NASr-21(11)

MEMORANDUM
RM-4749-NASA
JANUARY 1966

NASA CR70633

N66-18323	
(ACCESSION NUMBER)	(THRU)
51	1
(PAGES)	(CODE)
CR 70633	19
(NASA CR OR TMX OR AD NUMBER)	(CATEGORY)

FACILITY FORM 602

MAXIMUM LIKELIHOOD ESTIMATION AND CONSERVATIVE CONFIDENCE INTERVAL PROCEDURES IN RELIABILITY GROWTH AND DEBUGGING PROBLEMS

Richard E. Barlow, Frank Proschan and Ernest M. Scheuer

PREPARED FOR:
NATIONAL AERONAUTICS AND SPACE ADMINISTRATION

GPO PRICE $ ____

CFSTI PRICE(S) $ ____

Hard copy (HC) ____

Microfiche (MF) ____

ff 653 July 65

The RAND Corporation
SANTA MONICA • CALIFORNIA

NASr-21(11)

MEMORANDUM
RM-4749-NASA
JANUARY 1966

MAXIMUM LIKELIHOOD ESTIMATION AND CONSERVATIVE CONFIDENCE INTERVAL PROCEDURES IN RELIABILITY GROWTH AND DEBUGGING PROBLEMS

Richard E. Barlow, Frank Proschan and Ernest M. Scheuer

This research is sponsored by the National Aeronautics and Space Administration under Contract No. NASr-21. This report does not necessarily represent the views of the National Aeronautics and Space Administration.

The RAND Corporation
1700 MAIN ST. • SANTA MONICA • CALIFORNIA • 90406

Published by The RAND Corporation

PREFACE

This Memorandum derives from RAND's continuing interest in the assessment of reliability and related quantities. In part, it is complementary to work reported earlier in RM-4317-NASA.

The Memorandum is addressed to statisticians, test engineers, and managers concerned with assessing reliability. The investigation described was undertaken as a part of the reliability assessment study that RAND is conducting for the Apollo Reliability and Quality Office, Hq NASA, under contract NASr-21(11).

Two of the authors, Richard E. Barlow and Frank Proschan, have been consultants to The RAND Corporation.

SUMMARY

This study examines two problems. The first deals with estimating the reliability of a system that is undergoing developmental testing for the purpose of increasing its probability of successful operation, or increasing its time-to-failure, or decreasing its failure rate. If one or more of these events occur, we say reliability growth is taking place. Three models of reliability growth are formulated, and appropriate maximum likelihood estimates and conservative confidence bound procedures are derived for them.

The second problem treated here deals with the "debugging" of a new complex system during the initial period of its total life. During this period failures and errors are corrected as they occur, with resulting improvement in subsequent performance of the system. One mathematical idealization of this process leads to the assumption that system failure rate is decreasing with time. In practice, the debugging phase is considered completed when the failure rate reaches an equilibrium or constant value. Models are formulated for this phenomenon. Maximum likelihood estimates are obtained for relevant failure rate functions and for the end of the debugging period. A conservative upper confidence bound on the stable failure rate is obtained.

Both problems are treated from a point of view which lies between a completely nonparametric approach in which no information is assumed available concerning the form of the distribution, and a parametric outlook in which the form of the distribution is assumed known but a finite number of parameters need to be estimated.

Methods given in this paper are illustrated by numerical examples. The paper is arranged so that a reader not interested in the mathematical details can skip them, yet still understand the nature of the models and how to apply the techniques.

CONTENTS

1. INTRODUCTION

An important group of reliability problems consists of those in which the reliability of an evolving system is to be estimated at successive stages of its development. The evolution may be the result of changes in design which improve the reliability of the system, or of so-called "debugging" in which system weaknesses are gradually discovered through experience and removed. Most studies in the literature have assumed a priori knowledge of the form of the function governing reliability growth. (See, for example, Lloyd and Lipow (1962), Chap. 11; Wolman (1963); Bresenham (1964); and Corcoran, Weingarten, and Zehna (1964).) Unfortunately, in many cases the only a priori knowledge actually available is that the reliability at successive stages of evolution is monotonically increasing (not necessarily strictly).

Another group of reliability problems centers about the following. It is common practice after installing a new complex system such as that involving a missile, airplane, computer, etc., to "debug" it during the initial portion of its total life. During this debugging period, failures and errors are corrected as they occur, with resulting improvement in subsequent system performance. One mathematical idealization of this process leads to the assumption that system failure rate is decreasing with time. In practice, the debugging phase is considered completed when the failure rate reaches an equilibrium or constant value. An important problem is to determine when the constant failure rate condition has been achieved and to estimate the constant failure rate.

For the problems outlined above, we obtain maximum likelihood estimators (MLE's) and/or conservative confidence interval procedures. This is done without the customary assumptions concerning the form of the life distribution. The approach is intermediate between a completely nonparametric point of view (in which no information is assumed available concerning the form of the distribution) and a parametric outlook (in which the form of the distribution is assumed known, but a finite number of parameters are to be estimated).

The maximum likelihood estimators for our problems have, for the most part, been developed elsewhere (Ayer, et al (1955), Marshall and Proschan (1965)). We will exploit these known results and give a discussion of some properties of the estimators. One should point out that because of the constraints under which the MLE's are obtained, they do not necessarily enjoy all the desirable properties of MLE's in other situations. (See, for example, Cramér (1946), Chap. 33.)

Under the heading of conservative confidence bounds we seek methods which allow us to claim with specified (high) assurance that: a) the reliability of a system in its latest stage of development is at least a certain value; b) the cumulative distribution function, at a fixed time, of time-to-failure of a system in its latest stage of development lies below a certain value; and c) the "stable" failure rate of a system which is being debugged during development and initial use is no greater than a certain value.

If, as is customary, one assumes that the underlying failure distribution belongs to a specific family of distributions (e.g., exponential, Weibull, gamma, or normal), then standard methods are available for obtaining confidence bounds. (See Mood and Graybill (1963), Chap. 11; Lehmann (1959), Chap. 5; or Kendall and Stuart (1961), Chap. 20.) However, in many situations it may be unwarranted to assume a given form for the failure distribution. Possibly the most that one can say is that the failure rate of the system is increasing (corresponding, physically, to wearout) or decreasing (as in the case of system debugging). In other situations in which the system is evolving, it may not be reasonable to suppose that the evolution follows a specified functional form; but only that the reliability in successive stages of development has not deteriorated. In such cases, how does one obtain a confidence statement concerning reliability, failure rate, the distribution function, etc.?

Clearly, without a knowledge of the form of the distribution of a relevant statistic, one cannot hope to obtain exact confidence bounds. However, for the problems listed above, we obtain conservative confidence bounds. That is, our assurance is at least (instead of exactly equal to) a specified value that the reliability, failure rate, etc., falls in some confidence set determined from the observations. Of course, the price one pays is that the confidence sets tend to be larger than in the case in which the failure distribution is assumed to belong to a particular family of distributions. However, we shall show that the conservative confidence bounds obtained have the property that for a member of the class of distributions under consideration the confidence bounds are exact, not merely conservative.

2. MODELS

This section contains a precise specification of each mathematical model to be treated. The relevant maximum likelihood estimates and conservative confidence interval procedures are derived in Secs. 3 and 4, respectively. These procedures are summarized and illustrated in Sec. 5. Readers interested in the techniques, but not in the mathematical details, may, after reading the present section, bypass Secs. 3 and 4 and turn directly to Sec. 5.

2.1 RELIABILITY GROWTH MODELS

A. Success or Failure Observations

A system is being improved at successive stages of development. At stage i the system reliability (probability of success) is p_i. The model of reliability growth under which one obtains the MLE's of $p_1, p_2, \ldots, p_k$ assumes

(2.1) $$p_1 \leq p_2 \leq \ldots \leq p_k \ .$$

Condition (2.1) requires that reliability not degrade from stage to stage of development. No particular mathematical form of growth is imposed on reliability, however. In order to obtain a conservative lower confidence bound on p_k, we do not need as strong an assumption as specified by Condition (2.1). It suffices to require that

(2.1') $$p_k \geq \max_{i<k} p_i \ .$$

Condition (2.1') merely states that the reliability in the latest stage of development be at least as high as that achieved earlier in the development program. Condition (2.1') is clearly weaker than Condition (2.1).

Our data consist of x_i successes in n_i trials in stage i, $i = 1, 2, \ldots, k$.

A variation of this model is treated in Barlow and Scheuer (1964). There, two types of failure -- inherent and assignable cause -- are distinguished.

B. Life Length Observations - Ordered Distribution Functions

A system is being improved at successive stages of development. At stage i, the distribution of system life length is F_i. The model of reliability growth under which we obtain MLE's of $F_1(t)$, $F_2(t)$, ..., $F_k(t)$ for a single, fixed value of t, writing $\bar{F}_i(t) = 1 - F_i(t)$, is

(2.2) $$\bar{F}_1(t) \le \bar{F}_2(t) \le \ldots \le \bar{F}_k(t) \text{ for a fixed } t \ge 0.$$

In order to obtain a conservative, upper confidence curve on $F_k(t)$ and thereby, a conservative lower confidence curve on $\bar{F}_k(t)$ for all non-negative values of t, it suffices to require that

(2.2') $$\bar{F}_k(t) \ge \max_{i<k} \bar{F}_i(t) \quad \text{for all } t \ge 0.$$

Condition (2.2') states that the probability of system survival beyond any time t in the latest stage of development is at least as high as that achieved earlier in the development program.

Our data consist of independent life length observations $X_{i1}, \ldots, X_{in_i}$, $i = 1, \ldots, k$.

Note that (2.2) and (2.1) are equivalent if one calls the event $X_{ij} > t$ "success" and the event $X_{ij} \le t$ "failure," $j = 1, \ldots, n_i$; $i = 1, \ldots, k$.

C. Life Length Observations - Ordered Increasing Failure Rate Functions

We first define precisely "failure rate" in general and "increasing failure rate" in particular. For a distribution F with density f, the failure rate r(t) at time t is defined as $r(t) = f(t)/\bar{F}(t)$, where $\bar{F}(t) = 1 - F(t)$. Thus, $r(t) \cdot dt$ may be interpreted physically as the probability of failure in the interval $[t, t + dt]$

given survival to time t. If a unit is wearing out, it may very likely display an increasing failure rate. Note that when the failure rate is increasing, log $\bar{F}(t)$ is concave. This motivates the definition of increasing failure rate (IFR) in the general case where a density at each point is not assumed necessarily to exist. We say failure distribution F has increasing failure rate if log $\bar{F}(t)$ is concave; similarly, we say F has decreasing failure rate (DFR) if $\lim_{x \uparrow 0} F(x) = 0$ and log $\bar{F}(t)$ is convex on $[0, \infty)$. (See Barlow, Marshall, and Proschan (1963) for a discussion of the properties of distributions with increasing (decreasing) failure rate.)

