must first be calculated.

As shown in Appendix A (Eq. (A14)), the mean free path of a sample molecule moving at velocity V_s through a Maxwellian gas at density ρ with a thermal velocity C is

$$\frac{\lambda_s}{D} = \frac{2^{1/2} K n \mu_s}{\frac{\rho}{\rho_A} \left[\frac{\exp(-\mu_s^2)}{\pi^{1/2}} + (\operatorname{erf} \mu_s) \left(\mu_s + \frac{1}{2\mu_s} \right) \right]} \quad (10)$$

where μ_s is the velocity of the sample molecule nondimensionalized by the thermal velocity of the zone ($\mu_s = V_s/C$). Equation (10) can then be used in Equation (9) to obtain the distance to collision for the sample molecule in the zone.

Calculating New Sample-Molecule Velocity Components After Collision

The distribution of velocities of the target molecules colliding with the sample molecule is given in Appendix B follows the derivation in [8]. From this distribution, the velocity components of the target-molecule collision partner as given by Equation (B12) are chosen. The new velocity components of the sample molecule after collision with the target-molecule collision partner are given, following the derivation in [4].

$$V_{1s}^* = \frac{1}{2} (V_{1s} + V_{1t}) + V_R (1 - b^2)^{1/2} H \quad (11a)$$

$$V_{2s}^* = \frac{1}{2} (V_{2s} + V_{2t}) + \frac{1}{2} V_R (1 - 2b^2) \quad (11b)$$

$$V_{3s}^* = \frac{1}{2} (V_{3s} + V_{3t}) + V_R (1 - b^2)^{1/2} \Xi \quad (11c)$$

where

$$V_R = \left[(V_{1s} - V_{1t})^2 + (V_{2s} - V_{2t})^2 + (V_{3s} - V_{3t})^2 \right]^{1/2} \quad (11d)$$

The values of H and Ξ are obtained by picking two random numbers that are used in the following equations

$$H = 2R_H - 1 \quad \Xi = 2R_\Xi - 1 \quad \text{and} \quad b^2 = H^2 + \Xi^2 \quad (12)$$

where b^2 must be less than 1. If b^2 is greater than 1, a new set of random numbers must be chosen to find H and Ξ .

Macroscopic Flow Properties

The macroscopic fluid characteristics needed are density, temperature, and heat transfer across the channel. Scoring positions are located at various distances across the channel; as shown in Figure 1. The average quantity of Q transported across the scoring cross section p in the positive x_2 direction, can be written as

$$\frac{\left(\sum_{+} Q \right)_p}{S_{+,p}} \approx \frac{(\rho_+ \int Q V_2 f_+ d^3V)_p}{(\rho_+ \int V_2 f_+ d^3V)_p} = \frac{(\rho_+ \langle Q V_2 \rangle_+)_p}{(\rho_+ \langle V_2 \rangle_+)_p} \quad (13)$$

where $S_{+,p}$ is the number of sample molecules passing across the scoring cross section p in the positive x_2 direction and Q is some quantity that each sample molecule carries. Similarly, the average quantity Q transported in the negative x_2 direction is

$$\frac{\left(\sum_{-} Q \right)_p}{S_{-,p}} \approx \frac{(\rho_- \langle Q V_2 \rangle_-)_p}{(\rho_- \langle V_2 \rangle_-)_p} \quad (14)$$

Since there is no net flow across the channel

$$S_{+,p} = S_{-,p} \quad (15a)$$

and

$$(\rho_+ \int V_2 f_+ d^3V)_p = -(\rho_- \int V_2 f_- d^3V)_p \quad (15b)$$

Hence,

$$\frac{\rho \langle V_2 Q \rangle_p}{\rho_{+,p} \langle V_2 \rangle_{+,p}} = \frac{1}{S_{+,p}} \left(\sum_{+} Q - \sum_{-} Q \right) \quad (16)$$

The number of sample molecules that pass the scoring cross section at p in the positive V_2 direction, $S_{+,p}$, divided by the number of times the sample molecule leaves wall 0, N , can be related to the mass flux passing in the positive V_2 direction at p by

$$\frac{S_{+,p}}{N} \approx \frac{\rho_{+,p} \langle V \rangle_{+,p}}{\rho_{+,0} \langle V \rangle_{+,0}} \quad (17)$$

where $\rho_{+,0} \langle V \rangle_{+,0}$, the mass flux leaving wall 0, is equal to $\rho_{+,0} C_{w,0} / \pi^{1/2}$ [4]. Combining Equations (16) and (17) results in

$$\rho \langle V_2 Q \rangle_p = \frac{\rho_{+,0} C_{w,0}}{N \pi^{1/2}} \left(\sum_{+} Q - \sum_{-} Q \right) \quad (18)$$

