APPLICATION OF CONFORMAL MAPPING TO DIFFRACTION AND SCATTERING PROBLEMS[+]

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Summary

This paper describes an integral equation method, based on conformal mapping, which can be applied to scattering and diffraction problems of plane waves by an infinite cylinder with arbitrary cross section. This method proves to be particularly effective in dealing with fields diffracted by discontinuities both in curvature and in derivatives of curvature, and in showing the effects of smooth shallow corrugations of the boundary.

This method was first proposed by Garabedian (1955). Unlike that of Jones (1963), [Author] this method does not require analyticity of the mapping function on the boundary. Furthermore, the degree of singularity of the mapping function on the boundary is determined by the local behavior of geometry of the boundary (Warschawski, 1935). This property enables us to study the effect of a discontinuity in curvature or in derivatives of curvature on diffracted fields.

Consider a plane wave incident on a boundary on which the total field vanishes. Let $\tilde{U}(\underset{\rightarrow}{\rho})$ be the scattered field.

First the domain external to the given boundary is mapped into another domain external to a geometrically simpler boundary, where the Green's function satisfying the Dirichlet boundary condition is known.

For a closed boundary the mapping function is given as

$$F(z) = z + f(z) \quad \text{and} \quad f(z) \rightarrow O\left(\frac{1}{z}\right) \quad \text{as} \quad z \rightarrow \infty. \tag{1}$$

The new function $U(z) = \tilde{U}\left\{F(z)\right\}$ satisfies the transformed boundary value problem. Through application of Green's theorem U can be reduced to the solution of the following Fredholm integral equation.

$$U = \oint_{\text{boundary}} e^{ik\,\mathrm{Re}\,F(z)} \frac{\partial G}{\partial n} d\ell - \iint_{\substack{\text{exterior}\\ \text{domain}}} \left\{\left|\frac{dF}{dz}\right|^2 - 1\right\} G U k^2 dx dy \tag{2}$$

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The solution of (2) can be obtained by the method of successive approximations, provided that both original and transformed boundary have continuous tangent and the distance between two curves is sufficiently small compared to the wavelength. Under these conditions the normal of the kernel of (2) is small compared to unity.

Once $U(z)$ is known, the far scattered field $\tilde{U}(\rho)$ is easily obtainable since the mapping function becomes an identity as $|z| \to \infty$ (See Eq. (1)).

As a first example, let us consider the scattering body with periodic corrugation, such that the boundary is given by the polar equation

$$\rho = R + a \cos m\phi + b \sin m\phi \; . \tag{3}$$

If $k\sqrt{a^2+b^2} \ll 1$, then by mapping the given boundary into a circular one with the radius R the far scattered field is obtained.

$$\tilde{U}(\vec{\rho}) = \tilde{U}_o(\vec{\rho}) + \tilde{U}_p(\vec{\rho}) \tag{4}$$

where $\tilde{U}_o(\vec{\rho})$ is the field scattered by the circular cylinder with the radius R and

$$\tilde{U}_p(\vec{\rho}) \simeq \frac{1}{\pi}\sqrt{\frac{2}{\pi k \rho}}\, e^{ik\rho - i3\pi/4} \sum_{n=-\infty}^{\infty} \frac{e^{in\phi}}{H^{(1)}(kR)} \left\{ i^m \frac{a+ib}{RH^{(1)}_{n+m}(kR)} + i^{-m} \frac{a-ib}{RH^{(1)}_{n-m}(kR)} \right\} \tag{5}$$

is the perturbed field due to the corrugations.

The same result is obtained with different methods by Clemmow and Weston (1961) and by Mitzner (1964). In their papers the limit of the validity of the solution was conjectured. However, from the study of the equation (2), the conditions for validity of the solution are now clear that not only $k\sqrt{a^2+b^2}$ but also the norm of the kernel of (2) should be much less than unity.