Assume, then, that system life at the i-th stage of development has increasing failure rate. Because of improvement from stage to stage

$$r_1(t) \geq r_2(t) \geq \ldots \geq r_k(t) \text{ for } t \geq 0, \tag{2.3}$$

where $r_i(t)$ is the failure rate at time t at the i-th stage of development. This means that for each $t \geq 0$, the probability of system failure in the interval $(t, t + dt)$, given survival till time t, does not increase from stage to stage of testing.

Given life-length observations $X_{i1}, X_{i2}, \ldots, X_{in_i}$, we wish to obtain the maximum likelihood estimate of $r_1(t), r_2(t), \ldots, r_k(t)$, especially $r_k(t)$.

A similar model may be formulated in which system life has decreasing failure rate.

2.2. DEBUGGING MODELS

Debugging and reliability growth have a good deal in common. The main difference is that in the debugging models described below, system reliability improves continuously with time (i.e., failure rate is decreasing), while in the reliability growth models of Sec. 2.1, improvement in system reliability occurs only when a system design change occurs.

A. Decreasing Failure Rate, Followed by Constant Failure Rate

Suppose X_1, the time to the first failure, has distribution $F(t)$ having failure rate $r(t)$ which is decreasing for $0 \le t < t_0$ and constant for $t > t_0$. After failure, repair is performed so that the system operates again. Assume further that the system failure rate is restored to the value it had just prior to the failure. Specifically, assume that X_i, the time between the (i-1)st failure and the i-th failure has distribution

$$(2.4) \qquad F_{X_i}(x) = \frac{F(S_{i-1} + x) - F(S_{i-1})}{1 - F(S_{i-1})} \quad \text{for } x \ge 0$$

(where $S_{i-1} = X_1 + \ldots + X_{i-1}$), the conditional distribution of a system of age S_{i-1}.

Given observations $X_1, X_2, \ldots, X_b$, we seek: a) maximum likelihood estimates of t_o, the end of the debugging period, and $r(t_o)$, the equilibrium failure rate; and b) a conservative upper confidence bound on $r(t_o)$.

It is of interest to note the common-sense procedure often used in this situation to estimate t_o and $r(t_o)$. A graph is drawn in which the cumulative number of failures is plotted against elapsed operational time as in Fig. 2.1. (See Rosner (1961).) Debugging is terminated approximately at that point in time when the slopes $\frac{h - (h-1)}{S_h - S_{h-1}} = \frac{1}{X_h}$ of successive secants (shown by dashed lines) appear to have reached an equilibrium value. These slopes represent failure rates over successive time periods. System improvement corresponds to the situation in which the slopes, $\frac{1}{X_h}$, are decreasing with h. However, due to statistical fluctuations, some reversals will occur. The common-sense graphical procedure described above furnishes no precise way of taking into account these reversals. Our technique provides for this.

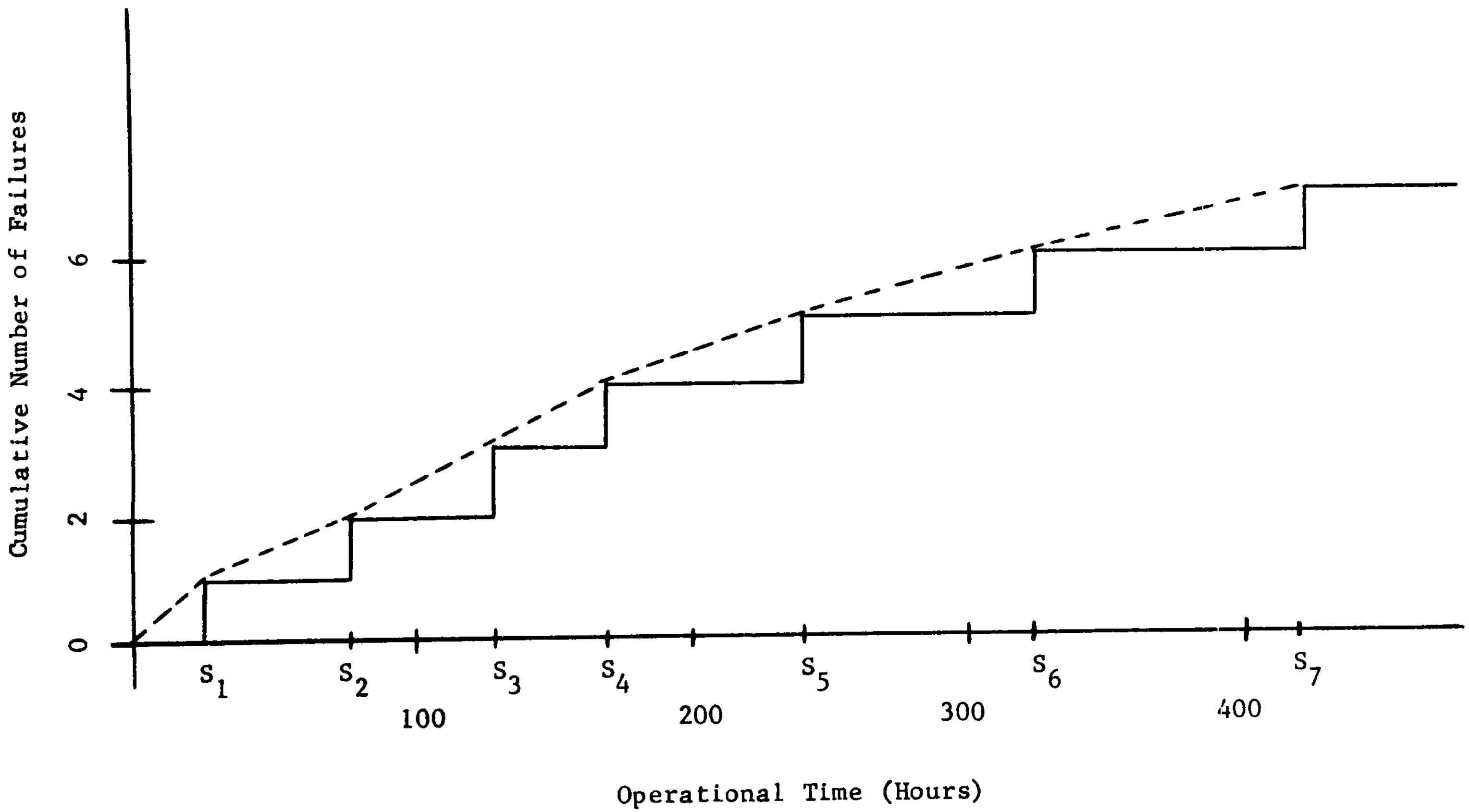

Fig.2.1—Cumulative number of failures versus time

B. Decreasing Failure Rate, Negligible Decrease Beyond a Point

In many practical situations it is not realistic to insist on determining the point t_o beyond which failure rate is constant. Rather, for pragmatic purposes it suffices to find the point t_1 such that $r(t_1) - \lim_{t \to \infty} r(t) = \epsilon$, for some specified $\epsilon > 0$. Thus, we wish to find the point beyond which further reliability improvement can decrease the failure rate by only ϵ.

Specifically, we use the same assumptions and notation as in 2.2A; there is the additional assumption that r(t) is continuous. We wish to obtain maximum likelihood estimates of t_1 and a conservative upper confidence bound on $r(t_1)$.

C. Debugging Not Completed During Period of Observation

1) The assumptions and notation are as in 2.2A. Assume, however, that constant failure rate is not attained during the period of observation. Finally, assume that observations $X_1, X_2, \ldots X_n$ are available from just one copy of the system. We wish to estimate r(t) for $0 \leq t \leq S_n = X_1 + \ldots + X_n$.

2) In a more general version of 1), we have k copies of the system, each copy independently operating as in 1). Observations X_{ij}, $i = 1, \ldots, k$; $j = 1, \ldots, n_i$, are obtained. The distribution of X_{ij} is

$$F_{X_{ij}}(x) = \frac{F(S_{i,j-1} + x) - F(S_{i,j-1})}{\bar{F}(S_{i,j-1})},$$

the conditional distribution of an item of age $S_{i,j-1} = X_{i1} + X_{i2} + \ldots + X_{i,j-1}$. Again we wish to estimate r(t) during the observation period.

3. MAXIMUM LIKELIHOOD ESTIMATION UNDER CONSTRAINTS

We may obtain a maximum likelihood estimator (MLE) in each of the models of Sec. 2 using methods developed in papers by Brunk (1955, 1958); by Ayer, Brunk, Ewing, Reid, and Silverman (1955); and by Van Eeden (1956, 1957, 1958). The key idea may be described as follows:

For Models 2.1C and 2.2A, B, and C of Section 2, to maximize the likelihood, the failure rate should be taken constant between observations. This is intuitively obvious from the fact that the likelihood depends only on the densities at the observations. Thus, the failure rate should be made as small as possible between observations so as to have as much probability as possible available at the observations. Since the failure rate is assumed monotone, the failure rate must be set constant between observations. This reduces the problem to the maximization of a function of a finite number of unknown parameters subject to constraints. In a similar manner, the problem of Model 2.1B may be reduced to the maximization of a function of a finite number of unknown parameters subject to constraints.

Brunk (1958) presents a procedure which may be used for maximizing a concave function $G(y_1, \ldots, y_k)$ of k real variables subject to N constraints. The i-th constraint requires that the vector of k variables lie in a convex region A_i of Euclidean k-space, i = 1, 2, ..., N. The first step is to determine the point at which G attains its unrestricted maximum. If this point lies in $A_1A_2 \ldots A_N$ (the intersection of $A_1, A_2, \ldots, A_N$), it is the solution. If not, one of the sets is selected in which it does not lie, and designated as A_1(relabeling perhaps being necessary). Next, we seek the point q_1 in A_1 at which G attains its maximum over A_1. If $B(A_i)$ represents the boundary of A_i, i = 1, 2, ..., N, we find that q_1 lies in $B(A_1)$. If q_1 lies in $A_1A_2 \ldots A_N$, it is the solution. If not, we designate as A_2 (relabeling, if necessary) one of the sets which does not contain q_1. Next, we maximize G subject to $y \in A_1A_2$. The maximizing value q_{12} lies on $B(A_2)$. Next , we find the point, q_2, where G is maximized subject to $y \in A_2$. If $q_2 \in A_1A_2$, then $q_2 = q_{12}$ is the solution of the present limited problem. Otherwise, $q_{12} \in B(A_1)B(A_2)$. We continue in this fashion until that point in $A_1A_2\ldots A_N$ is found at which the maximum of G is attained.