If Q is taken as $1/V_2$, Equation (18) becomes

$$\frac{\rho}{\rho_{+,0}} = \frac{C_{w,0}}{N \pi^{1/2}} \left(\sum_{+} \frac{1}{V_2} - \sum_{-} \frac{1}{V_2} \right) \quad (19)$$

The average density in the channel is then obtained by averaging the density of all the scoring cross sections from $p = 0$ to $p = p$.

$$\frac{\rho_A}{\rho_{+,0}} = \frac{1}{P_F} \left[\frac{1}{2} \left(\frac{\rho_0}{\rho_{+,0}} + \frac{\rho_{P_F}}{\rho_{+,0}} \right) + \sum_{p=1}^{P_F-1} \frac{\rho_p}{\rho_{+,0}} \right] \quad (20a)$$

As shown in [4], the continuum solution, $Kn \rightarrow 0$, is given by

$$\frac{\rho}{\rho_A} = \frac{1 + r_T^{1/2} + r_T}{3 \left[1 - (1 - r_T^{3/2}) x \right]^{2/3}} \quad (20b)$$

where $r_T = T_w / T_{w,0}$.

The temperature can be obtained in a similar manner. The undirected kinetic energy per unit mass of a monatomic gas is equal to $1/2(kT/M)$ times the degrees of freedom

$$\frac{3}{2} \frac{k}{M} T = \frac{1}{2} \langle V^2 \rangle \quad (21)$$

From Equation (18), this can be written as

$$\frac{T}{T_{w,0}} = \frac{2}{3 \left(\frac{\rho}{\rho_{+,0}} \right) N \pi^{1/2} C_{w,0}} \left[\sum_{+} \langle V^2 \rangle - \sum_{-} \langle V^2 \rangle \right]_p$$

where $\rho / (\rho_{+,0})$ is given by Equation (19). As shown in [4], the collisionless solution, $Kn \rightarrow \infty$, is given by

$$\left(\frac{T}{T_{w,0}} \right)_{Kn \rightarrow \infty} = r_T^{1/2} \quad (21a)$$

and the continuum solution, $Kn \rightarrow 0$, is given by

$$\frac{T}{T_{w,0}} = \left[1 - \left(1 - r_T^{3/2} \right) x \right]^{2/3} \quad (21b)$$

Finally, the heat transfer across the channel can be obtained from

$$q = \frac{\rho}{2} \langle v_z (v_z^2 + v_r^2) \rangle \\ = \frac{\rho_{+,0} C_{w,0}}{2\pi^{1/2} N} \left[\sum_{S=+}^{S_{+,p}} (v_z^2 + v_r^2) - \sum_{S=-}^{S_{-,p}} (v_z^2 + v_r^2) \right] \quad (22)$$

This can be nondimensionalized by $\frac{1}{2} [\rho \langle v_z (v_z^2 + v_r^2) \rangle]_{+,0} = \frac{1}{\pi^{1/2}} (\rho_{+,0} C_{w,0}^3)$, and then, by using the $\rho_{+,0}/\rho_A$ previously obtained, the following is found:

$$\frac{\pi^{1/2}}{\rho_A C_{w,0}^3} q = \frac{1}{2NC_{w,0}^2} \left[\sum_{S=+}^{S_{+,p}} (v_z^2 + v_r^2) - \sum_{S=-}^{S_{-,p}} (v_z^2 + v_r^2) \right] \left(\frac{\rho_{+,0}}{\rho_A} \right) \quad (23)$$

The heat transfer is then averaged across the channel as in the case of the density (Eq. (20b)). The heat transfer is compared to the collisionless result as derived in [4]:

$$\left(\frac{\pi^{1/2} q}{\rho_A C_{w,0}^3} \right)_{Kn \rightarrow \infty} = r_T^{1/2} - r_T \quad (24a)$$

and to the continuum solution as given by

$$\left(\frac{\pi^{1/2} q}{\rho_A C_{w,0}^3} \right)_{Kn \rightarrow 0} = \frac{25}{64} \pi \left(1 - r_T^{3/2} \right) Kn \quad (24b)$$

RESULTS AND CONCLUSIONS

The temperature and density distribution results are shown in Figure 2 for a Knudsen number of 2 and wall-temperature ratios of 0.25 and 0.75. These results are compared with the one case carried out by the Monte Carlo procedure of [4] and are in good agreement. The nonlinear moment solution results [2] have a significantly different slope than the Monte Carlo results.

The linearized solution [1] is also compared with the present results. For the smaller wall-temperature difference case ($T_{w,1}/T_{w,0} = 0.75$), the results agree with the linearized solution. For the larger wall-temperature differences, however, where $T_{w,1}/T_{w,0} = 0.25$, the temperature is different in magnitude from the present result, although the shapes of the curve agree. In the linearized solution the centerline temperature is a parameter taken midway between the wall temperatures. The present results have their centerline temperature significantly below this value for the Knudsen number of 2; hence the large disagreement between the results.

