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MIDCOURSE GUIDANCE FOR RETURN FROM THE MOON TO A GEOGRAPHICALLY FIXED LANDING SITE

by John D. McLean and Luigi S. Cicolani

Ames Research Center
Moffett Field, Calif.

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SUMMARY

This report describes a midcourse guidance system for return from the moon to a safe landing at a specific geographical site. The design is based on linear perturbations about a reference trajectory. The vehicle's ability to maneuver within the atmosphere is used to reduce the midcourse corrective velocity requirements.

The system is compared statistically on the basis of a digital computer simulation with one that uses fixed-time-of-arrival method of guidance. The trajectory estimation method used for both systems is that described in NASA TR R-135. The transearth injection errors, observation errors, velocity correction mechanization errors, and velocity correction measurement errors are specified by statistical distributions considered to be realistic in terms of present day capabilities.

The fixed-landing-site guidance system requires substantially less corrective velocity than a fixed-time-of-arrival system, but it is more sensitive to errors in the final velocity correction than is the fixed-time-of-arrival system. This sensitivity can be effectively compensated for by proper scheduling of observations and velocity corrections or by the use of a rocket engine having different error characteristics for midcourse corrections.

INTRODUCTION

The objectives of midcourse guidance for the return phase of a lunar mission are defined for this study as:

1. To place the vacuum perigee of the return trajectory at the center of the entry corridor.

2. To land the vehicle at a specified geographical landing site.

3. To reach this landing site without crossrange maneuvering during entry.

The third objective arises because the crossrange maneuvering capability for Apollo-class entry vehicles is much more limited than the downrange maneuver capability (ref. 1).

It is recognized that because of errors, from sources to be discussed later, the above objectives cannot be met exactly. However the fixed-time-of-arrival guidance system used in earlier studies at Ames (ref. 2) and elsewhere does not explicitly impose these objectives. (This system guides the vehicle to the position on a precomputed reference trajectory corresponding to the time of reference perigee.) It is true that the fixed-time-of-arrival system satisfies the desired constraints within acceptable tolerances for the range of initial errors studied in reference 2. Vacuum perigee differs from the reference value (center of the entry corridor) by small amounts and the desired landing site can be obtained with modest cross-range and downrange maneuvering during the entry flight. On the other hand, there may be some advantage in a guidance system which recognizes these constraints explicitly.

This report presents the development of a method for guidance to a fixed landing site. The performance of this system for a sample return trajectory is then compared statistically with that of a fixed-time-of-arrival system on the basis of a digital simulation.

NOTATION

C covariance matrix of velocity corrections actually made

D declination

G 3×6 matrix from guidance equation

H matrix of partial derivatives of observed angles with respect to Cartesian coordinates

I unit matrix

k_σ multiplying factor for initial covariance matrices

M 3×3 matrix from guidance equation

P covariance matrix of errors in trajectory estimation

Q covariance matrix of observation errors

R magnitude of $\bar{R}$

$\bar{R}$ position vector

$\bar{r}$ vector of small deviations from reference position

rms root mean square

RA right ascension

S covariance matrix of errors in making velocity corrections

S_M	covariance matrix of errors in measuring velocity corrections
t	time
$\bar{u}$	unit vector
V	magnitude of $\bar{V}$
$\bar{V}$	velocity vector
$\bar{v}$	vector of small deviations from reference velocity
$\bar{v}_a$, $\bar{v}_d$	corrective velocity increments
W	covariance matrix of deviations between actual and reference trajectories
x	state vector (6×1 matrix of vehicle's position and velocity deviations from reference)
α	azimuth angle from north of orbital plane at landing site
γ	one-half the subtended angle of earth or moon
ϵ, θ	pointing errors in corrective velocity
ζ	4×1 matrix of deviations in arrival parameters
φ	entry range angle
Φ	transition matrix
ψ	matrix of partial derivatives of Cartesian position and velocity with respect to arrival parameters
ω	earth's angular velocity

Superscripts

T	transpose of a matrix
$(\bar{\ })$	3×1 matrix (or vector)
$(\dot{\ })$	derivative with respect to time

Subscripts

a	actual
d	desired

E atmospheric entry

i,m integers

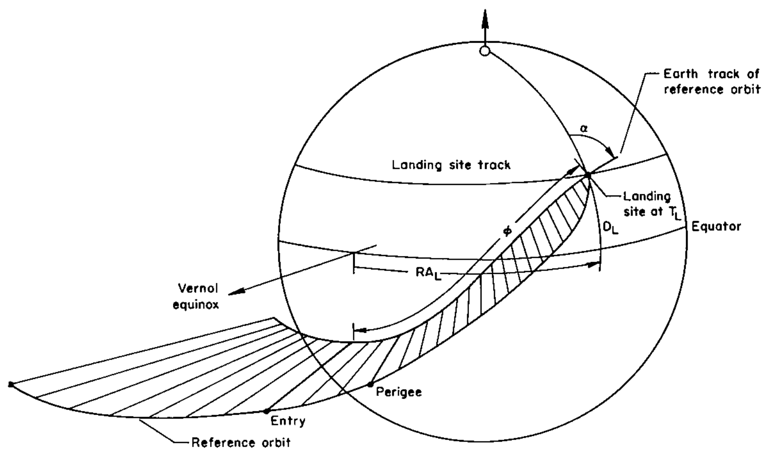
L landing site

p perigee

rms root-mean-square value

ANALYSIS

The development of the method of guidance to a fixed landing site follows the approach used in reference 3 for finding reference trajectories which return from the moon to a specified geographic site. Such a reference trajectory distorted for ease of illustration is pictured in sketch (a). The



Sketch (a)

point, P, represents either the transearth injection point or any point along the trajectory. The trajectory is divided into two phases: orbital and entry, entry being defined as a radius of 6500 km (about 400,000 ft altitude).

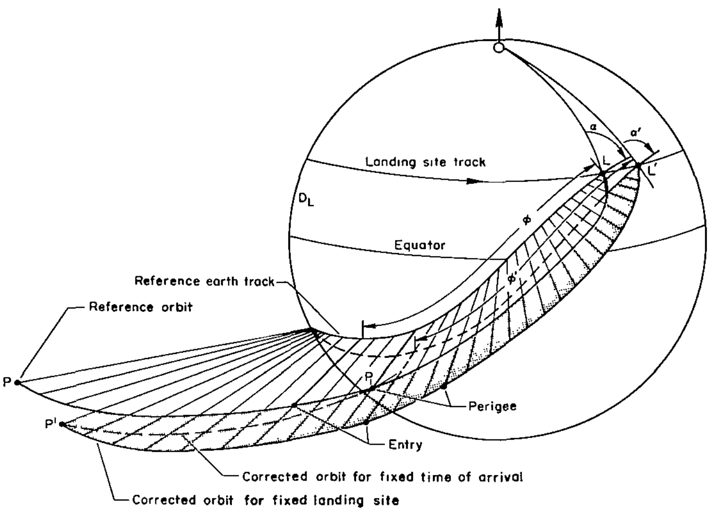
The point of view is taken that the trajectory travels backward in time from the landing site to point P. Hence, the orbital portion can be specified by the vacuum perigee position vector $\overline{R}_p$, perigee velocity vector $\overline{V}_p$, and time of perigee, t_p. If the trajectory is to terminate at the landing site, these quantities must satisfy certain functional relationships. These are:

$$\left.\begin{aligned}\overline{R}_p &= \overline{R}_p\ (\varphi,\alpha,t_L,D_L) \\ \overline{V}_p &= \overline{V}_p\ (\varphi,\alpha,t_L,D_L)\end{aligned}\right\} \qquad (1)$$

$$t_p = t_L - 0.00933\varphi - 0.00254 \qquad (2)$$

where φ is the angle from entry to landing; the azimuth angle, α, is measured from the meridian through the landing site to the orbital track; D_L is the declination of the landing site, and t_L is the time of landing. Equations (1) are geometrical relationships (derived in appendix A) which implicitly restrict the crossrange maneuver to zero. Equation (2) contains the dynamical relationship between the entry angle, φ, in radians and the time in mean solar days required to traverse it. This equation is of empirical origin (see ref. 3). Thus the complete trajectory may be specified in terms of the "arrival parameters" φ, α, t_L, V_p, R_p, and D_L.

Assume that a reference trajectory from the moon to the landing site has been defined. The point P in sketch (a) will now be regarded as lying on the reference trajectory somewhere in cislunar space. Because of injection errors, the vehicle arrives at point P' at time t instead of at P (see sketch (b)). Likewise, the velocity will generally be such that unless a velocity correction is made before entry the vehicle will not land at the desired site.



Sketch (b)

If a velocity correction is to be made at time t, then the function of the fixed-landing-site guidance system is to compute a new trajectory which leaves P' at time t and lands at the landing site at time t_L'. Because of the earth's rotation during the interval $(t_L' - t_L)$ the landing site will have moved to the point L'. The corrective velocity increment will be the difference between the vehicle's actual velocity and that of the corrected orbit.

For comparison the corrected trajectory which would be produced by the fixed-time-of-arrival method is shown as a dashed line in sketch (b). It intercepts the reference at P_1, corresponding to perigee on the reference orbit. Note that if the reference perigee point lies outside the orbital plane, a plane change will be required. On the other hand, if R_p and D_L are the only arrival parameters constrained, the fixed-landing-site guidance system will require a plane change only if the inclination of the incoming orbit is less than D_L (see ref. 3).