This concise description of the Brunk procedure for finding a maximum subject to constraints will become more meaningful when it is applied to the models of Sec. 2.

3.1 THE MLE'S FOR THE RELIABILITY GROWTH MODELS

A. Success or Failure Observations

The MLE's for this model are found by Ayer, Ewing, Brunk, Reid, and Silverman (1955). They are displayed, together with an example, in Sec. 5.1A.

B. Model 2.1B - Life Length Observations - Ordered Distribution Functions

Again, Ayer, Ewing, Brunk, Reid, and Silverman (1955) have found the MLE for this model. We display them, together with an example, in Sec. 5.2A.

C. Model 2.1C - Life Length Observations - Ordered, Increasing Failure Rate Functions

Grenander (1956), and Marshall and Proschan (1965) have given the MLE for a single failure rate function, $r(\cdot)$, under the assumption that $r(\cdot)$ is a monotone function. Here we seek the MLE of several failure rate functions, $r_1(\cdot), \ldots, r_k(\cdot)$, under the assumption that each $r_i(\cdot)$ is an increasing function and that $r_1(t) \geq r_2(t) \geq \ldots \geq r_k(t)$ for all $t \leq 0$. The log-likelihood, which is to be maximized by proper choice of the functions $r_1(\cdot), \ldots, r_k(\cdot)$ subject to the constraints, is

$$\sum_{i=1}^{k} \sum_{j=1}^{n_i} \log r(X_{ij}) - \sum_{i=1}^{k} \sum_{j=1}^{n_i - 1} (n_i - j)(X_{i,j+1} - X_{ij}) r(X_{ij}). \tag{3.1}$$

The MLE's $\hat{r}_1(\cdot)$, ..., $\hat{r}_k(\cdot)$ are step functions, constant between observations. In some instances it is convenient to determine the maximizing step functions by a "concave programming" procedure.

3.2 THE MLE'S FOR THE DEBUGGING MODELS

Using arguments similar to those in Marshall and Proschan (1965), we may establish that the MLE for $r(t)$, call it $\hat{r}_n(t)$, is constant on the intervals between observations. Explicit formulas for the estimate of $r(t)$ for the various models will be given in Sec. 5.2. Reference to that section will be helpful for the remainder of Sec. 3.2.

A. MLE of t_o and $r(t_o)$ for Model 2.2A

The MLE $\hat{t}_o$ of t_o is obtained by first going through the same procedure as in estimating $r(t)$ in Sec. 5.2 below. The estimate $\hat{t}_o$ is taken as the observation corresponding to the beginning of the last averaging interval. In the notation of Sec. 5.2, $\hat{t}_o = S_{n_k}$.

It turns out that in this case the MLE of t_o is a poor one since, as the number of observations increases to infinity, $\hat{t}_o$ converges almost surely to infinity.

The estimate of $r(t_o)$ is the value of $\hat{r}(t)$ in the last interval. In the notation of Sec. 5.2, $\hat{r}(t_o) = r_{n_k+1,n}$.

B. MLE of t_1 for Model 2.2B

Again we proceed as in Sec. 5.2 below to obtain the MLE of $r(t)$ under the assumption of decreasing failure rate. Let k^* be the smallest index k, such that $\hat{r}(S_k) - \hat{r}(S_n) \le \epsilon$. Then the MLE $\hat{t}_1$ for t_1 is S_{k^*} since $\hat{r}(t)$ is the MLE for $r(t)$, $0 \le t \le S_n$, and $\hat{r}(S_{k^*}) - \hat{r}(S_n) \ge \epsilon$ while $\hat{r}(S_{k^*}^+) - \hat{r}(S_n) < \epsilon$.

In this case, the MLE $\hat{t}_1$ does not converge almost surely to infinity when the sample size n grows indefinitely large, as in Model 2.2A; thus, the estimate does not suffer from the serious weakness possessed by $\hat{t}_o$ of Model 2.2A. On the other hand, $\hat{t}_1$ does not converge almost surely to t_1 as $n \to \infty$, since successive observations provide information about the failure rate at successively later points on the time axis while, roughly speaking, for consistency (convergence almost surely of $\hat{t}_1$ to t_1) one needs more and more information about t_1 as $n \to \infty$. This lack of consistency is no real criticism, however, since from the nature of the model, no consistent estimator of t_1 exists.

C. Debugging Not Completed During Period of Observation

Marshall and Proschan (1965) have provided a procedure for obtaining the MLE for monotone failure rate functions. This is precisely the technique needed for this model. The procedure is outlined and examples are given in Sec. 5.2.

4. CONSERVATIVE CONFIDENCE BOUNDS

In this section methods are presented which allow us to claim with specified (high) assurance that: a) the reliability of a system in its latest stage of development is at least a certain value; b) the distribution function of time-to-failure of a system in its latest stage of development everywhere lies below a certain curve; c) the "stable" failure rate of a system which is being debugged during development and initial use is no greater than a certain value.

4.1 A CONSERVATIVE CONFIDENCE BOUND ON THE STABLE FAILURE RATE OF A SYSTEM BEING DEBUGGED

In this section we show how to obtain a conservative confidence bound on the stable failure rate which is finally attained by a system being debugged during its development phase and early usage period. This particular example should make the basic idea clear. We then state the general theorem which exploits this basic idea. With this general theorem as a basis, conservative confidence bounds for our models are developed.

We consider Model 2.2A in which the failure rate of the system is a decreasing function of time for $0 \le t \le t_o$ and is a constant, say equal to r_o, for $t \ge t_o$. The system is put into operation at time zero and is run until failure at time X_1. Repair occurs in negligible time (e.g., the module containing the failed part is replaced immediately). The system resumes operation and continues until the next failure occurs X_2 units of time later, i.e., at times $S_2 = X_1 + X_2$. Again, repair occurs in negligible time, and the system runs until the next failure X_3 units of time later, at time $S_3 = X_1 + X_2 + X_3$. This continues until n successive life lengths $X_1, X_2, \ldots, X_n$ are obtained.

The basic idea in obtaining the conservative confidence bound on r_o, the stable failure rate, may be stated intuitively as follows. The observation, X_i, is a random variable from a distribution whose failure rate at each point of time is at least as great as r_o, $i = 1, 2, \ldots, n$. Therefore, if one uses the observations $X_1, X_2, \ldots, X_n$ to estimate a single failure rate (pretending that all the X_i are from a common exponential distribution), the estimate will tend to be higher than r_o. Similarly, an upper confidence bound for this common failure rate, calculated from the observations $X_1, X_2, \ldots, X_n$ as though they were a sample from a single exponential distribution, will constitute a conservative upper confidence bound for r_o. We make these ideas precise now.

Lemma 4.1. Let X_1 have distribution $F(x)$, X_2 have conditional distribution

$$\frac{F(X_1+x) - F(X_1)}{\bar{F}(X_1)}, \ldots,$$

X_i have conditional distribution

$$\frac{F(S_{i-1}+x) - F(S_{i-1})}{\bar{F}(S_{i-1})}, \ldots,$$

where F has failure rate $r(t) \geq r_o$ for all $t \geq 0$. Let $Y_1, Y_2, \ldots, Y_n$ be independent observations from the exponential distribution with failure rate r_o. Then $\Sigma_1^n X_i$ is stochastically smaller* than $\Sigma_1^n Y_i$.

Proof. First assume that F is continuous. Let the random variables $X_1, X_1 + X_2, \ldots, \Sigma_1^n X_i$ be simultaneously transformed into random variables $Y_1', Y_1' + Y_2', \ldots, \Sigma_1^n Y_i'$ under the transformation

$$(4.1) \qquad Y_1' + \ldots + Y_i' = -\frac{1}{r_o} \log \bar{F}(X_1 + \ldots + X_i), \; i = 1, \ldots, n.$$

---

*If $P(U \geq t) \leq P(V \geq t)$ for each t, then the random variable U is said to be stochastically smaller than the random variable V.

Then for $i = 1, \ldots, n$

$$P[Y_i' > u] = P[-\frac{1}{r_o} \log \bar{F}(X_1 + \ldots + X_i) + \frac{1}{r_o} \log \bar{F}(X_1 + \ldots + X_{i-1}) > u]$$

$$= P\left[\log \frac{\bar{F}(X_1 + \ldots + X_i)}{\bar{F}(X_1 + \ldots + X_{i-1})} < -r_o u\right]$$

$$= P\left[\frac{\bar{F}(X_1 + \ldots + X_i)}{\bar{F}(X_1 + \ldots + X_{i-1})} < e^{-r_o u}\right] = e^{-r_o u},$$

since the random variable

$$\frac{\bar{F}(X_1 + \ldots + X_i)}{\bar{F}(X_1 + \ldots + X_{i-1})}$$

is uniformly distributed on $[0,1]$.* Thus, the $Y_1', \ldots, Y_n'$ are independently distributed according to

$$G_{r_o}(x) = 1 - e^{-r_o x},$$

the exponential distribution with failure rate r_o.

Next observe that if $y = -\frac{1}{r_o} \log \bar{F}(x)$, then

$$\frac{dy}{dx} = \frac{r(x)}{r_o} \geq 1 \text{ for all } x \geq 0.$$

---

*If a random variable T has survival probability function $\bar{H}$, a continuous function, then the random variable $\bar{H}(T)$ is uniformly distributed on $[0,1]$; $\bar{F}(S_{i-1} + x)/\bar{F}(S_{i-1})$ is the conditional survival probability function of X_i given S_{i-1}.

Thus, under the transformation (4.1)

$$Y_1' + \dots + Y_n' \geq X_1 + \dots + X_n .$$

It follows from Lehmann (1959), Lemma 1, p. 73*, that $\Sigma_1^n Y_i$ is stochastically larger than $\Sigma_1^n X_i$.

If F is not continuous, the same result may be obtained by limiting arguments.||

We now apply Lemma 4.1 to obtain a conservative confidence bound on r_o from observations $X_1, \dots, X_n$.