Also shown in Figure 2 are the slip continuum results. The continuum equations are solved by using the fluid temperature (slip temperature) near the walls rather than the wall temperatures as the boundary temperature. By using the slip temperatures that give the best fit to both the temperature and density results from the present solution, the results shown are obtained. These results indicate that the slip continuum densities and temperature curves agree well with the present results. The present results show the magnitude of the temperature slip between the wall and the fluid near the wall as well as the Knudsen layer where the fluid temperature near the wall deviates from the continuum results toward the wall temperature.

In Figure 3, the temperature and density results for a range of Knudsen numbers are shown. These results are for a fixed value wall-temperature ratio of 0.25. They agree with the collisionless solution for the very large Knudsen number and approach the continuum solution for the values of the smaller Knudsen number. Figure 3 also shows the temperature and density distribution obtained from the aforementioned continuum slip solutions.

In Figure 4, the heat flux nondimensionalized by the continuum heat flux is plotted as a function of the inverse Knudsen number for the wall-temperature ratio of 0.25. The Monte Carlo result of [4] was solved for the inverse Knudsen number of 0.5 and found to agree with the present results and with the linearized solution of [1]. The nonlinear solution gave results above the linear solution, while the present results fell below the linear solution for small Knudsen numbers.

Also shown in Figure 4 is the heat transfer from the continuum solution with the slip boundary conditions. The slip boundary heat transfer is very high for larger Knudsen numbers, so that the slip approximation does not apply near the collisionless region. This could be because the noncontinuum conductivity, which is the first term in the series expansion of the second approximation in the Enskog method [4], is not appropriate in the present noncontinuum case. The results of Figure 4 indicate that if a conductivity lower than the continuum conductivity is used in the continuum solution with slip, the results would be in good agreement with the present case.

In conclusion, the present result agrees with the results obtained by the Monte Carlo method used in [4] for the one case that was carried out ($Kn = 2$, $T_{w,1}/T_{w,0} = 0.25$). The linear and the nonlinear solutions disagreed somewhat with the present result. This could be due partly to the approximation made in the other analysis and partly to the present assumption that the target molecules locally are in a Maxwellian distribution. The assumption of a local Maxwellian distribution for the target molecules would be most applicable near equilibrium conditions, that is, cases with small temperature gradients and short mean free paths. The present results are being extended to the more realistic two-sided Maxwellian as was used in [2].

The results indicate the continuum solution with slip boundaries, if the slip can be properly predicted, would give good results, except that for the larger Knudsen numbers, the continuum conductivity value is too high.

The Monte Carlo method seems to give reasonable results for this problem. There are certain drawbacks to the method, however. The error is

reduced by running a large number of samples, which run into large amounts of computing time. The computing time needed for a sample molecule was found to increase exponentially as the Knudsen number approached zero. On the IBM 7094, typical sample run times for 10,000 samples were: $Kn = 5$, 5 minutes; $Kn = 2$, 6 minutes; $Kn = 0.5$, 10 minutes; $Kn = 0.2$, 15 minutes; $Kn = 0.1$, 25 minutes; and $Kn = 0.05$, 45 minutes. However, for problems of the present type, which are difficult to solve by other methods without over simplifying assumptions, the present method proved very useful.

NOMENCLATURE

C	average thermal velocity, $(2kT/M)^{1/2}$
D	Channel height
erf(x)	error function $(2/\sqrt{\pi}) \int_0^x e^{-u^2} du$
f	probability distribution function
f_+, f_-	probability distribution function of molecules moving in positive, negative V_2 direction
Kn	Knudsen number $\lambda_m/D = M/\sqrt{2} S \rho_A D$
k	Boltzmann constant
M	mass of molecule
N	number of sample molecules leaving surface 0 in Monte Carlo run, proportional to flux of molecules leaving surface 0
P	zone or scoring-position number
P_f	last scoring-position number
Q	property of sample molecule
$\langle Q \rangle$	averaged quantity $\int Q f d^3V$
$\langle Q \rangle_+$	averaged quantity $\int Q f_+ d^3V \big _{V_2 > 0}$

R	random number between 0 and 1
r_T	wall temperature ratio $T_{w,1}/T_{w,0}$
S	mutual collision cross section $\pi\sigma^2$
S_+, S_-	number of sample molecules through scoring position in positive, negative x_2 direction
T	absolute temperature
V	molecular velocity
V_r	radial velocity; $(V_1^2 + V_3^2)^{1/2}$
V^*	velocity after collision
x_1, x_2, x_3	coordinates
γ	defined by Eqs. (B6) and (B7)
η	defined by Eqs. (B6) and (B7)
Θ	collision rate of sample molecule with target molecules
λ	path length to collision
λ_s	mean free path length
μ	dimensionless velocity, V/C
$\mu_{R,A}$	defined by Eq. (A11)
ρ	mass density
σ	diameter of hard-sphere molecule
Φ	angle between sample-molecule and target-molecule velocities

