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RECURSIVE COMPUTATION OF THE COEFFICIENTS OF THE TIME-DEPENDENT f AND g SERIES SOLUTION OF KEPLERIAN MOTION AND A STUDY OF THE CONVERGENCE PROPERTIES OF THE SOLUTION

by Victor R. Bond

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ABSTRACT

A method for recursively computing the coefficient of the time-dependent f and g series solution to the two-body problem is derived. To find the coefficients of the terms in the f and g series, time derivatives are taken of the function $h = \mu/r^3$. In this formulation another variable $\phi = h\dot{r}$ is introduced, and a recursive relation for the derivatives of h is found in terms of the derivatives of the radial magnitude and of the preceding derivatives of h. A recursive relation for the derivatives of r is then derived. After the time derivatives of h are determined, the coefficients in the f and g series are found from well-known formulas. The convergence properties of the f and g series are studied using up to 102 terms. The time interval of convergence is compared with the computed time radius of convergence of the series with excellent agreement.

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SUMMARY

This report formulates a method for recursively computing the coefficients of the time-dependent f and g series solution to the two-body problem. To find the coefficients of the terms in the f and g time series, time derivatives are taken of the function, $h = \mu/r^3$. This ordinarily becomes algebraically prohibitive after only a few derivatives. In the formulation presented in this paper another variable, $\phi = h\dot{r}$, is introduced, which gives a recursive relation for the derivatives of h in terms of the derivatives of the radial magnitude, r, and of preceding derivatives of h. Recursive relations for the derivatives of ϕ and r are then derived. After the time derivatives of h are determined, the coefficients in the f and g series are found from standard formulas.

The data presented provide a comparison of the non-dimensional angular momentum $f\dot{g} - g\dot{f}$ computed by this recursive method with the theoretical value of unity for the angular momentum. This comparison indicates the number of terms which should be used in the f and g series for them to represent accurately a solution to the two-body problem for a given set of initial conditions.

The results show that there is a point beyond which the use of additional terms in the series does not extend the time interval over which the series solution is valid. This time interval may be computed from the formula for the radius of convergence of the f and g series. The regions of convergence that are apparent from the results are in excellent agreement with the computed values of the radius of convergence. It is demonstrated by four examples that the use of 30 terms in the series maintains a high degree of accuracy for a time interval of approximately one-half the computed radius of convergence.

INTRODUCTION

The solution to the two-body problem, or Keplerian motion, may be expressed explicitly as a function of time through the use of the f and g series. As stated in reference 1, this solution has been known for some time, having first been used by Lagrange in 1869. The problem encountered in using this solution is that the coefficients in the series have been very difficult to obtain except for about the first six terms. In recent literature (refs. 2 and 3) authors comment that six terms are sufficient for short time intervals and that the higher terms are tedious or impractical to obtain. Reference 4 indicates that a general or recursive formula for generating the coefficients exists but is obscure; no attempt is made to present the recursive relations.

Reference 1 describes a digital computer program which generates up to 27 coefficients for use in the f and g series. This program uses a recursive formulation first discovered by Cipolletti in 1872. This formulation is recursive in only a limited sense because the coefficients of a given order depend upon the derivatives of the coefficients of the previous order. The coefficients beyond the sixth or seventh are still impractical to obtain by hand computation because there is no recursive formulation for the derivatives of the coefficients in Cipolletti's method.

This report includes the derivation of a recursive method for generating the coefficients which does not require derivative operations. It also discusses some of the convergence properties of the f and g series and presents several numerical examples.

SYMBOLS

a	semimajor axis
a_k	coefficients in equation for $f(t)$
b_k	coefficients in equation for $g(t)$
c_k	coefficients in equation for $h(t)$
E	eccentric anomaly
e	eccentricity
$f(t)$	series defined by equation (2)
$g(t)$	series defined by equation (3)
H	the hyperbolic equivalent of E
$h(t)$	series defined by equation (8)
i	index inferred from equations (16) and (17); $\sqrt{-1}$ in appendix A
j	index inferred from equations (4) and (5)

L angular momentum per unit mass

L' the non-dimensional angular momentum, $f\dot{g} - g\dot{f}$

ln natural logarithm

M mean anomaly

N the hyperbolic equivalent of M

n index inferred from equations (10), (16), and (17)

n_{max} maximum number of derivatives of h desired

R_B radius of attracting body

$\underline{r}$ position vector

r magnitude of $\underline{r}$

T time of peri-apsis passage

t time

$\underline{v}$ velocity vector

α parameter defined by equation (A7)

μ universal gravitational constant times mass of attracting body

ρ radius of convergence

ρ_H radius of convergence for hyperbolic orbits

ϕ variable defined by equation (12)

Other Notations:

$(\ )^{(n)}$ n^{th} derivative of ( ) with respect to time

$(\dot{\ })$ derivative of ( ) with respect to time

$(\ )_o$ ( ) evaluated at initial time t_o; or indicates the first coefficient in the series defined by equations (2), (3), and (8)

$\binom{N}{i}$ $\frac{N!}{i!(N - i)!}$, the binomial coefficients

$(\ )^*$ value of ( ) at first singular point of the f and g series

DERIVATION

This formulation of the f and g time series solution to the two-body problem will follow to a large extent that of Brouwer and Clemence (ref. 5), except that vector notation will be used for convenience.