Since $Y_1, \dots, Y_n$ are exponential with failure rate r_o, $\chi^2_{1-\alpha}(2n)/2\Sigma_1^n Y_i$ is an upper $100(1-\alpha)$ percent confidence bound on r_o, where $\chi^2_{1-\alpha}(2n)$ is the $100(1-\alpha)$ percentile of the chi-square distribution with 2n degrees of freedom. Hence

$$1-\alpha = P[r_o \leq \chi^2_{1-\alpha}(2n)/2\Sigma_1^n Y_i] \leq P[r_o \leq \chi^2_{1-\alpha}(2n)/2\Sigma_1^n X_i] .$$

Thus $\chi^2_{1-\alpha}(2n)/2\Sigma_1^n X_i$ is a conservative $100(1-\alpha)$ percent upper confidence bound on r_o. Note that if F is the exponential distribution, the confidence bound is exact.

---

*This Lemma states: "Let F_o and F_1 be two cumulative distribution functions on the real line. Then $F_1(x) \leq F_o(x)$ for all x if and only if there exist two nondecreasing functions f_o and f_1, and a random variable V, such that a) $f_o(v) \leq f_1(v)$ for all v; and b) the distributions of $f_o(v)$ and $f_1(v)$ are F_o and F_1 respectively." In our case, take $f_o(v) = v$, $f_1(v) = -\frac{1}{r_o} \log \bar{F}(v)$, $F_o = F$, $F_1 = G_{r_o}$.

4.2 GENERAL THEOREM FOR CONSERVATIVE CONFIDENCE BOUNDS

The ideas used in Sec. 4.1 lead to the following general theorem for obtaining conservative confidence bounds.

Theorem 4.1. Let

a) $\underline{Y}$ be an observation on a random variable (in general, vector-valued) having distribution function $G(\underline{y},\theta)$, with θ a one-dimensional parameter;

b) $\hat{\theta}(\underline{Y})$ be a one-dimensional statistic based on the observed vector $\underline{Y}$;

c) $\rho(\hat{\theta}(\underline{Y}))$ be a $100(1-\alpha)$ percent upper confidence bound on θ, where $\rho(u)$ is a decreasing function;

d) $\underline{X}$ be an observation on a random variable (vector-valued) having distribution function $F(\underline{x},\theta)$; and

e) $\hat{\theta}(\underline{Y})$ be stochastically larger than $\hat{\theta}(\underline{X})$.

Then

$$P[\rho(\hat{\theta}(\underline{X})) \geq \theta | F(\underline{x},\theta)] \geq 1-\alpha,$$

that is, $\rho(\hat{\theta}(\underline{X}))$ is a conservative $100(1-\alpha)$ percent upper confidence bound on θ, the parameter of the distribution F.

Proof. First, assume ρ is continuous and strictly decreasing. Let $u(\cdot)$ be the inverse of the function $\rho(\cdot)$. Then

$$1-\alpha = P[\rho(\hat{\theta}(\underline{Y})) \geq \theta | G(\underline{y},\theta)] = P[\hat{\theta}(\underline{Y}) \leq u(\theta) | G(\underline{y},\theta)],$$

the first equality following from c) and the second holding since u(·) is the inverse function to ρ(·), a decreasing function. By e)

$$P[\hat{\theta}(\underline{Y}) \leq u(\theta)\,|\,G(\underline{y},\theta)] \leq P[\hat{\theta}(\underline{X}) \leq u(\theta)\,|\,F(\underline{x},\theta)]$$

$$= P[\rho(\hat{\theta}(\underline{X})) \geq \theta\,|\,F(\underline{x},\theta)],$$

the last equality following again from the fact that u(·) is the inverse of ρ(·) and ρ(·) is decreasing. Combining results, we obtain

$$P[\rho(\hat{\theta}(\underline{X})) \geq \theta\,|\,F(\underline{x},\theta)] \geq 1-\alpha.$$

If ρ is not continuous or strictly decreasing, the same results may be obtained by limiting arguments.||

Other cases of interest are covered in:

Corollary 4.2.

1) If ρ(u) is an increasing function and $\hat{\theta}(\underline{X})$ is stochastically larger than $\hat{\theta}(\underline{Y})$, the same result follows.

2) If ρ(u) is a decreasing function, $\rho(\hat{\theta}(\underline{Y}))$ is a 100(1-α) percent lower confidence bound on the parameter θ of G, and $\hat{\theta}(\underline{X})$ is stochastically larger than $\hat{\theta}(\underline{Y})$, then $\rho(\hat{\theta}(\underline{X}))$ is a conservative 100(1-α) percent lower confidence bound on the parameter θ of F.

3) If ρ(u) is an increasing function, $\rho(\hat{\theta}(Y))$ is a 100(1-α) percent lower confidence bound on the parameter θ of G, and $\hat{\theta}(\underline{X})$ is stochastically smaller than $\hat{\theta}(\underline{Y})$, then $\rho(\hat{\theta}(X))$ is a conservative 100(1-α) percent lower confidence bound on the parameter θ of F.

The proof in each case is similar to that of Theorem 4.1.

4.3 RELIABILITY GROWTH WHEN ONLY SUCCESS OR FAILURE IS OBSERVED

In Model 2.1A, we considered a system being improved at successive stages of development. Specifically, in condition (2.1'), we require $p_k \geq \max_{i<k} p_i$; that is, that the reliability in the latest stage of development be at least as high as that achieved earlier in the development program. Suppose that x_i successes are observed in n_i trials in stage i, $i = 1, 2, \ldots, k$, where all trials are independent. From this set of observations, we wish to establish a conservative $100(1-\alpha)$ percent lower confidence bound on p_k, the reliability of the latest version of the system for which data are at hand. A variation of this model (in which two kinds of failure are distinguished) is treated by Barlow and Scheuer (1964).

Let X_i be a binomial random variable corresponding to n_i trials with underlying probability of success p_i, $X = \Sigma_1^k X_i$, and let Y be a binomial random variable corresponding to $n = \Sigma_1^k n_i$ trials with underlying probability of success p_k. Then Y is stochastically larger than X since Y is the sum of independent random variables each of which is stochastically larger than the corresponding random variables comprising X. Using Corollary 4.2(3), one may obtain a sharp conservative $100(1-\alpha)$ percent lower confidence bound for p_k by finding the value $p(x)$ satisfying

$$\Sigma_o^x \binom{n}{i} p^i (1 - p)^{n-1} = 1 - \alpha,$$

where $x = \Sigma_1^k x_i$. Then

$$P[p_k \geq p(x)] \geq 1 - \alpha;$$

that is, $p(x)$ is a conservative $100(1-\alpha)$ percent lower confidence bound for p_k, the reliability at the latest stage of development.

Note that the only information required to find the desired bound is the total number of successes and the total number of trials. The stage-by-stage history of the development program is not needed.

4.4 RELIABILITY GROWTH WHEN LIFE LENGTHS ARE OBSERVED

In Model 2.1B, we consider a system being improved at successive stages of development corresponding, say, to basic design changes. At stage i, the distribution of life length is F_i. No assumption is made about the form of $F_1, F_2, \ldots, F_k$, nor about the relation among them except that $F_k(t) \le \min_{i<k} F_i(t)$ for all t. This means that the probability of system life exceeding any fixed time is greatest at the last stage of system development. Independent life length observations $X_{i1}, \ldots, X_{in_i}$ are obtained at stage i, $i = 1, 2, \ldots, k$. From these $n = \Sigma_1^k n_i$ observations, we wish to obtain a conservative $100(1-\alpha)$ percent upper confidence curve on the entire failure distribution function, $F_k(t)$, of system life in the latest stage of development.

We first prove

Lemma 4.2. Let $X_{i1}, \ldots, X_{in_i}$ be independent observations from F_i, $i = 1, \ldots, k$, with $F_k(t) \le \min_{i<k} F_i(t)$ for all t, and let $\hat{F}(t)$ be the empirical distribution function formed from all the observations $X_{i1}, \ldots, X_{in_i}$, $i = 1, \ldots, k$. Let $Y_{i1}, \ldots, Y_{in_i}$, $i = 1, \ldots, k$ be independent observations from $F_k(t)$, and let $\hat{F}_k(t)$ be the empirical distribution function formed from $Y_{i1}, \ldots, Y_{in_i}$, $i = 1, \ldots, k$. Then given any function u(t)

$$P[\hat{F}(t) \ge u(t) \text{ for all } t] \ge P[\hat{F}_k(t) \ge u(t) \text{ for all } t].$$

Proof. First assume F_i continuous and strictly increasing, $i = 1, \ldots, k$. Define $Z_{ij} = F_i^{-1}F_k(Y_{ij})$, $j = 1, \ldots, n_i$; $i = 1, \ldots, k$. Then the set of random variables Z_{ij}, $j = 1, \ldots, n_i$; $i = 1, \ldots, k$ has the same joint distribution as the set X_{ij}, $j = 1, \ldots, n_i$, $i = 1, \ldots, k$. (This statement may be verified by using the following argument:

$$P[Z_{ij} \le z_{ij}] = P[F_i^{-1} F_k(Y_{ij}) \le z_{ij}] = P[F_k(Y_{ij}) \le F_i(z_{ij})]$$

$$= P[F_i(X_{ij}) \le F_i(z_{ij})] = P[X_{ij} \le z_{ij}]$$

Since each Y_{ij} has distribution F_k and each X_{ij} has distribution F_i, each of the $F_k(Y_{ij})$'s and $F_i(X_{ij})$'s is a uniform random variable on (0, 1).)

Moreover, since $F_i(t) \ge F_k(t)$ for $i = 1, \ldots, k$ and all t, $Z_{ij} \le Y_{ij}$, $j = 1, \ldots, n_i$; $i = 1, \ldots, k$. Thus, for each t, $\hat{H}(t) \ge \hat{F}_k(t)$, where $\hat{H}(t)$ is the empirical distribution function formed from Z_{ij}, $j = 1, \ldots, n_i$; $n = 1, \ldots, k$. Thus, for any function $u(t)$, $P[\hat{H}(t) \ge u(t)$ for all $t] \ge P[\hat{F}_k(t) \ge u(t)$ for all $t]$. Finally, since the Z_{ij}'s have the same joint distribution as the X_{ij}'s, $P[\hat{F}(t) \ge u(t)$ for all $t] \ge P[\hat{F}_k(t) \ge u(t)$ for all $t]$.

