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AN INTEGRAL MODEL FOR THE SIMILAR EXPANSION OF A GAS CLOUD INTO A VACUUM

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prepared for

NATIONAL AERONAUTICS AND SPACE ADMINISTRATION

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TOPICAL REPORT

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SUMMARY

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An integral expansion model is developed as a preliminary step in a theoretical model for the prediction of expansion resulting from a hypervelocity impact. The integral model is developed for the symmetrical (sphere, infinitely long cylinder or semi-infinite plane) expansion of a gas cloud into a vacuum. The mass, momentum and energy are conserved on a total integrated basis at all times during the expansion. Only the internal and kinetic energies are considered for the conservation of energy. The expansion is assumed to be isentropic.

The density, particle velocity and sound velocity variations with distance agree closely with those obtained from a finite difference solution for the initial times after the cloud is allowed to expand. After long periods of time, when the expansion flow becomes inertia dominated, the distributions of density, particle velocity, and sound velocity approach closely the long term self-similar results.

The integral model is a relatively fast method of obtaining the distributions of the density, particle velocity, and pressure with distance for different times during the expansion of an element. It takes less than a minute on an IBM 7040 computer to calculate 30 time increments for a cylindrical expansion. Author

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List of Symbols

a	velocity of sound
A	cross-sectional area
$B(t)$	power coefficient in density profile (Eq. 2.13)
$C(t)$	power coefficient in velocity profile (Eq. 2.14)
d	element diameter
$D(t)$	centre density decay coefficient (Eq. 4.12)
e	specific internal energy
E	total energy
f	function
I.E.	internal energy
j	similarity parameter (0 for plane, 1 for cylindrical 2 for spherical)
K.E.	kinetic energy
$\dot{m}$	mass rate of flow
M	total mass
p	pressure
$P(t)$	power coefficient in sound speed profile (Eq. B.3)
r	radial distance
r_i	radial position of expansion front
r_o	radial position of escape front
r'	radial position of escape front at the time the expansion front reaches the centre
R	$(r-r_i)/(r_o-r_i)$
t	time
t'	non-dimensional time $t/(d/2)/a_s$
t_c	time at which the expansion front reaches the center line.

List of Symbols - cont'd

u	particle velocity
V	volume
δx	element thickness
γ	expansion coefficient
Γ	gamma function
ρ	density
ψ	r/r_o
ψ_1	r'/r_o

Subscripts

c	refers to core conditions
es	refers to escape front
i	refers to expansion front
o	refers to escape front
r	refers to radial direction
R	refers to radial expansion region
s	shocked or high-energy condition

Table of Contents

1.0 Introduction:

The symmetrical expansion of a plane, cylindrical or spherical homogeneous mass of fluid into a vacuum is governed by the conservation equations in the following forms, plus an equation of state.

$$\frac{\partial \rho}{\partial t} + \rho \frac{\partial u}{\partial r} + u \frac{\partial \rho}{\partial r} + j \frac{\rho u}{r} = 0 \tag{1.1}$$

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial r} + \frac{1}{\rho} \frac{\partial p}{\partial r} = 0 \tag{1.2}$$

$$\frac{\partial e}{\partial t} + u \frac{\partial e}{\partial r} + p \frac{\partial \left(\frac{1}{\rho}\right)}{\partial t} = 0 \tag{1.3}$$

$$e = f(p, \rho) \tag{1.4}$$

Equation 1.2 for the conservation of momentum neglects shear and body forces and therefore equation 1.3, for the conservation of energy, assumes the expansion flow is isentropic (i.e. $\frac{ds}{dt} = 0$).

The four equations 1.1 to 1.4 may be solved for the four dependant variables ρ , u , p , and e as functions of the two independant variables r and t , if an equation of state (Eqn. 1.4) is specified. Closed form analytical solutions are available for the planar case (j =0) (Ref. 1), and the cylindrical and spherical cases (j = 1, 2) for certain specified conditions (Ref. 2).

Numerical methods may be used to solve the set of equations 1.1 to 1.4 for the general case (Ref. 3). However these methods require relatively long computer times to obtain close approximations for the expansion flow, and become unnecessary after long periods of time when the expansion becomes inertia dominated and self-similar solutions may be used.

An integral expansion model was developed in order to obtain a simple approximate solution for the variation of the dependant variables with r and t, for short and long periods of time. The integral model satisfies the conservation equations on a total integrated basis over the expansion region at any time. A form for the density and particle velocity distributions, with r, was specified. The forms assumed for these distributions (Sec. 2.0) were similar to the long term self-similar distributions (Ref. 1).

In the following sections the integral model will be developed for the expansion of an element of an infinitely long cylinder as an expansion of this type is required for the development of a theoretical model to predict the expansion resulting from a hypervelocity impact. Planar and spherical expansion solutions are developed in Appendices B and C.

2.0 Assumptions for the Cylindrical Model:

An infinitely long cylinder of diameter "d" was assumed to contain a homogeneous mass of fluid with a density "ρ_s" and a velocity of a sound "a_s". The particle velocities were assumed to be zero. At time t = 0 the mass of fluid was assumed to be unrestricted and allowed to expand into a surrounding vacuum. The resulting radial expansion of an element of the cylinder with thickness "δx" was considered. No axial expansion of the element will occur due to the infinite length of the cylinder. The model for the expansion of a finite cylinder with axial components of particle velocity will be presented in another report.

The material was assumed to expand as an ideal gas with:

$$\frac{P}{\rho^{\gamma}} = \text{const.} \tag{2.1}$$

The outside particles were assumed to move radially away from the centre axis at the constant escape velocity,

$$u_{es} = \frac{2}{\gamma - 1} a_s \tag{2.2}$$

A rarefaction wave was assumed to move radially into the element at the velocity of sound "a_s". Equation (2.2) fixes the outside radial boundary of the expanding cylindrical element as:

$$r_o = d/2 + u_{es} t \tag{2.3}$$

The position of the front of the inward moving rarefaction wave can be determined as a function of time knowing the sound velocity.

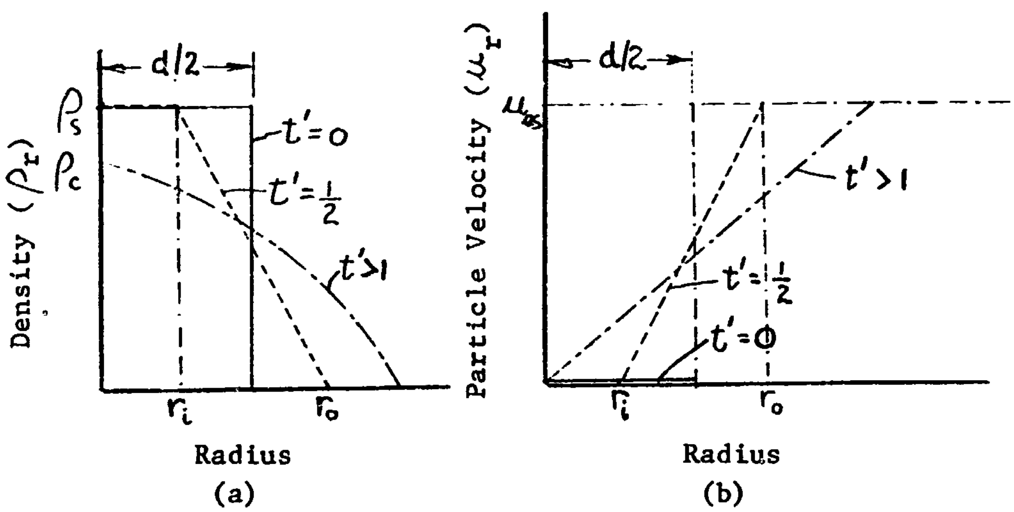
$$r_i = d/2 - a_s t \tag{2.4a}$$

Once the rarefaction wave has moved to within .0001 times the original radius of the cylinder it was assumed to remain at that position and

$$r_i = .0001 \; d/2 \tag{2.4b}$$

Figure 2.1 shows schematically the variation of density and particle velocity with respect to the radius for various non-dimensional times after release of the element. A non-dimensional time of 1 would occur when the ingoing rarefaction wave has reached the centre of the cylindrical element and can be expressed as

$$t' = \frac{t}{(d/2)a_s} \tag{2.5}$$



Density and Particle Velocity vs Radius

Figure 2.1

At $t'=0$, before expansion occurs, the density is constant and equal to ρ_s over the original radius d/2, and the particle velocity is zero throughout the cylinder.

At $t' = \frac{1}{2}$ the ingoing rarefaction wave has progressed into the cylinder $\frac{1}{2}$ the original radius to a position at r_i. The original outside particles of the cylinder have expanded outward to a position r_o which can be obtained from equations (2.2) and (2.5). Between the radii r_o and r_i the density and particle velocity will vary in some manner so that:

$$\text{at } r_o\,, \quad u_r = u_{es} \tag{2.6}$$

$$\text{and } \quad \rho_r = 0 \tag{2.7}$$

$$\text{at } r_i\,, \quad u_r = 0 \tag{2.8}$$

$$\rho_r = \rho_s \quad (t' \leq 1) \tag{2.9}$$

Equations (2.6) to (2.9) specify the required boundary conditions if the cylinder is expanding into a vacuum at a constant escape front velocity,and the rarefaction wave moves inward at the local sound velocity.

At $t' > 1$ the rarefaction wave will have reached the centre of the cylinder and the density at the centre will have decayed from the original value ρ_s to some reduced value ρ_c. Therefore the boundary condition for the density at r_i will become,

$$\text{at } r_i\,, \quad \rho_r = \rho_c \tag{2.10}$$

The other boundary conditions, equations (2.6) to (2.8) will remain the same. The value of ρ_c, which is a function of time is determined from a core model as discussed in section 4.

At any time during the expansion the total mass, momentum, and energy were conserved. The integral forms of the conservation equations for a cylindrical element are:

Mass $$M_R = 2\pi\delta x \int_{r_i}^{r_o} \rho_r \, r dr \qquad (2.11)$$

Momentum

The total momentum is automatically conserved by assuming the expansion to be symmetrical about the axis, as the change in momentum along any particle path is balanced by an equal change in momentum along another particle path in the opposite direction.

Energy
$$E_R = 2\pi\delta x \int_{r_i}^{r_o} \rho_r \left[\frac{a_r^2}{\gamma(\gamma-1)} + \frac{u_r^2}{2} \right] r dr \qquad (2.12)$$

where $\frac{a_r^2}{\gamma(\gamma-1)}$ is the specific internal energy

and $\frac{u_r^2}{2}$ is the specific kinetic energy

The conservation equations in their integral forms 2.11 and 2.12 require profile forms for the density and particle velocity variation with respect to the radius in order to evaluate the integrals for a particular time. The velocity of sound a_r was assumed to be a function of γ, and the density ρ_r. The profile forms assumed for the density and particle velocity variations with the radius are:

For $t' \leq 1$

$$\rho_r = \rho_s \left[1 - \frac{r - r_i}{r_o - r_i} \right]^{B(t)} \qquad (2.13)$$

$$u_r = u_{es} \left(\frac{r - r_i}{r_o - r_i} \right)^{C(t)} \qquad (2.14)$$

For $t' > 1$

$$\rho_r = \rho_c \left[1 - \left(\frac{r}{r_o} \right)^2 \right]^{B(t)} \tag{2.15}$$

$$u_r = u_{es} \left(\frac{r}{r_o} \right) \tag{2.16}$$

The profile forms for $t' > 1$, equations (2.15) and (2.16), are the same as those obtained for the long term self-similar flows when the expansion becomes inertia dominated and the pressure terms are neglected (Reference 1). The profile forms for $t' \leqslant 1$, equations (2.13) and (2.14) were obtained by modifying the long term expansion profiles in the light of some initial results obtained. For the initial times after expansion, r_i is not negligible compared to r_o and was therefore allowed for. It was found unnecessary to specify a linear velocity profile as the unknown value of $C(t)$ could be determined for $t' \leqslant 1$. The density was not made proportional to $\left(\frac{r}{r_o}\right)^2$ as the density decreased abruptly at r_i rather than in a continuous manner as it does for $t' > 1$ when the rarefaction has reached the centre.

3.0 Short Term Cylindrical Expansion:

The short term expansion is used here for the expansion of the element during the time $t' = 0$ to $t' = 1$. The unknown exponents $B(t)$ and $C(t)$ in equations 2.13 and 2.14 were determined from the 2 conservation equations 2.11 and 2.12. Once the exponents $B(t)$ and $C(t)$ had been determined the density and particle velocity profiles were determined from equations 2.13 and 2.14.