The solution to the two-body problem may be written

$$\underline{r}(t) = \underline{r}_o\, f(t) + \underline{v}_o\, g(t) \tag{1}$$

$\underline{r}_o$ and $\underline{v}_o$ are the initial position and velocity vectors. The functions $f(t)$ and $g(t)$ are given by

$$f(t) = \sum_{k=o}^{\infty} a_k \left(t - t_o\right)^k \tag{2}$$

$$g(t) = \sum_{k=o}^{\infty} b_k \left(t - t_o\right)^k \tag{3}$$

The a_k and b_k are obtained from the relations

$$a_{j+2} = \frac{-1}{(j+1)(j+2)} \sum_{k=o}^{j} c_k\, a_{j-k} \tag{4}$$

$$b_{j+2} = \frac{-1}{(j+1)(j+2)} \sum_{k=o}^{j} c_k\, b_{j-k} \tag{5}$$

for $j \geq o$. The coefficients a_o, a_1, b_o, b_1 are

$$\left.\begin{aligned} a_o &= 1 \\ a_1 &= 0 \\ b_o &= 0 \\ b_1 &= 1 \end{aligned}\right\} \tag{6}$$

The c_n are derivable from the Taylor's expansion for the term μ/r^3 which appears in the equations of motion

$$\ddot{\underline{r}} + \frac{\mu}{r^3}\underline{r} = 0 \tag{7}$$

Define the function

$$h(t) = \frac{\mu}{r^3} \tag{8}$$

by expanding $h(t)$ about the epoch t_o

$$h(t) = \sum_{n=o}^{\infty} c_n \left(t - t_o\right)^n \tag{9}$$

where

$$c_n = \frac{h(t_o)^{(n)}}{n!} \tag{10}$$

The subscript zero will now be dropped from the notation, since all coefficients and derivatives will be assumed to be evaluated at t_o.

Recursive Formula for Time Derivatives of h

The time derivatives of h must now be obtained in order to develop the solution.

The first derivative of h is

$$h^{(1)} = \frac{-3\mu}{r^4} r^{(1)} = -3\frac{h}{r} r^{(1)} \tag{11}$$

At this point introduce the variable

$$\phi = hr^{(1)} \tag{12}$$

Equation (11) becomes

$$h^{(1)} = -\frac{3\phi}{r} \tag{13}$$

Take the second derivative of h from (13) to obtain

$$h^{(2)} = -3\,\frac{r\phi^{(1)} - \phi r^{(1)}}{r^2} \tag{14}$$

Now use (13) to eliminate ϕ in (14)

$$h^{(2)} = -\frac{1}{r}\left(3\phi^{(1)} + r^{(1)}\, h^{(1)}\right) \tag{15}$$

In successive derivatives a similar elimination process is followed to obtain

$$h^{(3)} = -\frac{1}{r}\left(3\phi^{(2)} + 2r^{(1)}\, h^{(2)} + r^{(2)}\, h^{(1)}\right)$$

$$\vdots$$

The numerical coefficients of the terms in these expressions may be found from the binomial coefficient relation. For the nth derivative

$$h^{(n)} = \frac{-1}{r}\left[3\phi^{(n-1)} + \sum_{i=1}^{n-1} \binom{n-1}{i} r^{(i)}\, h^{(n-i)}\right] . \tag{16}$$

Recursive Formula for Time Derivatives of ϕ

The derivative $\phi^{(n-1)}$ must now be found in order to evaluate properly (16). Applying Leibnitz's rule for the differentiation of a product to equation (12), the nth derivative of ϕ becomes

$$\phi^{(n)} = hr^{(n+1)} + \sum_{i=1}^{n} \binom{n}{i} h^{(i)}\, r^{(n-i+1)} \tag{17}$$

Equation (17) may now be used to eliminate $\phi^{(n-1)}$ in equation (16). Substitute n-1 for n in (17) and obtain

$$\phi^{(n-1)} = hr^{(n)} + \sum_{i=1}^{n-1} \binom{n-1}{i} h^{(i)}\, r^{(n-i)} \tag{18}$$

Now substitute (18) into (16) and obtain

$$h^{(n)} = -\frac{1}{r}\left[3hr^{(n)} + \sum_{i=1}^{n-1} \binom{n-1}{i} \left(3h^{(i)}\, r^{(n-i)} + r^{(i)}\, h^{(n-i)}\right)\right] \tag{19}$$

