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CONCENTRATION PROFILE ESTABLISHMENT OF BINARY GAS MIXTURE IN SWIRL AND DUCT FLOWS

by Timothy W. Kao

Prepared under Grant No. NsG-586 *by*
THE CATHOLIC UNIVERSITY OF AMERICA
Washington, D. C.
for Lewis Research Center

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FOREWORD

The research described herein, which was conducted at the Catholic University of America, was performed under NASA Research Grant NsG-586 with Mr. Robert G. Ragsdale, Nuclear Reactor Division, NASA-Lewis Research Center, as Technical Manager.

CONCENTRATION PROFILE ESTABLISHMENT

OF BINARY GAS MIXTURE IN SWIRL

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ABSTRACT

A study is made of the establishment of density profile for a two-fluid single phase gas mixture under a body force from a uniformly mixed upstream condition. An inviscid hydrodynamical model is adopted. Two cases are considered. In the first case the flow is a swirling motion and the body force is provided by the centrifugal action of the swirl. In the second case the flow is a duct flow confined between two parallel walls with potential body force. Use is made of a perturbation procedure in terms of an expansion of the reciprocal of a diffusive "Reynolds" number to obtain a closed form zeroth order solution. The orders of the higher order correction terms are explicitly given. The interplay between Fickian and baro-diffusion is brought out, and asymptotic solutions far downstream of the inlet are explicitly calculated for a class of swirling flows. The problem bears on various phenomena where a knowledge of the concentration of the heavier component is important, such as in gaseous nuclear cavity reactor propulsion devices.

1. INTRODUCTION

For many industrial purposes it is desirable to know the establishment of stratification for a two-fluid single phase fluid system in the presence of strong body forces. In particular, it is very often necessary to know the concentration of the heavier species at various points downstream of the inlet where the two fluids are uniformly mixed, and the shape of the established density profile. This knowledge is needed, for example, in gaseous nuclear propulsion device[(1)] where the heavier fluid is uranium, and the critical concentration for the onset of reaction is of paramount importance. In most problems of this nature the body force is usually a centrifugal force, and for high flow velocities the effect of viscosity is generally negligible. The dominant effect is one of mass diffusion.

In this paper an inviscid, incompressible, hydrodynamical theory is proposed. Strictly speaking the thermodynamics of the system has to be considered together with the mechanical equations in order to obtain a complete set of equations (see for example Landau and Lifshitz[(2)]). However, when the change of density of the fluid mixture taken as a whole is assumed to be proportional to the change in concentration of the heavier fluid, a purely mechanical consideration suffices and thermodynamics can be left out of the analysis. This of course results in a major simplification of the problem. A perturbation scheme is then used to solve the problem, which is here considered as a two-dimensional flow.

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(1) The specific model envisaged is one of doughnut shape which is thus not affected by the viscous end-wall boundary layer which is important in cylindrical models.

(2) L. D. Landau and E. M. Lifshitz, Fluid Mechanics. Addison-Wesley Publishing Co., Inc., Reading, Mass. pp. 219-227.

Two cases are considered. In the first case the flow is a swirl motion and the body force is provided by the centrifugal action of the swirl. In the second case the flow is a horizontal duct flow with a potential body force in the vertical direction.

2. THE GOVERNING EQUATIONS

The equation of continuity for the total mass of fluid is

$$\frac{\partial \rho}{\partial t} + \nabla . (\rho \vec{v}) = 0 \quad , \tag{1}$$

where ρ is the total density of the fluid and $\vec{v}$ denotes the velocity. We note that velocity is here understood as the total momentum per unit mass of fluid, and the equations of motion are the Euler's equations

$$\rho \frac{D\vec{v}}{Dt} = - \nabla p + \rho \vec{g} \quad , \tag{2}$$

where p is the pressure, $\vec{g}$ is the body force and $\frac{D}{Dt} \equiv \left(\frac{\partial}{\partial t} + \vec{v} . \nabla \right)$ is the substantial derivative.

If we denote c to be the mass concentration of the heavier fluid, the equation of continuity for that species is

$$\frac{Dc}{Dt} = -\frac{1}{\rho} \nabla . \vec{i} \quad , \tag{3}$$

where $\vec{i}$ is the mass flux of that species. The mass flux is made up of three parts

$$\vec{i} = -D \left[\rho \nabla c + \left(\frac{\rho k_t}{T} \right) \nabla T + \left(\frac{\rho k_p}{p} \right) \nabla p \right] , \tag{4}$$

where D is the diffusion coefficient or mass transfer coefficient

k_t is the thermal diffusion ratio

k_p is the barodiffusion ratio

and T is the absolute temperature.

k_t and k_p are determined by thermodynamic properties alone. For the purpose of this analysis it can be shown (see Landau and Lifshitz[2]) that k_p is negative. In the present analysis we shall assume a uniform temperature distribution so that

$$\vec{i} = -\rho[D\nabla c - k_2 \nabla p] , \tag{5}$$

where $k_2 \equiv -\frac{k_p D}{p}$ is positive. From the above equation we can conclude at once that equilibrium is reached when the flux due to mass concentration is balanced by the pressure flux.

Substitution of (5) into (3) yields

$$\frac{Dc}{Dt} = \frac{1}{\rho}[\nabla \cdot (\rho D \nabla c) - \nabla \cdot (\rho k_2 \nabla p)]. \tag{6}$$

(1), (2), and (6) are five equations for the six unknowns, c, ρ, p, $\vec{v}$.

To complete the system we assume that the change of density of the fluid mixture taken as a whole is proportional to the change in concentration of the heavier fluid, i.e.

$$\frac{d\rho}{dc} = \beta^* , \tag{7}$$

where β^* is a constant of proportionality. Using the above equation, we have from (6),

$$\frac{D\rho}{Dt} = \frac{1}{\rho}[\nabla \cdot (\rho D \nabla \rho) - \nabla \cdot (\beta^* \rho k_2 \nabla p)]. \tag{8}$$

3. SWIRLING FLOW

The flow is assumed to be a swirling flow in a cylinder as shown in Figure 1. We further assume the flow to be independent of z, which is along the axis of the cylinder, but θ can increase indefinitely. r denotes the radial direction. The velocity components corresponding to (r, θ) are denoted by (u_r, u_θ). The equation of continuity can then be written as

$$\frac{1}{r}\frac{\partial}{\partial r}(r\rho u_r) + \frac{1}{r}\frac{\partial}{\partial\theta}(\rho u_\theta) = 0 , \tag{9}$$

The equations of motion are,

$$u_r \frac{\partial u_r}{\partial r} + \frac{u_\theta}{r}\frac{\partial u_r}{\partial \theta} - \frac{u_\theta^2}{r} = -\frac{1}{\rho}\frac{\partial p}{\partial r} , \tag{10}$$

$$u_r \frac{\partial u_\theta}{\partial r} + \frac{u_\theta}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{u_r u_\theta}{r} = -\frac{1}{\rho}\frac{1}{r}\frac{\partial p}{\partial \theta} . \tag{11}$$

The equation for the density of the mixture is

$$u_r \frac{\partial \rho}{\partial r} + \frac{u_\theta}{r}\frac{\partial \rho}{\partial \theta} = D\nabla^2\rho - \beta^* k_2 \nabla^2 p , \tag{12}$$

where we have taken ρD and ρk_2 to be constant, and

$\nabla^2 \equiv \frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}$ is the two-dimensional Laplacian in cylindrical co-ordinates.

We first non-dimensionalize our problem with a reference length L = radius of cylinder, a reference velocity U = maximum velocity at inlet, and a reference density $\bar{\rho}_0$ = density at inlet. We denote $u = u_r/U$ and $v = u_\theta/U$, $\beta = \frac{\beta^*}{\bar{\rho}_0}$, $\epsilon \equiv \frac{1}{R_D} \equiv \frac{D}{LU}$, $K_2 = \frac{k_2 U \bar{\rho}_0}{L}$ and the rest of the symbols are the same as the dimensional ones.

The non-dimensional forms of equations (9), (10), (11), and (12) are

$$\frac{1}{r}\frac{\partial}{\partial r}(\rho r u) + \frac{1}{r}\frac{\partial}{\partial\theta}(\rho v) = 0 , \tag{13}$$

$$u\frac{\partial u}{\partial r} + \frac{v}{r}\frac{\partial u}{\partial\theta} - \frac{v^2}{r} = -\frac{1}{\rho}\frac{\partial p}{\partial r} , \tag{14}$$

$$u\frac{\partial v}{\partial r} + \frac{v}{r}\frac{\partial v}{\partial \theta} + \frac{uv}{r} = -\frac{1}{\rho}\,\frac{1}{r}\frac{\partial p}{\partial \theta}\ , \tag{15}$$

and $$u\frac{\partial \rho}{\partial r} + \frac{v}{r}\frac{\partial \rho}{\partial \theta} = \epsilon\nabla^2\rho - \beta K_2\nabla^2 p \tag{16}$$

It is now to be noted that $R_D \equiv \frac{LU}{D}$ is a "diffusive Reynolds number". For all cases of practical interest this number is very large, that is ϵ is small. Physically this means that the mass diffusion rate is small compared with the velocity of the flow. Also for baro-diffusion to be significant $K_2 \sim \epsilon$. Normally k_2 is very small but K_2 can be seen, for high swirl velocity, to be of order ϵ. Equation (16) then indicates that $\frac{D\rho}{Dt} \sim \epsilon$. Thus (13) becomes:

$$\frac{1}{r}\frac{\partial}{\partial r}(ru) + \frac{1}{r}\frac{\partial v}{\partial \theta} = 0\ . \tag{17}$$

Since ϵ is associated with the highest order terms of the equation, we have a "singular" problem. Thus we assume an "outer" expansion of the form

$$f(r,\theta;\epsilon) = f_0(r,\theta) + \epsilon f_1(r,\theta) + \epsilon^2 f_2(r,\theta) + \dots$$

where f denotes any of the variables. Equations (13) to (16) are then to the zeroth order the non-diffusive equations of motion with solutions $u_0 = 0$, $v_0 = v_0(r)$, $\rho_0 = \rho_0(r)$ and $p_0 = p_0(r)$. It will be seen that the solution for ρ_0 cannot satisfy the boundary condition at $\theta = 0$ except for the trivial case where ρ_0 remains 1. Higher order terms of this expansion will not be able to satisfy all the boundary conditions of the problem by virtue of the singular nature of this problem.