The total mass and energy within the expansion region is equal to the mass and energy of the original element minus that which remains in the unexpanded region at any time.

$$M_R = \rho_s \pi \delta x \left[(d/2)^2 - r_i^2 \right] \tag{3.1}$$

$$E_R = M_R \frac{a_s^2}{\gamma(\gamma-1)} \tag{3.2}$$

where $$r_i = d/2 - a_s t \tag{2.4}$$

Substituting for ρ_r in equation 2.11 from equation 2.13,

$$M_R = 2\pi\delta x \int_{r_i}^{r_o} \rho_s \left[1 - \frac{r - r_i}{r_o - r_i} \right]^{B(t)} r\,dr \tag{3.3}$$

Integrating (3.3), (Appendix A)

$$M_R = 2\pi\delta x \rho_s \left[\frac{(r_o - r_i)^2}{(B(t)+1)(B(t)+2)} + \frac{r_i (r_o - r_i)}{(B(t)+1)} \right] \tag{3.4}$$

and $B(t)$ may be solved for at any time t knowing:

$$r_o = d/2 + \frac{2}{\gamma - 1} a_s t \tag{2.2}$$

$$r_i = d/2 - a_s t \tag{2.3}$$

Once $B(t)$ is obtained $C(t)$ may be determined from the energy equation (2.12). Substituting for ρ_r, a_r, u_r in Eqn. 2.12, and using the ideal gas relation:

$$a_r = a_s \left(\frac{\rho_r}{\rho_s}\right)^{\frac{\gamma-1}{2}} \tag{3.5}$$

$$E_R = 2\pi\delta x \int_{r_i}^{r_o} \rho_s \left[1 - \frac{r-r_i}{r_o-r_i}\right]^{B(t)} \left[\frac{a_s^2\left(1-\frac{r-r_i}{r_o-r_i}\right)^{B(t)(\gamma-1)}}{\gamma(\gamma-1)} + \frac{u_{es}^2}{2}\left(\frac{r-r_i}{r_o-r_i}\right)^{2C(t)} \right] r\, dr \tag{3.6}$$

Integrating equation 3.6 (Appendix A),

$$\begin{aligned} E_R = {} & K_1 \frac{a_s^2}{\gamma(\gamma-1)} \left[\frac{1}{(\gamma B+2)(\gamma B+1)}\right] \\ & + K_1 \frac{u_{es}^2}{2} \left[\frac{B \cdot 2C \cdot (2C+1)}{(B+2C+2)(B+2C+1)(B+2C)} \; \frac{\Gamma(B)\Gamma(2C)}{\Gamma(B+2C)}\right] \\ & + K_2 \frac{a_s^2}{\gamma(\gamma-1)} \left[\frac{1}{\gamma B+1}\right] \\ & + K_2 \frac{u_{es}^2}{2} \left[\frac{B \cdot 2C}{(B+2C+1)(B+2C)} \; \frac{\Gamma(B)\Gamma(2C)}{\Gamma(B+2C)}\right] \end{aligned} \tag{3.7}$$

where

$$\begin{aligned} K_1 &= 2\pi\delta x \rho_s (r_o - r_i)^2 \\ K_2 &= 2\pi\delta x \rho_s (r_o - r_i) r_i \end{aligned} \tag{3.8}$$

and r_o and r_i are $f(t)$ as before

E_R is given by Eqn. 3.2

From equations 3.7 and 3.8, $C(t)$ may be obtained for the same time chosen for the solution of B(t).

The distributions of ρ_r and u_r with respect to the radius, for any time during the expansion, can be obtained from the profiles given in equations 2.13 and 2.14, and the determined values of $B(t)$ and $C(t)$. Some calculated profiles are shown in section 6.

4.0 Centre Line Decay:

After the ingoing rarefaction front reaches the centre axis, a decay process takes place at the centre (Figure 2.1). This decay process introduces an additional unknown ρ_c at the centre. In order to determine how ρ_c varies with time, a simple core model was considered.

When the front of the ingoing rarefaction wave reaches a position a short distance away from the centre axis, it was assumed to remain there, forming a centre axis core with a radius r_c = .0001 d/2. This approach is consistant with other mathematical solutions where the position at the centre (r = 0 and u_r = 0) is avoided.

The volume and circumferential area of the core will remain constant with respect to time as r_c is assumed constant.

$$V_c = \pi r_c^2 \delta x \qquad (4.1)$$

$$A_c = 2\pi r_c \delta x \qquad (4.2)$$

At any time the mass within the core will be:

$$M_c = \rho_c V_c \qquad (4.3)$$

and the rate of mass flow outward from the core will be:

$$\dot{m} = A_c \rho_c u_c \qquad (4.4)$$

Equations (4.3) and (4.4) impose a limitation on the core radius r_c by assuming that the density (ρ_c) within the core is constant with respect to the radius. Immediately after the rarefaction wave reaches a distance r_c from the centre, ($\partial\rho/\partial r$) at r_c is very large requiring r_c to be small. After a period of time ($\partial\rho/\partial r$) at r_c becomes very small and the assumption that ρ_c is constant throughout the core is a good approximation.

The rate of mass outflow from the core may also be expressed as

$$\dot{m} = - \frac{dM_c}{dt} \tag{4.5}$$

Then combining equations (4.4) and (4.5)

$$\frac{\partial \rho_c}{\rho_c} = - \frac{A_c \, u_c}{V_c} \, dt \tag{4.6}$$

and substituting for A_c and V_c from equations (4.1) and (4.2)

$$\frac{\partial \rho_c}{\rho_c} = - 2 \, \frac{u_c}{r_c} \, dt \tag{4.7}$$

Now assuming a linear velocity profile from the center to the outside radius r_o (Equation 2.16)

$$u_c = u_{os} \left(\frac{r_c}{r_o} \right) \tag{4.8}$$

Equation (4.8) imposes a limit on r_c as it implies that at the same time the rarefaction front reaches r_c, the particle velocity increases from 0 at the centre.

Substituting (4.8) into (4.7) with

$$r_o = d/2 + u_{es}t$$

$$\frac{d\rho_c}{\rho_c} = -\frac{2u_{es}}{d/2 + u_{es}t}\, dt \qquad (4.9)$$

Integrating (4.9) with boundary conditions

$$\rho_c \rightarrow 0 \qquad t \rightarrow \infty$$

$$\rho_c \rightarrow \rho_s \qquad t = \frac{d/2 - r_c}{a_s}$$

gives

$$\rho_c = \rho_s \left(\frac{d/2 + u_{es}t_c}{d/2 + u_{es}t}\right)^2 \qquad (4.10)$$

$$\rho_c = \rho_s \left(\frac{r'}{r_o}\right)^2$$

where r' is the radius of the escape front at the time when the rarefaction would have reached the centre axis. Equation (4.10) gives the value of ρ_c at the centre as a function of time. After a long period of time r_o becomes large with respect to $d/2$ and from equations (2.3) and (4.10).

$$\rho_c \propto \frac{1}{t^2} \qquad (4.11)$$

Equation (4.11) is consistant with the long term self-similar solution for inertia dominated flow, (Reference 1). Equation (4.10) will not apply for early times, just after the rarefaction wave reaches the centre, due to the assumptions made in equations (4.3), (4.4) and (4.8), as previously discussed. It is not necessary to specify the density decay at the centre if a linear velocity profile is assumed as the two conservation equations may be used to obtain two unknowns. Equation (4.10) may be written as a general form for the density decay at the centre:

$$\rho_c = \rho_s \left(\frac{r'}{r_o} \right)^{D(t)} \tag{4.12}$$

5.0 Long Term Cylindrical Expansion:

For the long term expansion, after the rarefaction reaches the centre, a linear velocity profile was assumed and the two conservation equations (2.11) and (2.12) were used to solve for the two unknowns $B(t)$ and $D(t)$ equations (2.15) and (4.12).

The total mass and energy in the expansion region is equal to the original total mass and energy of the element.

$$M = \pi \left(d/2 \right)^2 \delta x \, \rho_s \tag{5.1}$$

$$E = M \frac{a_s^2}{\gamma(\gamma - 1)} \tag{3.2}$$

Let $\Psi = \dfrac{r}{r_o}$ and $\Psi_1 = \dfrac{r'}{r_o}$ then the relations to be substituted into the conservation equations become, from (2.15), (2.16) and (4.12)

$$\rho_r = \rho_c \left[1 - \Psi^2 \right]^{B(t)} \tag{5.2}$$

$$\rho_c = \rho_s \Psi_1^{\,2} \tag{5.3}$$

$$\mu_r = \mu_{os} \Psi \tag{5.4}$$

Substituting for ρ_r, a_r, μ_r and r in equations (2.11) and (2.12) (Appendix A)

$$M = 2\pi \delta x \int_0^1 \rho_s \Psi_1^{\,D(t)} \left[1 - \Psi^2 \right]^{B(t)} r_o \Psi \, d\Psi \tag{5.5}$$

and
$$E = 2\pi \delta x \int_0^1 \rho_s \psi_1^{D(t)} \left[1-\psi^2\right]^{B(t)} \left[\frac{a_s^2}{\gamma(\gamma-1)} \psi_1^{D(t)(\gamma-1)} \left(1-\psi^2\right)^{B(t)(\gamma-1)} + \frac{\mu_{es}^2}{2}\psi^2\right] r_0^2 \psi \, d\psi \qquad (5.6)$$

Integrating equations (5.5) and (5.6) (Appendix A),

$$M = \frac{K_1 \psi_1^{D(t)}}{2(B(t)+1)} \qquad (5.7)$$

$$E = K_1 \left\{ \frac{\psi_1^{\gamma D(t)} a_s^2}{\gamma(\gamma-1)} \left[\frac{1}{2(\gamma B(t)+1)}\right] + \psi_1^{D(t)} \frac{\mu_{es}^2}{2} \left[\frac{1}{2(B(t)+1)(B(t)+2)}\right]\right\} \qquad (5.8)$$

where $K_1 = 2\pi \delta x \rho_s r_0^2$

The two unknowns $B(t)$ and $D(t)$ were determined from the two simultaneous equations (5.7) and (5.8) at any time for $t' > 1$. The results are shown in the next section.

The linear particle velocity distribution is consistant with the long term self-similar assumption obtained from assuming the flow to be inertia dominated.

The density distribution exponent $B(t)$ approaches the same value as that for the long term self-similar solutions

$$B(t) = \frac{2}{\gamma - 1} \qquad \text{(Ref. 1)} \tag{6.3}$$

Consider the internal energy portion which is the first term of equation (5.8)

$$\text{I.E.} = \frac{2\pi\delta x \rho_s r_o^2}{\gamma(\gamma-1)} \left(\frac{r'}{r_o}\right)^{\gamma D(t)} a_s^2 \left[\frac{1}{2(\gamma B(t)+1)}\right] \tag{6.4}$$

$$\text{and I.E.} = \text{const.} \frac{1}{\gamma B(t)+1} \left[r_o^2 / r_o^{\gamma D(t)} \right] \tag{6.5}$$

The value of $D(t)$ approaches 2 and any variation in $B(t)$ is insignificant compared to the increase in r_o so

$$\text{I.E.} \approx \text{const.} \left(\frac{1}{r_o}\right)^{\gamma} \tag{6.6}$$

From (6.6) it can be seen that as r_o becomes large the internal energy approaches zero as long as $\gamma > 1$. The internal energy will approach zero more quickly as γ is increased. This is shown by the results in figure 6.5.

Considering only the kinetic energy term of equation (5.8), and substitute for Ψ_1 from equation (5.7), then

$$B(t) = \frac{M\, u_{ps}^2}{2E} - 2 \tag{6.7}$$

integrated mass. This error could also be part of the cause for the discrepancy in the centre line density decay (Figure 6.6). In any case the differences between the two models are small and would indicate that the integral model density distribution is a good approximation.

Figure 6.8 shows the comparison of particle velocity profiles. Initially the finite difference solution departs from the linearity approximation, but rapidly approaches a linear distribution.

The descrepancies between the particle velocity profiles are small and would indicate that the integral model would approximate the expansion flow closely.