Recursive Formula for Time Derivatives of r

The derivatives of the radial magnitude are now needed to complete the formulation. The first derivative of r is found from the initial conditions

$$r^{(1)} = \frac{\underline{r}_o \cdot \underline{v}_o}{r_o} \tag{20}$$

The second derivative is found from the equation of motion

$$r^{(2)} = \frac{L^2}{r^3} - \frac{\mu}{r^2} \tag{21}$$

where

$$L = \left| \underline{r}_o \times \underline{v}_o \right|$$

Equation (21) may be written by using (8)

$$r^{(2)} = \frac{L^2}{\mu} h - \frac{\mu}{r^2} \tag{22}$$

The third derivative of r is

$$r^{(3)} = \frac{L^2}{\mu} h^{(1)} + \frac{2\mu r^{(1)}}{r^3}$$

or by using (8) and (12)

$$r^{(3)} = \frac{L^2}{\mu} h^{(1)} + 2\phi \tag{23}$$

The higher derivatives of r follow directly

$$r^{(4)} = \frac{L^2}{\mu} h^{(2)} + 2\phi^{(1)}$$

$$\vdots$$

The $r^{(n+2)}$ derivative of r may now be written

$$r^{(n+2)} = \frac{L^2}{\mu} h^{(n)} + 2\phi^{(n-1)} \tag{24}$$

or by using equation (18)

$$r^{(n+2)} = \frac{L^2}{\mu} h^{(n)} + 2\left[\sum_{i=1}^{n-1} \binom{n-1}{i} h^{(i)} r^{(n-i)} + hr^{(n)}\right] \tag{25}$$

for $n \geq 1$.

Method of Computation

Given: $\underline{r}_o$, $\underline{v}_o$, μ, t_o, t, n_{max}

(a) Compute h, $r^{(1)}$, and $r^{(2)}$ from equations (8), (20), and (21).

(b) Set $n = 1$

Compute $h^{(1)}$ from (19)

Compute $r^{(3)}$ from (25).

(c) Set $n = 2$

Compute $h^{(2)}$ from (19)

Compute $r^{(4)}$ from (25)

(d) Continue this scheme until the desired number, n_{max}, of the derivatives of h are found.

(e) Compute the c_n, $n = 0$, to n_{max} from equation (10).

(f) Compute the coefficients a_k and b_k, $k = 0$ to $n_{max} + 2$ from equations (4) and (5).

(g) Compute $f(t)$ and $g(t)$ from equations (2) and (3).

(h) Compute the position vector $\underline{r}(t)$ from (1) and also the velocity vector from the time derivative of (1).

This completes the solution.

CONVERGENCE OF THE SERIES

In reference 6 Sconzo and Hale showed that the radius of convergence, that is, the time between epoch time t_o and the time of the nearest singular point of f and g, could be found from the expression

$$\rho = \sqrt{\frac{a^3}{\mu}} \left\{ M_o^{\,2} + \left[\ln \left(\frac{1 + \sqrt{1 - e^2}}{e} \right) - \sqrt{1 - e^2} \right]^2 \right\}^{\frac{1}{2}} \tag{26}$$

where

$$0 \leq e < 1, \; -\pi \leq M_o \leq \pi, \text{ and } \; a > 0$$

This relation is obviously limited to elliptical orbits, since for hyperbolic orbits, $a < 0$.

The semimajor axis a, eccentricity e, and the mean anomaly M_o may be computed from the formulas

$$\frac{1}{a} = \frac{2}{r_o} - \frac{\underline{v}_o \cdot \underline{v}_o}{\mu} \tag{27}$$

$$e \cos E_o = 1 - \frac{r_o}{a} \tag{28}$$

$$e \sin E_o = \frac{\underline{r}_o \cdot \underline{v}_o}{\sqrt{\mu a}} \tag{29}$$

$$M_o = E_o - e \sin E_o = \sqrt{\frac{\mu}{a^3}} \left(t_o - T \right) \tag{30}$$

The implication of equation (26) is that the convergence of the series depends not only upon the shape of the orbit as defined by e and a, but also upon the initial time as measured from the time of periapsis passage T.

The radius of convergence for hyperbolic orbits is given in appendix A.

NUMERICAL EXAMPLES

Four examples are presented to illustrate some of the results of the recursive formulation. In each example, the values of the quantity $L' = f\dot{g} - g\dot{f}$ are computed several times, each time with a different number of terms ranging from 6 terms to 102 terms. The quantity L' is a constant of the motion as shown in appendix B and should have a constant value of unity. Any deviation of L' from unity must result from either truncating the series before a sufficient number of terms have been taken, or from extending the time interval beyond the radius of convergence of the series.