If any F_i is not continuous or strictly increasing, the same result may be obtained by limiting arguments.||

We now use Lemma 4.2 to form a conservative $100(1-\alpha)$ percent upper confidence curve on the entire distribution F_k. Birnbaum and Tingey (1959) tabulate values $\epsilon_{n,\alpha}$ satisfying

$$P[G(t) \le \min(\hat{G}(t) + \epsilon_{n,\alpha}, 1) \text{ for all } t] \ge 1 - \alpha,$$

where $G(t)$ is a distribution function and $\hat{G}(t)$ is the corresponding empirical distribution function based on a sample of n from G. The value $\epsilon_{n,\alpha}$ is independent of $G(t)$. If in Lemma 4.2 we take $u(t) = F_k(t) - \epsilon_{n,\alpha}$, then

$$P[\hat{F}(t) + \epsilon_{n,\alpha} \ge F_k(t) \text{ for all } t] \ge P[\hat{F}_k(t) + \epsilon_{n,\alpha} \ge F_k(t) \text{ for all } t] \ge 1-\alpha;$$

that is, $\hat{F}(t) + \epsilon_{n,\alpha}$ is the desired conservative $100(1-\alpha)$ percent upper confidence curve on F_k.

5. SUMMARY OF PROCEDURES AND EXAMPLES

In this section we describe and illustrate by example the maximum likelihood estimation and conservative confidence interval procedures for the models of Sec. 2.

5.1 RELIABILITY GROWTH MODELS

5.1A Model 2.1A - Success or Failure Observations

Our data consist of x_i successes from n_i observations in stage i, $i = 1, \ldots, k$.

(i) To obtain the maximum likelihood estimates of $p_1, \ldots, p_k$ subject to the restriction that $p_1 \leq p_2 \leq \ldots \leq p_k$, first form the ratios $x_1/n_1, x_2/n_2, \ldots, x_k/n_k$. If $x_1/n_1 \leq x_2/n_2 \leq \ldots \leq x_k/n_k$, then x_i/n_i is the MLE $\hat{p}_i$ of p_i. If for some j ($j = 1, \ldots, k - 1$), $x_j/n_j > x_{j+1}/n_{j+1}$, combine the observations in the j-th and (j + 1)-st stages and examine the ratios,

$$x_1/n_1, \ldots, x_{j-1}/n_{j-1}, (x_j + x_{j+1})/(n_j + n_{j+1}),$$

$$x_{j+2}/n_{j+2}, \ldots, x_k/n_k,$$

for the (k - 1) stages thus formed. If these ratios are in nondecreasing order, they constitute the MLE's of $p_1, \ldots, p_k$ with $\hat{p}_j = \hat{p}_{j+1} = (x_j + x_{j+1})/(n_j + n_{j+1})$. If not, continue the process of combining stages until the ratios are in nondecreasing order. This process need be repeated at most (k - 1) times, and the result is independent of the order in which stages are combined to eliminate reversals in the sequence of ratios.

Example: The procedure is illustrated by the data in the following table.

Stage (i)	No. of Successes (x_i)	No. of Trials (n_i)	x_i/n_i
1	2	5	.400
2	3	7	.429
3	3	8	.375
4	2	6	.333
5	6	6	1.000

The process of combining stages to get a sequence of non-decreasing ratios is summarized below:

i	x_i	n_i	x_i/n_i	First Combination	Second Combination	Third Combination
1	2	5	.400	.400	.400	
2	3	7	.429			
3	3	8	.375	$\frac{6}{15}$ = .400	$\frac{8}{21}$ = .381	$\frac{10}{26}$ = .385
4	2	6	.333	.333		
5	6	6	1.000	1.000	1.000	1.000

Thus, we obtain the maximum likelihood estimates

$$\hat{p}_1 = \hat{p}_2 = \hat{p}_3 = \hat{p}_4 = .385, \; \hat{p}_5 = 1.000.$$

(ii) A conservative $100(1-\alpha)$ percent lower confidence bound on p_k, the reliability of the latest version of the system, is found by treating the data from the k stages of the development program as though they were homogeneous, then applying the standard binomial approach to get a lower confidence bound on a binomial parameter having observed $x = \Sigma_1^k x_i$ successes in $n = \Sigma_1^k n_i$ trials. (See Mood and Graybill (1963).) Thus, to obtain the conservative lower confidence bound on p_k, the stage-by-stage history of the development program is not needed; only the total number of successes and the total number of trials.

<u>Example</u>: In the example 5.1A(i) above, a total of 16 successes were observed in a total of 32 trials. A 95 percent lower confidence bound for a single binomial parameter based on these data is found from binomial tables to be .344. Thus, a conservative 95 percent lower confidence bound for p_5 is .344.

<u>5.1B Model 2.1B - Life Length Observations - Ordered Distribution Functions</u>

(i) As described in Sec. 2.1, Model A, let $X_{i1}, \ldots, X_{in_i}$ be the observations obtained from distribution F_i, where because of reliability growth, it is known that $F_1(t) \geq F_2(t) \geq \ldots \geq F_k(t)$ for a fixed $t \geq 0$. We obtain MLE's of $F_1(t), \ldots, F_k(t)$ for this fixed value of t as follows. For i = 1, 2, ..., k, obtain the empirical distribution function $F_{in_i}(t)$ from $F_{in_i}(t) = m_i(t)/n_i$, where $m_i(t)$ = number of observations among $X_{i1}, X_{i2}, \ldots, X_{in_i}$ not exceeding t. If $F_{1n_1}(t) \geq F_{2n_2}(t) \geq \ldots \geq F_{kn_k}(t)$, then these constitute MLE's of $F_1(t), F_2(t), \ldots, F_k(t)$ respectively. Suppose, on the contrary, the reversal,

$$\frac{m_i(t)}{n_i} < \frac{m_{i+1}(t)}{n_{i+1}} ,$$

occurs. Then the MLE is obtained by assuming a common value for $F_i(t)$ and $F_{i+1}(t)$. Under this assumption the MLE of this common value is obtained by pooling the observations from the two distributions to obtain

$$\frac{m_i(t) + m_{i+1}(t)}{n_i + n_{i+1}},$$

as the MLE of the common value $F_i(t) = F_{i+1}(t)$. We then examine

$$\frac{m_1(t)}{n_1}, \ldots, \frac{m_{i-1}(t)}{n_{i-1}}, \frac{m_i(t) + m_{i+1}(t)}{n_i + n_{i+1}}, \frac{m_i(t) + m_{i+1}(t)}{n_i + n_{i+1}},$$

$$\frac{m_{i+2}(t)}{n_{i+2}}, \ldots, \frac{m_k(t)}{n_k} .$$

If these are in decreasing order, they constitute MLE's of $F_1(t), \ldots, F_{i-1}(t), F_i(t), F_{i+1}(t), F_{i+2}(t), \ldots, F_k(t)$. If on the other hand, a reversal exists, we pool as before to eliminate the reversal (adding the various $m_i(t)$ involved in the reversal to obtain a new numerator and adding the corresponding n_i to obtain a new denominator). After each reversal has been eliminated, we test the resulting sequence of ratios to see if they are in decreasing order. When finally we obtain such a sequence of decreasing ratios, these constitute the MLE's $\hat{F}_1(t), \hat{F}_2(t), \ldots, \hat{F}_k(t)$.

An explicit expression for $\hat{F}_i(t)$ is given by Ayer, et al (1955):

$$\hat{F}_i(t) = \max_{s \geq i} \min_{r \leq i} \frac{m_r(t) + \ldots + m_s(t)}{n_r + \ldots + n_s}, \quad i=1,2,\ldots,k.$$

We remark finally that for the case $k = 2$, Brunk, et al (1965) have provided an algorithm for obtaining the MLE's of the functions $F_1(\cdot)$ and $F_2(\cdot)$ over the whole time axis and not merely at a fixed point, subject to $F_1(x) \geq F_2(x)$ for all x.

<u>Example:</u> A development program has two stages of development with four observations in Stage 1 and six observations in Stage 2. The data are:

X_{11} = 91 hours	X_{21} = 96 hours
X_{12} = 54 hours	X_{22} = 49 hours
X_{13} =120 hours	X_{23} =105 hours
X_{14} = 75 hours	X_{24} =125 hours
	X_{25} =101 hours
	X_{26} =115 hours.

Then

$$F_{14}(t) = \begin{cases} 0 , & t < 54 \\ 1/4 , & 54 \le t < 75 \\ 1/2 , & 75 \le t < 91 \\ 3/4 , & 91 \le t < 120 \\ 1 , & t \ge 120 \end{cases}$$

and

$$F_{26}(t) = \begin{cases} 0 , & t < 49 \\ 1/6 , & 49 \le t < 96 \\ 1/3 , & 96 \le t < 101 \\ 1/2 , & 101 \le t < 105 \\ 2/3 , & 105 \le t < 115 \\ 5/6 , & 115 \le t < 125 \\ 1 , & t \ge 125 \end{cases}$$

A graph of $F_{14}(t)$ and $F_{26}(t)$ is shown in Fig. 5.1, which indicates that there is a reversal in the intervals $49 \le t < 54$, and $115 \le t < 120$. Pooling to eliminate these reversals yields for the MLE's $\hat{F}_1(t)$ and $\hat{F}_2(t)$:

$$\hat{F}_1(t) = \begin{cases} 0 , & t < 49 \\ 1/10 , & 49 \le t < 54 \\ 1/4 , & 54 \le t < 75 \\ 1/2 , & 75 \le t < 91 \\ 3/4 , & 91 \le t < 115 \\ 8/10 , & 115 \le t < 120 \\ 1 , & t \ge 120 \end{cases}$$

and

$$\hat{F}_2(t) = \begin{cases} 0 , & t < 49 \\ 1/10 , & 49 \le t < 54 \\ 1/6 , & 54 \le t < 96 \\ 1/3 , & 96 \le t < 101 \\ 1/2 , & 101 \le t < 105 \\ 2/3 , & 105 \le t < 115 \\ 8/10 , & 115 \le t < 120 \\ 5/6 , & 120 \le t < 125 \\ 1 , & t \ge 125 . \end{cases}$$

A graph of $\hat{F}_1(t)$ and $\hat{F}_2(t)$ is shown in Fig. 5.2. Note that no reversals remain.

Note, again, that the procedure illustrated above yields the MLE for any one, fixed, predetermined value of t, not for the entire distribution function. The conservative confidence curve procedure, treated below, is valid for all $t \ge 0$ simultaneously, not merely for one value of t.