We now seek an "inner" expansion. To do that, we make the transformation(3)

$$\theta_1 = \epsilon\theta$$

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(3) Alternately, we can say that we make a "boundary-layer" type assumption that $\frac{1}{r}\frac{\partial}{\partial\theta} \sim \epsilon$ and $\frac{\partial}{\partial r} \sim 1$; an assumption which can be verified <u>a posteriori</u>.

and write a limit series expansion of the type

$$f(r,\theta_1;\epsilon) = g_0(r,\theta_1) + \mu_1(\epsilon) g_1(r,\theta_1) + \mu_2(\epsilon) g_2(r,\theta_1) + \dots$$

$\epsilon \to 0$, (r,θ_1) fixed. The inner solution is matched to the outer solution for large θ and is required to satisfy the boundary conditions at $\theta = 0$ and the condition of zero flux at the wall. This immediately indicates that $u_o = 0$, $v_o = v_o(r)$ as before. We thus take a perturbation expansion of the form

$$u = \mu_1(\epsilon)\, u_1(r,\theta_1) + \mu_2(\epsilon)\, u_2(r,\theta_1) + \dots$$

$$v = v_0(r) + \epsilon\, v_1(r,\theta_1) + \epsilon^2 v_2(r,\theta_1) + \dots$$

$$p = p_0(r,\theta_1) + \epsilon\, p_1(r,\theta_1) + \epsilon^2 p_2(r,\theta_1) + \dots$$

$$\rho = \rho_0(r,\theta_1) + \epsilon\, \rho_1(r,\theta_1) + \epsilon^2 \rho_2(r,\theta_1) + \dots$$

and see whether there exists a distinguished limit for $\mu_1(\epsilon)$ consistent with a limit series expansion of the type shown. From (17) we see that for the zeroth order solution, it is identically satisfied, and to the next order, we see that the first term is of order $\mu_1(\epsilon)$ and the second term is of order ϵ^2. Thus continuity demands that

$$\mu_1(\epsilon) \sim \epsilon^2$$

We thus write

$$u = \epsilon^2 u_1(r,\theta_1) + \mu_2(\epsilon)\, u_2(r,\theta_1) + \dots$$

where $\mu_2(\epsilon)$ is of order higher than ϵ^2.

Substituting the above series into (14) we have to the zeroth order in ϵ:

$$-\frac{v_0^2}{r} = -\frac{1}{\rho_0}\frac{\partial p_0}{\partial r}, \qquad (18)$$

and from (15) and (16), we have to the first order in ϵ:

$$0 = \frac{\partial p_0}{\partial \theta_1} \qquad (19)$$

and

$$\epsilon \frac{v_0}{r}\frac{\partial \rho_0}{\partial \theta_1} = \epsilon \nabla^2 \rho_0 - \epsilon \gamma \nabla^2 p_0 , \tag{20}$$

where $\gamma = \frac{\beta K_2}{\epsilon}$. Substitution of (18) and (19) into (20) then gives:

$$\frac{v_0}{r}\frac{\partial \rho_0}{\partial \theta_1} = \left(\frac{\partial^2 \rho_0}{\partial r^2} + \frac{1}{r}\frac{\partial \rho_0}{\partial r} \right) - \gamma \frac{1}{r}\frac{\partial}{\partial r}(\rho_0 v_0^2) . \tag{21}$$

Equation (21) is to be solved subject to the boundary conditions that the flux is zero at the wall of the cylinder, r = 1, i.e.,

$$\frac{\partial \rho_0}{\partial r} - \gamma \frac{\rho_0 v_0^2}{r} = 0 . \tag{22}$$

At $r = 0$, ρ_0 has to be bounded. At $\theta = 0$, $\rho_0 = 1$ and ρ_0 is bounded as $\theta \to \infty$.

The boundary value problem for ρ_0 is solvable by separation of variables. We note of $v_0(r)$ is arbitrary as one expects from the inviscid assumption. We let

$$\rho_0(r, \theta_1) = R(r) H(\theta_1) .$$

Substitution of the above into (21) and the boundary conditions yields

$$r R'' + (1 - \gamma v_0^2) R' - \left(\gamma \frac{d v_0^2}{dr} - \lambda^2 v_0\right) R = 0 , \tag{23}$$

with $$R'(1) - \gamma R(1)\, v_0^2(1) = 0 , \tag{24}$$

and $$|R(0)| < \infty ; \tag{25}$$

and $$H' + \lambda^2 H = 0 \tag{26}$$

where λ^2 is the separation constant, and prime denotes differentiation with respect to the argument.

Equation (26) yields at once the solution

$$H \sim e^{-\lambda^2 \theta_1} = e^{-\epsilon \lambda^2 \theta} , \tag{27}$$

which shows that $\frac{\partial \rho_0}{\partial \theta} \sim \epsilon$ and $\frac{\partial^2 \rho_0}{\partial \theta^2} \sim \epsilon^2$. It also shows that as $\theta \to \infty$, $H \to 0$ unless $\lambda^2 = 0$. The case $\lambda^2 = 0$ yields a particular solution which is independent of θ and is the asymptotic solution that is of interest in the present problem. This further indicates that matching is automatically achieved so that the inner solution is uniformly valid for all r and θ. Before examining these solutions we first investigate the solutions of the boundary value problem given by (23), (24) and (25). This system is an eigenvalue problem with λ as the eigenvalue. The nature of the solution depends on the nature of the function $v_o(r)$. In general if $v_o'(r)$ is an analytic function of r, then $r = 0$ is a regular singular point and series solutions can be easily obtained. Since any velocity profile can be approximated by polynomial functions, the present formulation is quite general.

Equation (23) may be written in self-adjoint form as

$$\frac{d}{dr}\left[r\, e^{-\int^r \frac{\gamma v_o^2}{r} dr} \frac{dR}{dr}\right] + \left[-\gamma \frac{dv_o^2}{dr} e^{-\int^r \frac{\gamma v_o^2}{r} dr} + \lambda^2 v_o\, e^{-\int^r \frac{\gamma v_o^2}{r} dr}\right] R = 0, \qquad (28)$$

with the weight function $= v_o(r)\, e^{-\int^r \frac{\gamma v_o^2}{r} dr}$. We now show that $\lambda = 0$ is an eigenvalue, which means that there exists a solution to the original problem which is independent of θ. In other words, there always exists an asymptotic solution for $\theta \rightarrow \infty$. Physically this is simply a statement that a balance is achieved between Fickian and baro-diffusion. Mathematically this can be easily shown. Indeed for $\lambda = 0$, the differential equation (23) becomes:

$$(rR')' - \gamma (v_o^2 R)' = 0 .$$

Therefore

$$R = c_o\, e^{\int^r \frac{\gamma v_o^2}{r} dr},$$

and the boundary conditions are identically satisfied. Hence $R = c_o\, e^{\int^r \frac{\gamma v_o^2}{r} dr}$ is an eigenfunction corresponding to $\lambda = 0$, which is thus an eigenvalue of the problem. If we denote $\varphi_j(r)$ as eigenfunctions corresponding to the eigenvalues

λ_j , then we can write our solution as

$$\rho_0(r,\theta) = c_0 e^{\int^r \frac{\gamma v_0^2}{r} dr} + \sum_{j=1}^{\infty} c_j e^{-\epsilon\lambda_j^2\theta} \varphi_j(r) . \tag{29}$$

At $\theta = 0$, $\rho_0 = 1$. Therefore the Fourier coefficients c_j are given by

$$c_j = \frac{\int_0^1 \varphi_j(r)\, v_0(r) \exp\left[-\int^r \frac{\gamma v_0^2}{r} dr\right] dr}{\int_0^1 \varphi_j^2(r)\, v_0(r) \exp\left[-\int^r \frac{\gamma v_0^2}{r} dr\right] dr} . \tag{30}$$

In particular:

$$c_0 = \frac{\int_0^1 v_0(r)\, dr}{\int_0^1 v_0(r) \exp\left[\int^r \frac{\gamma v_0^2}{r} dr\right] dr} , \tag{31}$$

has a physical meaning. It simply expresses the conservation of mass flux which of course should be satisfied.

Some explicit cases[4] will now be given for the sake of illustration:

<u>Case I</u> $v_0(r) = r^n$, $n = 0, 1, 2, 3, \ldots$

For $n = 0$, $v_0 = 1$, $c_0 = \gamma + 1$. Therefore $\rho_\infty = (\gamma+1)\, r^\gamma$. We shall henceforth denote the asymptotic form of ρ_0 by ρ_∞ . Equation (23) can be recognized as a modified form of Bessel's equation, with general solutions

$$R(r) = A r^{\frac{\gamma}{2}} J_\gamma(\alpha r^{1/2}) + B r^{\frac{\gamma}{2}} J_{-\gamma}(\alpha r^{\frac{1}{2}}) ,$$

where we have written $\alpha^2/4$ for λ^2 . For any $\gamma \geq 0$, $J_{\pm\gamma} \to \left(\frac{r^{\frac{1}{2}}}{2}\right)^{\pm\gamma}$ as $r \to 0$.

Thus both terms are bounded at $r = 0$. A more careful examination of the system here reveals that while the subsequent examples in this case belong to Weil's "limit point" case[5] , this particular example belongs to the "limit circle" case. If one demands the boundary condition $\rho_0 = 0$ at $r = 0$, one particular point on

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(4) The author is indebted to Ying M. Shy for the detail calculations.