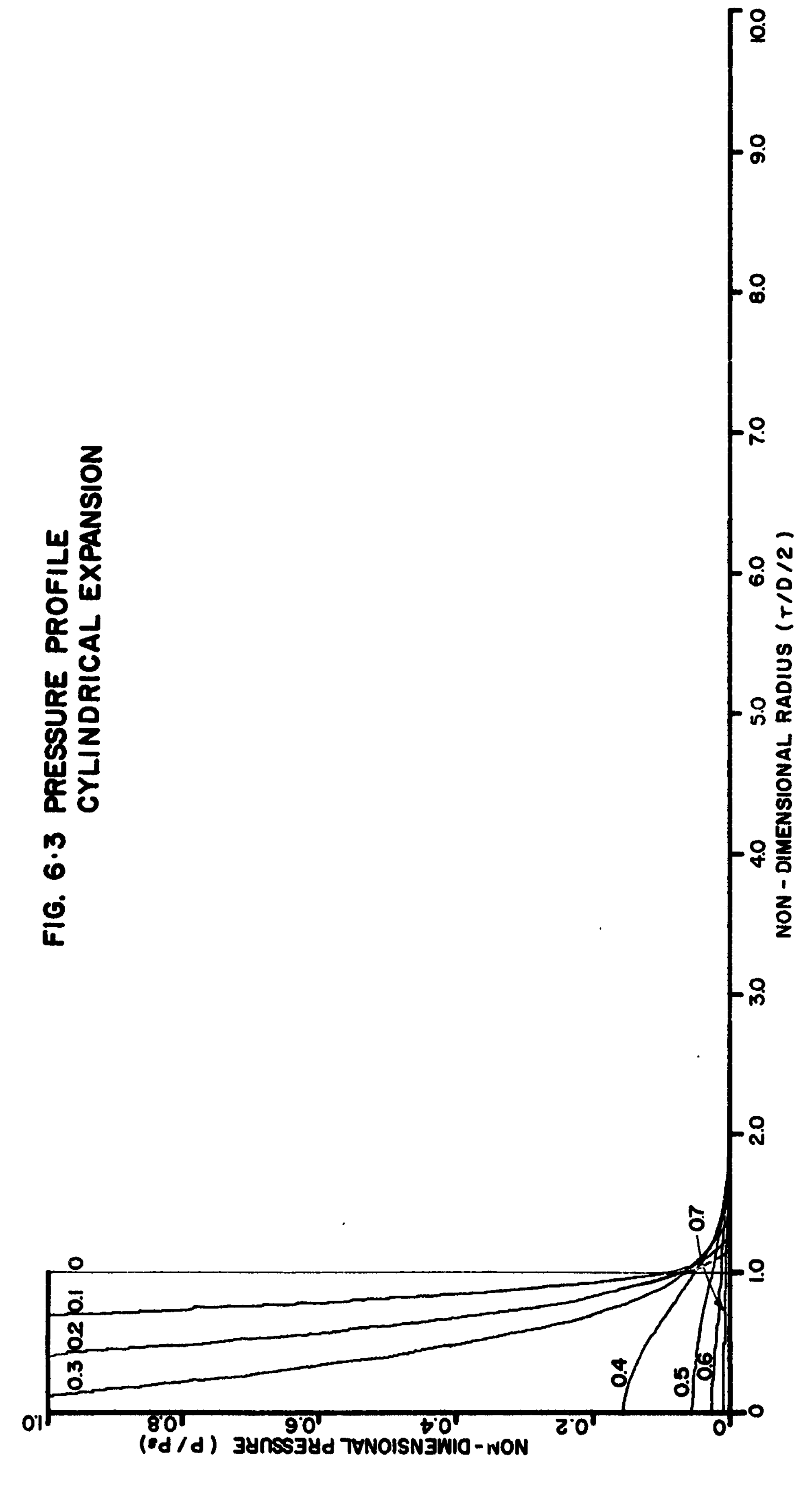


FIG. 6·3 PRESSURE PROFILE
CYLINDRICAL EXPANSION

6.0 Comparison of Results:

Figures 6.1 to 6.4 show the variations of sound velocity, particle velocity, pressure and density with radius for various times ($t' < 1$ and $t' > 1$). $B(t)$, $C(t)$ for the short term, and $B(t)$ and $D(t)$, with $C(t) = 1$, for the long term were first determined from the conservation equations. Then from the profile and decay forms, the distribution of density (Fig. 6.4) and particle velocity (Fig. 6.2) were determined as functions of radius for the times shown. Figures 6.1 and 6.3 were obtained from the density profile assuming the ideal gas relations:

$$a_r = a_s \left(\frac{\rho_r}{\rho_s} \right)^{\frac{\gamma-1}{2}} \tag{6.1}$$

$$P_r = P_s \left(\frac{\rho_r}{\rho_s} \right)^{\gamma} \tag{6.2}$$

The relationships shown in Figures 6.1 to 6.4 were calculated using $\gamma = 3$. Figure 6.5 shows the variation of the exponent $D(t)$ for the density decay vs time for different values of γ. It can be seen from figure 6.5 that the decay exponent very quickly approaches the long term self-similar value of 2 (Reference 1) for the higher values of γ. It appears, however, that the decay exponent ($D(t)$) does not quite reach the long term self-similar value of 2. This descrepancy could be due to the internal energy term which is not included in the long term self-similar solutions (Ref. 1).

Substituting for E , equation (6.7) becomes

$$E(t) = \frac{2}{\gamma - 1}$$

which is consistant with the long term self-similar density distribution exponent (equation 6.3).

In order to demonstrate the validity of the initial profiles of the dependant variables with radius and time, the results obtained from a finite difference solution are shown in figures 6.6 to 6.8. Figure 6.6 shows the centre line density ratio (ρ_c/ρ_s) vs non dimensional time (t') for both the finite difference and the integral solutions. It can be seen that in the integral model, the centre line density decays faster than in the finite difference solutions with the difference between the two solutions rapidly becoming small.

Figure 6.7 shows a comparison of the density profiles for the integral and finite difference solutions. There is virtually no difference between the two models until the ingoing rarefaction wave reaches the centre. In order to compare profiles, the centre line density of the integral model was made the same as that of the finite difference solution. It can be seen that the density obtained by the integral solution decreases more quickly with increasing radius than that obtained from the finite difference solution. This would indicate that an error has accumulated in the finite difference solution as it does not satisfy the total