Finally, it should be pointed out that the basic methods used in developing the fixed-landing-site guidance system can be used for landing on any rotating celestial body, provided the "entry" phase of the trajectory can be suitably approximated. This statement is true even if the destination body has no atmosphere (here "entry" would correspond to powered descent), but it would probably be pointless to apply the method when the rotation rate of the body is extremely low.

Fixed-Landing-Site Guidance Equation

For the computation of a new trajectory from P' to the landing site, R_p and D_L are constrained to the reference values. This leaves four arrival parameters (φ, α, t_L, and V_p) which may be changed from the reference values so that the trajectory will originate at P' instead of P. Since only three quantities are required to specify the position of P', there is an infinite family of possible corrected orbits. A unique trajectory may be selected from this family either by constraining an additional arrival parameter or by satisfying some other condition. For this study we require magnitude of the corrective velocity to be a minimum.

The procedure for finding the reference trajectory is a complex iterative one, so the problem of finding the corrected trajectory has been linearized in terms of small perturbations around the reference trajectory. The resulting linearized guidance equation is derived in appendix A and may be written as

$$\bar{v}_d = (I - \bar{u}\bar{u}^T)(M\bar{r} - \bar{v}) \tag{3}$$

where $\bar{v}_d$ is the desired corrective velocity vector, $\bar{u}$ is a unit vector, M is a 3×3 matrix, and $\bar{r}$ and $\bar{v}$ are vectors of position and velocity deviation from the reference trajectory.

Comparison with fixed time of arrival.- The fixed-time-of-arrival guidance equation in reference 2 is given by

$$\bar{v}_d = \Phi_2^{-1}\Phi_1\bar{r}-\bar{v} \tag{4}$$

where Φ_2 and Φ_1 are submatrices of the state transition matrix which relates position and velocity deviations at the time of reference perigee to those at the time of the correction.

If the vector $(M\bar{r} - \bar{v})$ in equation (3) is defined to be the nonoptimum correction, then both the fixed-time-of-arrival and the nonoptimum fixed-landing-site guidance corrections can be expressed in the form $(A\bar{r} - \bar{v})$. It can be seen that the magnitudes of both corrections are dependent on the direction of $\bar{r}$, but the correction will be zero only when $A\bar{r}$ is equal to $\bar{v}$.

Now consider the multiplying factor $(I - \bar{u}\bar{u}^T)$ in the landing-site guidance law. Since

$$(I - \bar{u}\bar{u}^T)\bar{u} = 0$$

any component of the nonoptimum correction lying in the $\bar{u}$ direction will be eliminated by this multiplying factor. Because of this fact, it is to be expected that the magnitude of the optimum correction for fixed-landing-site guidance will be more sensitive to the directions of $\bar{r}$ and $\bar{v}$ than is the case for fixed-time-of-arrival guidance. Thus, from a statistical point of view, the mean-square velocity correction for fixed-landing-site guidance will be more sensitive to the directional properties of the distribution of deviations from the reference trajectory.

DIGITAL SIMULATION

The fixed-landing-site guidance equation was compared on a statistical basis with that for fixed time of arrival by means of a digital computer simulation of the two systems under identical conditions.

Four sources of error were considered:

(1) Transearth injection errors

(2) Errors in knowledge of the vehicle position and velocity (trajectory estimation errors)

(3) Errors in mechanization of the desired velocity correction

(4) Errors in measurement of the velocity correction actually made

The injection and estimation errors were assumed to be identical for each system at injection time and to be described statistically by a known covariance matrix. The particular covariance matrices used in the study will be discussed in the section on results.

During the course of the flight, celestial observations are made in order to reduce the trajectory estimation errors. The resulting data are processed by an optimal filter (see ref. 4) which provides the best estimate of the spacecraft's position and velocity together with the covariance matrix of errors in that estimate. The type of celestial observations and the measurement error model assumed are identical with that used for the standard case in reference 2, as is the error model for item (4) above. These error models are described briefly in appendix B. While it is necessary to separate the errors of items (2) and (4) in the mathematical analysis, they will be lumped together under "trajectory estimation errors" in the remainder of this discussion.

The error model for the velocity correction, the one used in reference 5, is described in detail in appendix B. It is shown in the appendix that for small velocity corrections (of the order 1.0 m/sec) the vector error in the correction is due almost entirely to rocket engine cutoff error. This error lies nearly in the direction of the commanded velocity correction and has an rms value of 0.1 m/sec. On the other hand, as the magnitude of the commanded correction increases, the error normal to the commanded direction increases until the components of the error along and normal to the commanded correction are nearly equal. (The rms value of both components in this case depends on the magnitude of the correction.)

Schedule of Observations and Velocity Corrections

The schedule of observations and velocity corrections used is considered, on the basis of other Ames studies, to be nearly optimum for the fixed-time-of-arrival system. This schedule is outlined briefly as follows:

(1) Starting a half hour after transearth injection, ten observations are made followed by the first velocity correction, all at half-hour intervals.

(2) Starting 24 hours after transearth injection, nine observations are made followed by the second velocity correction, all at one-hour intervals.

(3) Starting 34 hours after transearth injection, 15 observations are made followed by the third and final velocity correction, all at one-hour intervals.

This last correction occurs about 8.6 hours before entry.

Computation of Statistical Information

The purpose of the digital computer simulation was to determine how well, and at what cost in corrective velocity, each guidance system fulfills the requirement of returning the spacecraft from the moon on a trajectory from which a satisfactory landing can be made. The two guidance systems are then compared on the basis of: (1) the rms values of the velocity corrections at the three different correction times and of the total correction, and (2) the rms deviations from the reference values of the "landing parameters," perigee altitude, entry range angle, crossrange adjustment, and time of landing.

During the orbital phase of the trajectory, the statistics of the velocity corrections and deviations from the reference trajectory were computed using the linear methods described in appendix C. However, it was necessary to use the Monte Carlo method for the entry phase of the flight. The computations involved for the entry phase are outlined in appendix D.

The first velocity correction attempts, on the basis of the estimated state vector at the time of the correction, to eliminate the effects of the injection errors. It is only because the estimate of the vehicle's position and velocity and the mechanization of the velocity correction are imperfect, and hence sources of error, that the second and third corrections are needed. It is shown in appendix C how the portions of the second and third corrections, for these sources of error, can be separated.

Finally, in the case of fixed-landing-site guidance the deviations from the reference values of entry range angle and time of landing arise from two sources and can be separated accordingly (see appendix C). Some of these deviations, as in the case of fixed time of arrival, result from errors in trajectory estimation and velocity correction mechanization and will be referred to as errors. The remaining deviations result from changes required by the guidance law and will be referred to as entry range and landing time adjustments.

RESULTS AND DISCUSSION

Data from the eight cases outlined below provide a comparison between the fixed-landing-site and fixed-time-of-arrival systems with regard to corrective velocity requirements, terminal accuracies and the effects of the three major error sources (injection errors, observation errors, and velocity correction mechanization errors). These data are intended to indicate the areas which must be considered for a particular application of the guidance law.

Two different covariance matrices of transearth injection errors were used. The first is considered, on the basis of unpublished work, to be realistic for an Apollo-type mission. This distribution has mean-square values of about 0.6 km^2 and 3.5 m^2/sec^2 in position and velocity, respectively. However, the errors are cross correlated and the complete statistical description is given by the covariance matrix presented in table I. This distribution was used for most of the data and will be referred to as the "standard" distribution.

The other covariance matrix of injection errors is of interest mainly from a theoretical point of view. This matrix is diagonal for Cartesian coordinates, with mean-square values of 3 km^2 and 3 m^2/sec^2 in position and velocity. The mean-square errors are equal for each Cartesian direction, so the distribution will be referred to as spherical.

The two covariance matrices of injection errors were multiplied by a scalar, k_σ, in order to assess the influence of the magnitude (as opposed to direction) of the initial error distribution. It is shown in reference 2 and is confirmed here that the effects of the magnitude of the injection errors on the estimation errors are of minor significance.

The effects of velocity correction mechanization errors were evaluated by comparison of data resulting from the use of the standard error model described in appendix B with the data resulting when the error was assumed to be zero.

With this background the eight cases for which data are to be presented can be summarized as follows:

Case	K	Velocity correction errors	Injection errors
1a	1	None	Standard
2a	10	None	Standard
3a	100	None	Standard
1b	1	Standard	Standard
2b	10	Standard	Standard
3b	100	Standard	Standard
1c	1	Standard	Spherical
3c	100	Standard	Spherical

Corrective Velocity Requirements

The ratio of the rms error in estimating the vehicle's position for case 3a to that for 1a is plotted in figure 1 as a function of the number of observations. Note that just before the first velocity correction the estimation error for case 3a is only about 1/3 larger than that for case 1a even though the initial rms errors differed by an order of magnitude. This increase in estimation error with initial error becomes progressively smaller as more observations are made, and the results are substantially the same for velocity estimation errors. Thus, it can be seen that the errors remaining to be corrected after the first velocity correction are mainly the result of observation errors and, where applicable, of velocity correction mechanization errors. The rms values of the individual velocity corrections and the total corrective velocity are presented in table II. Likewise, the ratio of the total rms corrective velocity for fixed landing site to that for fixed time of arrival is presented. As expected on the basis of the data in figure 1, the first velocity correction increases linearly with $\sqrt{k_\sigma}$ while subsequent corrections show only a small effect from initial errors.