(5) Coddington and Levinson, "Theory of Ordinary Differential Equations," McGraw Hill Co. 1955.

the limit circle is determined, and an expansion involving J_γ only is arrived at. Physically this boundary condition is an appropriate one since at r = 0 the centrifugal force is infinite. Using this boundary condition we thus have

$$R(r) = A\, r^{\frac{\gamma}{2}} J_\gamma(\alpha r^{\frac{1}{2}}) .$$

Substitution of the above into (24) yields

$$3\gamma J_\gamma(\alpha) - \alpha J'(\alpha) = 0 ,$$

which is the secular equation. The roots of this equation are the eigenvalues of the present problem. If we let α_j be the roots of this equation, then

$$\rho_0(r,\theta) = (1+\gamma)\, r^\gamma + \sum_{j=1}^{\infty} c_j\, e^{-\epsilon \alpha_j^2 \theta}\, r^{\frac{\gamma}{2}} J_\gamma(\alpha_j r^{\frac{1}{2}}) ,$$

where

$$c_j = \frac{2\alpha_j \left[\frac{(\alpha_j/2)^{\gamma-1}}{(\gamma-1)!} - J_{\gamma-1}(\alpha_j) \right]}{J_\gamma(\alpha_j)\left[\gamma(\gamma-1) J_\gamma(\alpha_j) - \alpha_j^2 J_\gamma''(\alpha_j)\right]} .$$

The graphs of ρ_∞ for various values of γ are shown in Figure 2.

For n = 1, $v_o(r) = r$, $c_o = \frac{\gamma}{2} \frac{1}{e^{\gamma/2} - 1}$.

Therefore

$$\rho_\infty = \frac{\gamma}{2} \frac{1}{e^{\gamma/2} - 1} e^{\frac{\gamma r^2}{2}} .$$

The graphs of ρ_∞ for various values of γ are shown in Figure 3. Figures 4, 5 show ρ_∞ for n = 2, 5 . The c_o's have not been explicitly calculated and for convenience of drawing suitably chosen constants are used. For n = 2, $\rho_\infty = c_o e^{\frac{\gamma r^4}{4}}$ and for n = 5, $\rho_\infty = c_o e^{\frac{\gamma r^{10}}{10}}$.

<u>Case II</u>. $v_o(r) = r^\ell (1 - r)^m \left[a - (a+b) r\right]^n$, ℓ , m , n are positive integers, and a, b 0 . This class of profiles exhibits reverse flows. Figure 6 is for $v_o = 9.48\, r^3(1-r)$, then $\rho_\infty = c_o e^{10 f}$ where $f = 4 r^2 \left(2 - \frac{8}{3} r + r^2\right)$.

Figure 7 is for $v_o = 9.48 r^3 (1-r)$, then $\rho_\infty = c_o e^{10 f}$ where $f = 89.87 r^6 \left(\frac{1}{6} - \frac{2}{7} r + \frac{r^2}{8}\right)$.

Figure 8 is for $v_o = 10.4\, r(1-r)(1-2r)$, then $\rho_\infty = c_o e^{10 f}$ where $f = 108\, r^2 \left(\frac{1}{2} - 2r + \frac{13}{4} r^2 - \frac{12}{5} r^3 + \frac{2}{3} r^4\right)$.

Figure 9 is for $v_o = 7.4 r^2 (1-r)(3-4r)$, then $\rho_\infty = c_o e^{10 f}$ where $f = 55 r^4 \left(\frac{9}{4} - \frac{42}{5} r - \frac{73}{6} r^2 - \frac{8}{3} r^3 + 2 r^4\right)$.

It may be noted that with reverse flow, the region of high concentration can be made to spread out and thus reduce the concentration in the immediate neighborhood of the wall. In all cases c_o has been assigned a value for convenience of drawing.

Case III. $v_o = r^{1/2}$. This case is of special interest since it corresponds to the case where the centripetal acceleration is constant along the radial direction. In this case equation (23) becomes

$$r R'' + (1-\gamma r) R' - (\gamma - \lambda^2 \sqrt{r}) R = 0 \, .$$

The eigensolution for $\lambda^2 = 0$ is $c_a e^{\gamma r}$. By setting $x = \sqrt{r}$, the above equation becomes

$$x R'' + (1 - 2\gamma x^2) R' - 4x(\gamma - \lambda^2 x) R = 0 \, ,$$

and the boundary condition becomes

$$R'(1) - 2\gamma R(1) = 0 \qquad \text{at} \quad x = 1 \, .$$

The eigenvalue problem can be solved as in the previous cases. The asymptotic solution for large θ is an exponential density profile which will be seen to be the same as the duct flow to be considered in detail in the next Section. The graphs for ρ_∞ for this case are plotted in Figure 10. Again the constants are chosen for convenience and have not been explicitly calculated.

4. DUCT FLOW

It is seen from Section 3 that the actual numerical calculations of the eigenvalues and eigenfunctions in most cases involve considerable computational labor. In order to illustrate some additional features of this problem an explicit calculation is made for the case of a duct flow with a constant body force. This situation corresponds most closely to the case $v_0(r) = r^{1/2}$ in the previous Section. The flow is assumed to be steady and two-dimensional and bounded by two parallel walls at $y = 0$ and $y = L$ as shown in Figure 11. The body force acceleration $\vec{g}$ is taken to be in the negative y-direction and may be much larger than the acceleration of gravity.

Following the same procedure as for the swirling flow we arrive at the following dimensionless equation governing ρ_0 by grouping terms of like orders in ϵ(6).

$$\frac{\partial^2 \rho_0}{\partial y^2} + \gamma \frac{\partial \rho_0}{\partial y} = \frac{\partial \rho_0}{\partial x_1} \tag{32}$$

where $\epsilon = \frac{D}{LU}$, $\gamma = \frac{\beta K_2}{\epsilon}$, $K_2 = \frac{k_2 g \bar{\rho}_0}{U}$, in which U is the velocity at inlet, assumed constant, and $x_1 = \epsilon x$. The boundary conditions are: $\rho_0 = 1$ at $x = 0$, ρ_0 is bounded as $x \to \infty$, and at $y = 0, 1$,

$$\frac{\partial \rho_0}{\partial y} + \gamma \rho_0 = 0 . \tag{33}$$

By separation of variables, the solution to the above problem is

$$\rho_0(x,y) = c_0 e^{-\gamma y} + \sum_{n=0}^{\infty} c_n \varphi_n(y) \exp\left[-\left(\frac{\gamma^2}{4} + n^2\pi^2\right)\epsilon x\right] , \tag{34}$$

where $\varphi_n(y) = \exp\left(-\frac{\gamma y}{2}\right)\left(\sin n\pi y - \frac{2n\pi}{\gamma}\cos n\pi y\right)$ are eigenfunctions of the Sturm-Liouville system from separation of variables and are orthogonal

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(6) Reference can be made to Kao (1965): "On the Establishment of Density Profiles for the Flow of a Two-Fluid Single Phase Gas Mixture." Tech. Rep. No. 65-005. Dept. of Space Science & Appl. Physics, The Catholic University of America, Washington, D. C.

with respect to the weight function $e^{\gamma y}$ in the interval $(0, 1)$. C_n are the Fourier coefficients given by

$$C_n = \frac{\int_0^1 e^{\gamma y} \varphi_n(y)\, dy}{\int_0^1 e^{\gamma y} \varphi_n^2(y)\, dy} = \frac{n\pi \gamma^2 [(-1)^{n+1} e^{\frac{\gamma}{2}} + 1]}{[\frac{\gamma^2}{4} + n^2\pi^2]^2}, \quad n = 1, 2, 3, \ldots$$

and

$$C_0 = \frac{\gamma}{1 - e^{-\gamma}}$$

From the solution it is immediately clear that as $x \rightarrow \infty$, $\rho_0 \rightarrow \frac{\gamma}{1-e^{-\gamma}} e^{-\gamma y}$. It is also seen that the important parameter of the problem is of course the ratio γ. If γ is fairly small then ρ_0 will remain essentially constant since baro-diffusion is not effective. If γ is large then the heavier gas sinks to the bottom. For some physically realistic value there is of course a balance between baro-diffusion and mass diffusion and the asymptotic form of ρ_0 above indicates the equilibrium distribution. It has an exponential behavior and the approach to equilibrium is also exponential. It is seen that the series converge rather rapidly for all $x > 0$.

Figure 12 shows the asymptotic density profile for various values of γ. Figure 13 shows the density profiles for various values of γ at $x_1 = 0.1$ and Figure 14 shows the evolution of the density profile at various points downstream from the inlet for a typical case ($\gamma = 3.0$). Figure 15 shows the distances downstream of inlet where 90, 95, 98 per cent of asymptotic profile is established as function of γ. It exhibits a maximum distance for γ approximately equal to 2.

