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LIAPUNOV FUNCTIONS FOR THE PROBLEM OF LURIE

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The search for Liapunov functions for the system of differential equations (1) known as the Lurie system has been extensive in recent years. A significant change in the course of this research occurred after the appearance of the 1961 paper of Popov [1]. Popov obtained a criteria for the asymptotic stability of the Lurie system without the use of Liapunov functions. Moreover, he proved that his criteria is satisfied if there exists a positive definite Liapunov function of the type quadratic form plus the integral of the nonlinearity; that is, a function of the form (3) whose derivative along the trajectories of (1) is negative definite.

Several authors have proved partial converses to Popov's theorem on the existence of Liapunov functions. Two noteworthy papers are those of Yacubovich [2] and Kalman [3]. Yacubovich was able to prove a partial converse by strengthening the Popov criteria (see (2)) below from $P(i\omega) \geq 0$ to $P(i\omega) > 0$ and Kalman was able to establish a partial converse by requiring that the system be completely controllable and completely observable (defined below). This note is to announce the result that the Popov criteria implies the existence of a positive definite Liapunov function of the type quadratic form plus integral of the nonlinearity that can be used to prove asymptotic stability in the large without the above restrictions. Also the lemma 1 which is used to prove this result can also be used to establish further results such as Theorem 2.

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Let E^n denote Euclidean n-space and I the $n \times n$ identity matrix. Let A be a real $n \times n$ matrix and b a real n-vector. The vector b will be considered as a column vector and b' will be its transpose. The cyclic subspace generated by b relative to A will be denoted by $[A, b]$; that is $[A, b] = \{x \in E^n: \alpha_o b + \alpha_1 Ab + \ldots + \alpha_{n-1}A^{n-1}b$ where the α_j's are real numbers$\}$. Let $[A, b]^o$ denote the orthogonal complement of $[A, b]$ in E^n. The pair (A, b) is said to be completely controllable if $[A, b] = E^n$ and (A, b') is said to be completely observable provided (A', b) is completely controllable.

The main result is the following lemma:

Lemma 1: Let A be a real $n \times n$ matrix all of whose characteristic roots have negative real parts; let τ be a real nonnegative number and let b, k be two real n-vectors. If

$$\tau + 2 \operatorname{Re} k'(i\omega I - A)^{-1}b \geq 0$$

for all real ω then there exists two real $n \times n$ symmetric matrices B D, and a real n-vector q such that

(a) $A'B + BA = -qq' - D$

(b) $Bb - k = \sqrt{\tau}q$

(c) D is positive semi definite and B is positive definite

(d) $\{x \in E^n: x'Dx = 0\} \cap [A', q]^o = \{0\}$

(e) $q \notin [A, b]^o$

(f) if $i\omega$, ω real, is a root of $-q'(zI - A)^{-1}b + \sqrt{\tau}$ then it is a root of $b'(zI - A)^{-1} D(zI - A)^{-1}b$.

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The beginning of the proof of this lemma is similar to the proof given by Kalman in [3]. A second lemma which is easy to establish shows a generalization of Popov's theorem to the critical cases. It is

Lemma 2: Let A be a real $n \times n$ matrix all of whose characteristic roots are simple, distinct and have zero real parts. If the residues of $k'(zI - A)^{-1}b$ are all positive then there exists a positive definite matrix B such that

$$A'B + BA = 0 \quad \text{and} \quad Bb - k = 0 .$$

The two lemmas above can be used in the analysis of several differential systems that occur in control theory. For example consider

$$\text{(1)} \qquad \dot{x} = Ax - b\phi(\sigma) \quad , \quad \sigma = c'x$$

where x, b and c are real n-vectors; A is a real $n \times n$ matrix and $\phi(\sigma)$ is a real continuous scalar function of the real scalar σ such that $\sigma\phi(\sigma) > 0$ for $\sigma \neq 0$. Both x and σ are functions of the real variable t and $\dot{x} = \frac{dx}{dt}$. Let (1) be such that through each $x_0 \in E^n$ there exists a unique trajectory of (1). The system (1) is said to be completely controllable and completely observable provided that (A, b) and (A, c') are respectively completely controllable and completely observable. For this system, the lemmas 1 and 2 can be used to prove:

Theorem 1. Let there exist two constants $\alpha \geq 0$, $\beta \geq 0$, $\alpha + \beta > 0$ such that

$$\text{(2)} \qquad P(i\omega) = (\alpha + i\omega\beta)c'(i\omega I - A)^{-1}b \geq 0$$

for all real ω and if $i\omega_o$, ω_o real, is a characteristic root of A then the pole of $P(z)$ at $i\omega_o$ is simple and has positive residues. Then there exists a Liapunov function V of the form

$$V = x'Bx + \beta \int_0^{\sigma} \phi(\tau)d\tau \tag{3}$$

where B is a symmetric matrix such that V is positive definite for all x and $\dot{V} = (\text{grad } V)\cdot(Ax - b\phi(\sigma)) \leq 0$ for all x. If the linear system $\dot{x} = (A - \mu bc')x$ is asymptotically stable for all $\mu > 0$ then no non zero trajectory of (1) remains in the set where $\dot{V} = 0$.

Moreover, if when $\alpha = 0$ and A is singular $\int_0^{\sigma} \phi(\tau)d\tau$ tends to $+\infty$ with $|\sigma|$ then V tends to $+\infty$ with $\| x \|$. Thus, under the above conditions, the solution $x = 0$ of (1) is asymptotically stable in the large.

The lemmas 1 and 2 can equally well be used to consider the system (1) when $\phi(\sigma)$ is restricted to $0 < \sigma\phi(\sigma) < k\sigma^2$ and a sharper result can be obtained. Another example of the consequences of lemma 1 is the following:

Theorem 2. Let A be any real $n \times n$ matrix and b, c as before. Let λ be any real number that is strictly greater than the real part of all the characteristic roots of A. If

$$\text{Re } c'(zI - A)^{-1}b \geq 0$$

for all $z = i\omega + \lambda$, ω real, then there exists a nonnegative function K defined on $[0, \infty)$ such that

$$\|x(t)\| \leq K(\|x_o\|)e^{\lambda t}$$

for all $t \geq 0$ where $x(t)$ is the solution of (1) such that $x(0) = x_o$.

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