Subscripts:

A	averaged
p	increment number
P_f	last increment
R	relative velocity
s	sample molecule
t	target molecule
$w, 0; w, 1$	at wall 0, 1
0, 1	evaluated next to wall 0, 1
1, 2, 3	coordinate directions
+, -	positive or negative direction

APPENDIX A

EVALUATION OF MEAN FREE PATH

The number of collisions per unit time $d\Theta$ of a sample molecule moving at velocity V_s through target molecules in velocity volume space d^3V_t for hard-sphere molecules is given in [8] as

$$d\Theta = \rho f_t V_R S d^3V_t \quad (A1)$$

where V_R is the velocity of the target molecules relative to the sample-molecule velocity before collision, $V_{R,i} = V_{t,i} - V_{s,i}$, and S is the mutual-collision cross section $\pi\sigma^2$ where σ is the diameter of the molecule. The relative velocity V_R can then be written as

$$V_R = \left[(V_{1t} - V_{1s})^2 + (V_{2t} - V_{2s})^2 + (V_{3t} - V_{3s})^2 \right]^{1/2} \quad (A2)$$

For a Maxwellian distribution of target molecules, the collision rate can be written as

$$d\Theta = \frac{\rho S}{M \pi^{3/2} C_p^3} \left(\exp - \frac{V_{1t}^2 + V_{2t}^2 + V_{3t}^2}{C_p^2} \right) \times V_R dV_{1t} dV_{2t} dV_{3t} \quad (A3)$$

Then transforming variable by a rotation, using Eulerian angles as given in [9], yields

$$V_{it} = A_{ij}^{-1} V'_{jt} \quad (A4a)$$

where

$$A_{ij}^{-1} = \begin{vmatrix} \frac{V_{2s}}{V_{rs}} & \frac{V_{1s}}{V_s} & -\frac{V_{3s}}{V_{rs}} \frac{V_{1s}}{V_s} \\ -\frac{V_{1s}}{V_{rs}} & \frac{V_{2s}}{V_s} & -\frac{V_{3s}}{V_{rs}} \frac{V_{2s}}{V_s} \\ 0 & \frac{V_{3s}}{V_s} & \frac{V_{rs}}{V_s} \end{vmatrix} \quad (A4b)$$

where the inverse transformation is

$$V_{it}' = A_{ij} V_{jt} \quad (A5a)$$

where

$$A_{ij} = \begin{vmatrix} \frac{V_{2s}}{V_{rs}} & -\frac{V_{1s}}{V_{rs}} & 0 \\ \frac{V_{1s}}{V_s} & \frac{V_{2s}}{V_s} & \frac{V_{3s}}{V_s} \\ -\frac{V_{3s}}{V_{rs}} \frac{V_{1s}}{V_s} & -\frac{V_{2s}}{V_{rs}} \frac{V_{3s}}{V_s} & \frac{V_{rs}}{V_s} \end{vmatrix} \quad (A5b)$$

and where

$$V_r = (V_1^2 + V_2^2)^{1/2} \quad \text{and} \quad V = (V_r^2 + V_3^2)^{1/2} \quad (A6a)$$

Then, the following is obtained:

$$d\theta = \frac{\rho S}{M \pi^{3/2} C_p^3} \left(\exp - \frac{V_{1t}'^2 + V_{2t}'^2 + V_{3t}'^2}{C_p^2} \right) \times [V_{1t}'^2 + (V_{2t}' - V_s)^2 + V_{3t}'^2]^{1/2} dV_{1t}' dV_{2t}' dV_{3t}' \quad (A7b)$$

By transforming into spherical coordinates

$$\begin{aligned} V_{1t}' &= V_t \cos \theta \sin \varphi \\ V_{2t}' &= V_t \cos \varphi \\ V_{3t}' &= V_t \sin \theta \sin \varphi \end{aligned} \quad (A7)$$

The following is obtained:

$$d\theta = \frac{\rho S}{M \pi^{3/2} C_p^3} \left(\exp - \frac{V_t^2}{C_p^2} \right) V_R V_t^2 \sin \varphi d\varphi d\theta dV_t \quad (A8a)$$

where

$$V_R = (V_t^2 + V_s^2 - 2V_s V_t \cos \varphi)^{1/2} \quad (A8b)$$

This can be written as

$$d\theta = \frac{C_p \rho S}{M \pi^{3/2}} \left[\exp(-\mu_t^2) \right] \mu_t^2 \mu_R \sin \varphi d\varphi d\theta d\mu_t \quad (A9)$$

where μ is the nondimensionalized velocity V/C . To obtain the total collision rate for the target molecules over all velocities, Equation (A9) must be integrated over φ , θ , and μ_t . This can be integrated over θ from 0 to 2π and over φ from 0 to π to give