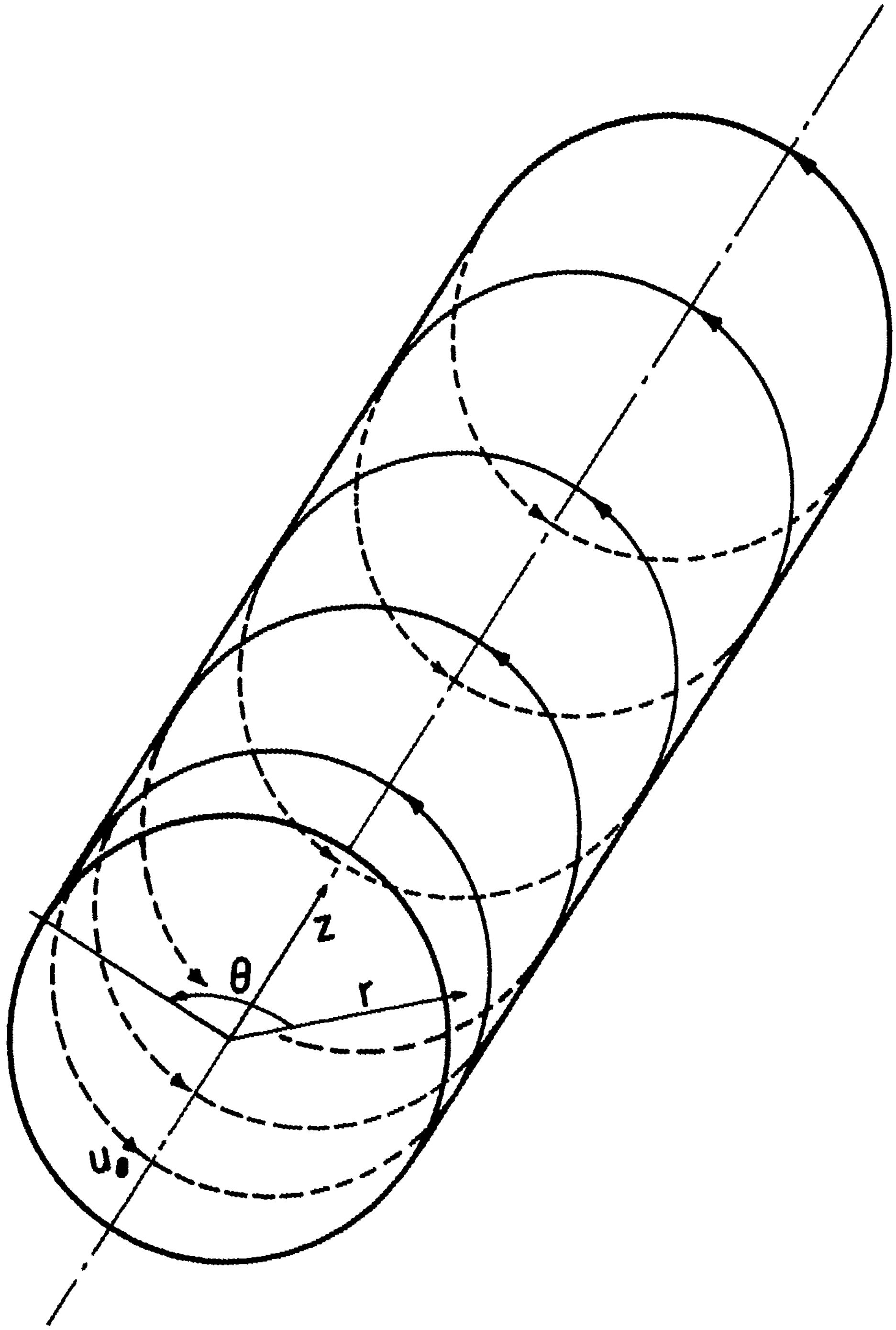


Figure 1

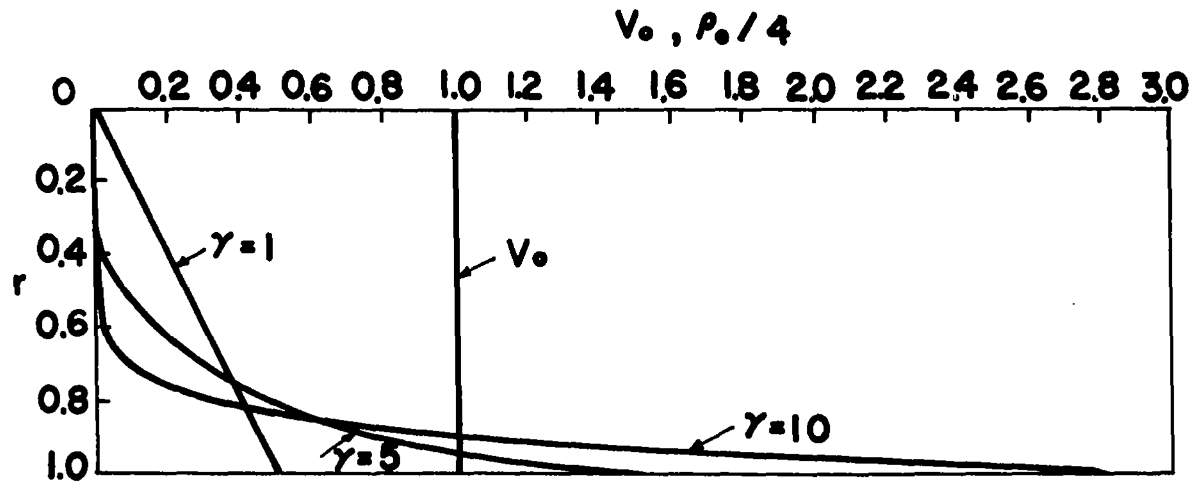


Figure 2

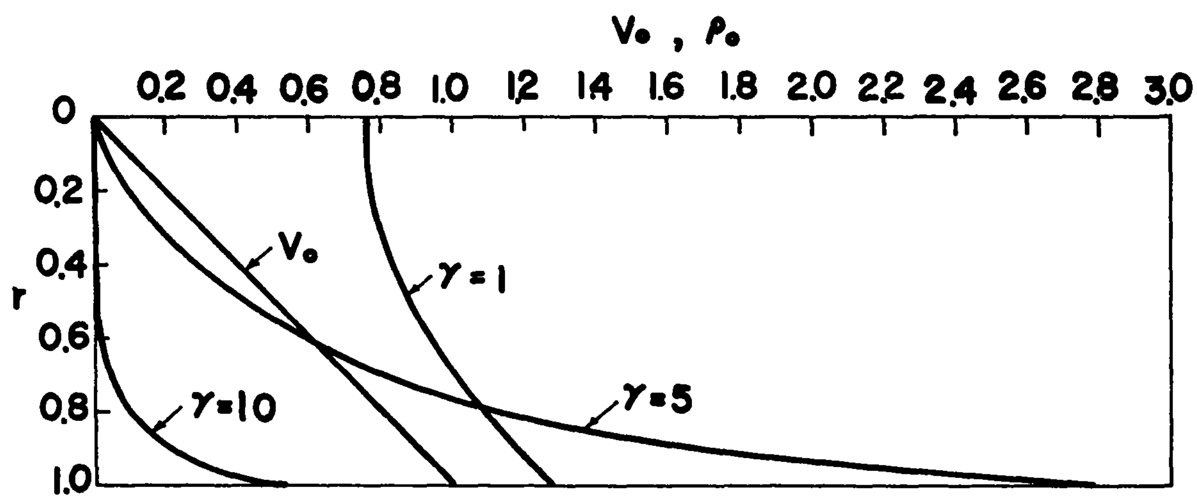
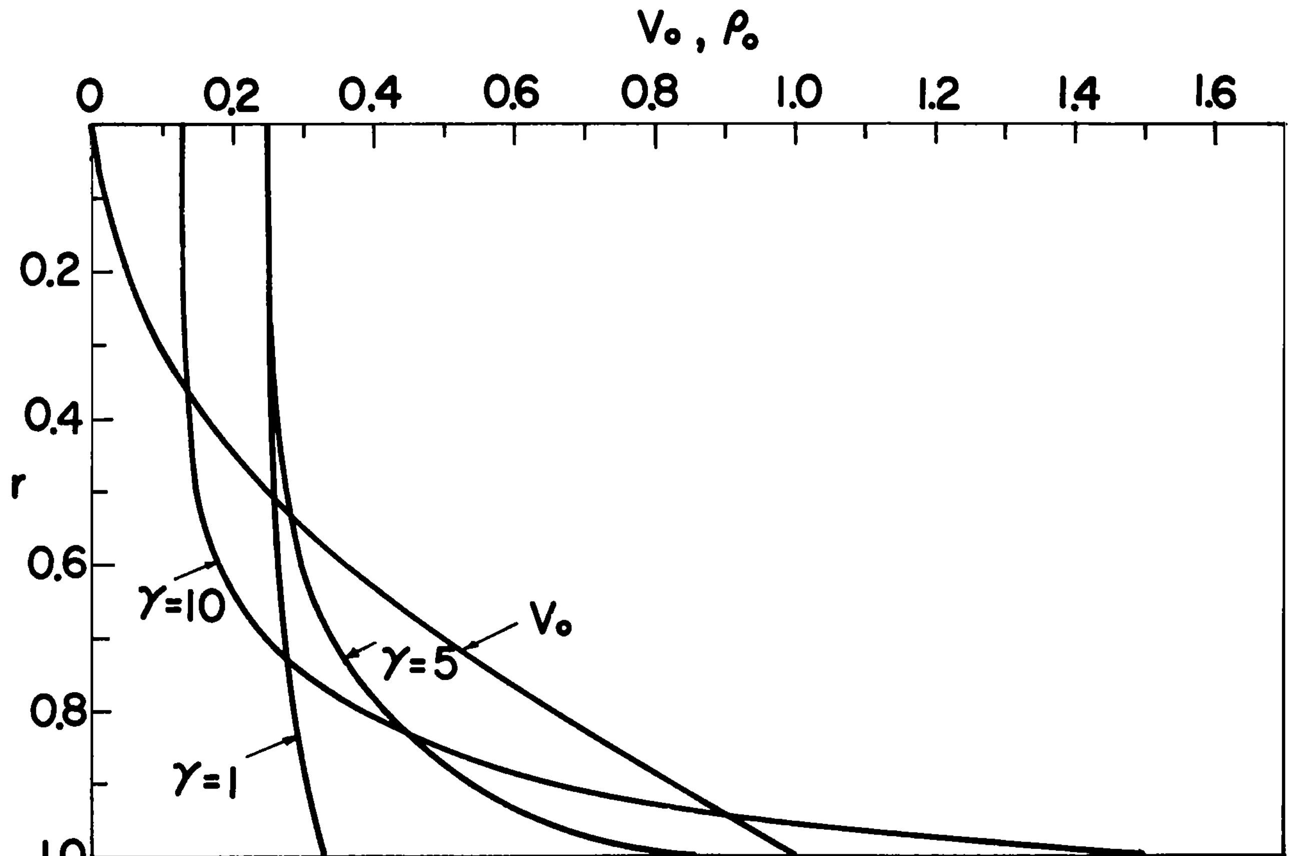
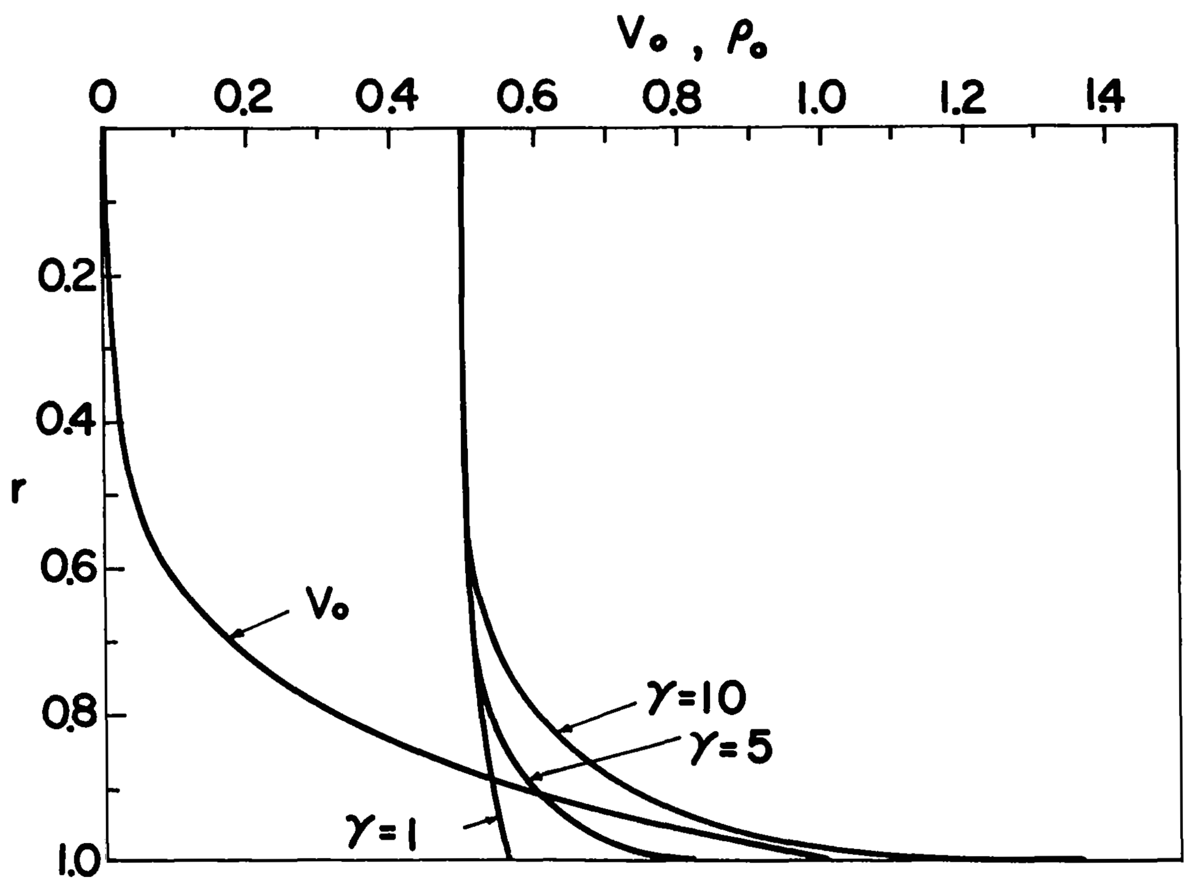