Example 1, low altitude nearly circular orbit about the moon:

$$\underline{r}_o = (-1012.4370, -51.263872, -20.120039) \text{ n. mi.}$$

$$\underline{v}_o = (-287.96060, 4914.2673, 1967.3377) \text{ ft/sec.}$$

Example 2, at insertion into translunar orbit from earth parking orbit:

$$\underline{r}_o = (3091.8028, 1633.2175, 883.5347) \text{ n. mi.}$$

$$\underline{v}_o = (-14646.307, 27051.882, 17921.139) \text{ ft/sec.}$$

Example 3, approximately 46 hours from insertion into a translunar trajectory:

$$\underline{r}_o = (-161265.14, -20351.149, -5044.6929) \text{ n. mi.}$$

$$\underline{v}_o = (-3132.8173, -1023.4259, -501.70741) \text{ ft/sec.}$$

Example 4, at lunar sphere of influence on a translunar trajectory, going toward the moon:

$$\underline{r}_o = (25135.706, -20187.383, -11280.829) \text{ n. mi.}$$

$$\underline{v}_o = (-3090.2697, 2125.8987, 1198.1489) \text{ ft/sec}$$

In examples 2 and 3, $\underline{r}_o$ and $\underline{v}_o$ are referenced to the geocentric system; and in examples 1 and 4, $\underline{r}_o$ and $\underline{v}_o$ are referenced to the selenocentric system.

All four examples, seen in tables I to IV, show that the accuracy of the series solution improves as the number of terms in the f and g series increases for a given time from epoch, as would be expected. It is also seen from tables I to IV that the series solution diverges rather sharply near the computed values of the radius of convergence. The addition of more terms to the series will not increase the validity range of the solution near the computed value of the radius of convergence.

An interesting point that can be seen from tables II to IV is that L' may actually be closer to unity with only a few terms than it is with a larger number of terms; but this occurs only at values of time greater than the computed radius of convergence. The numbers for L' beyond the radius of convergence have little if any meaning.

The radius of convergence for example 1 was found from equation (26) to be $\rho = 3.69$ hours. The eccentricity of this orbit is 0.00001032, so that it is very nearly circular. For a purely circular orbit where the eccentricity is exactly zero, the radius of convergence would approach infinite time.

The radii of convergence ρ, for examples 2 and 3 (both are examples of nearly parabolic trajectories), were computed from equation (26). For example 2 it was found that $\rho = 0.2280$ hour, and for example 3 it was found that $\rho = 45.71$ hours. These times are consistent with the results shown in tables II and III. For example 2 the epoch chosen was 0.03941 hour after perigee, and, for example 3 the epoch chosen was 45.71 hours after perigee. The contrast of these two examples points out the implication of equation (26), that is, that the radius of convergence depends upon the relation of epoch to the time of periapsis passage. The radius of convergence increases as $t_o - T$, or M_o increases.

Example 4 is a case representing a hyperbolic orbit. The epoch chosen was 13.51 hours before pericynthion passage, and the radius of convergence was 13.51 hours.

The number of terms required for the solution is of course dictated by the accuracy requirements of the problem to be solved. It should be noted from tables I to IV that for 30 terms, eight-digit accuracy is maintained for a time equal to about one-half of the computed radius of convergence. The time interval over which the series converges increases with the addition of more terms, but the change in the time interval decreases. This means, of course, that the addition of more terms will increase accuracy, but not always by a significant amount. For example, table I shows that eight-digit accuracy is maintained to a maximum of 2.20 hours by using 54 terms or by using up to 102 terms. The accuracy that is obtained, of course, depends to a great extent on the accuracy of the computer that is used in the computation. The numerical results presented here were obtained on a 10-digit electronic computer.

All four examples were computed by using the non-dimensionalized method such that the initial position was in units of the radius R_B of the attracting body; the initial velocity in units of the circular satellite velocity $\left(\frac{\mu}{R_B}\right)^{1/2}$; and time in units of $\left(\frac{R_B^3}{\mu}\right)^{1/2}$. This allows the gravitational parameter μ to be set equal to unity in the equations.

CONCLUDING REMARKS

A recursive scheme for computing the coefficients in the f and g series solution to the two-body problem has been presented. Four examples demonstrate the time interval over which the series solution is valid from 6 terms up to 102 terms. This time interval is compared with the radius of convergence of the f and g series, and excellent agreement is obtained. It is demonstrated by four examples that a high degree of accuracy is maintained for a time interval of approximately one-half the computed radius of convergence by using 30 terms in the series.

APPENDIX A

RADIUS OF CONVERGENCE FOR HYPERBOLIC CASES

In reference 6, it was shown that the radius of convergence of the f and g series for the elliptical case could be expressed by

$$\rho = \sqrt{\frac{a^3}{\mu}} \left| M^* - M_o \right| \tag{A1}$$

where M^* is the value of the mean anomaly at the first singular point of the f and g series. By analogy, the radius of convergence for the hyperbolic case may be derived by expressing equation (A1) as

$$\rho_H = \sqrt{\frac{-a^3}{\mu}} \left| N^* - N_o \right| \tag{A2}$$

where N is the hyperbolic equivalent of the elliptical mean anomaly M, and is given by

$$N = e \sinh H - H \tag{A3}$$

The value of N at the first singular point of the f and g series is N^*, and the value of N at epoch time is N_o. The hyperbolic equivalent of the elliptical eccentric anomaly E is H.