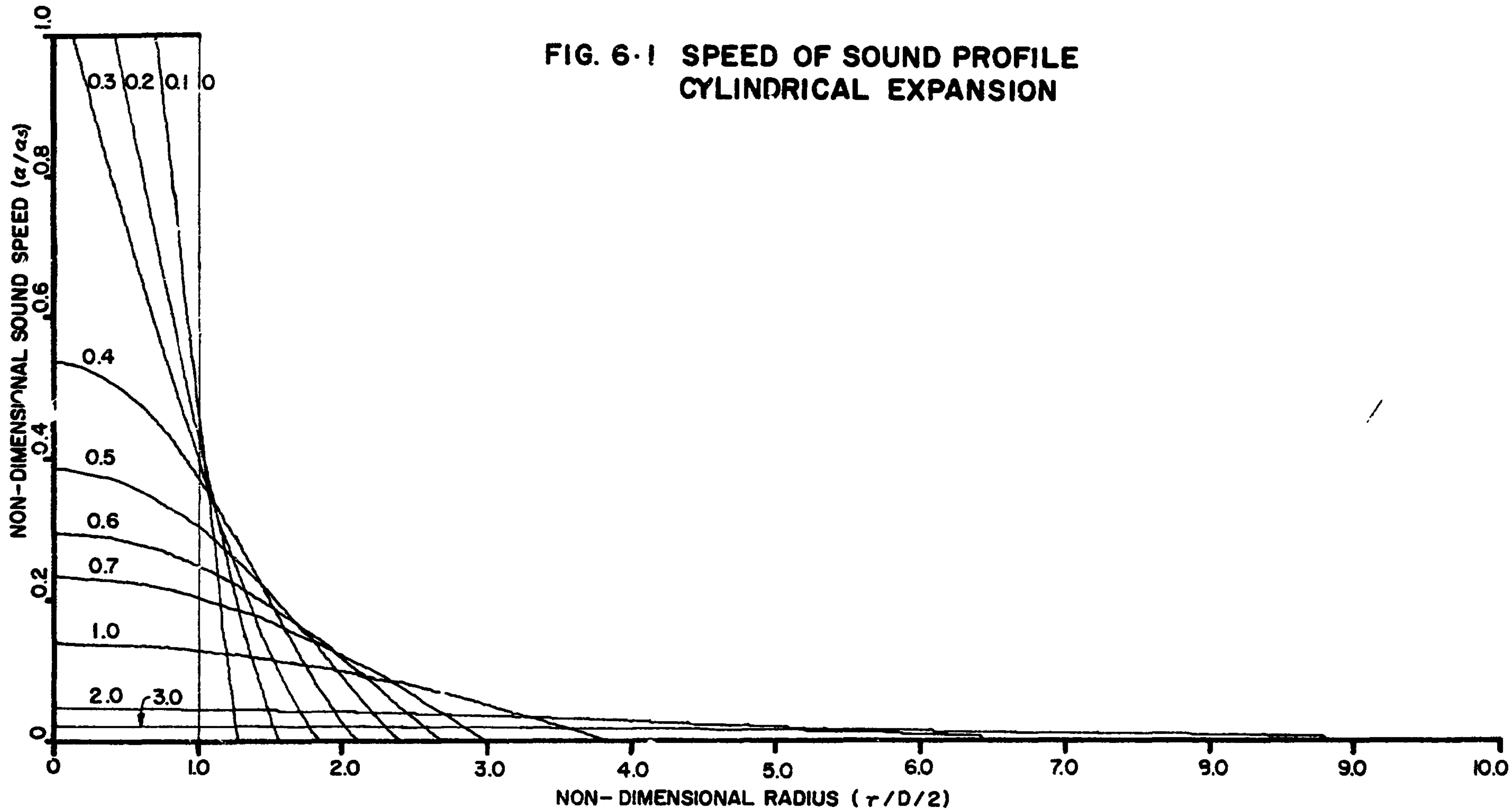


FIG. 6·1 SPEED OF SOUND PROFILE
CYLINDRICAL EXPANSION

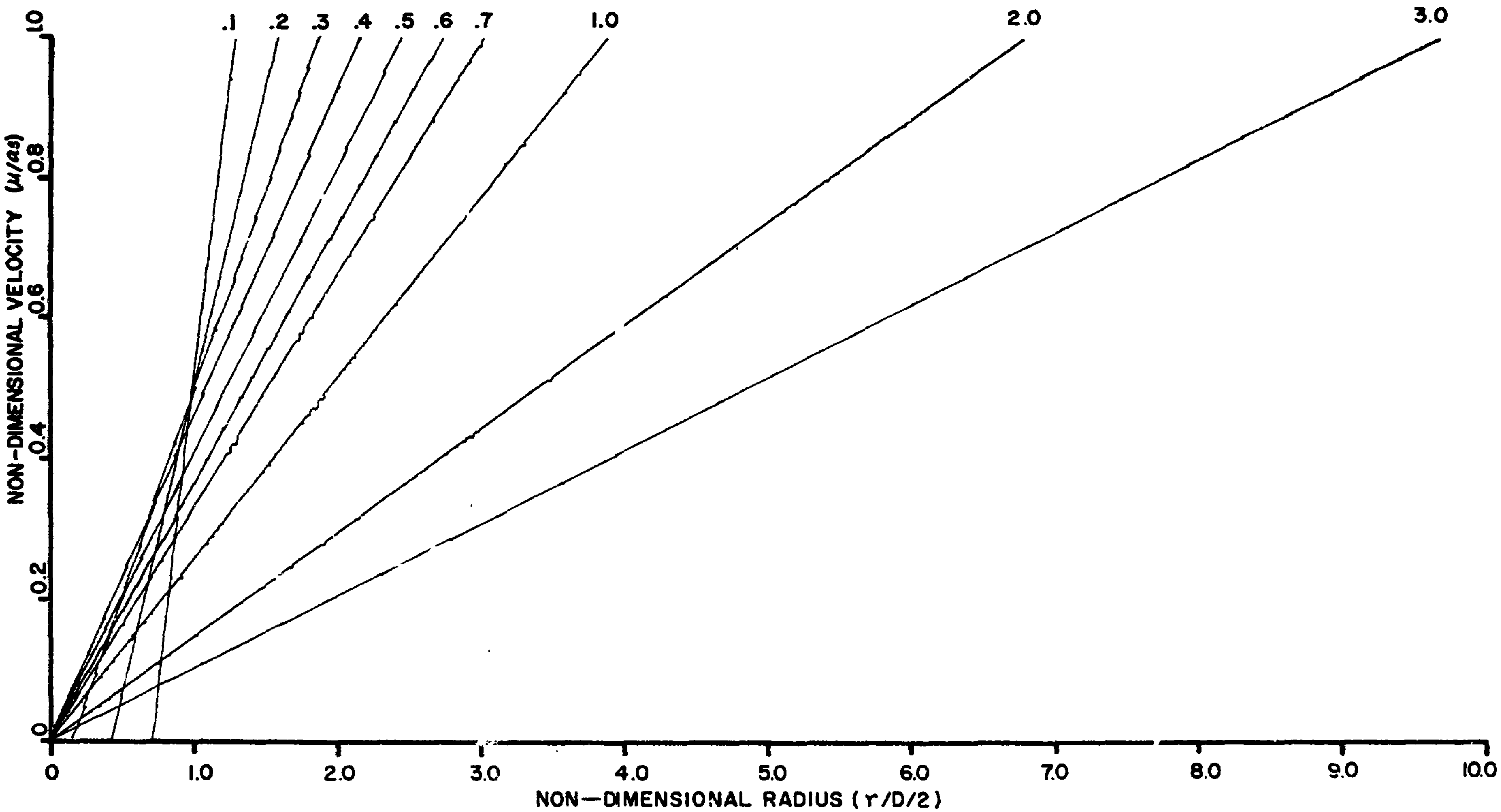


FIG. 6·2 VELOCITY PROFILE CYLINDRICAL EXPANSION

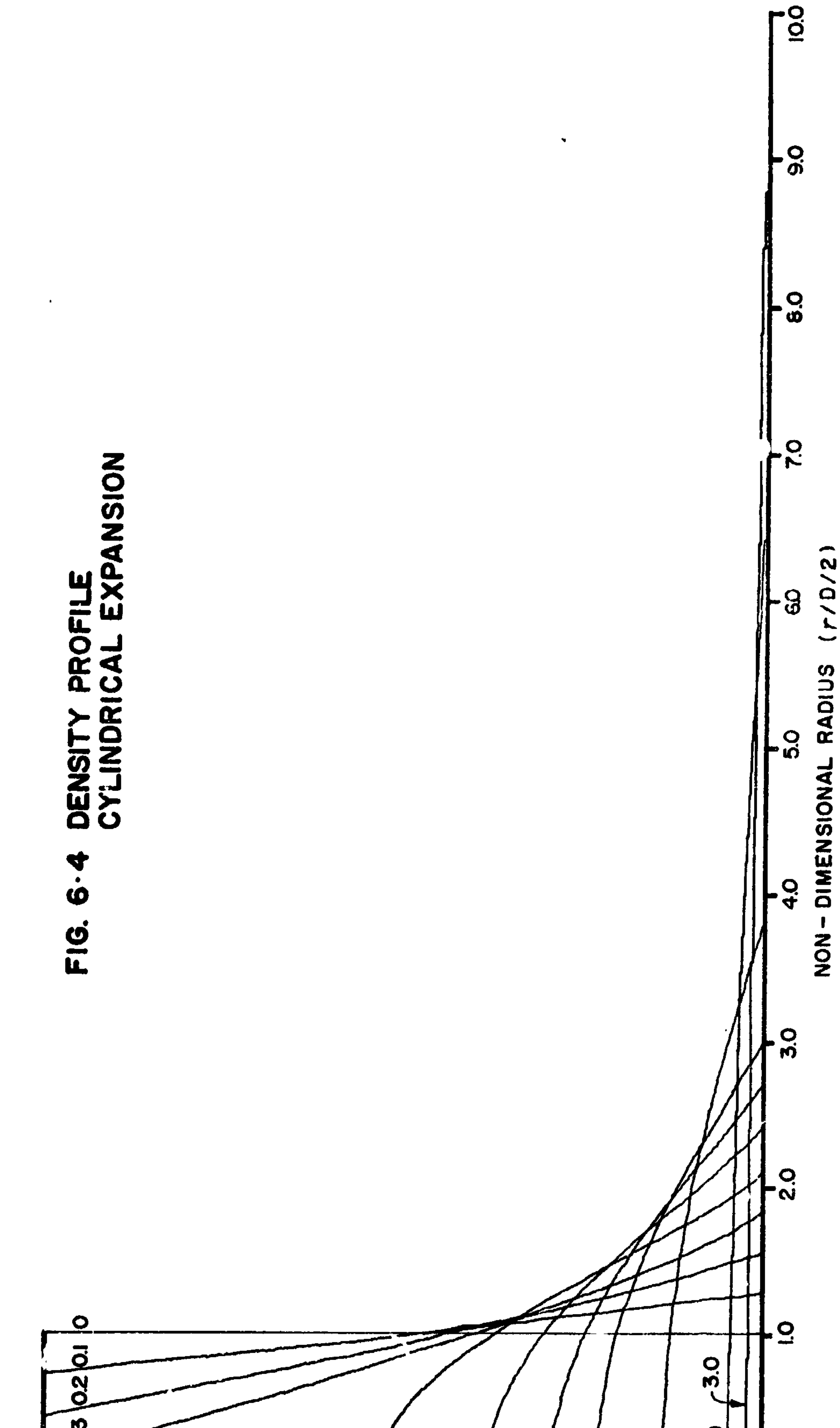


FIG. 6·4 DENSITY PROFILE
CYLINDRICAL EXPANSION

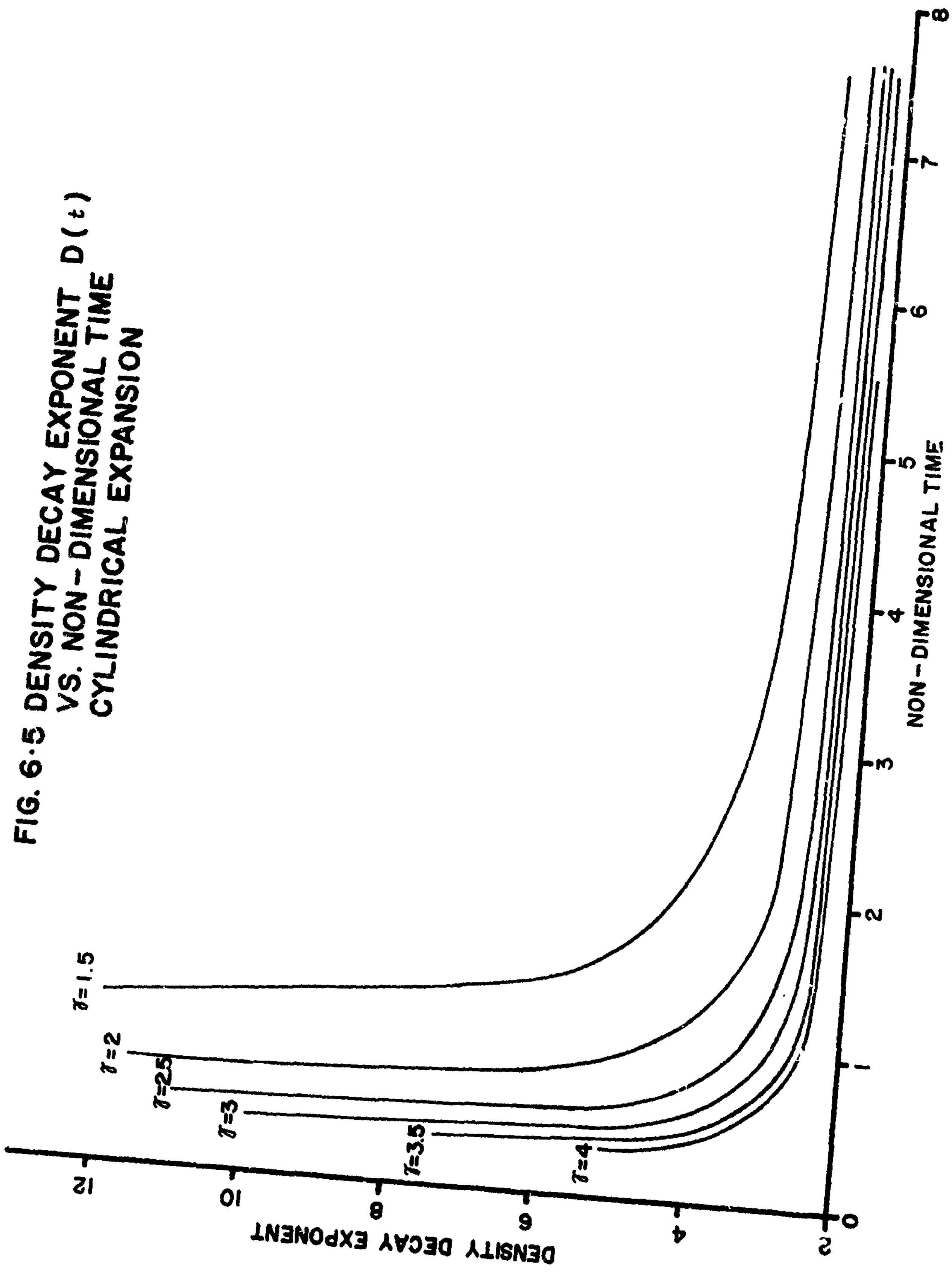


FIG. 6·5 DENSITY DECAY EXPONENT D(t)
VS. NON-DIMENSIONAL TIME
CYLINDRICAL EXPANSION

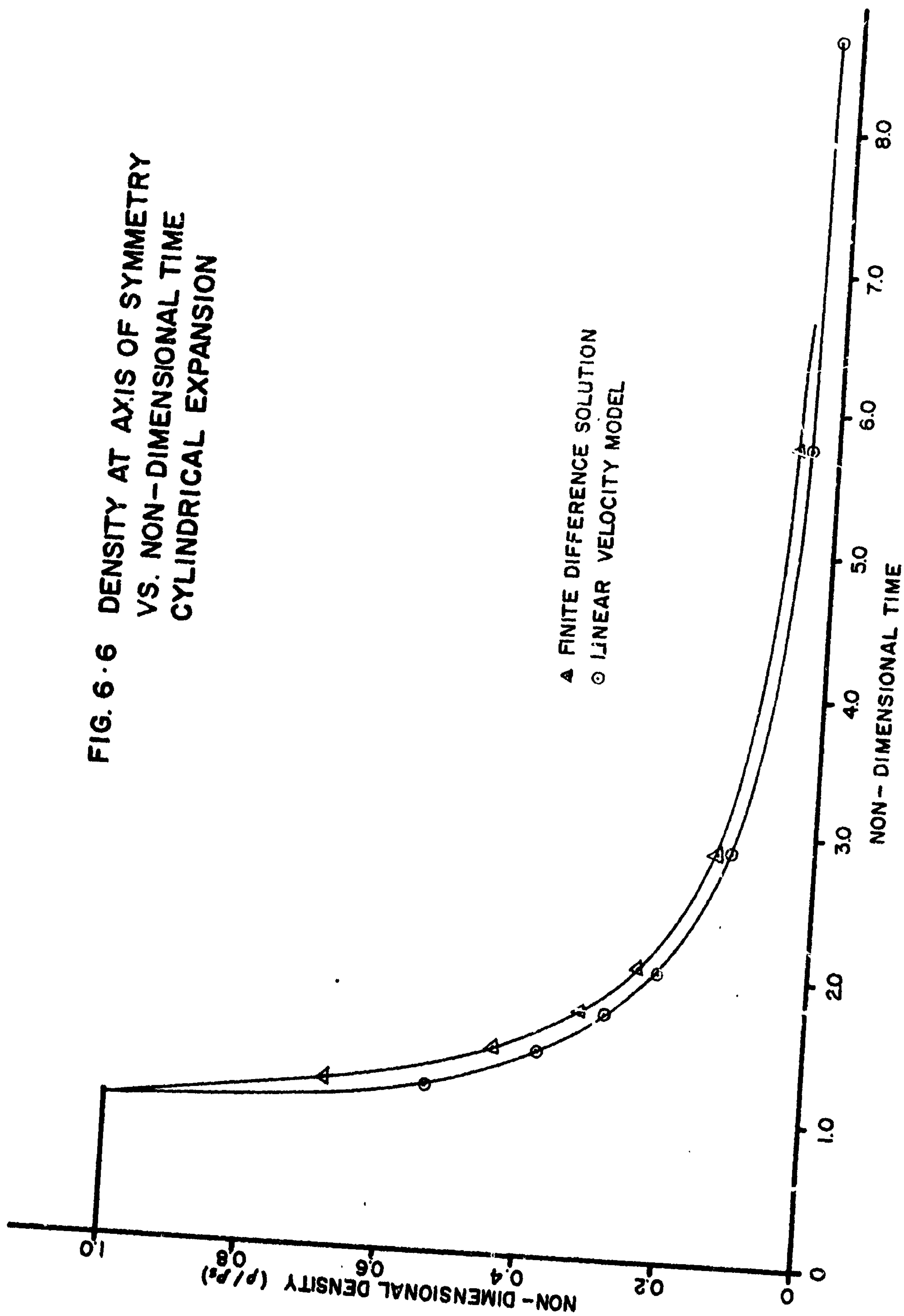


FIG. 6·6 DENSITY AT AXIS OF SYMMETRY VS. NON-DIMENSIONAL TIME CYLINDRICAL EXPANSION

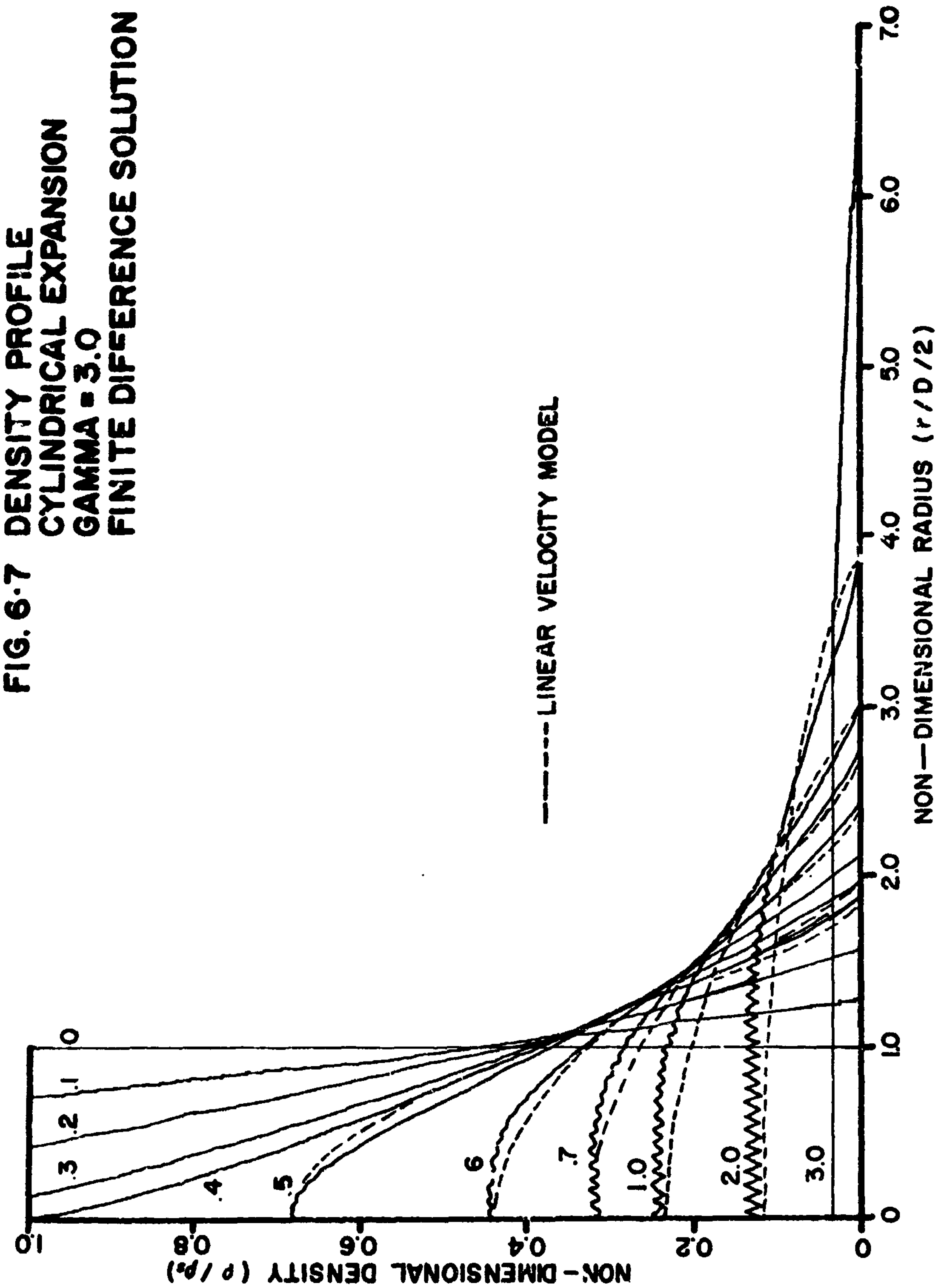


FIG. 6·7 DENSITY PROFILE
CYLINDRICAL EXPANSION
GAMMA = 3.0
FINITE DIFFERENCE SOLUTION

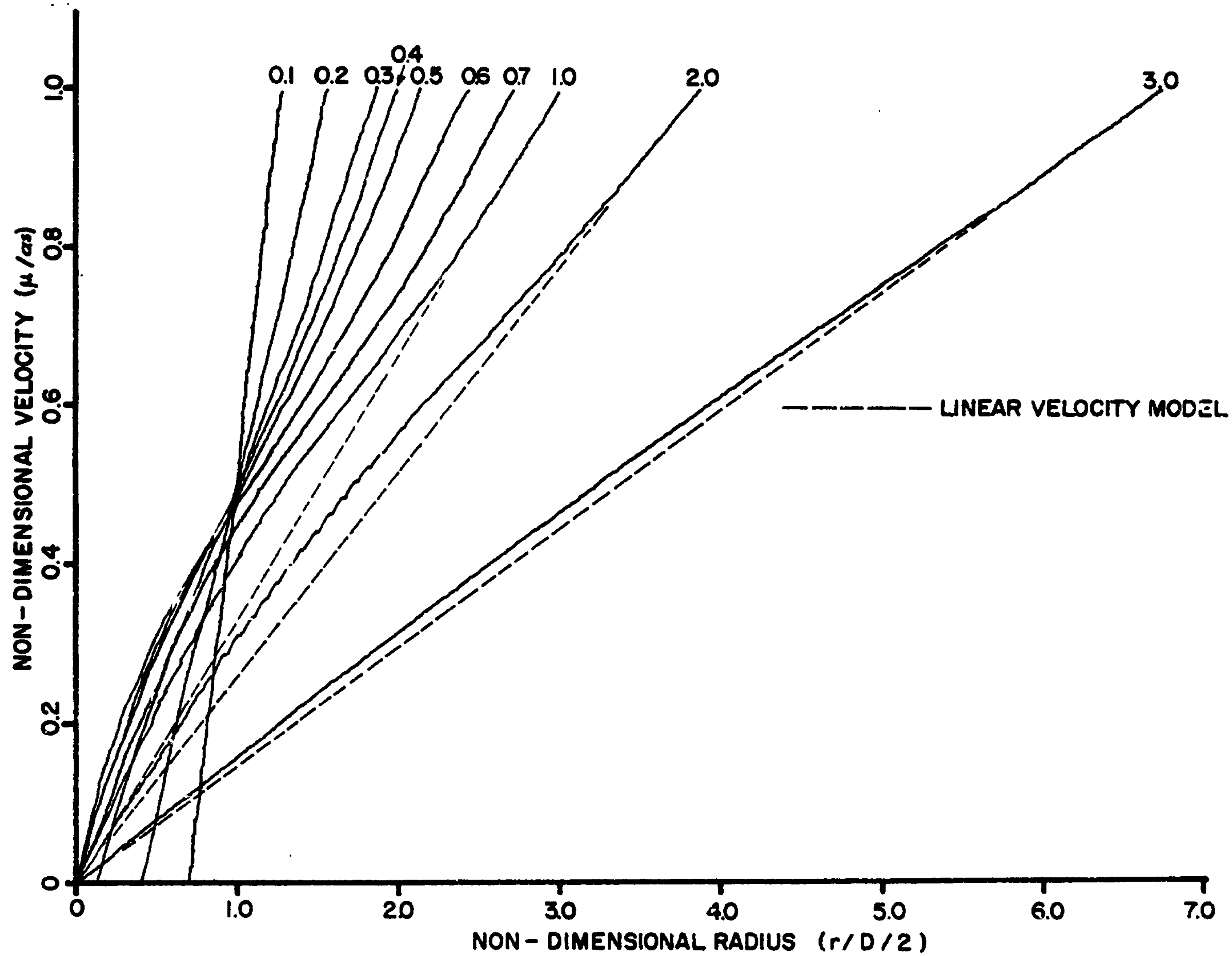


FIG. 6.8 VELOCITY PROFILE CYLINDRICAL EXPANSION GAMMA = 3.0
FINITE DIFFERENCE SOLUTION

7.0 Conclusions:

It may be concluded from the comparison of results that the integral model will give a good approximation for the distribution of density, sound velocity, particle velocity and pressure vs radius, at any time, resulting from the radial expansion of a cylindrical element of an infinitely long cylinder of fluid into a vacuum.

The integral model is a simple and fast method of calculating the expansion parameters. It takes less than a minute to calculate 30 time increments on an IBM 7040.

The plane and spherical expansion results given in Appendices B and C, indicate that the integral model may be equally well applied to these types of expansions.

8.0 References:

1. Mirels, H. Mullen, J.F. — Expansion of Gas Clouds and Hypersonic Jets Bounded by a Vacuum, AIAA Journal Vol. 1, No. 3, 596, 1963

2. Keller, J.B. — Spherical, Cylindrical and One-Dimensional Gas Flows, Quarterly of Applied Math. Vol. 14, No. 2, 171, 1956

3. Shanfield, I. Lee, J.H.S. Bach, G.G. — A Finite Difference Solution for the Cylindrical Expansion of a Gas Cloud into Vacuum, NASA CR-54254, March 1965

APPENDIX A

CYLINDRICAL EXPANSION

a) Short Term Integral Expansion

The forms assumed for the density and velocity distributions are:

$$\rho_r = \rho_s \left[1 - \frac{r - r_i}{r_o - r_i}\right]^{B(t)} \qquad \text{A.1}$$

$$u_r = u_{es} \left(\frac{r - r_i}{r_o - r_i}\right)^{C(t)} \qquad \text{A.2}$$

$B(t)$ and $C(t)$ may be determined from the conservations of mass and momentum in their total integral forms:

$$M_R = \int_{r_i}^{r_o} 2\pi \delta x \rho_r \, r \, dr \qquad \text{A.3}$$

$$E_R = \int_{r_i}^{r_o} 2\pi \delta x \rho_r \left[\frac{a_r^2}{\gamma(\gamma - 1)} + \frac{u_r^2}{2}\right] r \, dr \qquad \text{A.4}$$

Substituting A.1 into A.3, $B(t)$ may be determined.

Let $\quad R = \dfrac{r - r_i}{r_o - r_i}$

then $\quad r = r_i + (r_o - r_i) R$

$$dr = (r_o - r_i) \, dR$$

$$\rho_r = \rho_s (1 - R)^{B(t)}$$

and the limits for A.3 become

$$r = r_i \qquad R = 0$$

$$r = r_o \qquad R = 1$$

$$\therefore M_R = 2\pi \delta x \rho_s \int_0^1 (1 - R)^{B(t)} [r_i + (r_o - r_i) R] (r_o - r_i) \, dR \qquad \text{A.5}$$

re-arranging A.5