Total corrective velocity requirements.- The total rms corrective velocity for fixed-landing-site guidance ranges from 52 percent of that for fixed time of arrival in case 1a to 64 percent in case 3a. Including the velocity correction mechanization errors changes the result for the first case to 54 percent but has negligible effect on the third case because of the dominant influence of injection errors. These data indicate that the fixed-landing-site guidance system requires substantially less corrective velocity than fixed time of arrival for a realistic set of injection errors. However, for cases 1c and 3c the corrective velocity requirements for fixed-landing-site guidance increase to 70 and 89 percent of those for fixed time of arrival, indicating that the initial distribution of errors must be considered carefully in comparing the two systems.

The importance of the directional distribution of the deviations being corrected for is demonstrated by comparison of the data in table II for cases 1c and 3c with those for cases 1b and 3b. The spherical distribution of injection errors requires an initial correction for fixed-landing-site guidance which is about 95 percent of that for fixed time of arrival as compared to 69 percent for the standard distribution. The remaining corrections are in the same ratio as for the standard distribution, since the first correction eliminates most of the effects of initial errors.

Penalties from velocity correction mechanization errors.- Table III lists the rms values of the portions of the second and third velocity corrections which result from errors in making the previous corrections. The portion of the second correction due to errors in the first increases with initial errors because the magnitude dependent portion of the velocity correction mechanization error in the first correction becomes significant for cases 2b and 3b. On the other hand, since the second correction is small the rms error in the correction is essentially constant and the rms corrective velocity it requires at the time of the third correction is constant.

Fixed-landing-site guidance requires only about half as much corrective velocity at the time of the third correction because of errors in the second correction as does fixed time of arrival even though the rms mechanization errors are the same. However, from the data in table II (cases 1a, 2a, and 3a) it is seen that the ratio of the corrections caused by trajectory estimation errors at the time of the second correction was about 1/5. This difference is due largely to the fact that for small corrections, the mechanization error lies approximately along the commanded velocity correction. For fixed-landing-site guidance this means the error has no component along the unit vector $\overline{u}$ while one would not expect this to be the case with the estimation errors. (It should be emphasized here that the magnitude of the third correction is rather insignificant and it is used only to demonstrate a directional sensitivity which could be important in other applications of the fixed-landing-site guidance scheme.)

Terminal Deviations

Perigee altitude.- The rms deviations from reference perigee altitude are summarized in table IV. Only the errors in the final correction and in the observational data have an appreciable effect on deviations from the desired terminal conditions. Therefore the deviations for the first three cases and the last three, respectively, are nearly constant. In addition, the deviations for the two guidance systems are nearly equal in the absence of velocity correction mechanization errors.

The data in table IV also indicate that perigee altitude is much more sensitive to errors in the final velocity correction for fixed-landing-site guidance than for fixed time of arrival. It was pointed out earlier that for velocity corrections of the order of 1 m/sec, the mechanization error results almost entirely from the rocket engine cutoff error. Therefore, if the final velocity correction is small, the rms perigee altitude error will be determined almost entirely by the magnitude of the cutoff error and the time of the final correction. The perigee altitude error due to errors in the final correction is much larger for the landing-site-guidance system than for fixed-time-of-arrival guidance. For this reason the use of a rocket engine with a smaller cutoff error would improve the accuracy of the former scheme relative to fixed-time-of-arrival guidance. It can be shown analytically (on a two-body basis) that perigee altitude becomes less sensitive to velocity changes as the time to go (to perigee) decreases. Therefore, since the mechanization error is nearly constant, the perigee error could be reduced, at the expense of more corrective fuel, by delaying the final correction. This approach will be discussed in more detail later.

Crossrange.- The rms crossrange deviations for the two systems are presented in table V. (See appendix D.) The fixed-time-of-arrival system does not attempt to eliminate the necessity for crossrange adjustment during entry so that the deviations for this system increase with the initial errors. This increase is nonlinear as is to be expected since the computation of the crossrange deviation from the state vector at the time of entry is nonlinear.

The fixed-landing-site guidance system does constrain crossrange adjustment, and the rms crossrange deviation is determined mainly by errors in the final velocity correction and trajectory estimation errors at the time of this correction. The crossrange deviations for fixed-landing-site guidance are much more sensitive to errors in the final velocity correction than those for fixed time of arrival. However, the total deviation for landing-site guidance is so small compared to that for fixed time of arrival that this increased sensitivity to the velocity correction errors is not significant.

Downrange.- The rms entry-range deviations for the two systems and the rms entry-range adjustment for fixed-landing-site guidance are presented in table VI. As in the case of perigee altitude and crossrange deviations, the entry-range deviation for the fixed-time-of-arrival system is much less sensitive to errors in the final velocity correction. The entry-range adjustment for fixed-landing-site guidance results from adjusting the entry-range angle and azimuth angle to compensate for deviations from the reference position. Hence, the rms value of the adjustment increases linearly with the square root of k_σ. The total deviation in table VI is the root sum square of the entry-range deviation and the downrange adjustment. Under the assumption that the two components are independent random variables this quantity represents the total rms change from the reference entry range required of the vehicle. It can be seen by comparison of cases 1a, 2a, and 3a with cases 1b, 2b, and 3b that the contribution of the entry-range deviation, particularly the part caused by velocity correction errors, is relatively unimportant.

Landing time.- The rms landing-time deviations for the two systems and the rms landing-time adjustment for fixed-landing-site guidance are given in table VII. Like the entry-range deviation, the landing-time deviation is due principally to errors in the final velocity correction and in trajectory estimation at the time of that correction. In this case, in contrast to the other landing parameters, velocity correction errors have negligible effect on the fixed-landing-site guidance system. In addition, for fixed-landing-site guidance the landing-time deviations are negligible compared to the landing time adjustment. Since the landing-time adjustment and the entry-range adjustment are closely related, the landing-time adjustment also increases approximately linearly with the square root of k_σ.

Effect of changing observation and velocity correction schedule.- The terminal accuracy of the fixed-landing-site guidance system with the standard schedule of observations and velocity corrections is adequate; however, it was desired to compare the total corrective velocity requirements needed when two systems provide approximately the same terminal accuracy. As was pointed out earlier, the perigee altitude error can be reduced by delaying the final velocity correction, but the magnitude of the final correction increases rapidly as the delay is increased. It was found empirically that the perigee altitude error can be reduced by making additional observations during the delay time. A few trials with this procedure resulted in the following revised schedule: Starting one hour after the last observation for the standard schedule, eight observations were made of the earth followed by the third velocity correction, all at half-hour intervals. This revision delayed the final correction until about 4.6 hours before entry.

When $k_\sigma = 1$ was used with the new schedule, the terminal errors of the fixed-landing-site system were comparable to those of the fixed-time-of-arrival system for case 1b (e.g., the rms perigee altitude error was reduced from 2.11 to 1.57 km). This improved accuracy required an increase from 0.26 to 0.53 m/sec in the rms value of the final velocity correction. The total rms corrective velocity for fixed-landing-site guidance ranged from about 62 percent of fixed-time-of-arrival guidance with the standard schedule in case 1b to 66 percent in case 3b. Thus, even with the increase in the final correction, the fixed-landing-site guidance system still saves a substantial portion of the midcourse fuel.

Computer requirements.- The fixed-time-of-arrival system used the same computer program as was used for the data in reference 2. A simple modification of this program was required to obtain data for the fixed-landing-site guidance system. The increase in program size was about equivalent to the amount required to compute the multiplying factor $(I - \bar{u}\bar{u}^T)$ and multiply the matrix $(M \quad -I)$ by it. No noticeable change in computation time resulted. The initial conditions required for the fixed-landing-site guidance system are the same as those for fixed-time-of-arrival system with the addition of the matrix ψ_o discussed in appendix A.

CONCLUDING REMARKS

The fixed-landing-site guidance system has been shown to require only a little over half as much corrective velocity as the fixed-time-of-arrival system in the presence of injection errors considered to be realistic. This ratio increases to about 2/3 for an order of magnitude increase in injection errors, but because the total corrective velocity also increases, the actual saving in fuel is greater.

The fixed-landing-site system is more sensitive to errors in the final velocity correction than the fixed-time-of-arrival system, but this increased sensitivity will not significantly affect the objective of making the desired landing. It is possible to improve the terminal accuracy to that of the fixed-time-of-arrival system by revision of the observation and velocity correction schedule. This improvement increases the corrective velocity required to about 2/3 of that of the fixed-time-of-arrival system. An alternate method would be to use a rocket engine for the midcourse correction having a smaller cutoff error than the one assumed in this study. This approach would not degrade the total corrective velocity requirement, but would reduce the error caused by errors in making the final velocity correction.

The errors assumed in the study are considered to be realistic, but it must be noted that the relative rms corrective velocity required by the two systems is strongly dependent on the distribution of errors. The relative merits of fixed-landing-site guidance can only be determined on the basis of the type of observed data and the statistical distribution of injection errors, observation errors, and the velocity correction errors.

The computations required are of about the same complexity for the two systems. However, the computation of the input data is slightly more complicated for fixed-landing-site guidance.

The example presented here was an application to return to earth from the moon, but the basic method can be applied to direct landing on other rotating bodies.