The singularities of equation (A3) are given by

$$\frac{dN}{dH} = e \cosh H - 1 = 0 \tag{A4}$$

This follows since the singularities of the function $H(N)$ are given by

$$\frac{dH}{dN} \rightarrow \infty \tag{A5}$$

Since for the hyperbolic case $e > 1$, and also since $\cosh H \geq 1$, the solutions of equation (A4) for H^* can only be in imaginary. Setting

$$H^* = i\alpha \tag{A6}$$

where $i = \sqrt{-1}$

and substituting this expression into (A4),

$$e \cosh (i\alpha) - 1 = 0$$

or, since $\cosh (i\alpha) = \cos \alpha$

$$\cos \alpha = \frac{1}{e} \tag{A7}$$

Substituting equation (A6) and equation (A7) into equation (A3), the value of N^* at the first singularity is

$$N^* = \frac{1}{\cos \alpha} \sinh (i\alpha) - i\alpha$$

But since $\sinh (i\alpha) = i \sin \alpha$

$$N^* = i(\tan \alpha - \alpha) \tag{A8}$$

Substituting equation (A8) into equation (A2) yields

$$\rho_H = \sqrt{\frac{-a^3}{\mu}} \left| i(\tan \alpha - \alpha) - N_o \right|$$

The absolute value of the complex variable ρ_H is given by

$$\rho_H = \sqrt{\frac{-a^3}{\mu}} \left[N_o^{\,2} + (\tan \alpha - \alpha)^2 \right]^{\frac{1}{2}} \tag{A9}$$

The semimajor axis a may be computed as in the elliptical case from equation (27). The eccentricity and N_o are computed from

$$e \cosh H_o = 1 - \frac{r_o}{a} \tag{A10}$$

$$e \sinh H_o = \frac{\underline{r}_o \cdot \underline{v}_o}{\sqrt{-\mu a}} \tag{A11}$$

$$N_o = e \sinh H_o - H_o = \sqrt{\frac{-\mu}{a^3}} \left(t_o - T\right) \tag{A12}$$

The parameter α is computed from equation (A7).

APPENDIX B

THE NON-DIMENSIONAL ANGULAR MOMENTUM, $f\dot{g} - g\dot{f}$

The quantity $f\dot{g} - g\dot{f}$ may be shown to be a constant of the motion by taking the cross product of equation (1) with its time derivative

$$\underline{v}(t) = \underline{r}_o\dot{f}(t) + \underline{v}_o\dot{g}(t) \tag{B1}$$

Taking the cross product,

$$\underline{r} \times \underline{v} = \left(\underline{r}_o f + \underline{v}_o g\right) \times \left(\underline{r}_o \dot{f} + \underline{v}_o \dot{g}\right)$$

$$= \left(\underline{r}_o \times \underline{v}_o\right)\left(f\dot{g} - g\dot{f}\right)$$

Since the angular momentum $\underline{r}_o \times \underline{v}_o$ must remain constant,

$$\underline{r} \times \underline{v} = \underline{r}_o \times \underline{v}_o$$

Therefore,

$$f\dot{g} - g\dot{f} = 1 \tag{B2}$$

It is seen also that the ratio

$$\frac{\left|\underline{r} \times \underline{v}\right|}{\left|\underline{r}_o \times \underline{v}_o\right|} = f\dot{g} - g\dot{f} \tag{B3}$$

This quantity is defined by L' and is called the non-dimensional angular momentum.

Manned Spacecraft Center
National Aeronautics and Space Administration
Houston, Texas, September 9, 1965

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TABLE I.- NON-DIMENSIONAL ANGULAR MOMENTUM AS A FUNCTION OF TIME (EXAMPLE 1) FOR A LOW ALTITUDE NEARLY CIRCULAR ORBIT ABOUT THE MOON

[a = 1013.93 n. mi.; e = 0.00001032; $t_o - T$ = 0.8017 hr; ρ = 3.694 hr]