$$M_R = 2\pi\delta x \rho_s \left\{ (r_o - r_i)^2 \int_0^1 (1-R)^{B(t)} R\,dR + r_i(r_o - r_i) \int_0^1 (1-R)^{B(t)} dR \right\} \qquad \text{A.6}$$

using a Gamma function (Γ) form for the solution

$$\int_0^1 (1-R)^x R^y dR = \frac{\Gamma(x+1)\Gamma(y+1)}{\Gamma(x+y+2)}, \qquad \text{A.7}$$

equation A.6 may be integrated to give:

$$M_R = 2\pi\delta x \rho_s \left\{ (r_o - r_i)^2 \frac{\Gamma(B(t)+1)\Gamma(2)}{\Gamma(B(t)+3)} + r_i(r_o - r_i) \frac{\Gamma(B(t)+1)\Gamma(1)}{\Gamma(B(t)+2)} \right\} \qquad \text{A.8}$$

Simplifying A.8

$$M_R = 2\pi\delta x \rho_s \left[\frac{(r_o - r_i)^2}{(B(t)+2)(B(t)+1)} + \frac{r_i(r_o - r_i)}{B(t)+1} \right] \qquad \text{A.9}$$

$B(t)$ was solved for numerically from equation A.9 at any given time where:

$$M_R = \rho_s \pi \delta x \left[(d/2)^2 - r_i^2 \right]$$
$$r_i = d/2 - a_s t$$
$$r_o = d/2 + \frac{2}{\gamma - 1} a_s t \qquad \text{A.10}$$

$C(t)$ was determined knowing $B(t)$ and using the profile forms A.1 and A.2 together with the conservation of energy A.4.

For an ideal gas:

$$a_r^2 = a_s^2\left(\frac{\rho_r}{\rho_s}\right)^{\gamma-1}$$

and

$$a_r^2 = a_s^2(1-R)^{B(t)(\gamma-1)} \qquad \text{A.11}$$

Then substituting for ρ_r, a_r, and μ_r in the energy equation A.4 with

$$R = \frac{r-r_i}{r_o-r_i}$$

$$E_R = \int_0^1 2\pi\delta x\rho_s(1-R)^{B(t)}\left[\frac{a_s^2(1-R)^{B(t)(\gamma-1)}}{\gamma(\gamma-1)} + \frac{u_{es}^2}{2}R^{2C(t)}\right]\left[r_i+(r_o-r_i)R\right](r_o-r_i)\,dR \qquad \text{A.12}$$

Re-arranging A.12

$$\begin{aligned} E_R = {} & 2\pi\delta x\rho_s(r_o-r_i)^2\frac{a_s^2}{\gamma(\gamma-1)}\int_0^1(1-R)^{\gamma B(t)}R\,dR \\ & + 2\pi\delta x\rho_s(r_o-r_i)^2\frac{u_{es}^2}{2}\int_0^1(1-R)^{B(t)}R^{2C(t)+1}\,dR \\ & + 2\pi\delta x\rho_s r_i(r_o-r_i)\frac{a_s^2}{\gamma(\gamma-1)}\int_0^1(1-R)^{\gamma B(t)}\,dR \qquad \text{A.13} \\ & + 2\pi\delta x\rho_s r_i(r_o-r_i)\frac{u_{es}^2}{2}\int_0^1(1-R)^{B(t)}R^{2C(t)}\,dR \end{aligned}$$

Integrating A.13 using the Γ function form (A.7) and letting:

$$K_1 = 2\pi\delta x\rho_s(r_o-r_i)^2 \qquad \text{A.14}$$

$$K_2 = 2\pi\delta x\rho_s r_i(r_o-r_i) \qquad \text{A.15}$$

$$E_R = K_1 \frac{a_s^2}{\gamma(\gamma-1)}\left[\frac{\Gamma(\gamma B(t)+1)\,\Gamma(2)}{\Gamma(\gamma B(t)+3)}\right] + K_1 \frac{\mu_{es}^2}{2}\left[\frac{\Gamma(B(t)+1)\,\Gamma(2C(t)+2)}{\Gamma(B(t)+2C(t)+3)}\right] + K_2 \frac{a_s^2}{\gamma(\gamma-1)}\left[\frac{\Gamma(\gamma B(t)+1)\,\Gamma(1)}{\Gamma(\gamma B(t)+2)}\right] + K_2 \frac{\mu_{es}^2}{2}\left[\frac{\Gamma(B(t)+1)\,\Gamma(2C(t)+1)}{\Gamma(B(t)+2C(t)+2)}\right] \qquad \text{A.16}$$

Simplifying A.16

$$E_R = K_1 \frac{a_s^2}{\gamma(\gamma-1)}\left[\frac{1}{(\gamma B(t)+2)(\gamma B(t)+1)}\right] + K_1 \frac{\mu_{es}^2}{2}\left[\frac{B(t)(2C(t)+1)(2C(t))\,\Gamma(B(t))\,\Gamma(2C(t))}{(B(t)+2C(t)+2)(B(t)+2C(t)+1)(B(t)+2C(t))\,\Gamma(B(t)+2C(t))}\right] + K_2 \frac{a_s^2}{\gamma(\gamma-1)}\left[\frac{1}{(\gamma B(t)+1)}\right] + K_2 \frac{\mu_{es}^2}{2}\left[\frac{B(t)(2C(t))\,\Gamma(B(t))\,\Gamma(2C(t))}{(B(t)+2C(t)+1)(B(t)+2C(t))\,\Gamma(B(t)+2C(t))}\right] \qquad \text{A.17}$$