Ames Research Center
National Aeronautics and Space Administration
Moffett Field, Calif., Nov. 3, 1965

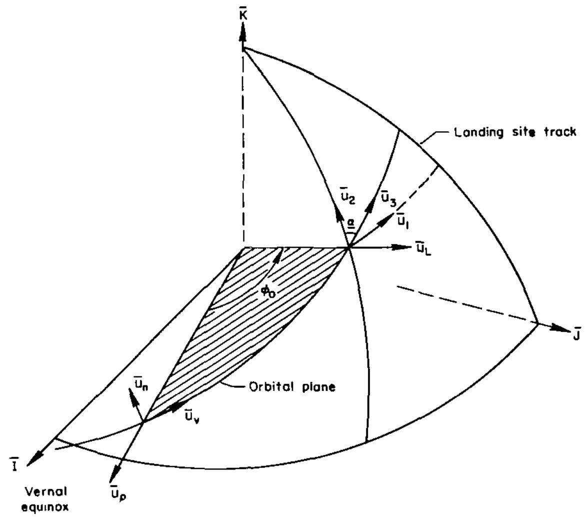
APPENDIX A

DERIVATION OF THE FIXED-LANDING-SITE GUIDANCE EQUATION

The first step in obtaining the linearized fixed-landing-site guidance equation is the derivation of the relationships given in generalized form in equations (1). For this purpose the angle from vacuum perigee to landing is defined as φ_p. It is assumed that the angle from entry to vacuum perigee is fixed at 12^o ($\varphi_p = \varphi - 12^o$) and that it is traversed in 122 seconds. Thus, from equation (2) the time of vacuum perigee is

$$t_p = t_L - 0.0093\varphi_p - 0.00240 \tag{A1}$$

We now define three coordinate frames, illustrated in sketch (c) by three sets of unit vectors: (1) $\overline{u}_p$ along the radius vector to vacuum perigee, $\overline{u}_v$ along the perigee velocity vector, and $\overline{u}_h$ normal to the orbital plane; (2) $\overline{u}_L$ along the radius vector to the location of the landing site at the time of landing, $\overline{u}_1$ eastward from the landing site, and $\overline{u}_2$ northward from the landing site (an auxiliary unit vector to this set is $\overline{u}_3$ which is normal to $\overline{u}_L$ and in the orbit plane); (3) an inertial Cartesian set with $\overline{I}$ toward the vernal equinox, $\overline{K}$ northward along the earth's axis, and $\overline{J}$ completing a right-handed orthogonal system.



Sketch (c)

The unit vectors $\overline{u}_p$, $\overline{u}_L$, and $\overline{u}_3$ lie in the orbital plane and it can be seen from the sketch that $\overline{u}_p$ can be resolved into components $\cos \varphi_p$ and $\sin \varphi_p$ along $\overline{u}_L$ and $\overline{u}_3$, respectively. Now $\overline{u}_1$, $\overline{u}_2$, and $\overline{u}_3$ lie in a plane normal to $\overline{u}_L$ (tangent to the earth's surface) so that the component of $\overline{u}_p$ along $\overline{u}_3$ may be further resolved into components along $\overline{u}_1$ and $\overline{u}_2$ to give

$$\overline{u}_p = \overline{u}_L \cos \varphi_p + \overline{u}_1 \sin \varphi_p \sin \alpha + \overline{u}_2 \cos \varphi_p \cos \alpha \tag{A2}$$

Similarly, it can be shown that

$$\overline{u}_v = -\overline{u}_L \sin \varphi_p + \overline{u}_1 \cos \varphi_p \sin \alpha + \overline{u}_2 \cos \varphi_p \cos \alpha \tag{A3}$$

From sketch (c) it can be seen that the $\overline{u}_L$, $\overline{u}_1$, $\overline{u}_2$ system is related to the Cartesian system by the transformation.

$$\begin{bmatrix} \bar{u}_L \\ \bar{u}_1 \\ \bar{u}_2 \end{bmatrix} = \begin{bmatrix} \cos D_L \cos RA_L & \cos D_L \sin RA_L & \sin D_L \\ -\sin RA_L & \cos RA_L & 0 \\ -\sin D_L \cos RA_L & -\sin D_L \sin RA_L & \cos D_L \end{bmatrix} \begin{bmatrix} \bar{I} \\ \bar{J} \\ \bar{K} \end{bmatrix} \tag{A4}$$

Equations (A2), (A3), and (A4) can be combined to give

$$\begin{aligned} \bar{u}_p = {} & \bar{I}(\cos D_L \cos RA_L \cos \varphi_p - \sin RA_L \sin \varphi_p \sin \alpha \\ & - \sin D_L \cos RA_L \sin \varphi_p \cos \alpha) + \bar{J}(\cos D_L \sin RA_L \cos \varphi_p \\ & + \cos RA_L \sin \varphi_p \sin \alpha - \sin D_L \sin RA_L \sin \varphi_p \cos \alpha) \\ & + \bar{K}(\sin D_L \cos \varphi_p + \cos D_L \sin \varphi_p) \end{aligned} \tag{A5}$$

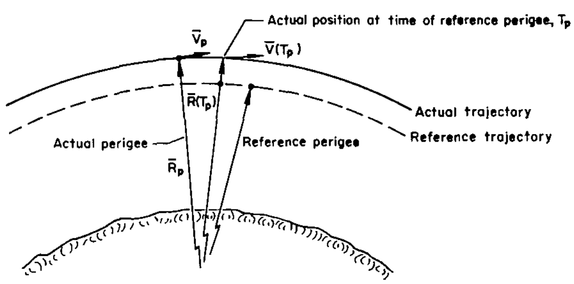
$$\begin{aligned} \bar{u}_V = {} & \bar{I}(-\cos D_L \cos RA_L \sin \varphi_p - \sin RA_L \cos \varphi_p \sin \alpha \\ & - \sin D_L \cos RA_L \cos \varphi_p \cos \alpha) + \bar{J}(-\cos D_L \sin RA_L \sin \varphi_p \\ & + \cos RA_L \cos \varphi_p \sin \alpha - \sin D_L \sin RA_L \cos \varphi_p \cos \alpha) \\ & + \bar{K}(-\sin D_L \sin \varphi_p + \cos D_L \cos \varphi_p \cos \alpha) \end{aligned} \tag{A6}$$

Rewriting equations (1) we have

$$\left.\begin{aligned} \bar{R}_p &= R_p \bar{u}_p \\ \bar{V}_p &= V_p \bar{u}_V \end{aligned}\right\} \tag{A7}$$

which together with equations (A5) and (A6) relate perigee position and velocity to the arrival parameters. Note that the assumption that $\bar{u}_p$ and $\bar{u}_L$ both lie in the orbital plane implicitly restricts the incoming orbit to one that requires no crossrange adjustment.

We now wish to linearize the problem by considering first-order perturbations about a reference trajectory subject to the constraint that the perturbed trajectories have the same values of R_p and D_L as the reference. Differentiating equations (A7) will result in a matrix of the first partial derivatives of perigee position and velocity with respect to the arrival parameters. However, we wish to obtain first partial derivatives of position and velocity at the time of reference perigee. The distinction between these quantities is illustrated with the aid of sketch (d).



Sketch (d)

Here $\overline{R}_p$ and $\overline{V}_p$ are position and velocity vectors of perigee on the perturbed trajectory which occurs at some time t_a. On the other hand, $\overline{R}(t_p)$ and $\overline{V}(t_p)$ are the position and velocity vectors on the perturbed trajectory at the time, t_p, when reference perigee occurs. If t_a is nearly equal to t_p, then we can write

$$\left.\begin{aligned}\overline{R}(t_p) &= \overline{R}_p - \overline{V}_p(t_a - t_p) = \overline{R}_p - \overline{V}_p(\delta t_1 - 0.00933\,\delta\varphi)\\ \overline{V}(t_p) &= \overline{V}_p - \dot{\overline{V}}_p(t_a - t_p) = \overline{V}_p - \dot{\overline{V}}_p(\delta t_1 - 0.00933\,\delta\varphi)\end{aligned}\right\} \tag{A8}$$

Define ψ_p as the matrix of first partial derivatives of position and velocity at the time of reference perigee with respect to the arrival parameters. We now compute ψ_p by differentiating equations (A8), but we must recognize that the terms involving $(t_a - t_p)$ are already in the form of first-order perturbations. Therefore,

$$\psi_p = \begin{bmatrix} \dfrac{\partial\overline{R}_p}{\partial\varphi_o} & \dfrac{\partial\overline{R}_p}{\partial\alpha} & \dfrac{\partial\overline{R}_p}{\partial t_L} & \dfrac{\partial\overline{R}_p}{\partial V_p} \\[2ex] \dfrac{\partial\overline{V}_p}{\partial\varphi_o} & \dfrac{\partial\overline{V}_p}{\partial\alpha} & \dfrac{\partial\overline{V}_p}{\partial t_L} & \dfrac{\partial\overline{V}_p}{\partial V_p} \end{bmatrix} + \begin{bmatrix} 0.00933\ \overline{V}_p & 0 & -\overline{V}_p & 0 \\[2ex] 0.00933\ \dot{\overline{V}}_p & 0 & -\dot{\overline{V}}_p & 0 \end{bmatrix} \tag{A9}$$

The computation of the partial derivatives in equation (A9) is simplified by expressing $\overline{R}_p$ and $\overline{V}_p$ as in equations (1).