Figure 3



V_o , ρ_o
0
0.2
0.4
0.6
0.8
1.0
1.2
1.4
1.6
r
0.2
0.4
0.6
0.8
1.0
γ=10
γ=5
γ=1
V_o

Figure 4



V_o , ρ_o
0
0.2
0.4
0.6
0.8
1.0
1.2
1.4
r
0.2
0.4
0.6
0.8
1.0
V_o
γ=10
γ=5
γ=1

Figure 5

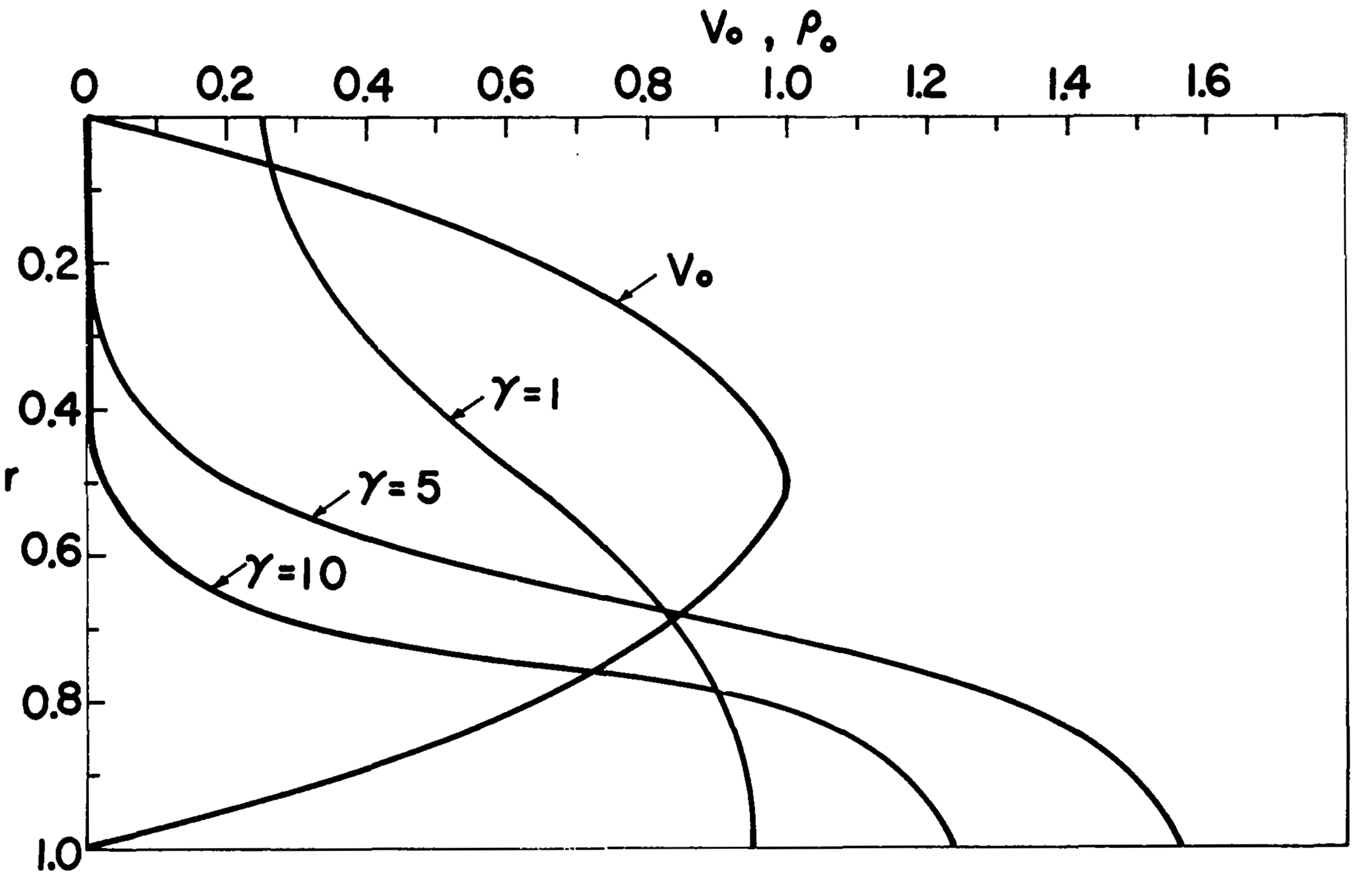
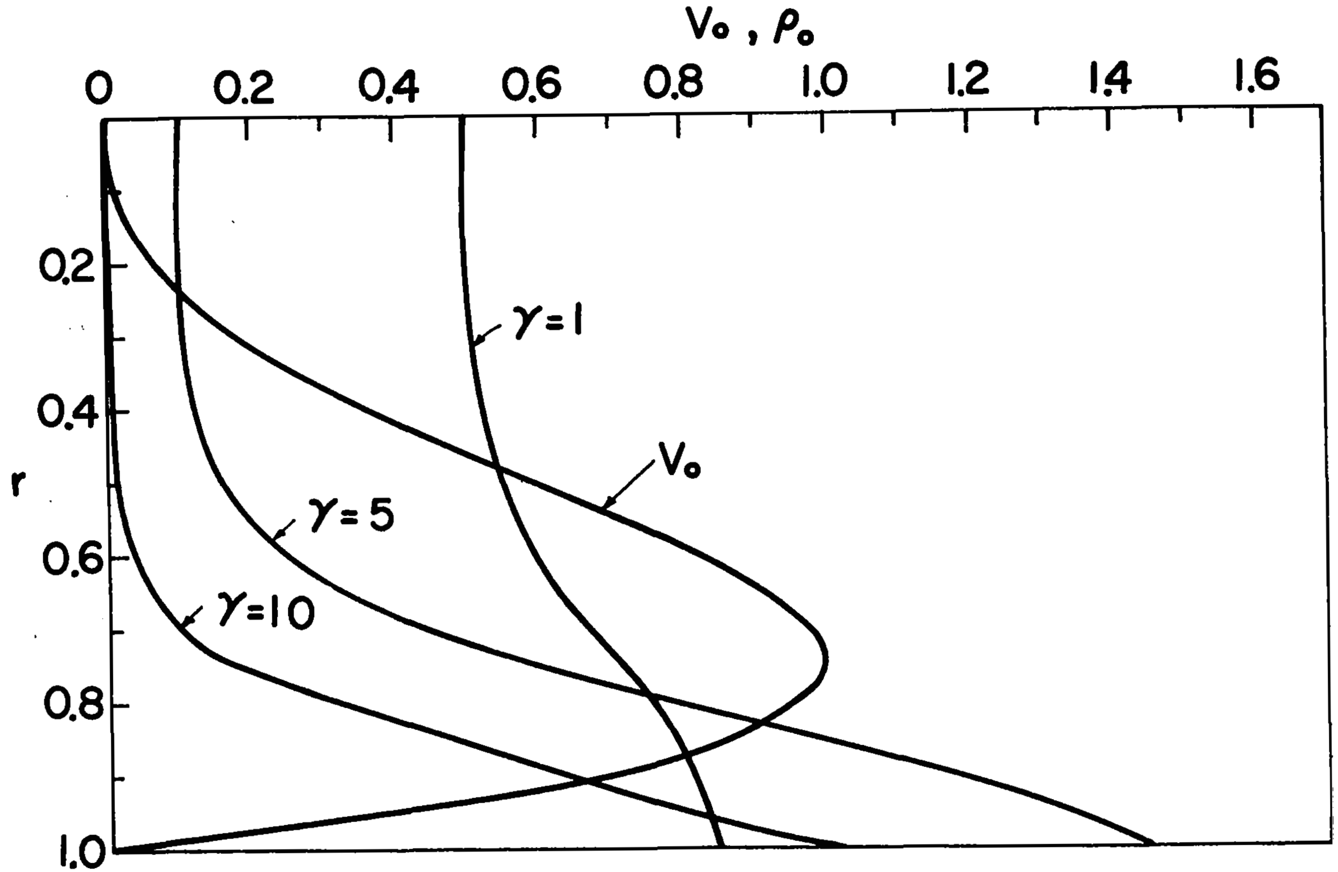