Time, hr	[a]L′ (6 terms)	L′ (18 terms)	L′ (30 terms)	L′ (42 terms)	L′ (54 terms)	L′ (66 terms)	L′ (78 terms)	L′ (90 terms)	L′ (102 terms)
0.00	1.00000000	1.00000000	1.00000000	1.00000000	1.00000000	1.00000000	1.00000000	1.00000000	1.00000000
0.20	0.99964353	1.00000000	1.00000000	1.00000000	1.00000000	1.00000000	1.00000000	1.00000000	1.00000000
0.40	0.98182759	1.00000000	1.00000000	1.00000000	1.00000000	1.00000000	1.00000000	1.00000000	1.00000000
0.60	0.82045322	1.00000000	1.00000000	1.00000000	1.00000000	1.00000000	1.00000000	1.00000000	1.00000000
0.80	0.20720302	1.00000000	1.00000000	1.00000000	1.00000000	1.00000000	1.00000000	1.00000000	1.00000000
1.00	-0.96464808	0.99999967	1.00000000	1.00000000	1.00000000	1.00000000	1.00000000	1.00000000	1.00000000
1.20	-0.99897217	0.99999165	1.00000000	1.00000000	1.00000000	1.00000000	1.00000000	1.00000000	1.00000000
1.40	7.48513006	0.99990544	1.00000000	1.00000000	1.00000000	1.00000000	1.00000000	1.00000000	1.00000000
1.60	45.08339798	0.99951540	0.99999999	1.00000000	1.00000000	1.00000000	1.00000000	1.00000000	1.00000000
1.80	158.45595713	1.00026474	0.99999885	1.00000000	1.00000000	1.00000000	1.00000000	1.00000000	1.00000000
2.00	440.47155398	1.02147412	0.99996624	1.00000003	1.00000000	1.00000000	1.00000000	1.00000000	1.00000000
2.20	1059.92557659	1.16384599	0.99949041	1.00000072	1.00000000	1.00000000	1.00000000	1.00000000	1.00000000
2.40	2303.41483879	1.70519094	0.99590675	1.00000269	1.00000013	1.00000001	1.00000001	1.00000001	1.00000001
2.60	4632.25594861	2.77697663	0.99267234	0.99947866	1.00001480	0.99999974	0.99999997	0.99999997	0.99999997
2.80	8757.63835345	1.65990198	1.23112709	0.98096202	1.00079465	0.99997926	1.00000018	0.99999999	0.99999997
3.00	15737.50566854	-19.16049293	4.20531117	0.65112437	1.02244277	0.99945287	0.99992753	1.00001273	0.99999881
3.20	27096.96546936	-149.92193010	24.85781020	-2.65787912	1.22616860	1.05622703	0.97741093	1.00469948	0.99932236
3.40	44990.32841110	-921.60317027	129.45372920	-9.72831443	-5.43368079	7.51120066	-1.84211140	1.84966776	0.85383668
3.60	72367.18373871	-5717.42793164	921.95825434	732.92421770	-92.84171057	415.70947206	-152.31261474	78.66362500	24.51076870
3.80	Diverged								

[a]Theoretical value of L′ is one.

TABLE II.- NON-DIMENSIONAL ANGULAR MOMENTUM AS A FUNCTION OF TIME (EXAMPLE 2) FROM INSERTION INTO TRANSLUNAR ORBIT FROM AN EARTH PARKING ORBIT

[a = 134 467 n. mi.; e = 0.9375; $t_o - T$ = 0.03941 hr; ρ = 0.2280 hr]