$$\overline{R}_p = R_p\overline{u}_p$$

$$\overline{V}_p = V_p\overline{u}_V$$

so that

$$\left.\begin{aligned}\frac{\partial\overline{R}_p}{\partial\varphi_p} &= R_p\,\frac{\partial\overline{u}_p}{\partial\varphi_p}\\ \frac{\partial\overline{V}_p}{\partial\varphi_p} &= V_p\,\frac{\partial\overline{u}_V}{\partial\varphi_p}\end{aligned}\right\} \tag{A10}$$

Since differentiation with respect to φ_p is equivalent to a small rotation about $\overline{u}_H$, from sketch (c)

$$\left.\begin{aligned}\frac{\partial\overline{R}_p}{\partial\varphi_o} &= -R_p\overline{u}_V\\ \frac{\partial V_p}{\partial\varphi_o} &= V_p\overline{u}_p\end{aligned}\right\} \tag{A11}$$

A change in the angle α represents a rotation of the orbital plane about $\bar{u}_L$ so that

$$\left.\begin{aligned} \frac{\partial \bar{R}_p}{\partial \alpha} &= R_p \frac{\partial \bar{u}_p}{\partial \alpha} = \bar{R}_p \times \bar{u}_L \\ \frac{\partial \bar{v}_p}{\partial \alpha} &= V_p \frac{\partial \bar{u}_V}{\partial \alpha} = \bar{V}_p \times \bar{u}_L \end{aligned}\right\} \tag{A12}$$

A change in the time of landing rotates the orbital plane about the earth's axis. If ω is the angular velocity of the earth then

$$\left.\begin{aligned} \frac{\partial \bar{R}_p}{\partial t_L} &= R_p \frac{\partial \bar{u}_p}{\partial t_L} = R_p(\omega\bar{K} \times \bar{u}_p) \\ \frac{\partial \bar{v}_p}{\partial t_L} &= v_p \frac{\partial \bar{u}_V}{\partial t_L} = v_p(\omega\bar{K} \times \bar{u}_V) \end{aligned}\right\} \tag{A13}$$

Finally,

$$\left.\begin{aligned} \frac{\partial \bar{R}_p}{\partial v_p} &= 0 \\ \frac{\partial \bar{V}_p}{\partial V_p} &= \bar{u}_V \end{aligned}\right\} \tag{A14}$$

Substituting equations (A11) through (A14) in equation (A3) with the unit vectors in terms of $\bar{R}_p$ and $\bar{v}_p$ gives

$$\psi_p = \begin{bmatrix} \left(\frac{-R_p}{V_p} + 0.0093\right)\bar{V}_p & \bar{R}_p \times \bar{u}_V & (\bar{K} \times \bar{R}_p)\text{-}\bar{V}_p & 0 \\ \frac{V_p\bar{R}_p}{R_p} + 0.0093\,\dot{\bar{V}}_p & \bar{V}_p \times \bar{u}_L & (\bar{K} \times \bar{v}_p)\text{-}\dot{\bar{V}}_p & \frac{\bar{v}_p}{v_p} \end{bmatrix}$$

This expression can be verified by straightforward differentiation of equations (A5) and (A6) followed by considerable algebraic manipulation.

We will now use ψ_p to derive the linearized fixed-landing-site guidance equation. For this purpose we define $\bar{r}$ and $\bar{v}$ as small vector deviations from the reference position and velocity and x as the 6×1 state vector given by

$$x = \begin{bmatrix} \bar{r} \\ \bar{v} \end{bmatrix} \tag{A15}$$

The state vectors at different times are related by the state transition matrix Φ as follows:

$$x(t_2) = \Phi(t_2,t_1)x(t_1) \tag{A16}$$

If t_2 is the present time, t, and t_1 the time of reference perigee, t_p, then equation (A16) becomes

$$x(t) = \Phi(t,t_p)x(t_p) \tag{A17}$$

Let ζ be defined as a 4×1 matrix of deviations from the reference arrival parameters. Then

$$x(t_p) = \psi_p\zeta$$

substituting for $x(t_p)$ in equation (A17)

$$x(t) = \Phi(t,t_p)\psi_p\zeta \tag{A18}$$

or

$$x = \psi\zeta \tag{A19}$$

where

$$\psi = \Phi(t,t_p)\psi_p \tag{A20}$$

Equation (A19) may be written in partitioned form as

$$\begin{bmatrix} \bar{r}' \\ \bar{v}' \end{bmatrix} = \begin{bmatrix} \psi_1 & \bar{\rho}_1 \\ \psi_2 & \bar{\rho}_2 \end{bmatrix} \begin{bmatrix} \zeta_1 \\ \zeta_2 \\ \zeta_3 \\ \zeta_4 \end{bmatrix} \tag{A21}$$

Here ψ_1 and ψ_2 are 3×3 matrices, $\bar{\rho}_1$ and $\bar{\rho}_2$ are 3×1 matrices and the ζ_1 indicate deviations from the reference values of the arrival parameters. The primes have been added to $\bar{r}$ and $\bar{v}$ to indicate that they are the deviations from the reference position and velocity which would be produced by changes of ζ_1, ζ_2, ζ_3, ζ_4 in the arrival parameters.

Expansion of equation (A21) gives

$$\bar{r}' = \psi_1 \begin{bmatrix} \zeta_1 \\ \zeta_2 \\ \zeta_3 \end{bmatrix} + \bar{\rho}_1\zeta_4 \tag{A22}$$

$$\bar{v}' = \psi_2 \begin{bmatrix} \zeta_1 \\ \zeta_2 \\ \zeta_3 \end{bmatrix} + \bar{\rho}_2\zeta_4 \tag{A23}$$

If $\bar{r}'$ is taken to be the actual deviation of the vehicle's position from the reference trajectory, the changes in the first three arrival parameters necessary to produce a trajectory from the present position to the landing site are

$$\begin{bmatrix} \zeta_1 \\ \zeta_2 \\ \zeta_3 \end{bmatrix} = \psi_1^{-1}\,\bar{r} - \psi_1^{-1}\,\bar{\rho}_1\zeta_4 \tag{A24}$$

The velocity deviation of this trajectory from that of the reference trajectory is found by substituting equation (A24) into (A23)

$$\bar{v}' = \psi_2\psi_1^{-1}\,\bar{r} - (\psi_2\psi_1^{-1}\,\bar{\rho}_1 - \bar{\rho}_2)\zeta_4$$

The total difference between the desired velocity and the present velocity is

$$\bar{v}_c = \bar{v}' - \bar{v}$$

or if

$$M = \psi_2\psi_1^{-1}$$

and

$$\bar{u} = \frac{-(\psi_2\psi_1^{-1}\,\bar{\rho}_1 - \bar{\rho}_2)}{|\psi_2\psi_1^{-1}\,\bar{\rho}_1 - \bar{\rho}_2|} = \frac{-(M\bar{\rho}_1 - \bar{\rho}_2)}{k}$$

then

$$\bar{v}_c = M\bar{r} + k\bar{u}\zeta_4 - \bar{v} \tag{A25}$$

It can be seen from equation (A25) that $\bar{v}_c$ is not uniquely determined but is a function of ζ_4. It is desired to choose ζ_4 so as to minimize $|\bar{v}_c|$, and this optimum is $\bar{v}_d$ in equation (3)

$$|\bar{v}_c|^2 = (M\bar{r} + K\bar{u}\zeta_4 - \bar{v})^T(M\bar{r} + k\bar{u}\zeta_4 - \bar{v})$$

$$\frac{d(|v_c|^2)}{d\zeta_4} = 2k^2\zeta_4 + 2k\bar{u}^T(M\bar{r}) - k\bar{u}^T\bar{v}$$

If this derivative is set equal to zero, the optimum ζ_4 is found to be

$$\zeta_4 = \frac{-(\bar{u}^T M\bar{r}) - \bar{u}^T\bar{v}}{k} \tag{A26}$$

Substituting in equation (A5) gives

$$\bar{v}_d = M\bar{r} - \overline{uu}^T M\bar{r} + \overline{uu}^T\bar{v} - \bar{v}$$

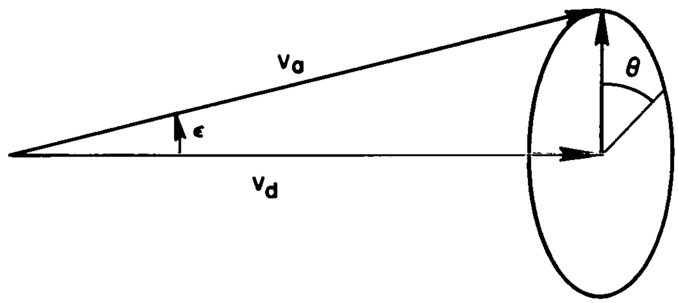
or

$$\bar{v}_d = (I - \overline{uu}^T)[M \quad (-I)]x$$

APPENDIX B

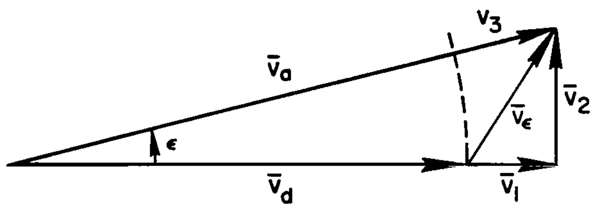
ERROR MODEL FOR THE DIGITAL COMPUTER SIMULATION

The error model assumed for mechanizing the velocity correction can be explained briefly with the aid of sketch (e) in which $\overline{v}_d$ is the desired velocity correction while $\overline{v}_a$ is the correction actually made. The directional displacement between $\overline{v}_d$ and $\overline{v}_a$ can be specified by the angles ϵ and θ. The angle, ϵ, between the two vectors may be either positive or negative and has Gaussian distribution, zero mean, and 1° standard deviation. The angle, θ, gives the orientation of the plane of the two vectors with respect to some fixed reference and has uniform distribution between zero and π. There is also an error in $|\overline{v}_a|$ which has two components. There is a cutoff error having Gaussian distribution, zero mean, and a standard deviation of 0.1 m/sec. The other component due to thrust level also has Gaussian distribution with zero mean, but its standard deviation is 1 percent of $|\overline{v}_d|$.