Figure 6



V_o , ρ_o
0
0.2
0.4
0.6
0.8
1.0
1.2
1.4
1.6
r
0.2
0.4
0.6
0.8
1.0
$\gamma=1$
V_o
$\gamma=5$
$\gamma=10$

Figure 7

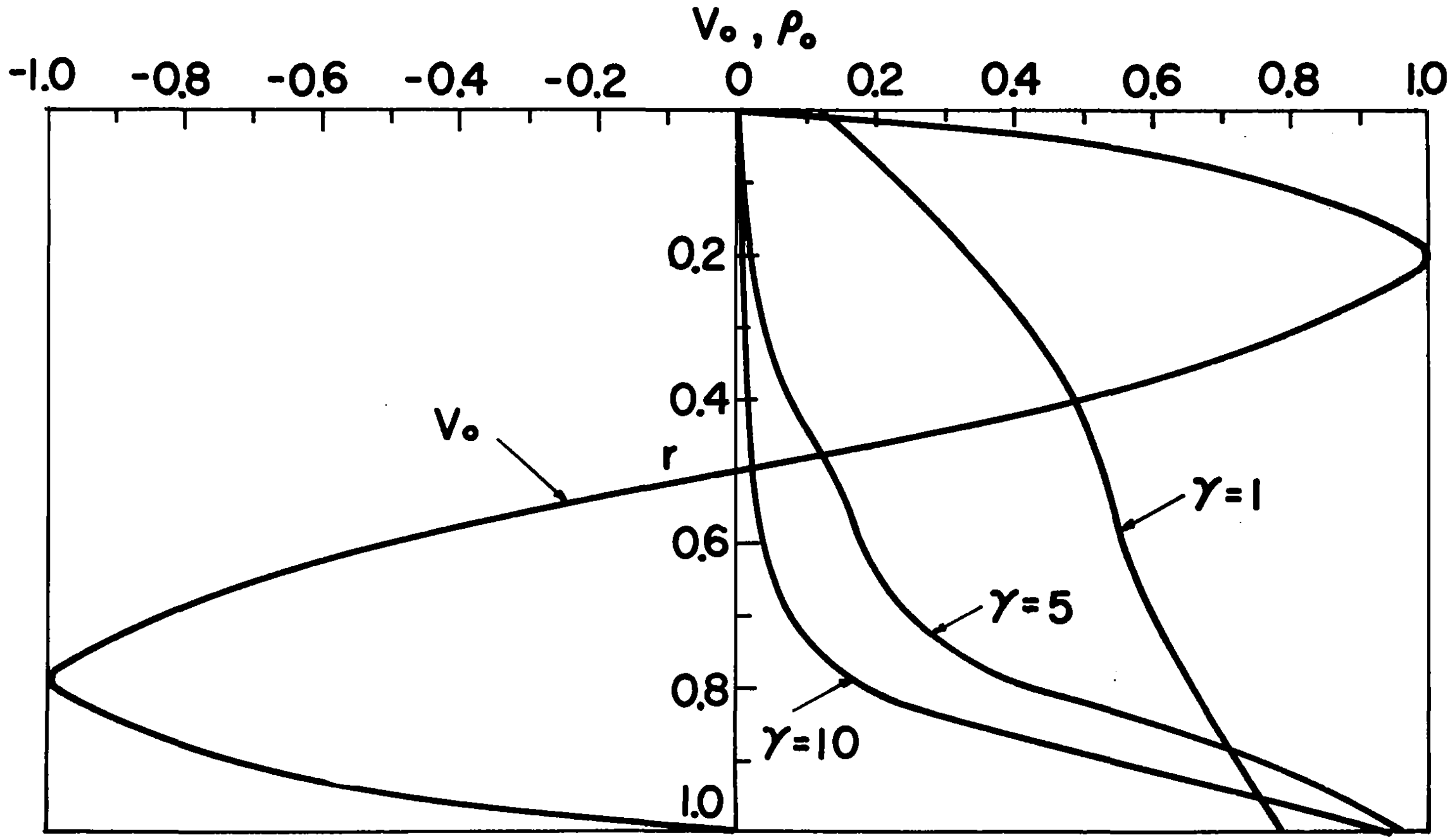


Figure 8

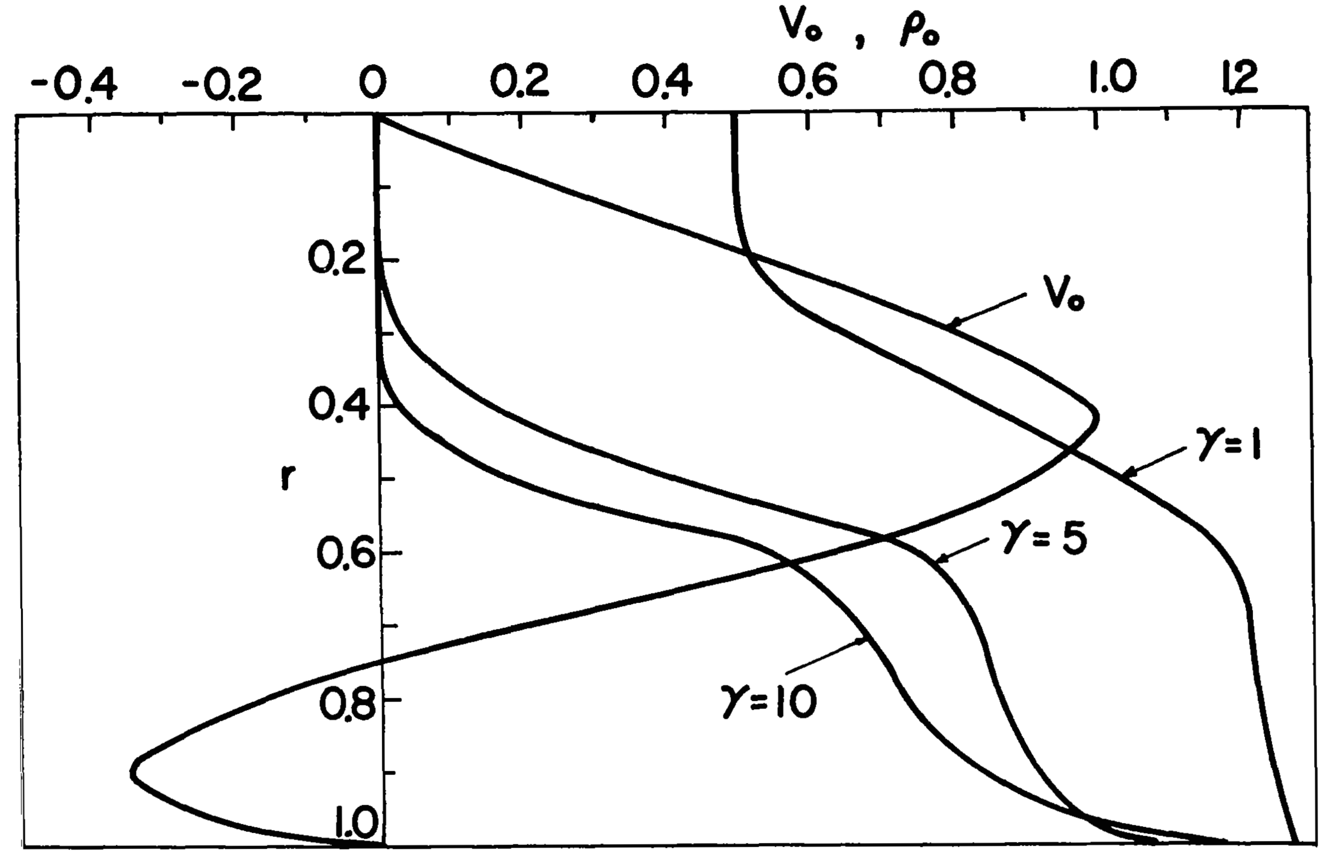


Figure 9

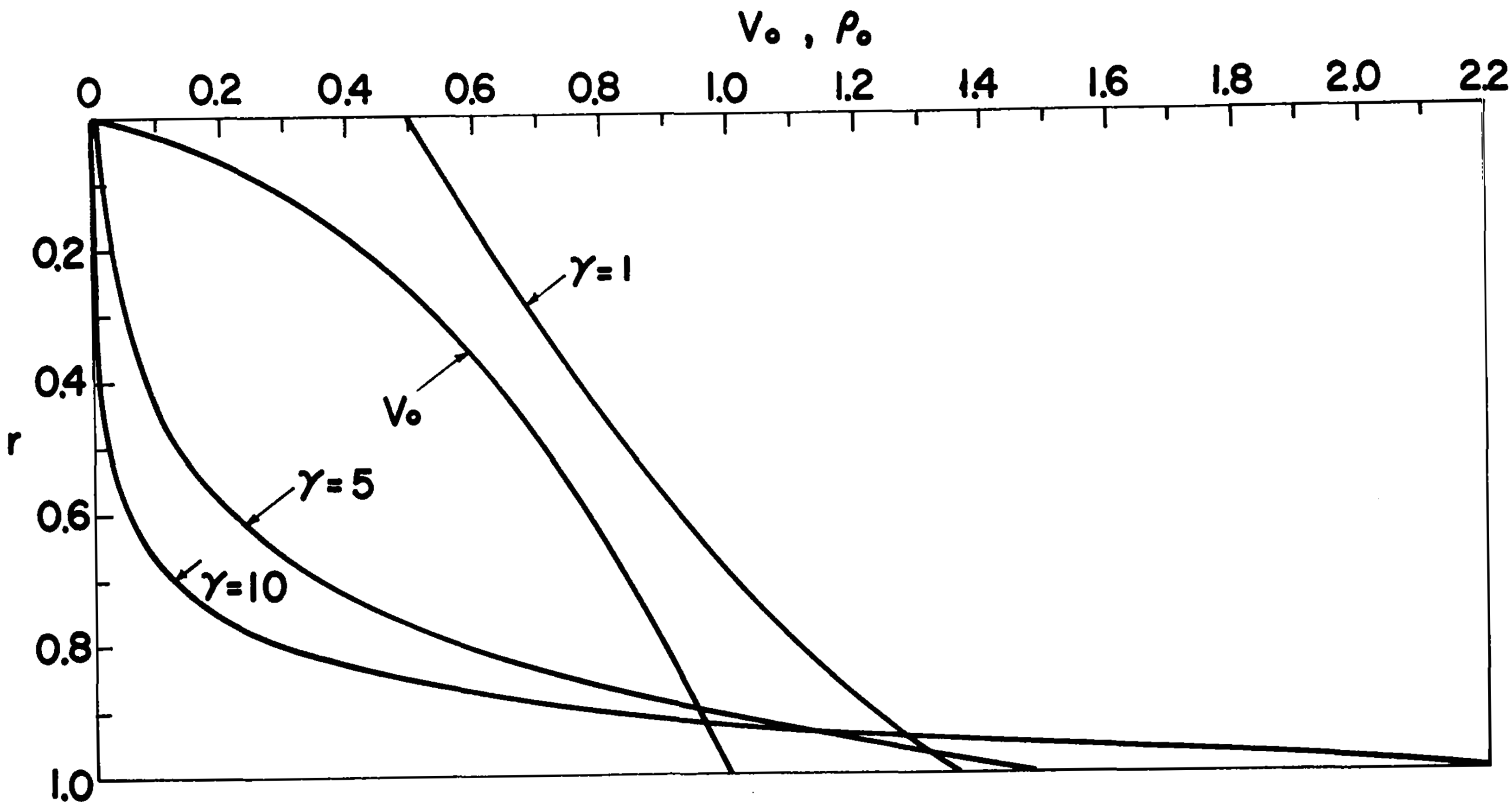