The second example deals with the effect of the discontinuity in curvature in high frequency scattering. A scattering body with two discontinuities in curvature is obtained by smoothly joining together a semi-circle of radius a and a semi-ellipse of the eccentricity ϵ and the minor axis $2a$. At the two junction points, curvatures of the circle and the ellipse are $1/a$ and $1-\epsilon^2/a$ respectively.

Transforming this boundary into a circular one and substituting the asymptotic form of the Green's function for high frequency in (2), it is found that the integrand of the first term of (2) has saddle points and branch points. These branch points are due to the singularity of the mapping function on the boundary, which is in turn due to the geometrical singularity of the boundary.

We can obtain the field diffracted by the point of discontinuity in curvature by properly deforming the contour of integration and evaluating the branch cut integrals corresponding to the branch points of the mapping function.

Various forms of the diffracted field are obtainable by changing both the observation point and the position of the discontinuity in curvature. When the point of the discontinuity lies in the illuminated region or near the shadow boundary and if this point can be directly seen from the observation point, the field diffracted by the discontinuity in curvature is,

$$\tilde{U}_d(\vec{\rho}) \sim \frac{(\frac{1}{a} - \frac{1-\epsilon^2}{a})(\cos\phi_d - 2i\sin\phi_d)\cdot\cos(\phi-\phi_d)}{k2\left[\sin(\phi-\phi_d)+\sin\phi_d\right]^3}\cdot\sqrt{\frac{2}{\pi k\rho}}\, e^{ik\rho - i\pi/4} \quad (6)$$

Here $\phi_d - \pi/2$ is the angle between the direction of incident field and the tangent line to the boundary at the discontinuity. Equation (6) represents a direct ray from the discontinuity.

When the observation point moves across the tangential line to the boundary at the discontinuity, then the diffracted field becomes

$$\tilde{U}_d \sim \left[\frac{(\frac{1}{a} - \frac{1-\epsilon^2}{a})(2i\sin\phi_d - \cos\phi_d)}{2k(1-\sin\phi_d)^3}\right]\cdot\left[\frac{\tilde{f}\left\{(\frac{kR}{2})^{1/3}(\phi_d - \frac{\pi}{2} - \phi)\right\}}{(\frac{kR}{2})^{1/3}}\cdot e^{ikR(\phi_d - \frac{\pi}{2} - \phi)}\right]$$

$$\cdot\sqrt{\frac{2}{\pi k\rho}}\, e^{ik\rho - i\,3\pi/4} \quad (7)$$

where $\tilde{f}$ is the usual Fock function and is defined by

$$\tilde{f}(\zeta) = \frac{1}{\sqrt{\pi}}\int_{\Gamma}\frac{e^{i\zeta t}}{w_1(t)}dt\,. \quad (8)$$

Equation (6) represents the contribution of the creeping waves launched by the point of discontinuity in curvature.

When the two discontinuities in curvature lie at the shadow boundary, the effect of discontinuities in back scattering direction becomes

$$\tilde{U}_d \sim \left[\frac{(\frac{1}{a} - \frac{1-\epsilon^2}{a})}{16k}\right]\cdot\left[\frac{\tilde{f}\left\{(\frac{kR}{2})^{1/3}(\pi-\phi)\right\}\cdot e^{ikR(\pi-\phi)} + \tilde{f}\left\{(\frac{kR}{2})^{1/3}(\phi-\pi)\right\} e^{ikR(\phi-\pi)}}{(\frac{kR}{2})^{1/3}}\right.$$

$$\cdot\sqrt{\frac{2}{\pi k\rho}}\, e^{ik\rho - i\pi/4}\,. \quad (9)$$

Generally when the boundary has a discontinuity in the $(n-1)$'th derivative of curvature, the effect of this discontinuity is of the order of $k^{-n}\cdot(k\rho)^{-1/2}$. Extension to the case of Neumann boundary condition is possible. It is still an open question how to apply Eq. (2) in the case of diffraction by the wedge even though the formulation of (2) does not require the boundary to have continuous tangents.

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