$$d\Theta(\mu_t) = \frac{4\rho C_p S}{M \pi^{1/2}} \left[\exp(-\mu_t^2) \right] \mu_t^2 \mu_{R,A} d\mu_t \quad (A10a)$$

where

$$\mu_{R,A} = \int_0^\pi (\mu_t^2 + \mu_s^2 - 2\mu_s \mu_t \cos \varphi)^{1/2} \frac{\sin \varphi}{2} d\varphi \quad (A10b)$$

or

$$\mu_{R,A} = \begin{cases} \mu_s + \mu_t^2/3\mu_s & \text{if } \mu_s > \mu_t \\ \mu_t + \mu_s^2/3\mu_t & \text{if } \mu_t > \mu_s \end{cases} \quad (A11)$$

Integrating over μ_t from 0 to ∞ to obtain the total collision frequency for a sample molecule moving at velocity μ_s through a Maxwellian gas will then produce

$$\Theta = \frac{\rho S C_p}{M} \left[\frac{\exp(-\mu_s^2)}{\pi^{1/2}} + (\text{erf } \mu_s) \left(\mu_s + \frac{1}{2\mu_s} \right) \right] \quad (A12)$$

The nondimensional mean free path is then found, as discussed in [8], by dividing the sample-molecule velocity by the total collision rate to give

$$\frac{\lambda_{\mu_s}}{D} = \frac{V_s}{\Theta} = \frac{\mu_s M}{D \rho S \left\{ \frac{1}{\pi^{1/2}} \left[\exp(-\mu_s^2) \right] + (\text{erf } \mu_s) \left(\mu_s + \frac{1}{2\mu_s} \right) \right\}} \quad (A13)$$

By using the definition of Knudsen number given by Equation (11) the following can be written:

$$\frac{\lambda_{\mu_s}}{D} = \text{Kn} 2^{1/2} \mu_s \left(\frac{\rho}{p_A} \right)^{-1} \left\{ \frac{\exp(-\mu_s^2)}{\pi^{1/2}} + (\text{erf } \mu_s) \left(\mu_s + \frac{1}{2\mu_s} \right) \right\}^{-1} \quad (A14)$$

APPENDIX B

PICKING THE TARGET-MOLECULE COLLISION PARTNER

The velocity distribution of target molecules that the sample molecule will collide with as discussed in [1] is given by

$$\frac{d\theta}{\theta} = \frac{\left[\exp(-\mu_t^2) \right] \mu_t^2 \mu_R \sin \varphi d\varphi d\theta d\mu_t}{\pi \left[\exp(-\mu_s^2) \right] + \pi^{3/2} (\text{erf } \mu_s) \left(\mu_s + \frac{1}{2\mu_s} \right)} \quad (B1)$$

The distribution in θ for the target molecules is readily seen to be

$$f_\theta d\theta = \frac{d\theta}{2\pi} \quad (B2)$$

Then, the θ can be picked from this distribution by

$$\theta = 2\pi R_0 \quad (B3)$$

The distribution of μ_t for the target molecules is obtained from the marginal distribution

$$f_{\mu_t} d\mu_t = \frac{4 \left[\exp\left(-\frac{\mu_t^2}{\mu_s^2}\right) \right] \mu_t^2 \mu_{R,A} d\mu_t}{\left[\exp\left(-\frac{\mu_s^2}{\mu_s^2}\right) \right] + \pi^{1/2} (\text{erf } \mu_s) \left(\mu_s + \frac{1}{2\mu_s} \right)} \quad (B4)$$

We can then pick from this distribution as before

$$R_{\mu_t} = \frac{\gamma(\mu_s, \mu_t) + \eta(\mu_s, \mu_t)}{\left[\exp\left(-\frac{\mu_s^2}{\mu_s^2}\right) \right] + \pi^{1/2} (\text{erf } \mu_s) \left(\mu_s + \frac{1}{2\mu_s} \right)} \quad (B5)$$

where, when $\mu_s > \mu_t$,

$$\begin{aligned} \gamma(\mu_s, \mu_t) &= -2\mu_s \mu_t \left[\exp\left(-\frac{\mu_t^2}{\mu_s^2}\right) \right] + \mu_s \sqrt{\pi} (\text{erf } \mu_t) \\ &\quad - \frac{2}{3} \frac{\mu_t^3}{\mu_s} \left[\exp\left(-\frac{\mu_t^2}{\mu_s^2}\right) \right] - \frac{\mu_t}{\mu_s} \left[\exp\left(-\frac{\mu_t^2}{\mu_s^2}\right) \right] + \frac{\sqrt{\pi}}{2\mu_s} (\text{erf } \mu_t) \end{aligned} \quad (B6)$$

and

$$\eta = 0$$

and, when $\mu_s < \mu_t$,

$$\left. \begin{aligned} \gamma(\mu_s, \mu_t) &= \gamma(\mu_s, \mu_s) \\ \text{and} \\ \eta &= -2 \left[\exp\left(-\frac{\mu_t^2}{\mu_s^2}\right) \right] \left(\frac{\mu_s^2}{3} + 1 + \mu_t^2 \right) \\ &\quad + \left[2 \exp\left(-\frac{\mu_s^2}{\mu_s^2}\right) \right] \left(\frac{4\mu_s^2}{3} + 1 \right) \end{aligned} \right\} \quad (B7)$$