Figure 10

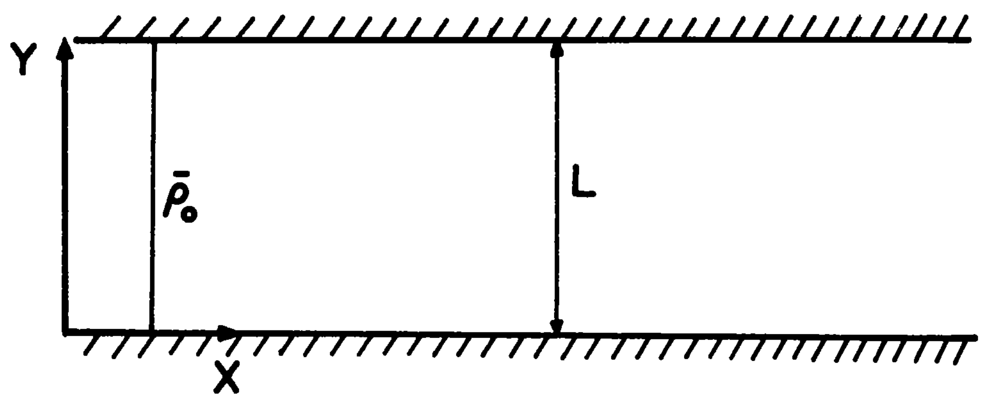


Figure 11

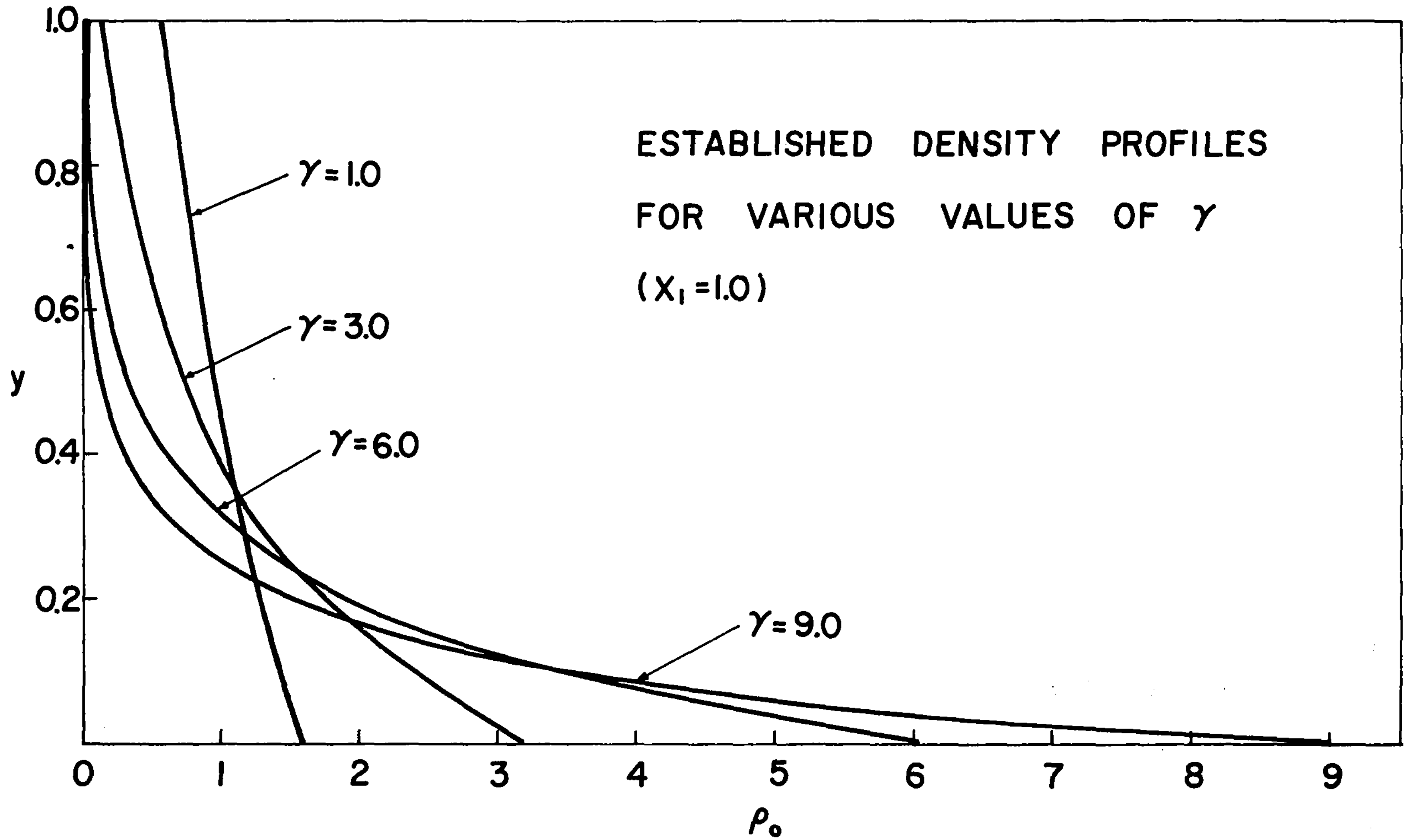

ESTABLISHED DENSITY PROFILES
FOR VARIOUS VALUES OF γ
(X₁ = 1.0)
γ = 1.0
γ = 3.0
γ = 6.0
γ = 9.0
y
1.0
0.8
0.6
0.4
0.2
0
1
2
3
4
5
6
7
8
9
ρ₀