$C(t)$ was solved for numerically from equation A.17 at any given time knowing:

$$B(t),\ K_1,\ K_2,\ M_e \text{ and } E_R = f(t)$$

b) Long Term Integral Expansion

The assumed forms for the density and velocity profiles with respect to and the density decay at the centre with respect to time are:

$$\rho_r = \rho_c \left[1 - \left(\frac{r}{r_o}\right)^2\right]^{B(t)} \qquad \text{A.18}$$

$$\mu_r = \mu_{es}\left(\frac{r}{r_o}\right) \qquad \text{A.19}$$

$$\rho_c = \rho_s\left(\frac{r'}{r_o}\right)^{D(t)} \qquad \text{A.20}$$

$$\text{where } r' = d/2 + \mu_{es} t_c \qquad \text{A.21}$$

and t_c is the time taken for the head of the rarefaction wave to reach a position

$$r_c = .0001\,(d/2) \qquad \text{A.22}$$

The conservation of mass and energy for $t' \geq 1$ become

$$M = \int_0^{r_o} 2\pi\,\delta x\,\rho_r\, r\,dr \qquad \text{A.23}$$

$$E = \int_0^{r_o} 2\pi\,\delta x\,\rho_r\left[\frac{a_r^2}{\gamma(\gamma-1)} + \frac{\mu_r^2}{2}\right] r\,dr \qquad \text{A.24}$$

$$\text{and } M = \pi (d/2)^2\,\delta x\,\rho_s \qquad \text{A.25}$$

$$E = M\frac{a_s^2}{\gamma(\gamma-1)} \qquad \text{A.26}$$

Substituting for ρ_r, μ_r and a_r (equations A.18 - A.20) in the conservation equations A.23 and A.24 with:

$$\psi_1 = \frac{r'}{r_0} \quad \text{and} \quad \psi = \frac{r}{r_0} \qquad \text{A.27}$$

$$a_r^2 = a_s^2\left[\psi_1^{D(t)}(1-\psi^2)^{B(t)}\right]^{\gamma-1} \qquad \text{A.28}$$

and,

$$M = 2\pi\delta x \int_0^1 \rho_s \psi_1^{D(t)}(1-\psi^2)^{B(t)} r_0^2 \psi \, d\psi \qquad \text{A.29}$$

$$E = 2\pi\delta x \int_0^1 \rho_s \psi_1^{D(t)}(1-\psi^2)^{B(t)}\left[a_s^2\left(\psi_1^{D(t)}(1-\psi^2)^{B(t)}\right)^{\gamma-1} + \frac{\mu_{es}^2}{2}\psi^2\right] r_0^2 \psi \, d\psi \qquad \text{A.30}$$

Simplifying A.29 and A.30

$$M = 2\pi\delta x \rho_s r_0^2 \psi_1^{D(t)} \int_0^1 (1-\psi^2)^{B(t)} \psi \, d\psi \qquad \text{A.31}$$

$$E = 2\pi\delta x \rho_s r_0^2 \frac{a_s^2}{\gamma(\gamma-1)} \psi_1^{\gamma D(t)} \int_0^1 (1-\psi^2)^{\gamma B(t)} \psi \, d\psi + 2\pi\delta x \rho_s r_0^2 \frac{\mu_{es}^2}{2} \psi_1^{D(t)} \int_0^1 (1-\psi^2)^{B(t)} \psi^3 \, d\psi \qquad \text{A.32}$$

Integrating A.31 and A.32

$$M = K_4 \psi_1^{D(t)} \frac{1}{B(t)+1} \qquad \text{A.33}$$

$$E = K_4 \frac{a_s^2}{\gamma(\gamma-1)} \psi_1^{\gamma D(t)}\left[\frac{1}{B(t)+1}\right] + K_4 \frac{\mu_{es}^2}{2} \psi_1^{D(t)}\left[\frac{1}{(B(t)+2)(B(t)+1)}\right] \qquad \text{A.34}$$

where $K_4 = \pi\delta x \rho_s r_0^2$

and combining equations A.33 and A.34 to eliminate $D(t)$.

$$E = K_4 \frac{a_s^2}{\gamma(\gamma-1)} \left[\frac{M\left(B(t)+1\right)}{K_4}\right]^{\gamma} \frac{1}{\left(\gamma B(t)+1\right)} + \frac{u_{es}^2}{2} M \frac{1}{B(t)+2} \qquad \text{A.35}$$

$B(t)$ was solved numerically from equation A.35 at any time t with

$$E, \; K_4, \; M \; = f(t)$$

Knowing $B(t)$ then $D(t)$ was solved for from equation A.33.

APPENDIX B

Following the analysis in the main body of this report (Sections 3 to 5), the solution for the long term plane expansion is obtained in this appendix.

For the initial expansion process, the following density velocity and speed of sound profiles are assumed:

$$\rho_r = \rho_s \left[1 - \left(\frac{r-r_i}{r_o-r_i}\right)\right]^{B(t)} \qquad \text{B.1}$$

$$\mu_r = \mu_{es}\left(\frac{r-r_i}{r_o-r_i}\right)^{C(t)} \qquad \text{B.2}$$

$$a_r = a_s\left[1 - \left(\frac{r-r_i}{r_o-r_i}\right)\right]^{P(t)} \qquad \text{B.3}$$

Letting $$R = \frac{r-r_i}{r_o-r_i} \qquad \text{B.4}$$

The profiles may be written as

$$\rho_r = \rho_s\left[1-R\right]^{B} \qquad \text{B.5}$$

$$\mu_r = \mu_{es} R^{C} \qquad \text{B.6}$$

$$a_r = a_s\left[1-R\right]^{P} \qquad \text{B.7}$$

Both the total mass and total energy within the radial expansion region are conserved. The total mass and energy integrals in this region may be written as

$$M_R = \int_{r_i}^{r_o} \rho_r A\, dr \qquad \text{B.8}$$

$$E_R = \int_{r_i}^{r_o} \rho_r \left[\frac{a_r^2}{\gamma(\gamma-1)} + \frac{u_r^2}{2} \right] A\,dr \qquad \text{B.9}$$

Substituting Eq. B.4 and B.5 into Eq. B.8 and integrating one obtains

$$M_R = \frac{A \rho_s (r_o - r_i)}{B+1} \qquad \text{B.10}$$

Substituting Eqs. B.4 to B.7 into Eq. B.9 and integrating one obtains

$$E_R = A\rho_s(r_o - r_i)\,\frac{a_s^2}{\gamma(\gamma-1)}\;\frac{1}{B+2P+1}$$

$$+A\rho_s(r_o - r_i)\,\frac{u_{es}^2 \cdot B.C}{(B+2C+1)(B+2C)}\;\frac{\Gamma(B)\Gamma(2C)}{\Gamma(B+2C)} \qquad \text{B.11}$$

The isentropic relation is

$$\frac{a_r}{a_s} = \left(\frac{\rho_r}{\rho_s}\right)^{\frac{\gamma-1}{2}} \qquad \text{B.12}$$

A comparison of Eqs. B.5 and B.7 now shows that

$$P = \frac{B(\gamma-1)}{2} \qquad \text{B.13}$$

Substitution of Eq. B.13 in Eq. B.11 now gives the final form of the energy equation

$$E_R = A\rho_s(r_o - r_i)\left\{\frac{a_s^2}{\gamma(\gamma-1)}\;\frac{1}{(\gamma B+1)}\right.$$

$$\left. + u_{es}^2\,\frac{B \cdot C}{(B+2C+1)(B+2C)}\;\frac{\Gamma(B)\Gamma(2C)}{\Gamma(B+2C)}\right\} \qquad \text{B.14}$$

Equations B.10 and B.14 are the expressions for the total mass and energy in the radial expansion region as long as the center line value of density does not change, $(r_i \neq 0)$, and as such, are only valid until the initial expansion wave has reached the center-line.

For times greater than that time which is required for the initial expansion to reach the center, other expressions must be derived. The density and velocity profiles assumed are

$$\rho_r = \rho_c \left[1 - \frac{r - r_i}{r_o - r_i}\right]^B \qquad \text{B.15}$$

$$\mu_r = \mu_{os} \left(\frac{r - r_i}{r_o - r_i}\right) \qquad \text{B.16}$$

Since $r_i \cong 0$ * one may write

$$\rho_r = \rho_c \left[1 - \frac{r}{r_o}\right]^B \qquad \text{B.17}$$

$$\mu_r = \mu_{os} \left(\frac{r}{r_o}\right) \qquad \text{B.18}$$

---

* A finite core radius r_c is assumed to exist at the center line within which the properties are decaying.

The center core decay is assumed to be of the following form:

$$\rho_c = \rho_s \left(\frac{r'}{r_o}\right)^{D(t)} \tag{B.19}$$

where ρ_c is the center line density, ρ_s is the initial shocked density, r' is the escape front radius at the time the rarefaction first reaches the center-line, and r_o is the escape front radius.

$$\text{Writing } \frac{r}{r_c} = \psi \tag{B.20}$$

$$\frac{r'}{r_o} = \psi_1$$

and the total mass integral as

$$M = \int_0^{r_o} A \rho_r \, dr \tag{B.22}$$

and substituting Eqs. B.17, B.19, B.20 and B.21 into Eq. B.22 and integrating one obtains

$$M = \frac{A r_o \rho_s}{B+1} \psi_1^{\,D} \tag{B.23}$$

$$\psi_1^{\,D} = \frac{M}{A r_o \rho_s} (B+1) \tag{B.24}$$

The energy integral is written as

$$E = \int_0^{r_o} \rho_r \left[\frac{a_r^2}{\gamma(\gamma-1)} + \frac{u_r^2}{2}\right] A \, dr \tag{B.25}$$

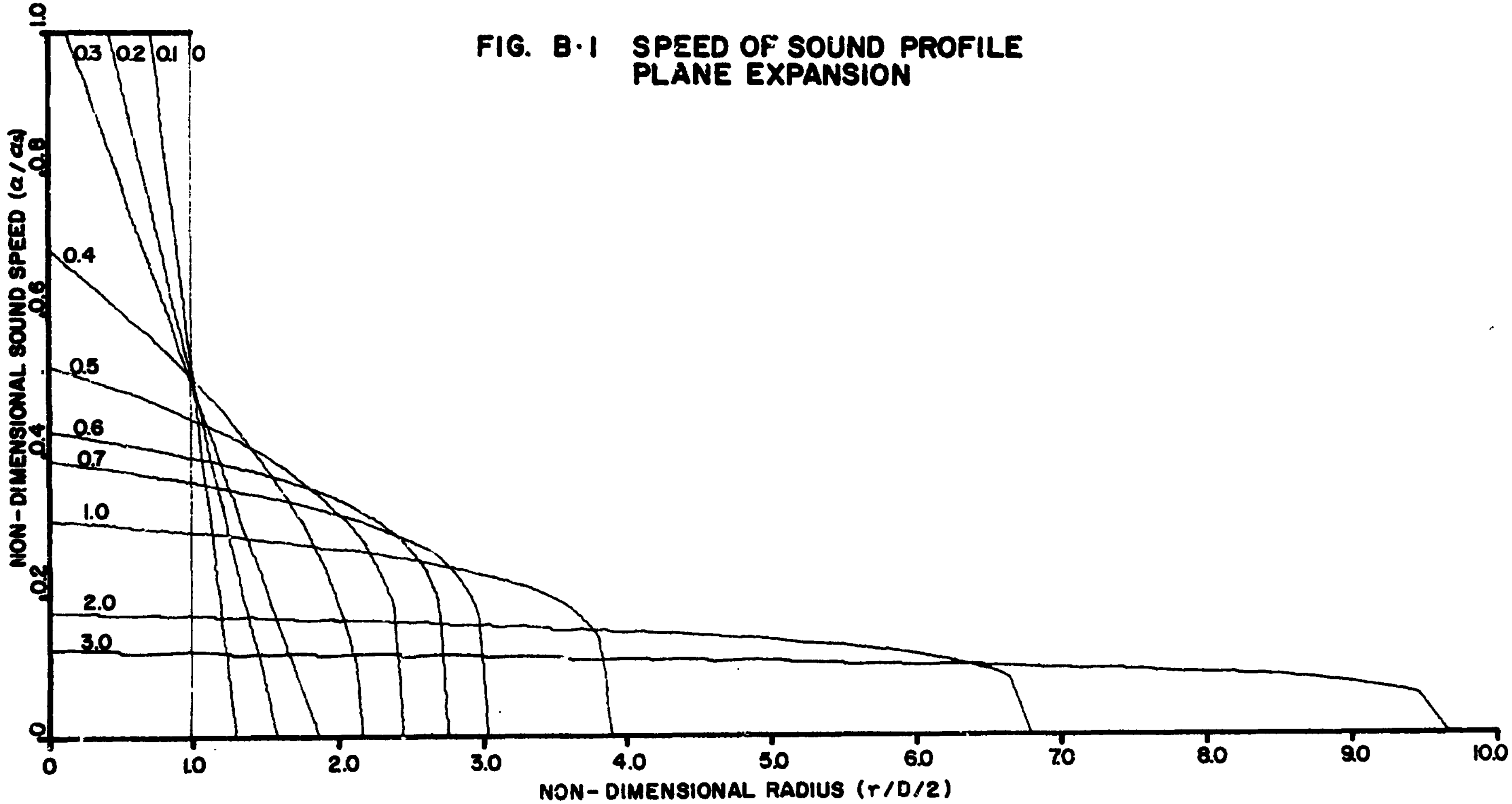
Substituting Eqs. B.17 to B.21 and Eqs. B.12 and B.13 into Eq. B.25 and integrating the resulting energy equation is

$$E = A r_o \rho_s \left\{ \left[\eta_1^{D} \right]^{\gamma} \frac{a_s^2}{\gamma(\gamma-1)} \frac{1}{\gamma B+1} + \eta_1^{D} \frac{u_{es}^2}{2} \frac{1}{(B+3)(B+2)(B+1)} \right\} \qquad B.26$$

Eqs. B.24 and B.26, total conservation of mass and energy respectively, are the valid expressions for all times greater than the time required for the initial expansion to reach the center-line.