To find ϕ for the target molecule, the distribution of target molecules as a product of a marginal distribution times a conditional distribution can be written:

$$\frac{d\theta(\mu_t, \phi)}{\theta} = f(\mu_t) f(\phi | \mu_t) \quad (B8)$$

Then, ϕ can be picked from the conditional distribution

$$f(\phi | \mu_t) = \frac{(\mu_t^2 + \mu_s^2 - 2\mu_t \mu_s \cos \phi)^{1/2} \sin \phi d\phi}{\mu_{R,A}} \quad (B9)$$

Picking ϕ for a given μ_t is found as before from

$$R_\phi = \frac{(\mu_t + \mu_s - 2\mu_t \mu_s \cos \phi)^{3/2} - |\mu_t - \mu_s|^3}{(\mu_t^2 + \mu_s^2 + 2\mu_t \mu_s)^{3/2} - |\mu_t - \mu_s|^3} \quad (B10)$$

After picking ϕ , μ_t , and θ , the velocity components before transformation must be found. These are obtained from

$$V'_{1t} = \mu_t C_p \cos \theta \sin \phi \quad (B11a)$$

$$V'_{2t} = \mu_t C_p \cos \phi \quad (B11b)$$

$$V'_{3t} = \mu_t C_p \sin \theta \sin \phi \quad (B11c)$$

and from Equation (A4)

$$V_{1t} = V'_{1t} \left(\frac{V_{2s}}{V_{rs}} \right) + V'_{2t} \left(\frac{V_{1s}}{V_s} \right) - V'_{3t} \left(\frac{V_{3s}}{V_{rs}} \frac{V_{1s}}{V_s} \right) \quad (B12a)$$

$$V_{2t} = -V'_{1t} \left(\frac{V_{1s}}{V_{rs}} \right) + V'_{2t} \left(\frac{V_{2s}}{V_s} \right) - V'_{3t} \left(\frac{V_{2s}}{V_{rs}} \frac{V_{3s}}{V_s} \right) \quad (B12b)$$

$$V_{3t} = 0 + V'_{2t} \left(\frac{V_{3s}}{V_s} \right) + V'_{3t} \left(\frac{V_{rs}}{V_s} \right) \quad (B12c)$$

REFERENCES

1. Gross, E. P.; and S. Ziering: "Heat Flow Between Parallel Plates," *Phys. Fluids*, vol. 2, No. 6, November-December, 1959, pp. 701-712.
2. Levin, M. L.; and J. K. Haviland: "Application of a Moment Method to Heat Transfer in Rarefied Gases," *Phys. Fluids*, vol. 5, No. 3, March, 1962, pp. 274-279.
3. Levin, M. L.: "A Monte Carlo Solution for Heat Transfer in Rarefied Gases," *Massachusetts Inst. of Tech. Rep. No. 61-3, AFOSR Rept. No. 627, May, 1961.*
4. Haviland, J. K.: "Monte Carlo Application to Molecular Flows," *Massachusetts Inst. of Tech., Rep. No. 61-5, AFRL Scientific Rep. 648, May, 1961.*
5. Haviland, J. K.; and M. L. Levin: "Application of the Monte Carlo Method to Heat Transfer in a Rarefied Gas," *Phys. Fluids*, vol. 5, No. 11, November, 1962, pp. 1399-1405.
6. Perlmuter, M.: "Monte Carlo Solution for the Characteristics of a Highly Ionized Gas Flowing Through a Channel with a Transverse Magnetic Field," *NASA TN D-2211, October, 1964.*
See also vol. 2 of *Fourth Symposium on Rarefied Gas Dynamics*, J. H. deLeeuw, ed., Academic Press, New York, to be published 1966.
7. Patterson, G. N.: "A State-of-the-Art Survey of Some Aspects of the Mechanics of Rarefied Gases and Plasmas," *Toronto University, AFARL 64-60, DDC No. AD-600873, April, 1964.*
8. Kennard, E. H.: *Kinetic Theory of Gases with and Introduction to Statistical Mechanics*, McGraw-Hill Book Co., Inc., New York, 1938.
9. Goldstein, H.: *Classical Mechanics*, Addison-Wesley Press, Cambridge, Massachusetts, 1950.