Time, hr	L' (6 terms)[a]	L' (18 terms)	L' (30 terms)	L' (42 terms)	L' (54 terms)	L' (66 terms)	L' (78 terms)	L' (90 terms)	L' (102 terms)
0.00	1.00000000	1.00000000	1.00000000	1.00000000	1.00000000	1.00000000	1.00000000	1.00000000	1.00000000
0.01	1.00000003	1.00000000	1.00000000	1.00000000	1.00000000	1.00000000	1.00000000	1.00000000	1.00000000
0.02	1.00000102	1.00000000	1.00000000	1.00000000	1.00000000	1.00000000	1.00000000	1.00000000	1.00000000
0.03	1.00000763	1.00000000	1.00000000	1.00000000	1.00000000	1.00000000	1.00000000	1.00000000	1.00000000
0.04	1.00003169	1.00000000	1.00000000	1.00000000	1.00000000	1.00000000	1.00000000	1.00000000	1.00000000
0.05	1.00009536	1.00000000	1.00000000	1.00000000	1.00000000	1.00000000	1.00000000	1.00000000	1.00000000
0.06	1.00023390	1.00000000	1.00000000	1.00000000	1.00000000	1.00000000	1.00000000	1.00000000	1.00000000
0.07	1.00049819	1.00000000	1.00000000	1.00000000	1.00000000	1.00000000	1.00000000	1.00000000	1.00000000
0.08	1.00095694	1.00000000	1.00000000	1.00000000	1.00000000	1.00000000	1.00000000	1.00000000	1.00000000
0.09	1.00169857	1.00000001	1.00000000	1.00000000	1.00000000	1.00000000	1.00000000	1.00000000	1.00000000
0.10	1.00283279	1.00000006	1.00000000	1.00000000	1.00000000	1.00000000	1.00000000	1.00000000	1.00000000
0.11	1.00449202	1.00000032	1.00000000	1.00000000	1.00000000	1.00000000	1.00000000	1.00000000	1.00000000
0.12	1.00683246	1.00000141	1.00000000	1.00000000	1.00000000	1.00000000	1.00000000	1.00000000	1.00000000
0.13	1.01003503	1.00000562	0.99999999	1.00000000	1.00000000	1.00000000	1.00000000	1.00000000	1.00000000
0.14	1.01430614	1.00002013	0.99999991	1.00000000	1.00000000	1.00000000	1.00000000	1.00000000	1.00000000
0.15	1.01987834	1.00006609	0.99999933	1.00000000	1.00000000	1.00000000	1.00000000	1.00000000	1.00000000
0.16	1.02701088	1.00020097	0.99999563	1.00000002	1.00000000	1.00000000	1.00000000	1.00000000	1.00000000
0.17	1.03599024	1.00057129	0.99997476	1.00000029	1.00000001	1.00000000	1.00000000	1.00000000	1.00000000
0.18	1.04713077	1.00152979	0.99986804	1.00000294	1.00000019	0.99999998	1.00000000	1.00000000	1.00000000
0.19	1.06077532	1.00388382	0.99936927	1.00002636	1.00000332	0.99999941	1.00000003	1.00000000	1.00000000
0.20	1.07724611	1.00439948	0.99721876	1.00021091	1.00005083	0.99998349	1.00000127	1.00000036	0.99999988
0.21	1.09709581	1.02178656	0.98859584	1.00152308	1.00068187	0.99960797	1.00005312	1.00002833	0.99998387
0.22	1.12060894	1.04455794	0.95622234	1.01002099	1.00810513	0.99197037	1.00186145	1.00179572	0.99823757
0.23	1.14830367	1.10443420	0.84174949	1.06057271	1.08629204	0.85623690	1.05562255	1.09464784	0.84371270
0.24	1.18068403	1.21741223	0.45857647	1.33887684	1.83074814	-1.27562261	2.43728604	5.21477953	-10.44409756
0.25	1.21829285	1.43933618	-0.76075607	2.76759534	8.30096441	-31.11421387	33.82119273	163.75213523	-689.15565288

[a]Theoretical value of L' is one.

TABLE III.- NON-DIMENSIONAL ANGULAR MOMENTUM AS A FUNCTION OF TIME (EXAMPLE 3) FOR INITIAL CONDITIONS TAKEN APPROXIMATELY 46 HOURS FROM INSERTION INTO TRANSLUNAR ORBIT

[a = 133 311 n. mi.; e = 0.9762; t_o - T = 45.71 hr; ρ = 45.71 hr]

Time, hr	[a]L′ (6 terms)	L′ (18 terms)	L′ (30 terms)	L′ (42 terms)	L′ (54 terms)	L′ (66 terms)	L′ (78 terms)	L′ (90 terms)	L′ (102 terms)
0.00	1.00000000	1.00000000	1.00000000	1.00000000	1.00000000	1.00000000	1.00000000	1.00000000	1.00000000
2.00	0.99999996	1.00000000	1.00000000	1.00000000	1.00000000	1.00000000	1.00000000	1.00000000	1.00000000
4.00	0.99999885	1.00000000	1.00000000	1.00000000	1.00000000	1.00000000	1.00000000	1.00000000	1.00000000
6.00	0.99999133	1.00000000	1.00000000	1.00000000	1.00000000	1.00000000	1.00000000	1.00000000	1.00000000
8.00	0.99996366	1.00000000	1.00000000	1.00000000	1.00000000	1.00000000	1.00000000	1.00000000	1.00000000
10.00	0.99988968	1.00000000	1.00000000	1.00000000	1.00000000	1.00000000	1.00000000	1.00000000	1.00000000
12.00	0.99972691	1.00000000	1.00000000	1.00000000	1.00000000	1.00000000	1.00000000	1.00000000	1.00000000
14.00	0.99941283	1.00000000	1.00000000	1.00000000	1.00000000	1.00000000	1.00000000	1.00000000	1.00000000
16.00	0.99886116	1.00000000	1.00000000	1.00000000	1.00000000	1.00000000	1.00000000	1.00000000	1.00000000
18.00	0.99795840	0.99999997	1.00000000	1.00000000	1.00000000	1.00000000	1.00000000	1.00000000	1.00000000
20.00	0.99656038	0.99999985	1.00000000	1.00000000	1.00000000	1.00000000	1.00000000	1.00000000	1.00000000
22.00	0.99448900	0.99999924	1.00000000	1.00000000	1.00000000	1.00000000	1.00000000	1.00000000	1.00000000
24.00	0.99152896	0.99999670	1.00000000	1.00000000	1.00000000	1.00000000	1.00000000	1.00000000	1.00000000
26.00	0.98742477	0.99998724	0.99999999	1.00000000	1.00000000	1.00000000	1.00000000	1.00000000	1.00000000
28.00	0.98187768	0.99995545	0.99999988	1.00000000	1.00000000	1.00000000	1.00000000	1.00000000	1.00000000
30.00	0.97454288	0.99985738	0.99999915	0.99999999	1.00000000	1.00000000	1.00000000	1.00000000	1.00000000
32.00	0.96502657	0.99957668	0.99999465	0.99999993	1.00000000	1.00000000	1.00000000	1.00000000	1.00000000
34.00	0.95288378	0.99882432	0.99996873	0.99999914	0.99999998	1.00000000	1.00000000	1.00000000	1.00000000
36.00	0.93761484	0.99692144	0.99983760	0.99999118	0.99999951	0.99999997	1.00000000	1.00000000	1.00000000
38.00	0.91866363	0.99235083	0.99922893	0.99991990	0.99999156	0.99999910	0.99999990	0.99999999	1.00000000
40.00	0.89541575	0.98186867	0.99662169	0.99935094	0.99987349	0.99997515	0.99999510	0.99999903	0.99999981
42.00	0.86719430	0.95881007	0.98623365	0.99525338	0.99833935	0.99941445	0.99979260	0.99992635	0.99997381
44.00	0.83325970	0.90996386	0.94747330	0.96836997	0.98066990	0.98809280	0.99263181	0.99542841	0.99715928
46.00	0.79280656	0.80498129	0.81123358	0.80634705	0.79833747	0.78830374	0.77674350	0.76391472	0.74996452
48.00	0.74496192	0.61164566	0.35780251	-0.09721476	-0.90327617	-2.32846288	-4.84811209	-9.30337182	-17.18123839
50.00	0.68878331	0.22735975	-1.07757166	-4.78963840	-15.38392773	-45.74771256	-133.01677686	-384.27132685	-1108.34246802