Sketch (e)

The relative significance of the different components of the velocity correction error can be better explained with the aid of sketch (f) showing the plane of $\overline{v}_d$ and $\overline{v}_a$. The error, $\overline{v}_\epsilon$, in the velocity correction can be resolved into two components, v_1, in the direction of $\overline{v}_d$ and v_2, normal to $\overline{v}_d$. Since ϵ is very small a good approximation to v_2 is given by



Sketch (f)

$$v_2 \approx v_d \epsilon$$

Likewise, v_1 is approximately equal to the error, v_3, in v_a so that

$$v_1 \approx \sqrt{0.01 + (0.01\ v_d)^2}$$

Since the rms value of ϵ is 1° (about 1/60 radian), it is seen that for v_d of order 1 m/sec, v_2 is nearly an order of magnitude smaller than v_1, and the rms value of v_1 consists almost entirely of the cutoff error. On the other hand, if $\overline{v}_d$ is of order 10 m/sec, the rms value of v_2 is about 0.17 m/sec while that of v_1 is about 0.14 m/sec.

The method of calculating the covariance matrix, S, of velocity correction mechanization errors from the error model just described is given in reference 5.

The errors in measuring the individual Cartesian components of velocity correction were assumed to be random and independent with Gaussian distribution, zero mean, and standard deviation of these measurements is

$$S_M = \sigma^2 I$$

As mentioned in the text it was assumed that certain celestial observations (angles) are made for the purpose of estimating trajectory. These measured angles consist of the subtense angle of either earth or moon together with the azimuth and elevation angles of center of the same body with respect to some inertial reference.

The errors in the observations were assumed to be random with Gaussian distribution and zero mean. Also, the errors in different angles and measurements made at different times were assumed to be uncorrelated. The standard deviation was the same for the three angles measured and is assumed to be given in seconds of arc by

$$\sigma = \sqrt{100 + (0.001\gamma)^2}$$

where γ is half the subtended angle of the body observed. These statistical characteristics result in the covariance matrix, Q, of observation errors being a diagonal 3×3 matrix

$$Q = \sigma^2 I$$

APPENDIX C

COMPUTATION OF STATISTICAL INFORMATION FOR ORBITAL PHASE OF FLIGHT

During the orbital phase of the flight the statistical performance of both guidance systems was evaluated by means of linear theory. This evaluation required the knowledge of three covariance matrices:

(1) The covariance matrix, W, of position and velocity deviations from the reference trajectory,

(2) The covariance matrix, P, of errors in estimating those quantities,

(3) The covariance matrix, C, of velocity correction actually made.

This third matrix is separated into two parts so that the penalties due to velocity correction errors may be evaluated. A supplementary covariance matrix is needed, for reasons to be discussed later, to account for adjustments made in entry range and landing time by the landing-site guidance system.

The covariance matrices S, S_M, and Q of velocity correction mechanization errors, velocity correction measurement errors, and errors in celestial observations are used in calculating W, P, and C. The sources of these matrices are given in appendix B.

Initially, P and W are both set equal to the covariance matrix of injection errors, and they must be updated as the flight proceeds. This updating which takes place either as a result of the passage of time or because of operations (observations, velocity corrections, or termination of the flight) performed by the system proceeds as follows:

Let Φ be the transition matrix giving deviations from the reference position and velocity at time t due to deviations at some earlier time. Then between operations, P and W are updated by

$$\left.\begin{aligned} P &= \Phi P_o \Phi^T \\ W &= \Phi W_o \Phi^T \end{aligned}\right\} \tag{C1}$$

If H is the matrix of partial derivatives of the observed angles with respect to the Cartesian components of position and velocity, then after an observation

$$P = P_o - P_o H^T [H P_o H^T + Q]^{-1} H P_o \tag{C2}$$

while W remains unchanged. The derivation of this relationship is quite complex and will not be presented here. (See ref. 4, eqs. (13) and (14).)

The only estimation errors incurred in making a velocity correction are those in measuring the correction actually made. Hence, after the correction

$$P = P_o + S_M \tag{C3}$$

It is shown in reference 2 that straightforward computation of $E(xx^T)$ after a velocity correction gives

$$W = \begin{bmatrix} I & 0 \\ G_1 & (G_2 + I) \end{bmatrix} (W_o - P_o) \begin{bmatrix} I & 0 \\ G_1 & (G_2 + I) \end{bmatrix}^T + P_o + S \tag{C4}$$

where I is a 3×3 unit matrix, 0 represents a 3×3 null matrix, and

$$G = (G_1 \quad G_2)$$

The significance of the different terms on the right side of equation (C4) can be explained by the use of equations (C1) to update W to the time of reference perigee. Then

$$W(t_p) = W_1 + W_2$$

where by definition

$$W_1 = \Phi \begin{bmatrix} I & 0 \\ G_1 & (G_2 + I) \end{bmatrix} (W_o - P_o) \begin{bmatrix} I & 0 \\ G_1 & (G_2 + I) \end{bmatrix}^T \Phi^T$$

and

$$W_2 = \Phi(P_o + S)\Phi^T$$

It is also shown in reference 2 that (W-P) is the covariance matrix of the estimated deviations from reference position and velocity. Since the velocity correction would cause the guidance requirements to be satisfied if the estimate were correct, W_1 represents a distribution of deviations from the reference trajectory for which the guidance requirements are still satisfied. On the other hand W_2 represents deviations, from members of this family of satisfactory trajectories, which are caused by errors in trajectory estimation at the time of the velocity correction and by errors in making the correction.

From this analysis it appears that the first term on the right side of equation (C4) could be eliminated, with no loss of significant information, to give

$$W = P_o + S \tag{C5}$$

However, the information contained in the term eliminated from equation (C4) is necessary for computing the statistics of the landing parameters. The above reasoning can be confirmed analytically for the case of fixed-time-of-arrival guidance as follows:

For the fixed-time-of-arrival guidance equation stated in equation (2)

$$\left.\begin{aligned} G_1 &= -\Phi_2^{-1}\Phi_1 \\ G_2 &= -I \end{aligned}\right\} \tag{C6}$$

Here I is the unit matrix and Φ_1 and Φ_2 arise from partitioning Φ as follows:

$$\Phi = \begin{bmatrix} \Phi_1 & \Phi_2 \\ \Phi_3 & \Phi_4 \end{bmatrix} \tag{C7}$$

Substitution from equations (C6) and (C7) shows that

$$\Phi \begin{bmatrix} I & 0 \\ G_1 & (G_2 + I) \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ (\Phi_3 - \Phi_4\Phi_2^{-1}\Phi_1) & 0 \end{bmatrix} \tag{C8}$$

It is shown in reference 2 that

$$\Phi^{-1} = \begin{bmatrix} \Phi_4^T & -\Phi_2^T \\ -\Phi_3^T & \Phi_1^T \end{bmatrix} \tag{C9}$$

and this relationship can be used to show that

$$(\Phi_3 - \Phi_4\Phi_2^{-1}\Phi_1) = (-\Phi_2^{-1})^T \tag{C10}$$

Therefore, for the fixed-time-of-arrival guidance equation

$$W_1 = \begin{bmatrix} 0 & 0 \\ (\Phi_2^{-1})^T & 0 \end{bmatrix} (W_0 - P_0) \begin{bmatrix} 0 & \Phi_2^{-1} \\ 0 & 0 \end{bmatrix} \tag{C11}$$

Hence, only the lower right 3×3, the velocity components, of W_1 are not zero. These are the deviations in velocity at the time of reference perigee which must be accepted in order to have zero position deviation. Since the landing parameters are functions of perigee velocity as well as position, it is necessary to retain the information contained in W_1. This information was retained in the case of fixed-time-of-arrival system by using equations (C1) to update W. (Since, to a first-order approximation, Φ_2 is proportional to the time over which it is computed, the effects of W_1 on the landing parameters will be small.)

In the case of fixed-landing-site guidance it would be very difficult to reduce W_1 mathematically to a simple form. However, as in the case of fixed-time-of-arrival system, W_1 will make no contribution to the deviations which the guidance law requires to be zero. For this system these are deviations in perigee altitude, declination of the landing site, and crossrange adjustment. In this case the information lost by the elimination of W_1 will be the statistical distribution of the adjustments in entry-range angle, landing time, and perigee velocity made by the guidance system in order to achieve a satisfactory trajectory. It was desired to know the first two of these three quantities which are defined in the text as downrange adjustment and landing-time adjustment, respectively, but preliminary results using equations (C1) with the fixed-landing-site guidance system showed that round-off errors cause unacceptable inaccuracy for large initial values of W. For this reason equation (C5) was used with this system and the statistics of the downrange adjustment and landing-time adjustment were computed separately as described in the following paragraphs.