(ii) To obtain a conservative $100(1-\alpha)$ percent upper confidence curve on $F_k(t)$ for all $t \ge 0$, combine the $n = \Sigma_1^k n_i$ observations from all the stages and form the empirical cumulative $\hat{F}(t)$ therefrom. Obtain the value $\epsilon_{n,\alpha}$ from Birnbaum and Tingey (1951) reproduced below. Then $\hat{F}(t) + \epsilon_{n,\alpha}$ is the desired confidence curve.

n \ α	Values of $\epsilon_{n,\alpha}$ .100	.050	.010	.001
5	.4470	.5094	.6271	.7480
8	.3583	.4096	.5065	.6130
10	.3226	.3687	.4566	.5550
20	.23155	.26473	.3285	.4018
40	.16547	.18913	.2350	.2877
50	.14840	.16959	.2107	.2581
Large n	$\sqrt{\frac{1}{2n} \log \frac{1}{\alpha}}$			

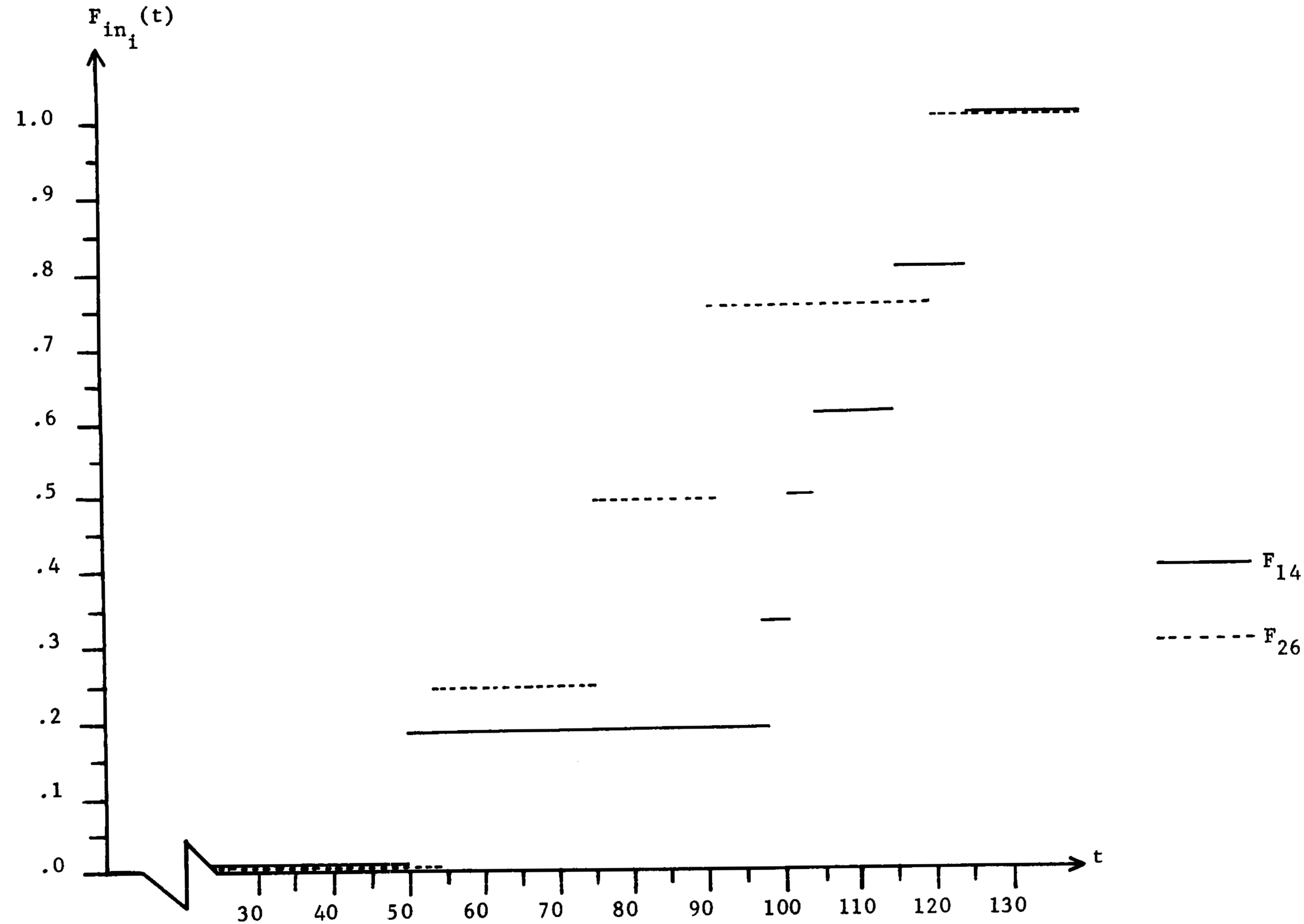

Fig.5.1—Graph of $F_{14}(t)$ and $F_{26}(t)$

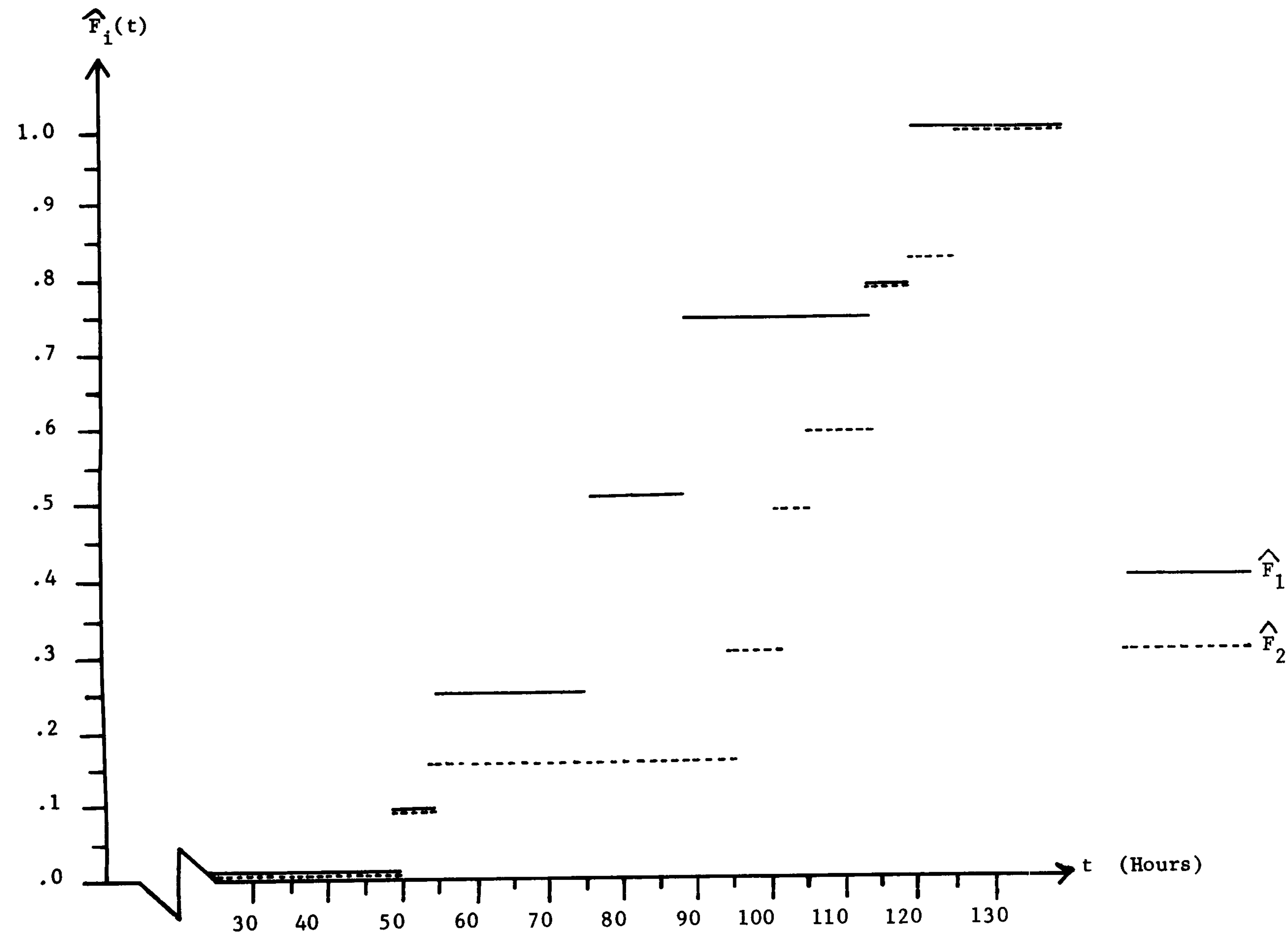

Fig.5.2—Graph of $\hat{F}_1(t)$ and $\hat{F}_2(t)$

<u>Example</u>: Take the data from example 5.1B(i) above, choosing $\alpha = .05$. The empirical cumulative $\hat{F}(t)$ formed from all the data is

$$\hat{F}(t) = \begin{cases} 0 & , \quad t < 49 \\ .1 & , \quad 49 \le t < 54 \\ .2 & , \quad 54 \le t < 75 \\ .3 & , \quad 75 \le t < 91 \\ .4 & , \quad 91 \le t < 96 \\ .5 & , \quad 96 \le t < 101 \\ .6 & , \quad 101 \le t < 105 \\ .7 & , \quad 105 \le t < 115 \\ .8 & , \quad 115 \le t < 120 \\ .9 & , \quad 120 \le t < 125 \\ 1.0 & , \quad t \ge 125 . \end{cases}$$

In this example $k = 2$, $n = n_1 + n_2 = 10$, $\alpha = .05$, $\epsilon_{10,.05} = .3687$, so that the conservative 95 percent upper confidence curve on $F_2(t)$ is

$$\hat{F}_k(t) + \epsilon_{10,.05} = \begin{cases} .3687 & , \quad t < 49 \\ .4687 & , \quad 49 \le t < 54 \\ .5687 & , \quad 54 \le t < 75 \\ .6687 & , \quad 75 \le t < 91 \\ .7687 & , \quad 91 \le t < 96 \\ .8687 & , \quad 96 \le t < 101 \\ .9687 & , \quad 101 \le t < 105 \\ 1.0 & , \quad t \ge 105 . \end{cases}$$

The upper confidence curve for this example is not very satisfactory because of the relatively small number of observations.