[a]Theoretical value of L′ is one.

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TABLE IV.- NON-DIMENSIONAL ANGULAR MOMENTUM AS A FUNCTION OF TIME (EXAMPLE 4) FOR INITIAL CONDITIONS TAKEN AT LUNAR SPHERE OF INFLUENCE (GOING TOWARDS MOON)

[$a = -2059.46$ n. mi.; $e = 1.658$; $t_o - T = -13.51$ hr; $\rho_H = 13.51$ hr]

Time, hr	[a]L′ (6 terms)	L′ (18 terms)	L′ (30 terms)	L′ (42 terms)	L′ (54 terms)	L′ (66 terms)	L′ (78 terms)	L′ (90 terms)	L′ (102 terms)
0.00	1.00000000	1.00000000	1.00000000	1.00000000	1.00000000	1.00000000	1.00000000	1.00000000	1.00000000
1.00	1.00000016	1.00000000	1.00000000	1.00000000	1.00000000	1.00000000	1.00000000	1.00000000	1.00000000
2.00	1.00000524	1.00000000	1.00000000	1.00000000	1.00000000	1.00000000	1.00000000	1.00000000	1.00000000
3.00	1.00003980	1.00000000	1.00000000	1.00000000	1.00000000	1.00000000	1.00000000	1.00000000	1.00000000
4.00	1.00016769	1.00000000	1.00000000	1.00000000	1.00000000	1.00000000	1.00000000	1.00000000	1.00000000
5.00	1.00051289	1.00000001	1.00000000	1.00000000	1.00000000	1.00000000	1.00000000	1.00000000	1.00000000
6.00	1.00127761	1.00000019	1.00000000	1.00000000	1.00000000	1.00000000	1.00000000	1.00000000	1.00000000
7.00	1.00276449	1.00000260	1.00000000	1.00000000	1.00000000	1.00000000	1.00000000	1.00000000	1.00000000
8.00	1.00539607	1.00002533	1.00000006	1.00000000	1.00000000	1.00000000	1.00000000	1.00000000	1.00000000
9.00	1.00973554	1.00018894	1.00000191	1.00000002	1.00000000	1.00000000	1.00000000	1.00000000	1.00000000
10.00	1.01650765	1.00114254	1.00004114	1.00000125	1.00000003	1.00000000	1.00000000	1.00000000	1.00000000
11.00	1.02661982	1.00583227	1.00066456	1.00006402	1.00000558	1.00000045	1.00000003	1.00000000	1.00000000
12.00	1.04118351	1.02590279	1.00848225	1.00234740	1.00058814	1.00013588	1.00002894	1.00000560	1.00000094
13.00	1.06153592	1.10244032	1.08918116	1.06574330	1.04396422	1.02719069	1.01558661	1.00819230	1.00382623
14.00	1.08926206	1.36744593	1.80011782	2.49128756	3.55409557	5.11645155	7.30789565	10.23325536	13.93487235
15.00	1.12621709	2.21353360	7.33652369	30.51489329	136.40132928	653.94844055	3496.40093994	21382.31518555	
16.00	1.17454914	4.73764381	47.01826665	578.36602402	9174.39428711		Diverged		

[a]Theoretical value of L′ is one.

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