The generalized inverse of the matrix ψ in equation (A19) (derived in appendix E) can be used to compute these statistics. This inverse will be written ψ^* so that

$$\zeta = \psi^* x \tag{C12}$$

where ζ is the result of applying the guidance equation.

At the time of any given velocity correction

$$\left.\begin{aligned} E[\zeta\zeta^T] &= E[\psi^* x x^T (\psi^*)^T] \\ E[\zeta\zeta^T] &= \psi^* W (\psi^*)^T \end{aligned}\right\} \tag{C13}$$

Note that this is the covariance matrix of ζ due to a specific velocity correction, and W is evaluated immediately before that correction. Under the assumption that the various velocity corrections are uncorrelated, the covariance matrix for the total change in arrival parameters can be found by summing the individual covariance matrices.

The mean-square velocity correction for the nth correction time is given by the trace of the covariance matrix.

$$C_n = G(W_n - P_n)G^T + S_n \tag{C14}$$

In equation (C14) W_n and P_n are evaluated just before the correction. If W_n is obtained by updating W from the time of the previous correction then

$$C_n = G(\Phi W_{n-1}\Phi^T - P_n)G^T + S_n \tag{C15}$$

Substitution for W_{n-1} from equation (C4) yields

$$C_n = G\Phi(W_{n-1} - P_{n-1})\Phi^T G^T + G\Phi P_{n-1}\Phi^T G^T + G\Phi S_{n-1}\Phi^T G^T + S_n \tag{C16}$$

where W_{n-1} and P_{n-1} are evaluated immediately before the n-1st correction.

(Also note that the first term on the right side of equation (C16) should be zero.) The trace of the matrix $G\Phi S_{n-1}\Phi^T G^T$ gives the mean-square value of that portion of the nth velocity correction resulting from errors in making the previous correction.

APPENDIX D

COMPUTATION OF STATISTICS OF THE LANDING CONDITIONS

The application of the Monte Carlo method to the computation of the landing conditions is as follows:

1. Starting immediately after the final velocity correction, two-body formulas were used to compute the perigee radius of the reference trajectory and its position and velocity vectors at atmospheric entry.

2. The formulas derived below were used to compute the reference landing conditions.

3. A set of random position and velocity deviations from the reference trajectory was obtained from the statistical distribution described by W (ref. 6). These deviations were added to the reference position and velocity to obtain a perturbed position and velocity.

4. Steps 1 and 2 were repeated for the perturbed trajectory and the differences between perturbed and reference landing conditions were computed.

5. A sample of 1000 different perturbed trajectories was found to give sufficiently accurate values of the statistical quantities of interest.

Standard two-body formulas were used for Step 1 and will not be presented here. The entry position, velocity, and time were then used to compute the landing conditions, and the formulas used for this computation are derived in the following paragraphs.

The relationship of the unit vector $\bar{u}_L$, along the radius vector to the landing site to the Cartesian coordinate system is shown in appendix A to be

$$\bar{u}_L = \bar{I} \cos D_L \cos RA_L + \bar{J} \cos D_L \sin RA_L + \bar{K} \sin D_L \tag{D1}$$

Similarly, the unit vector $\bar{u}_E$ along the radius vector to the entry point is given by

$$\bar{u}_E = \bar{I} \cos D_E \cos RA_E + \bar{J} \cos D_E \sin RA_E + \bar{K} \sin D_E \tag{D2}$$

Since

$$\cos \varphi = \bar{u}_E \cdot \bar{u}_L$$

it follows that

$$\cos \varphi = \cos D_L \cos D_E \cos (RA_L - RA_E) + \sin D_L \sin D_E \tag{D3}$$

If RA_{LE} is the right ascension of the landing site at the time of entry, then

$$RA_L = RA_{LE} + \omega t_F \tag{D4}$$

where ω is the earth's angular velocity and t_F is the time of flight from entry to landing. Substitution in equation (D3) gives

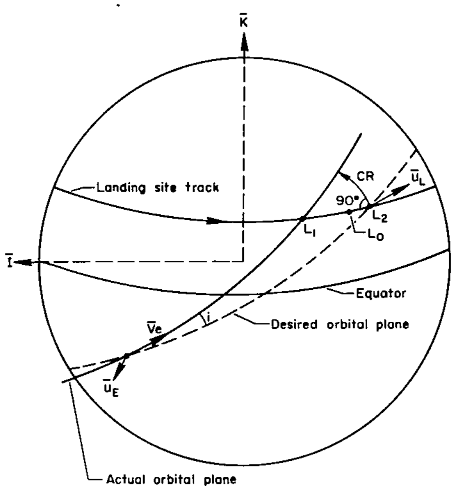
$$\cos \varphi = \cos D_E \cos D_L \cos (RA_{LE} + \omega t_F - RA_E) \tag{D5}$$

and from equation (2)

$$t_F = 0.00933\varphi + 0.00254$$

Equations (2) and (D5) must be solved by iteration (see ref. 1) to find φ and t_F. The time of landing, t_L, is

$$t_L = t_E + t_F$$



Sketch (g)

The computation of the crossrange deviation can be explained with the aid of sketch (g).

If the vehicle were allowed to continue in its incoming orbital plane the "actual orbital plane" in the sketch, it would cross the landing site track at point L_1. At the same time the landing site would be at point L_0. If the vehicle were on a satisfactory trajectory, it would pass through the entry point in the "desired orbital plane" and arrive at L_2 at the same time as the landing site. The crossrange deviation is defined as the normal arc, CR, from the landing site to the actual orbital plane.

Let $\bar{n}_a$ be a unit vector normal to the actual orbital plane and $\bar{n}_d$ a unit vector normal to the desired plane. Then

$$\bar{n}_a = \frac{\bar{R}_E \times \bar{V}_E}{|\bar{R}_E \times \bar{V}_E|}$$

and

$$\bar{n}_d = \frac{\bar{u}_E \times \bar{u}_L}{|\bar{u}_E \times \bar{u}_L|}$$

The relative inclination angle, i, can be obtained from

$$\cos i = \overline{n}_a \cdot \overline{n}_d$$

and from spherical trigonometry

$$\tan (CR) = \tan i \sin \varphi$$

Multiplying by a constant factor converts CR from angular units to kilometers.

APPENDIX E

DERIVATION OF THE GENERALIZED INVERSE OF ψ

It is shown in reference 7 that every rectangular matrix, A, has a generalized inverse, A*, which is unique. Furthermore, it is a property of A* that if

$$b = Ay \tag{E1}$$

has a solution, then that solution is given by

$$y = A^*b \tag{E2}$$

The guidance law was derived to insure that for every state vector x there exists a unique ζ. Therefore, it is known that

$$\zeta = \psi^* x \tag{E3}$$

This fact will be used to solve for ψ^*.

Equation (E3) can be written in partitioned form as

$$\begin{bmatrix} \zeta_1 \\ \zeta_2 \\ \zeta_3 \\ \zeta_4 \end{bmatrix} = \begin{bmatrix} \psi_1{}^* & \psi_2{}^* \\ (\bar{\rho}_1{}^*)^T & (\bar{\rho}_2{}^*)^T \end{bmatrix} \begin{bmatrix} \bar{r} \\ \bar{v} \end{bmatrix} \tag{E4}$$

where $\psi_1{}^*$ and $\psi_2{}^*$ are 3×3 matrices. From equation (A26)

$$\zeta_4 = \frac{-1}{k} (\bar{u}^T M\bar{r} - \bar{u}^T \bar{v})$$

so that

$$\left.\begin{aligned} (\bar{\rho}_1{}^*)^T &= \frac{-1}{k} \bar{u}^T M \\ (\bar{\rho}_2{}^*)^T &= \frac{1}{k} \bar{u}^T \end{aligned}\right\} \tag{E5}$$

If the optimum ζ_4 from equation (A26) is substituted into equation (A22) (the prime is omitted from $\bar{r}$) then

$$\bar{r} = \psi_1 \begin{bmatrix} \zeta_1 \\ \zeta_2 \\ \zeta_3 \end{bmatrix} - \frac{1}{k} \bar{\rho}_1 (\bar{u}^T M\bar{r} - \bar{u}^T \bar{v})$$

so that

$$\begin{bmatrix} \zeta_1 \\ \zeta_2 \\ \zeta_3 \end{bmatrix} = \psi_1^{-1}\left(I + \frac{1}{k}\,\bar{\rho}_1\bar{u}^T M\right)\bar{r} - \psi_1^{-1}\,\frac{1}{k}\bar{\rho}_1\bar{u}^T\bar{v} \tag{E6}$$

From equation (E6)

$$\left.\begin{aligned} \psi_1^* &= \psi_1^{-1}\left(I + \frac{1}{k}\,\bar{\rho}_1\bar{u}^T M\right) \\ \psi_2^* &= -\psi_1^{-1}\,\frac{1}{k}\,\bar{\rho}_1\bar{u}^T \end{aligned}\right\} \tag{E7}$$

Substitution from equations (E7) and (E5) into equation (E4) gives

$$\psi^* = \begin{bmatrix} \psi_1^{-1}\left(I + \frac{1}{k}\,\bar{\rho}_1\bar{u}^T M\right) & -\frac{1}{k}\,\psi_1^{-1}\bar{\rho}_1\bar{u}^T \\ \frac{-1}{k}\,(\bar{u}^T M) & \frac{1}{k}\,\bar{u}^T \end{bmatrix}$$

It can be shown that

$$\psi^*\psi = I$$

where I is a 4×4 unit matrix and

$$\psi\psi^* = I$$

where I is a 6×6 unit matrix.