5.1C Model 2.1C - Life Length Observations - Ordered, Increasing Failure Rate Functions

(i) We seek the MLE's of several failure rate functions $r_1(\cdot)$, $r_2(\cdot)$, ..., $r_k(\cdot)$ under the assumption that each $r_i(\cdot)$ is an increasing function and that $r_1(t) \geq r_2(t) \geq \ldots \geq r_k(t)$ for all $t \geq 0$. Given observations X_{i1}, X_{i2}, ..., X_{in_i} from $r_i(\cdot)$, the log-likelihood which is to be maximized by proper choice of the functions $r_1(\cdot)$, ..., $r_k(\cdot)$, subject to the constraints, is

$$\sum_{i=1}^{k} \sum_{j=1}^{n_i} \log r_i(X_{ij}) - \sum_{i=1}^{k} \sum_{j=1}^{n_i - 1} (n_i - j)(X_{i,j+1} - X_{ij}) r_i(X_{ij}).$$

The MLE's $\hat{r}_1(\cdot)$, ..., $\hat{r}_k(\cdot)$ are step functions, constant between observations. A computer program is being written to calculate these functions.*

(ii) It would be desirable if there were a procedure which yielded a conservative confidence bound for $r_k(\cdot)$, the failure rate function for the latest stage of development. Such procedure has not as yet been found.

5.2 DEBUGGING MODELS

5.2.1 MLE for a Decreasing Failure Rate

We begin by giving the MLE, $\hat{r}_n(t)$, for a decreasing failure rate function based on a sample of size n : X_1, X_2, ..., X_n from one copy of the system. Recall the notation $S_i = X_1 + X_2 + \ldots + X_i$, with the convention that $S_o = 0$. The MLE $\hat{r}_n(t)$ is constant on the intervals

*Lawrence E. Bodin, University of California, Berkeley, is writing this program. A description of it, together with examples, will appear separately.

$(S_i, S_{i+1}]$ for $i = 0, 1, \ldots, n-1$. The MLE for $r(t)$ on $(S_i, S_{i+1}]$ is X_{i+1}^{-1} before taking account of the fact that the distribution is DFR. If it turns out that $X_1^{-1} \geq X_2^{-1} \geq \ldots \geq X_n^{-1}$, then we conclude that $\hat{r}_n(t) = X_i^{-1}$ for $S_i \leq t < S_{i+1}$, $i = 0, 1, \ldots, n-1$. If a reversal occurs, say $X_i^{-1} < X_{i+1}^{-1}$, then we must average to obtain a common estimate of failure rate, $\left\{ \frac{1}{2} (X_i + X_{i+1}) \right\}^{-1}$, for $S_i \leq t < S_{i+2}$. Next we examine $X_1^{-1}, \ldots, X_{i-1}^{-1}, \left\{ \frac{1}{2} (X_i + X_{i+1}) \right\}^{-1}, \left\{ \frac{1}{2} (X_i + X_{i+1}) \right\}^{-1}, X_{i+2}^{-1}, \ldots, X_n^{-1}$ to see if these estimates of the failure rate on the successive intervals are decreasing. If so, they constitute the MLE's of the failure rates on the successive intervals. If not, we continue to average until no reversals remain. At the end of this process, we obtain MLE's $r_{1,n_1} \geq r_{n_1+1,n_2} \geq \ldots \geq r_{n_k+1,n}$ satisfying:

$$r_{1,n_1} = \left\{ \frac{1}{n_1} (X_1 + \ldots + X_{n_1}) \right\}^{-1},$$

$$r_{n_1+1,n_2} = \left\{ \frac{1}{n_2 - n_1} (X_{n_1+1} + \ldots + X_{n_2}) \right\}^{-1},$$

$$\vdots$$

$$r_{n_k+1,n} = \left\{ \frac{1}{n - n_k} (X_{n_k+1} + \ldots + X_n) \right\}^{-1},$$

and

$$\hat{r}_n(t) = \begin{cases} r_{1,n_1} & \text{for } 0 \leq t \leq S_{n_1} \\ r_{n_1+1,n_2} & \text{for } S_{n_1} < t \leq S_{n_2} \\ \vdots & \\ r_{n_k+1,n} & \text{for } S_{n_k} < t \leq S_n . \end{cases}$$

No estimate of $r(t)$ is made for $t > S_n$ since no data are available for that time interval.

Example: In Fig. 2.1 the cumulative number of failures versus times of failure is graphed for the following data

Time of Failure	Time Between Successive Failures
S_1 = 25 hours	X_1 = 25 hours
S_2 = 75 hours	X_2 = 50 hours
S_3 = 125 hours	X_3 = 50 hours
S_4 = 165 hours	X_4 = 40 hours
S_5 = 240 hours	X_5 = 75 hours
S_6 = 310 hours	X_6 = 70 hours
S_7 = 410 hours	X_7 = 100 hours

We are assuming that $r(t)$ is decreasing in t so that if there were no reversals in the observed failure rates on successive intervals, the estimate of $r(t)$ would be

$$\hat{r}(t) = \frac{1}{X_i}, \quad S_{i-1} < t \le S_i, \quad i = 1, \ldots, 7;$$

i.e.,

$$\frac{1}{25}; \frac{1}{50}; \frac{1}{50}; \frac{1}{40}; \frac{1}{75}; \frac{1}{70}; \frac{1}{100} .$$

Since $\frac{1}{50} < \frac{1}{40}$ and $\frac{1}{75} < \frac{1}{70}$, we have two reversals.

By combining the second, third,and fourth estimates (adding numerators of the three estimates to obtain a new numerator, and adding denominators to obtain a new denominator), we obtain as our new, tentative estimate of $r(t)$:

$$\frac{1}{25}; \frac{3}{140}; \frac{3}{140}; \frac{3}{140}; \frac{1}{75}; \frac{1}{70}; \frac{1}{100} .$$

The reversal $\frac{1}{75} < \frac{1}{70}$ is left. Combining these as before, we obtain finally as the MLE of r at the observations:

$$\hat{r}(25) = \frac{1}{25}$$

$$\hat{r}(75) = \hat{r}(125) = \hat{r}(165) = \frac{3}{140}$$

$$\hat{r}(240) = \hat{r}(310) = \frac{2}{145}$$

$$\hat{r}(410) = \frac{1}{100} .$$

Between successive observations, $\hat{r}$ is, of course, constant. Using this "smoothed" data, we obtain a new graph in Fig. 5.3, in which the slopes (failure rates) of Fig. 2.1 are smoothed.

5.2.2 MLE for a Decreasing Failure Rate from k Copies of the System

First the actual failure times (not interval between failures) for all n systems are pooled and ordered. Call these ordered observations $T_1 \le T_2 \le \ldots \le T_n$, where $n = \sum_{i=1}^{k} n_i$. Between successive T_i, the failure rate estimate is constant as above. Our initial estimate of the failure rate in an interval before imposing the constraint that the failure rate be decreasing is computed as the reciprocal of the total test time observed in that interval. Thus, on $[0, T_1]$, the initial estimate is $(n\, T_1)^{-1}$, on $(T_1, T_2]$, the initial estimate is $\{N_1(T_2-T_1)\}^{-1}$, on $(T_2,T_3]$, the initial estimate is $\{N_2(T_3-T_2)\}^{-1}$, ..., on $(T_{n-1}, T_n]$, the initial estimate is $(T_n - T_{n-1})^{-1}$ where N_i is the number of systems simultaneously in operation during $(T_i, T_{i+1}]$. On (T_n,∞), no estimate of failure rate is made since no failures are observed.

The initial estimates are then compared; if they are in decreasing order, they constitute a MLE of r(t) on $[0, T_n]$. If a reversal occurs, we average as above to eliminate it. For example, if $N_{i-1}(T_i-T_{i-1}) > N_i(T_{i+1}-T_i)$, the revised estimate of the failure rate on $(T_{i-1}, T_{i+1}]$ is $\left\{ \frac{1}{2} [N_{i-1}(T_i-T_{i-1}) + N_i(T_{i+1}-T_i)]\right\}^{-1}$. We continue averaging in this fashion until all reversals are eliminated. The resulting estimate is the MLE of r(t) on $[0, T_n]$ under the assumption that r(t) is a decreasing function.