The speed of sound, velocity, pressure and density profiles were obtained at various times as functions of non-dimensionalized radius by using the relations developed in this appendix, and are shown in Figs. B.1 to B.4. The times indicated on each profile is the time (in microseconds) from the initiation of the expansion.

FIG. B·1 SPEED OF SOUND PROFILE PLANE EXPANSION



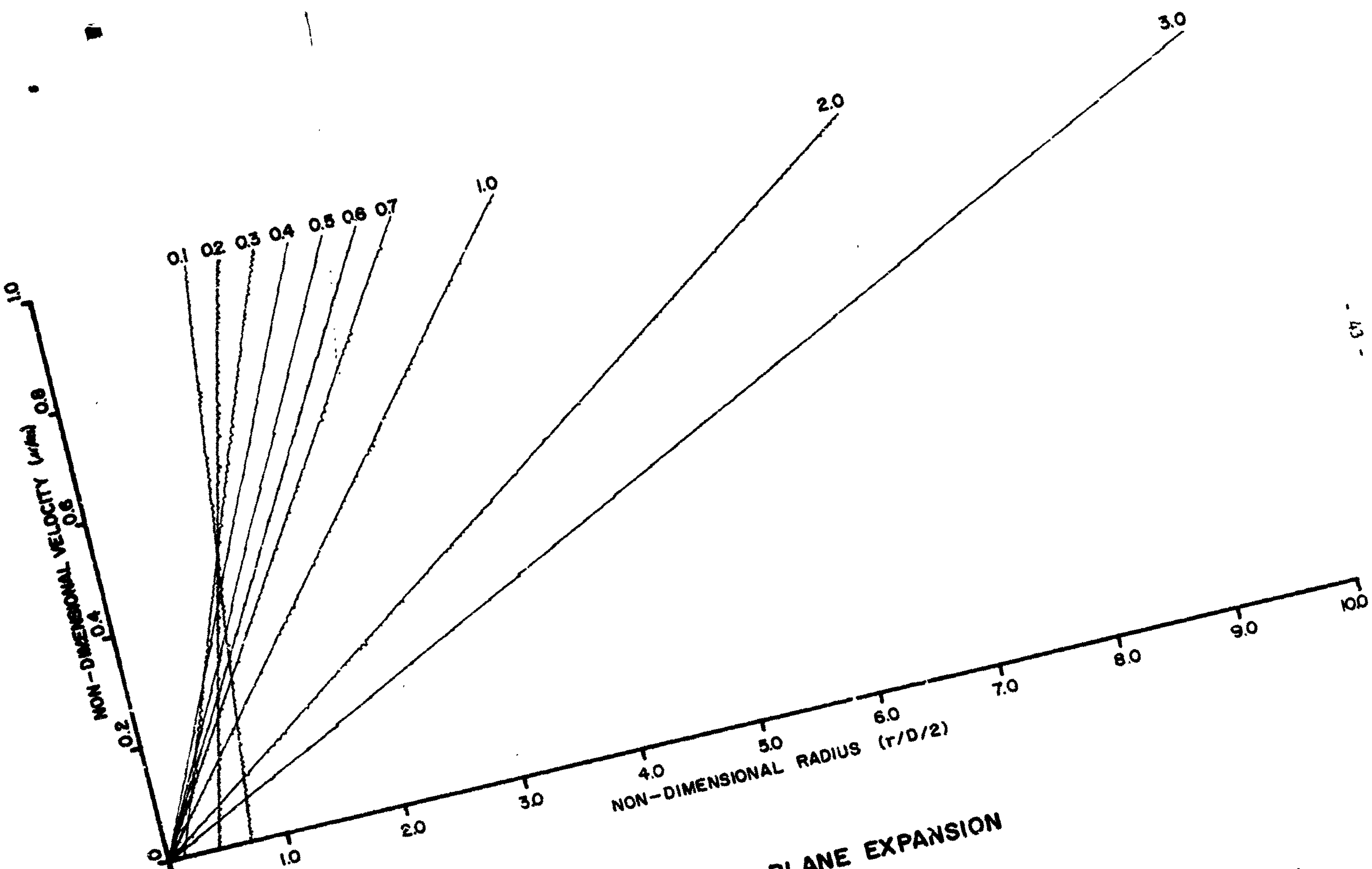


FIG. B·2 VELOCITY PROFILE PLANE EXPANSION

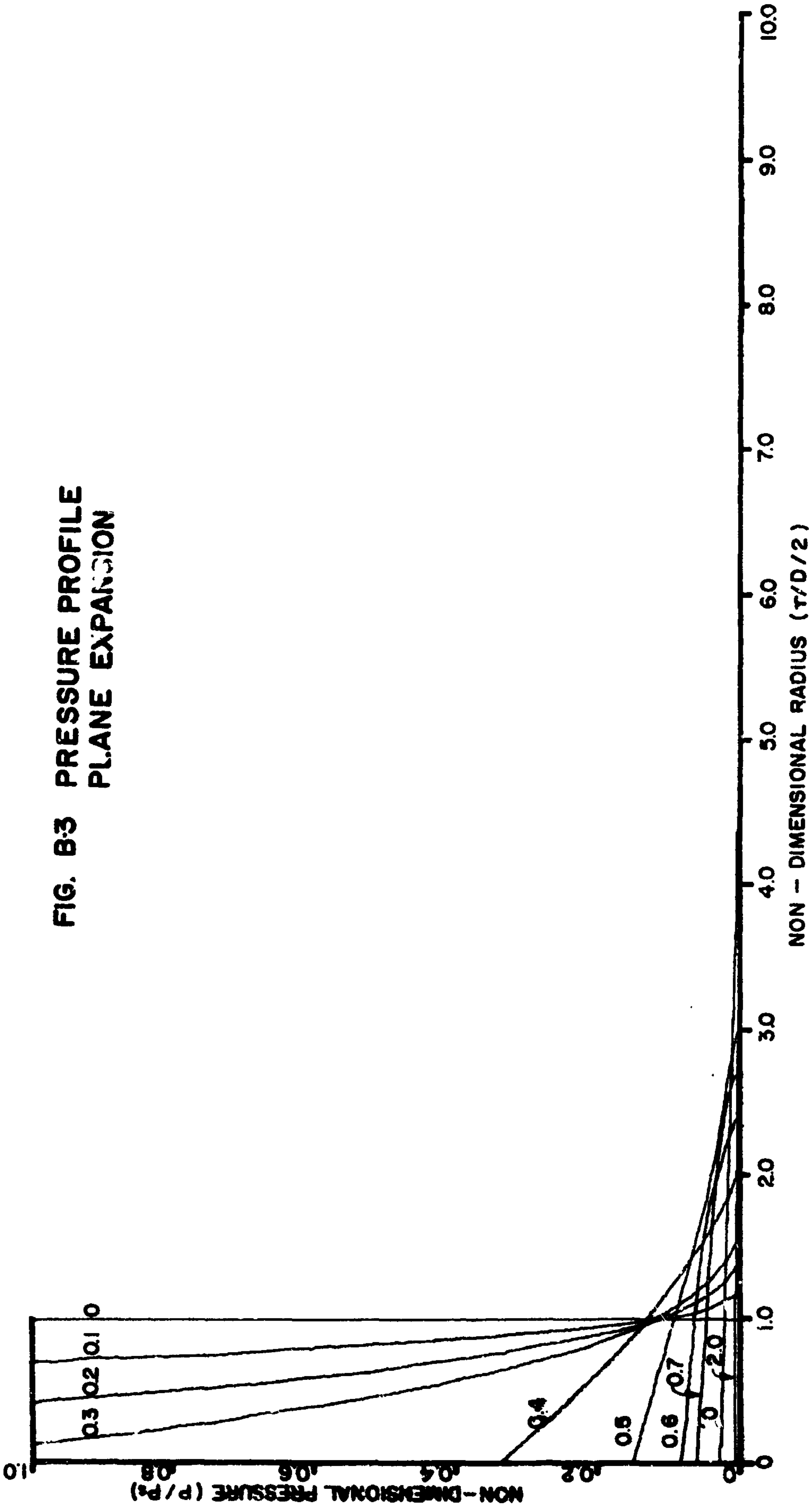
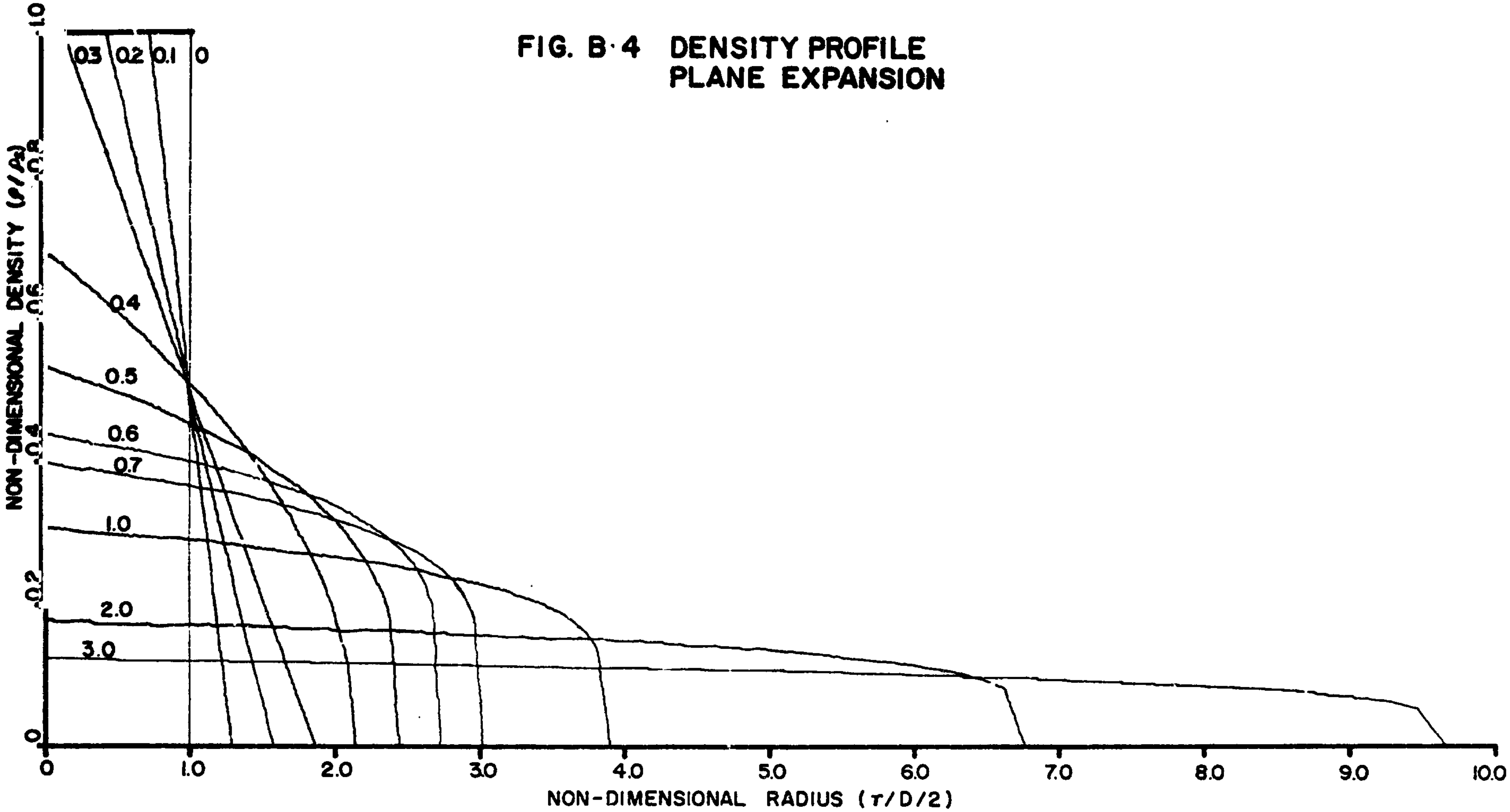


FIG. B-3 PRESSURE PROFILE
PLANE EXPANSION

FIG. B·4 DENSITY PROFILE PLANE EXPANSION



APPENDIX C

The solution for the long term spherical expansion of a gas cloud into a vacuum is developed in this appendix.