Figure 12

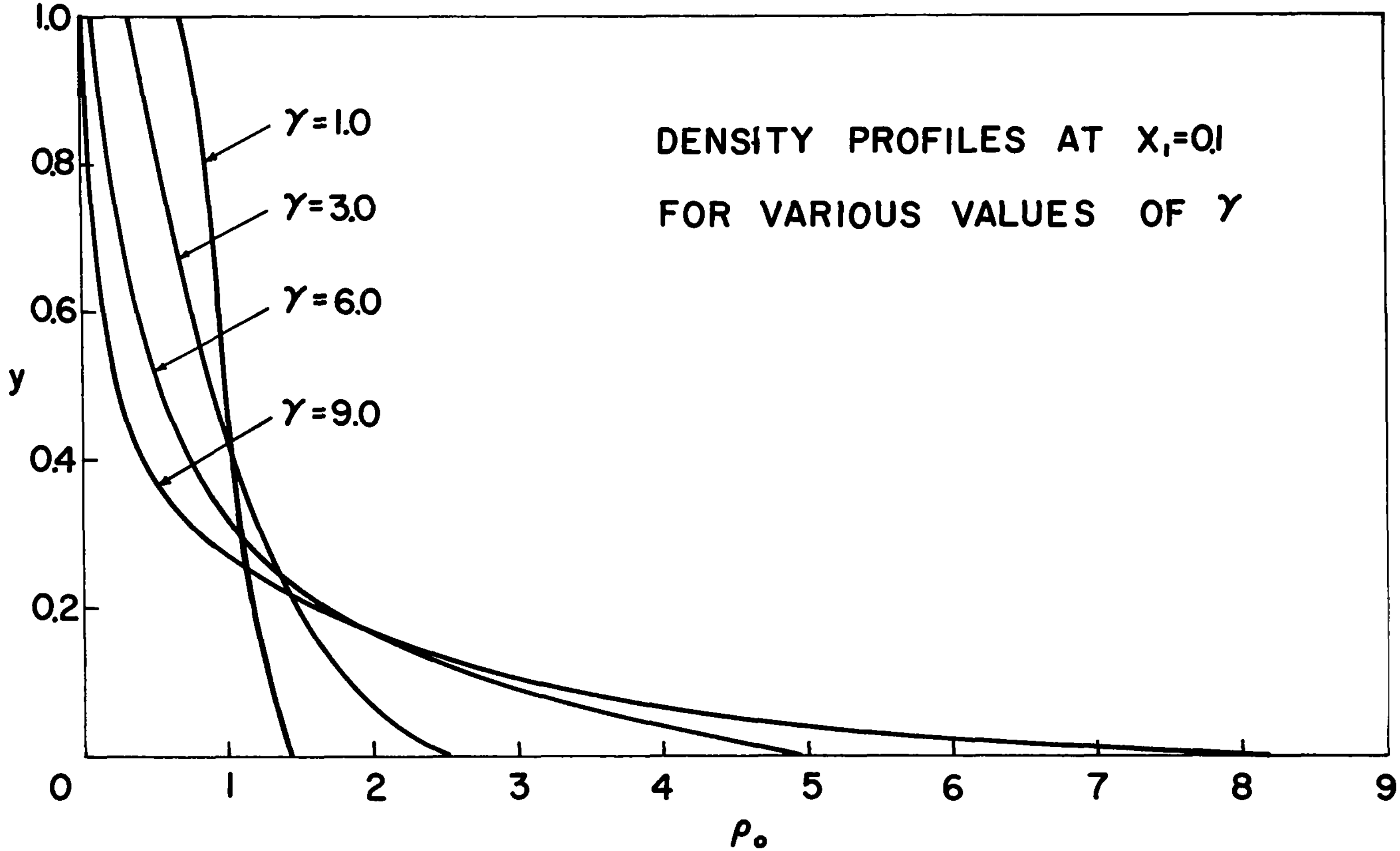

DENSITY PROFILES AT X₁=0.1
FOR VARIOUS VALUES OF γ
γ = 1.0
γ = 3.0
γ = 6.0
γ = 9.0
y
ρ₀
1.0
0.8
0.6
0.4
0.2
0
1
2
3
4
5
6
7
8
9

Figure 13

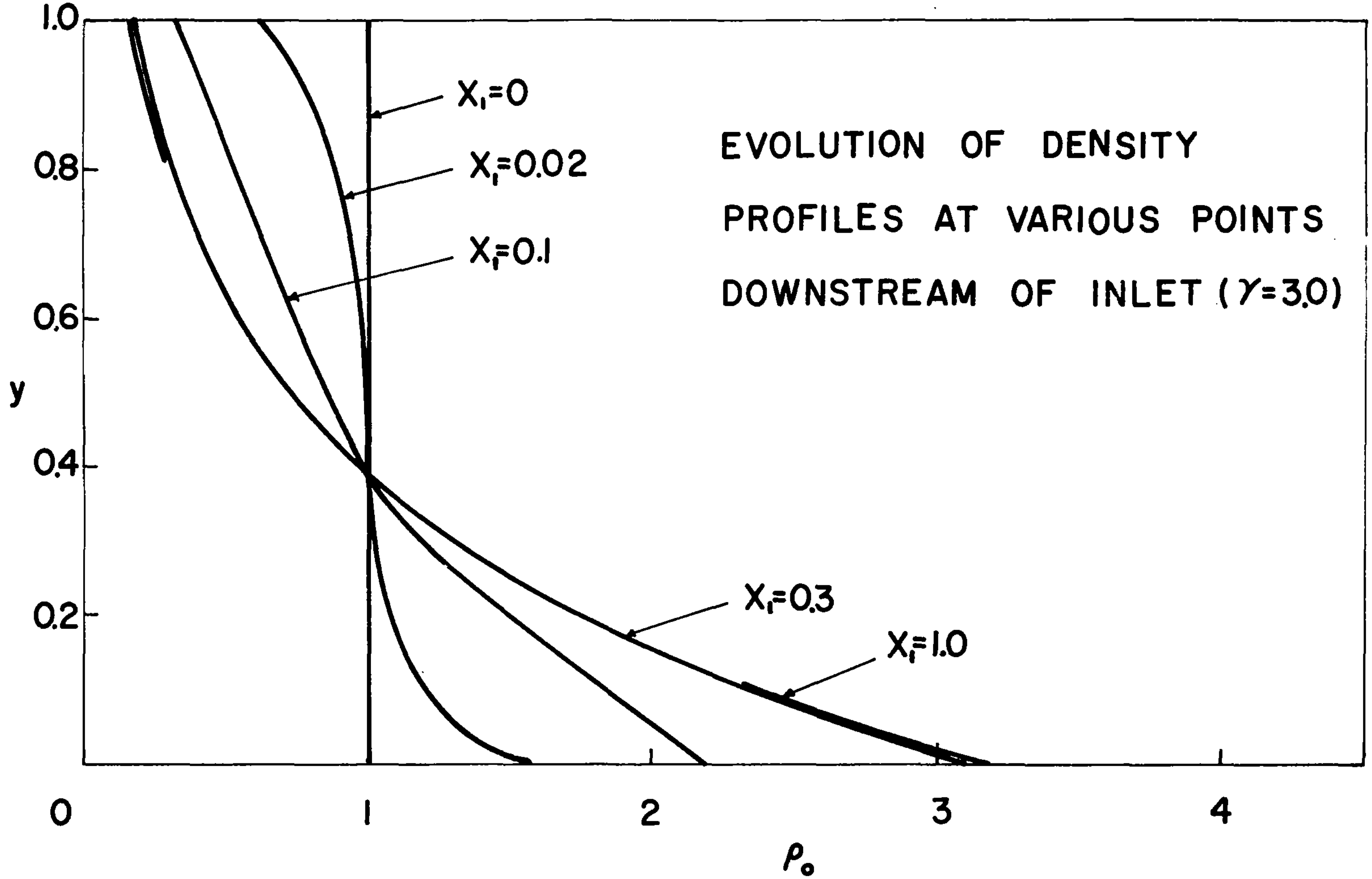

EVOLUTION OF DENSITY
PROFILES AT VARIOUS POINTS
DOWNSTREAM OF INLET (γ=3.0)
X_1=0
X_1=0.02
X_1=0.1
X_1=0.3
X_1=1.0
1.0
0.8
0.6
0.4
0.2
y
0
1
2
3
4
ρ_o

Figure 14

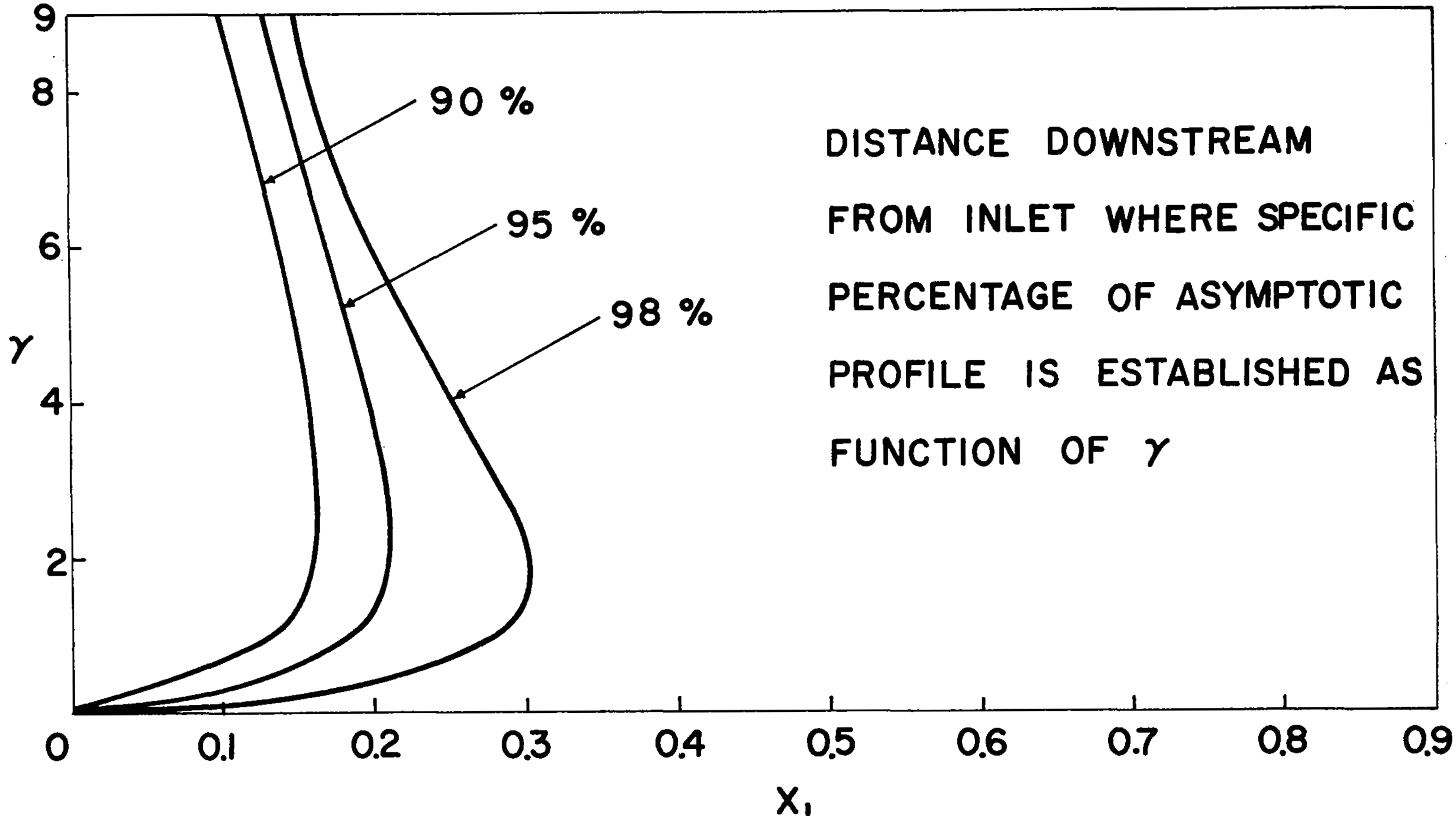


Figure 15

NASA-Langley, 1966 E-2612