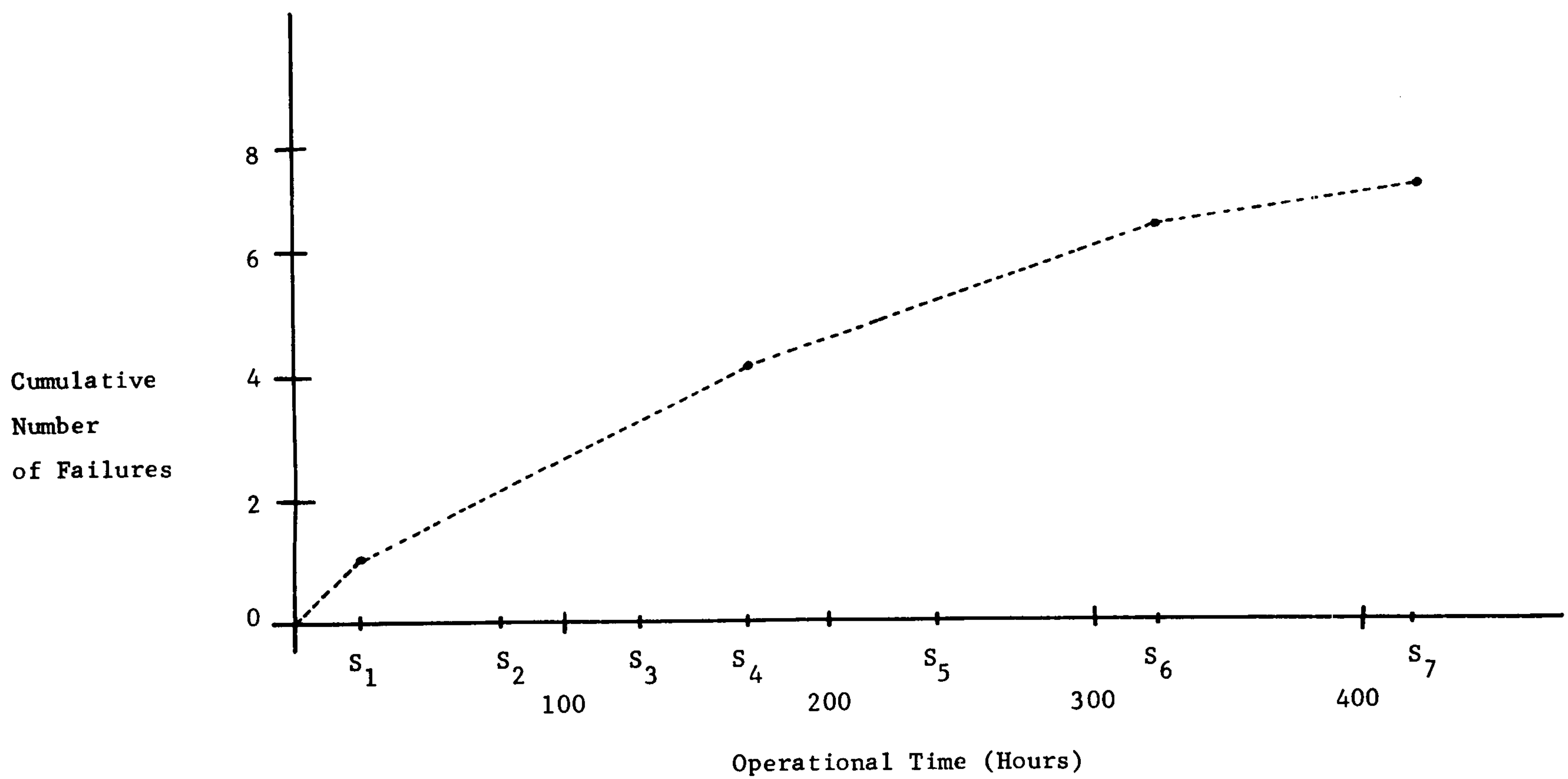

Fig.5.3—"Smoothed" graph of cumulative number of failures versus time

Note: See Fig.2.1 for unsmoothed graph

<u>Example:</u> The method for obtaining the MLE of r(t) in the system debugging model is illustrated for two copies of the same system. Suppose System 1 is debugged at times

$$S_{11} = 25 \text{ hrs.} \leq S_{12} = 125 \leq S_{13} = 240 ,$$

and that System 2 is debugged at times

$$S_{21} = 75 \leq S_{22} = 165 \leq S_{23} = 310 \leq S_{24} = 410.$$

If the failure times are pooled and are denoted by T_i, as before, our estimate of r(t), assuming no reversal, would be

$$\hat{r}(t) = \begin{cases} \dfrac{1}{2(T_i - T_{i-1})}, T_{i-1} < t \leq T_i, & i \leq 5 \\ \dfrac{1}{T_i - T_{i-1}}, \ T_{i-1} < t \leq T_i, & i > 5 \ ; \end{cases}$$

i.e., $\frac{1}{50}$; $\frac{1}{100}$; $\frac{1}{100}$; $\frac{1}{80}$; $\frac{1}{150}$; $\frac{1}{70}$; $\frac{1}{100}$. Since $\frac{1}{100} < \frac{1}{80}$ and $\frac{1}{150} < \frac{1}{70}$, we have two reversals. By combining the second, third, and fourth estimates as before, we obtain

$$\frac{1}{50}; \frac{3}{280}; \frac{1}{150}; \frac{1}{70}; \frac{1}{100} .$$

By combining the estimates $\frac{1}{150}$ and $\frac{1}{70}$, which represent a reversal, we obtain

$$\frac{1}{50}; \frac{3}{280}; \frac{2}{220}; \frac{1}{100} .$$

The reversal $\frac{2}{220} < \frac{1}{100}$ is still left. Combining as before, we finally obtain

$$\hat{r}(t) = \begin{cases} \dfrac{1}{50}, & 0 \leq t \leq 25 \\ \dfrac{3}{280}, & 25 < t \leq 165 \\ \dfrac{3}{320}, & 165 < t \leq 410 \ . \end{cases}$$

5.2.3 MLE for the Time Debugging Ends and for the Stable Failure Rate

To estimate the end point of the debugging period, one first computes the MLE for the failure rate as above.

For Model 2.2A, the MLE for the end point of the debugging period is the beginning of the last averaging interval. In the notation of Section 5.2.1, $t_o = S_{n_k}$. The MLE for $r(t_o)$ is the value of $\hat{r}(t)$ in the last interval, that is $\hat{r}(t_o) = r_{n_k+1,n}$.

For Model 2.2.B, in which we wish to estimate t_1, the point beyond which the failure rate does not decrease by more than ϵ, denote by k^* the smallest index k such that $\hat{r}(S_k) - \hat{r}(S_n) \leq \epsilon$. Then the MLE for t_1 is $\hat{t}_1 = S_{k^*}$.

5.2.4 MLE for the Failure Rate in Model 2.2C

In Model 2.2C debugging is not completed during the period of observation; that is, the failure rate function does not achieve an equilibrium value during the observation period. The estimation, by maximum likelihood, of the failure rate for this model is then precisely what was discussed in Sec. 5.2.1 for observations from one copy of the system and in Sec. 5.2.2 for several copies of the system.

5.2.5 Conservative Upper Confidence Bounds on the Stable Failure Rate

In the debugging models the failure rate function is bounded below. Let r_o denote the greatest lower bound. It was shown in Sec. 4 that a conservative 100(1-α) percent upper confidence bound on r_o can be obtained by treating the data as though they were from an exponential population with failure rate r_o. Thus, the desired bound on r_o based on observations $X_1, \ldots, X_n$ is

$$\chi^2_{1-\alpha}(2n)/2\Sigma_1^n X_i,$$

where $\chi^2_{1-\alpha}(2n)$ is the 100(1-α) percentile of the chi-square distribution with 2n degrees of freedom.

Example

(i) In the example in Sec. 5.2.1, $n = 7$ and $\Sigma X_i = 410$. Choosing $\alpha = .05$, we find from tables that $\chi^2_{.95}(14) = 23.7$. Thus, a conservative 95 percent upper confidence bound on r_o is $23.7/820 = .0289$. That is $P[r_o \le .0289] \ge .95$.

The data in the example in Sec. 5.2.2 come from two copies of the system. The procedure for more than one copy of the system is essentially the same as that for precisely one system, viz. a conservative $100(1-\alpha)$ percent upper confidence bound on r_o is $\chi^2_{1-\alpha}(2n)/2\Sigma_1^k\Sigma_1^{n_i}X_{ij}$, where $n = \Sigma_1^k n_i$, k is the number of copies, and n_i is the number of observations on the i-th copy.

(ii) For the data in the example in Sec. 5.2.2, $k = 2$, $n_1 = 3$, $n_2 = 4$ (so that $n = 7$), and $\Sigma\,\Sigma\, X_{ij} = 650$. Choosing $\alpha = .05$, a conservative 95 percent upper confidence bound on r_o is $23.7/1300 = .018$. That is $P[r_o \le .018] \ge .95$.

BIBLIOGRAPHY

1. Ayer, Miriam, H. D. brunk, G. M. Ewing, W. T. Reid, and Edward Silverman (1955), "An Empirical Distribution Function for Sampling with Incomplete Information," Ann. Math. Stat., Vol. 26, pp. 641-647.

2. Barlow, R. E., A. W. Marshall, and F. Proschan (1963), "Properties of Probability Distributions with Monotone Hazard Rate," Ann. Math. Stat., Vol. 34, pp. 375-389.

3. Barlow, R. E., and Ernest M. Scheuer (1965), Reliability Growth During a Development Testing Program, The RAND Corporation, RM-4317-1-NASA.

4. Birnbaum, Z. W., and F. H. Tingey (1951), "One-Sided Confidence Contours For Probability Distribution Functions," Ann. Math. Stat., Vol. 22, pp. 592-596.

5. Bresenham, Jack E. (1964), Reliability Growth Models, Stanford University, Department of Statistics, Technical Report No. 74.

6. Brunk, H. D. (1955), "Maximum Likelihood Estimates of Monotone Parameters," Ann. Math. Stat., Vol. 26, pp. 607-616.

7. -----, (1958), "On the Estimation of Parameters Restricted by Inequalities," Ann. Math. Stat., Vol. 29, pp. 437-454.

8. -----, W. E. Franck, D. L. Hanson, and R. V. Hogg (1965), Maximum Likelihood Estimation of Two Stochastically Ordered Distributions, University of Missouri, Math. Sci. Tech. Report No. 11.

9. Corcoran, W. J., H. Weingarten, and P. W. Zehna (1964), "Estimating Reliability after Corrective Action," Management Sci., Vol. 10, pp. 786-795.

10. Cramér, H. (1946), Mathematical Methods of Statistics, Princeton University Press, Princeton, N. J.

11. Grenander, Ulf (1956), "On the Theory of Mortality Measurement, Part II," Skan. Aktuarietidskr., Vol. 39, pp. 125-153.

12. Kendall, M. G., and A. Stuart (1961), The Advanced Theory of Statistics, Vol. II, Charles Griffin and Company, Ltd., London.

13. Lehmann, E. L. (1959), Testing Statistical Hypotheses, John Wiley and Sons, New York.

14. Lloyd, David K., and Myron Lipow (1962), Reliability: Management, Methods, and Mathematics, Prentice-Hall, Englewood Cliffs, N. J.

15. Marshall, A. W., and F. Proschan (1965), "Maximum Likelihood Estimation for Distributions with Monotone Failure Rate," Ann. Math. Stat., Vol. 36, pp. 69-77.

16. Mood, A. M., and F. A. Graybill (1963), Introduction to the Theory of Statistics, 2d ed., McGraw-Hill Book Company, New York.

17. Rosner, N. (1961), "System Analysis - Non-Linear Estimation Techniques," Proceedings of the 7th National Symposium on Reliability and Quality Control, Philadelphia, January 9-11, pp. 203-207.

18. Van Eeden, C. (1956), "Maximum Likelihood Estimation of Ordered Probabilities," Indag. Math., Vol. 18, pp. 444-455.

19. -----, (1957), "Maximum Likelihood Estimation of Partially or Completely Ordered Parameters," Indag. Math., Vol. 19, pp. 128-136, 201-211.

20. -----, (1958), Testing and Estimating Ordered Parameters of Probability Distributions, Doctoral Thesis, University of Amsterdam, Holland.

21. Wolman, William (1963), "Problems in System Reliability Analysis," in Statistical Theory of Reliability, Marvin Zelen (ed.), University of Wisconsin Press, Madison, Wisconsin.