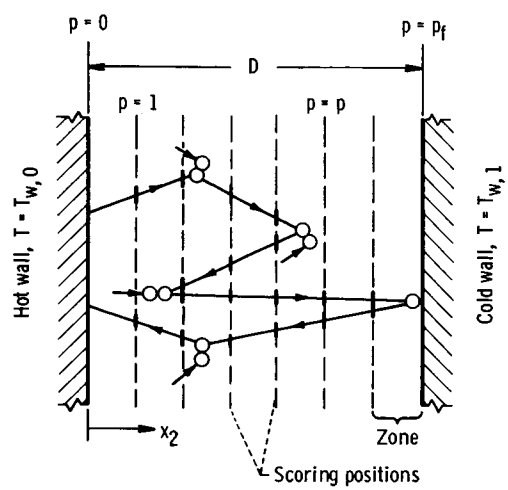


Figure 1. - Model.

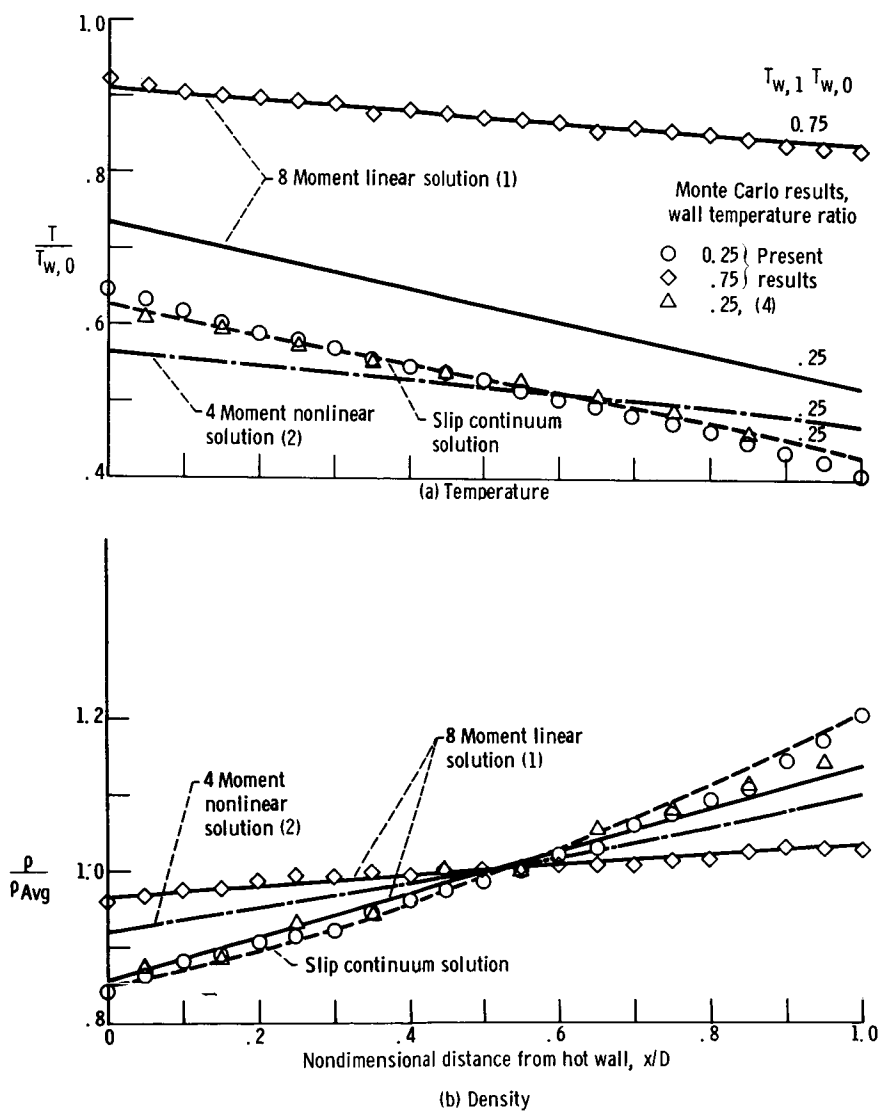


Figure 2. - Distributions across channel for Knudsen number 2.