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TABLE I.- COVARIANCE MATRIX OF INJECTION ERRORS FOR STANDARD DISTRIBUTION

	X, km	Y, km	Z, km	$\dot{X}$, km/sec	$\dot{Y}$, km/sec	$\dot{Z}$, km/sec
X	0.50443614	-0.1053457	-0.16956842	-0.73958537×10^{-4}	-0.47001117×10^{-3}	-0.22777452×10^{-3}
Y	-.10534570	$.36457942\times10^{-1}$	$.30887450\times10^{-1}$	$.17144754\times10^{-5}$	$.15240230\times10^{-3}$	$.59077529\times10^{-4}$
Z	-.16956842	$.30887450\times10^{-1}$	$.84712863\times10^{-1}$	$.29243600\times10^{-4}$	$.17300981\times10^{-3}$	$.13020790\times10^{-3}$
$\dot{X}$	$-.73958537\times10^{-4}$	$.17144755\times10^{-5}$	$.29243600\times10^{-4}$	$.22371290\times10^{-6}$	$-.11326999\times10^{-6}$	$.25554670\times10^{-6}$
$\dot{Y}$	$-.47001117\times10^{-3}$	$.15240228\times10^{-3}$	$.17300981\times10^{-3}$	$-.11326999\times10^{-6}$	$.18436915\times10^{-5}$	$.25527313\times10^{-6}$
$\dot{Z}$	$.59077530\times10^{-4}$	$.59077530\times10^{-4}$	$.13020790\times10^{-3}$	$.25554670\times10^{-6}$	$.25526313\times10^{-5}$	$.15039492\times10^{-5}$

TABLE II.- RMS VELOCITY CORRECTIONS

				Fixed-time-of-arrival correction no.				Fixed-landing-site correction no.				Ratio of total for F.L.S. to that for F.T.A.
Case	k_σ	Velocity correction errors	Injection errors	1, m/sec	2, m/sec	3, m/sec	Total, m/sec	1, m/sec	2, m/sec	3, m/sec	Total, m/sec	
1a	1	None	Standard	1.37	1.08	0.72	3.17	0.94	0.55	0.14	1.63	0.52
2a	10	None	Standard	4.58	1.32	.75	6.65	3.15	.66	.15	3.96	.59
3a	100	None	Standard	14.61	1.64	.84	17.09	10.04	.77	.15	10.96	.64
1b	1	Standard	Standard	1.37	1.11	.83	3.31	.94	.58	.26	1.78	.54
2b	10	Standard	Standard	4.58	1.36	.86	6.80	3.15	.69	.27	4.11	.60
3b	100	Standard	Standard	14.61	1.76	.94	17.31	10.04	.83	.27	11.14	.64
1c	1	Standard	Spherical	2.79	1.47	.93	5.19	2.68	.71	.27	3.66	.70
3c	100	Standard	Spherical	28.62	2.47	.99	32.08	27.22	1.15	.27	28.64	.89

TABLE III.- CORRECTIONS DUE TO ERRORS IN PREVIOUS CORRECTION

	Fixed time of arrival		Fixed landing site	
Case	Second correction, m/sec	Third correction, m/sec	Second correction, m/sec	Third correction m/sec
1b	0.24	0.40	0.16	0.20
2b	.31	.40	.18	.20
3b	.63	.41	.30	.20

TABLE IV.- RMS PERIGEE ALTITUDE DEVIATIONS

Case	Rms perigee altitude deviations, km	
	Fixed time of arrival	Fixed landing site
1a	1.45	1.44
2a	1.51	1.49
3a	1.54	1.51
1b	1.50	2.11
2b	1.56	2.17
3b	1.58	2.20

TABLE V.- RMS CROSSRANGE DEVIATION

Case	Crossrange deviation, km	
	Fixed time of arrival	Fixed landing site
1a	3.82	0.59
2a	5.61	.61
3a	12.26	.62
1b	3.85	.92
2b	5.67	.93
3b	12.37	.93

TABLE VI. - RMS ENTRY-RANGE DEVIATIONS

	Fixed time of arrival	Fixed landing site		
Case	Total deviation, km	Entry-range deviation, km	Entry-range adjustment, km	Total deviation, km
1a	12.8	12.8	23.2	26.5
2a	13.3	13.1	72.2	73.3
3a	13.7	13.3	227.8	227.8
1b	13.3	19.1	23.2	30.0
2b	13.9	19.6	72.2	74.8
3b	14.3	19.8	227.8	227.8

TABLE VII. - RMS LANDING-TIME DEVIATIONS

	Fixed time of arrival	Fixed landing site		
Case	Total deviation, km	Landing-time deviation, sec	Landing-time adjustment, sec	Total deviation, sec
1a	3.74	3.72	69	70
2a	3.80	3.79	215	215
3a	3.96	3.94	680	680
1b	3.83	3.73	69	70
2b	3.87	3.82	215	215
3b	4.02	3.96	680	680

NASA-Langley, 1966 A-2117

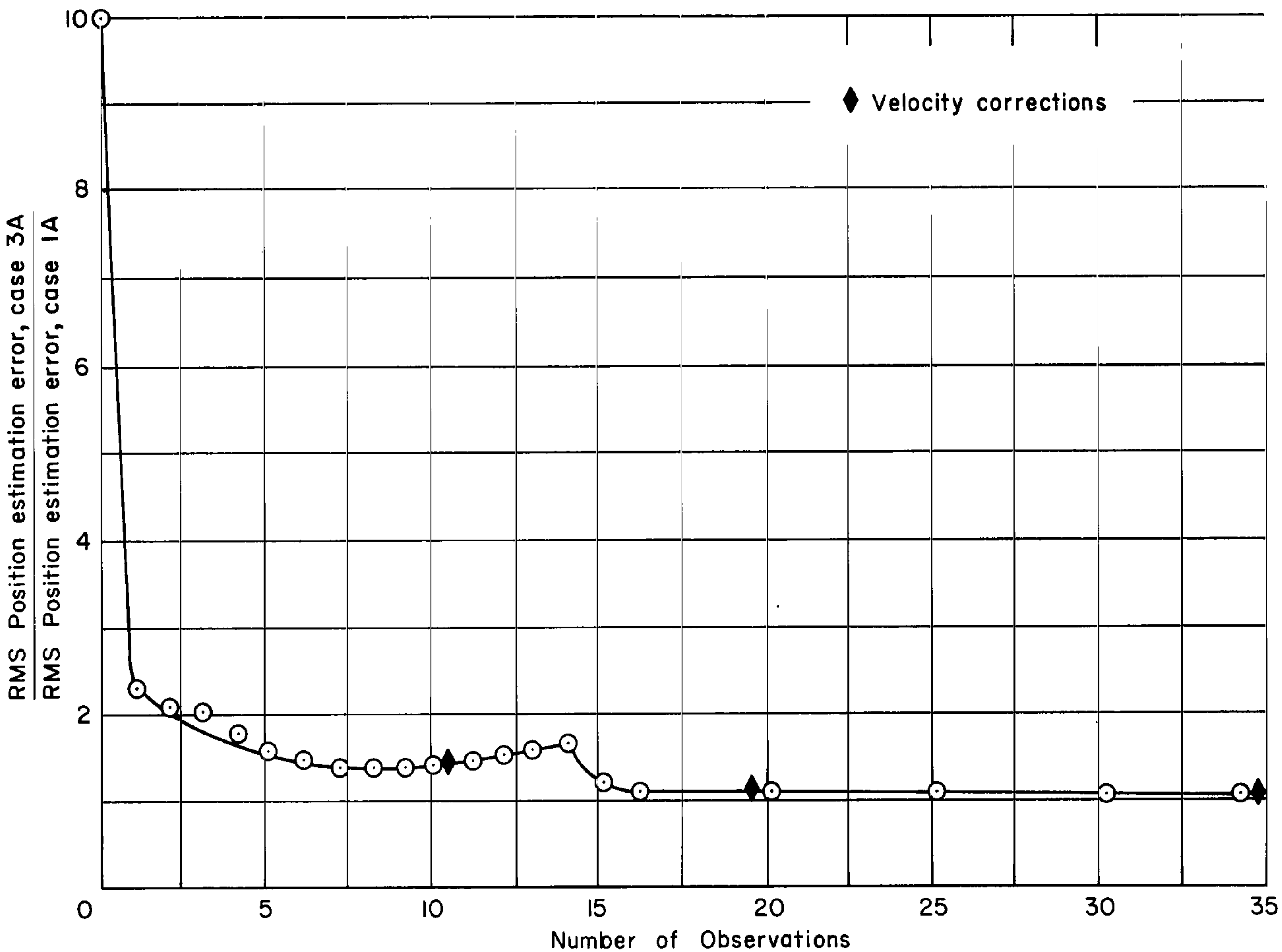

Figure 1.- RMS position estimation errors.

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