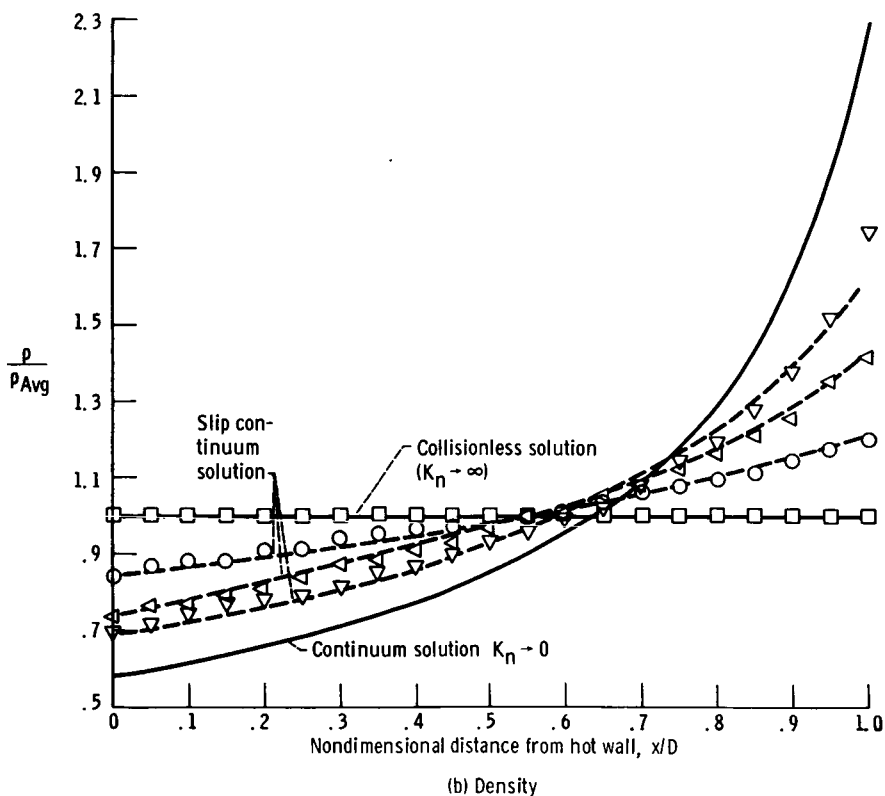
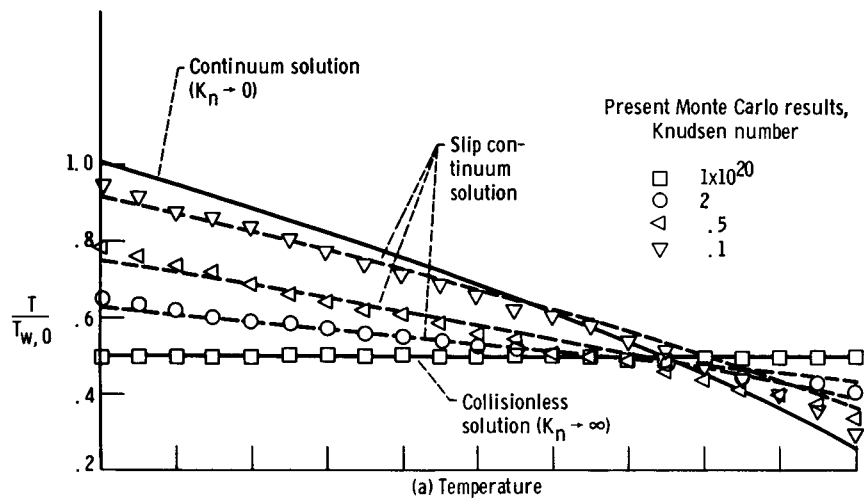


Figure 3. - Distributions for various Knudsen numbers; wall temperature ratio, 0.25.

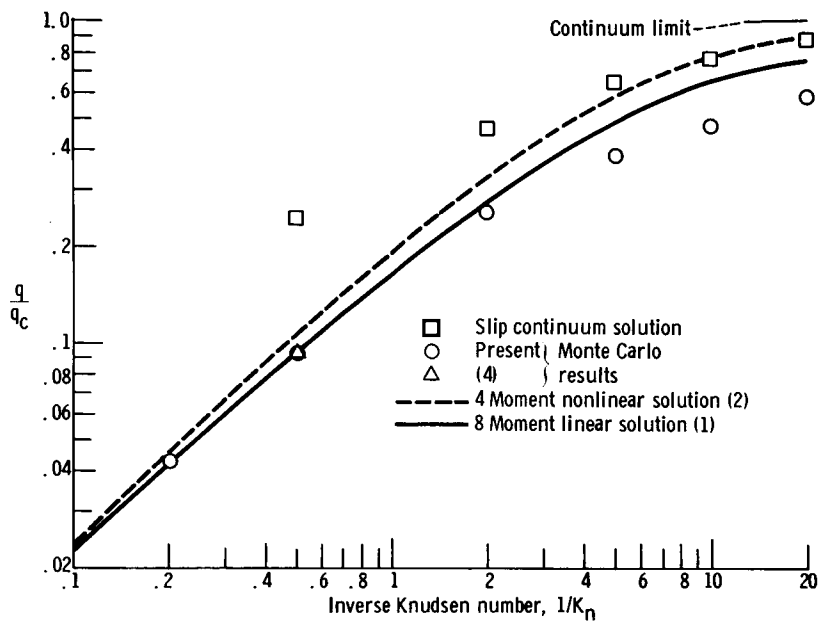


Figure 4. - Dimensionless heat flux for wall-temperature ratio of 0.25.