The following density, velocity and speed of sound profiles are assumed for the initial expansion process (before radial rarefraction reaches the center).

$$\rho_r = \rho_s \left[1 - \left(\frac{r - r_i}{r_o - r_i} \right) \right]^{B(t)} \qquad \text{C.1}$$

$$u_r = u_{es} \left(\frac{r - r_i}{r_o - r_i} \right)^{C(t)} \qquad \text{C.2}$$

$$a_r = a_s \left[1 - \left(\frac{r - r_i}{r_o - r_i} \right) \right]^{P(t)} \qquad \text{C.3}$$

Letting $R = \dfrac{r - r_i}{r_o - r_i}$ C.4

The profiles are written as

$$\rho_r = \rho_s \left[1 - R \right]^{B} \qquad \text{C.5}$$

$$u_r = u_{es} R^{C} \qquad \text{C.6}$$

$$a_r = a_s \left[1 - R \right]^{P} \qquad \text{C.7}$$

Both the total mass and total energy within the radial expansion region are conserved. The mass and energy integrals in this region are

$$M_R = \int_{r_i}^{r_o} \rho_r \, 4\pi r^2 \, dr \qquad \text{C.8}$$

$$E_R = \int_{r_i}^{r_o} \rho_r \left[\frac{a_r^2}{\gamma(\gamma-1)} + \frac{u_r^2}{2} \right] \cdot 4\pi r^2 dr \qquad \text{C.9}$$

Substituting Eqs. C.4 and C.5 into Eq. C.8 and integrating, one obtains:

$$M_R = 4\pi\rho_s (r_o - r_i)^3 \frac{1}{(B+3)(B+2)(B+1)} + 8\pi\rho_s r_i (r_o - r_i)^2 \frac{1}{(B+2)(B+1)} + 4\pi\rho_s r_i^2 (r_o - r_i) \frac{1}{(B+1)} \qquad \text{C.10}$$

Substituting Eqs. C.4 to C.7 into Eq. C.9 and integrating (with use of the isentropic relation expressed by Eq. B.13) one obtains:

$$E_R = 4\pi\rho_s \frac{a_s^2}{\gamma(\gamma-1)} \left\{ (r_o - r_i)^3 \frac{2}{(\gamma B+3)(\gamma B+2)(\gamma B+1)} + 2r_i (r_o - r_i)^2 \frac{1}{(\gamma B+2)(\gamma B+1)} + r_i^2 (r_o - r_i) \frac{1}{\gamma B+1} \right\}$$
$$+ 4\pi\rho_s \frac{u_{es}^2}{2} \left\{ \frac{(r_o - r_i)^3 \cdot 2C(2C+2)(2C+1)B}{(B+2C+3)(B+2C+2)(B+2C+1)(B+2C)} \frac{\Gamma(2C)\Gamma(B)}{\Gamma(B+2C)} \right.$$
$$+ \frac{2r_i (r_o - r_i)^2 B(2C+1)(2C)}{(B+2C+2)(B+2C+1)(B+2C)} \frac{\Gamma(2C)\Gamma(B)}{\Gamma(B+2C)}$$
$$\left. + \frac{r_i^2 (r_o - r_i) \cdot 2 \cdot B \cdot C}{(B+2C+1)(B+2C)} \frac{\Gamma(2C)\Gamma(B)}{\Gamma(B+2C)} \right\} \qquad \text{C.11}$$

Equations C.10 and C.11 remain valid as long as the center value of density does not change ($r_i \neq 0$), and are therefore applicable only until the expansion head has reached the center of symmetry.

For times greater than that required for the expansion to reach the center, the following derivation is made:

The density and velocity profiles are assumed to be:

$$\rho_r = \rho_c \left[1 - \left(\frac{r - r_i}{r_o - r_i} \right)^2 \right]^{B(t)} \qquad \text{C.12}$$

$$\mu_r = \mu_{es} \left(\frac{r - r_i}{r_o - r_i} \right) \qquad \text{C.13}$$

Since $r_i \cong 0$ one may write

$$\rho_r = \rho_c \left[1 - \left(\frac{r}{r_o} \right)^2 \right]^{B} \qquad \text{C.14}$$

$$\mu_r = \mu_{es} \left(\frac{r}{r_i} \right) \qquad \text{C.15}$$

The center core decay is assumed to be the following form:

$$\rho_c = \rho_s \left(\frac{r'}{r_o} \right)^{D} \qquad \text{C.16}$$

Writing $\frac{r}{r_o} = \Psi$ C.17

and $\frac{r'}{r_o} = \Psi_1$ C.18

and the total mass integral as

$$M = \int_0^{r_o} \rho_r \cdot 4\pi r^2 dr \qquad \text{C.19}$$

and substituting Eqs. C.14, C.16, C.17 and C.18 into Eq. C.19 and integrating, one obtains

$$M = 4\pi \rho_s r_0^3 \nu_1^D \frac{1}{4} \frac{B}{\left(B+\frac{3}{2}\right)\left(B+\frac{1}{2}\right)} \frac{\Gamma\left(\frac{1}{2}\right)\Gamma(B)}{\Gamma\left(B+\frac{1}{2}\right)} \qquad \text{C.20}$$

The equation of total energy conservation in integral form is

$$E = \int_0^{r_0} \rho_r \left[\frac{a_r^2}{\gamma(\gamma-1)} + \frac{u_r^2}{2}\right] \cdot 4\pi r^2 dr \qquad \text{C.21}$$

Substitution of Eqs. C.14 to C.18, Eqs. B.12 and B.13 into Eq. C.21 and integrating yields:

$$E = 4\pi r_0^3 \rho_s \left\{ \left[\nu_1^D\right]^{\gamma} \frac{a_s^2}{\gamma(\gamma-1)} \frac{1}{4} \frac{\gamma B}{\left(\gamma B+\frac{3}{2}\right)\left(\gamma B+\frac{1}{2}\right)} \frac{\Gamma(\gamma B)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(B+\frac{1}{2}\right)} \right.$$

$$\left. + \nu_1^D \frac{u_{es}^2}{2} \frac{3}{8} \frac{B}{\left(B+\frac{5}{2}\right)\left(B+\frac{3}{2}\right)\left(B+\frac{1}{2}\right)} \frac{\Gamma\left(\frac{1}{2}\right)\Gamma(B)}{\Gamma\left(B+\frac{1}{2}\right)} \right\} \qquad \text{C.22}$$

Equations C.20 and C.22 are the forms of total mass and energy conservation equations which are valid for all times greater than the time required for the initial expansion to reach the center of symmetry.

The speed of sound, velocity, pressure and density profiles were obtained at various times as functions of non-dimensionalized radius by using the relations developed in this appendix, and are shown in Figs. C.1 to C.4. The times indicated on each profile is the time (in microseconds) from the initiation of the expansion.

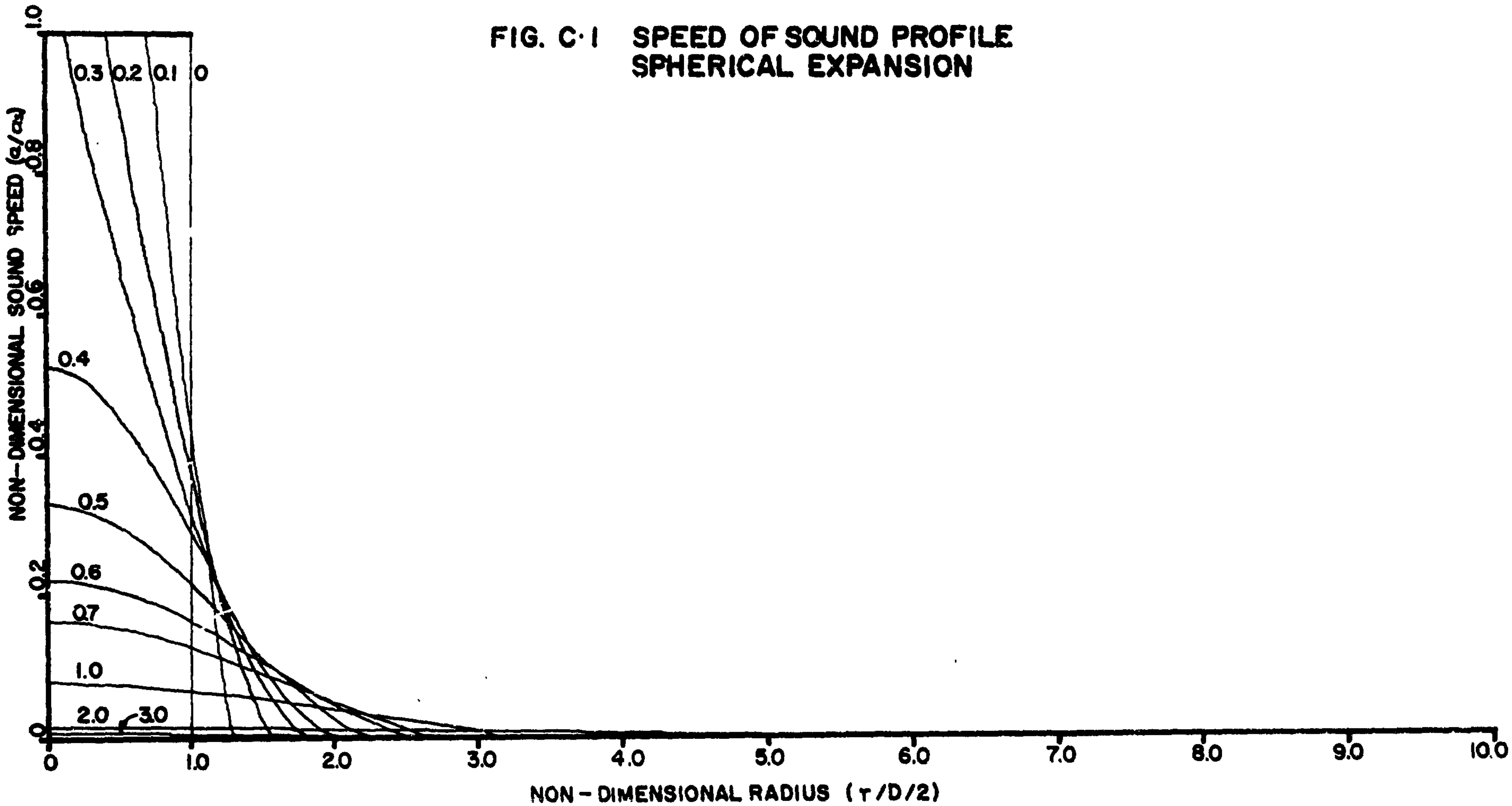


FIG. C·1 SPEED OF SOUND PROFILE
SPHERICAL EXPANSION

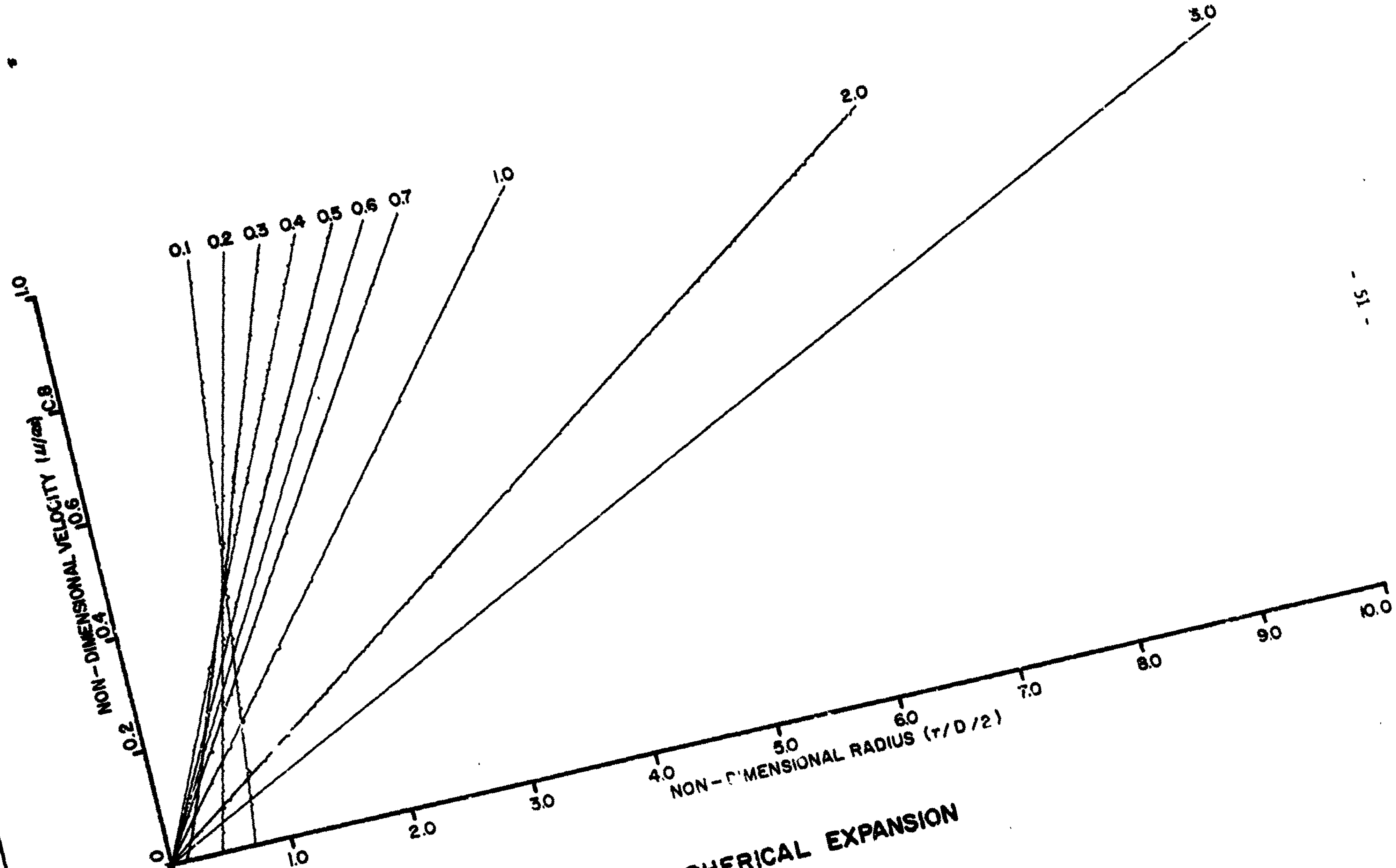


FIG. C·2 VELOCITY PROFILE SPHERICAL EXPANSION

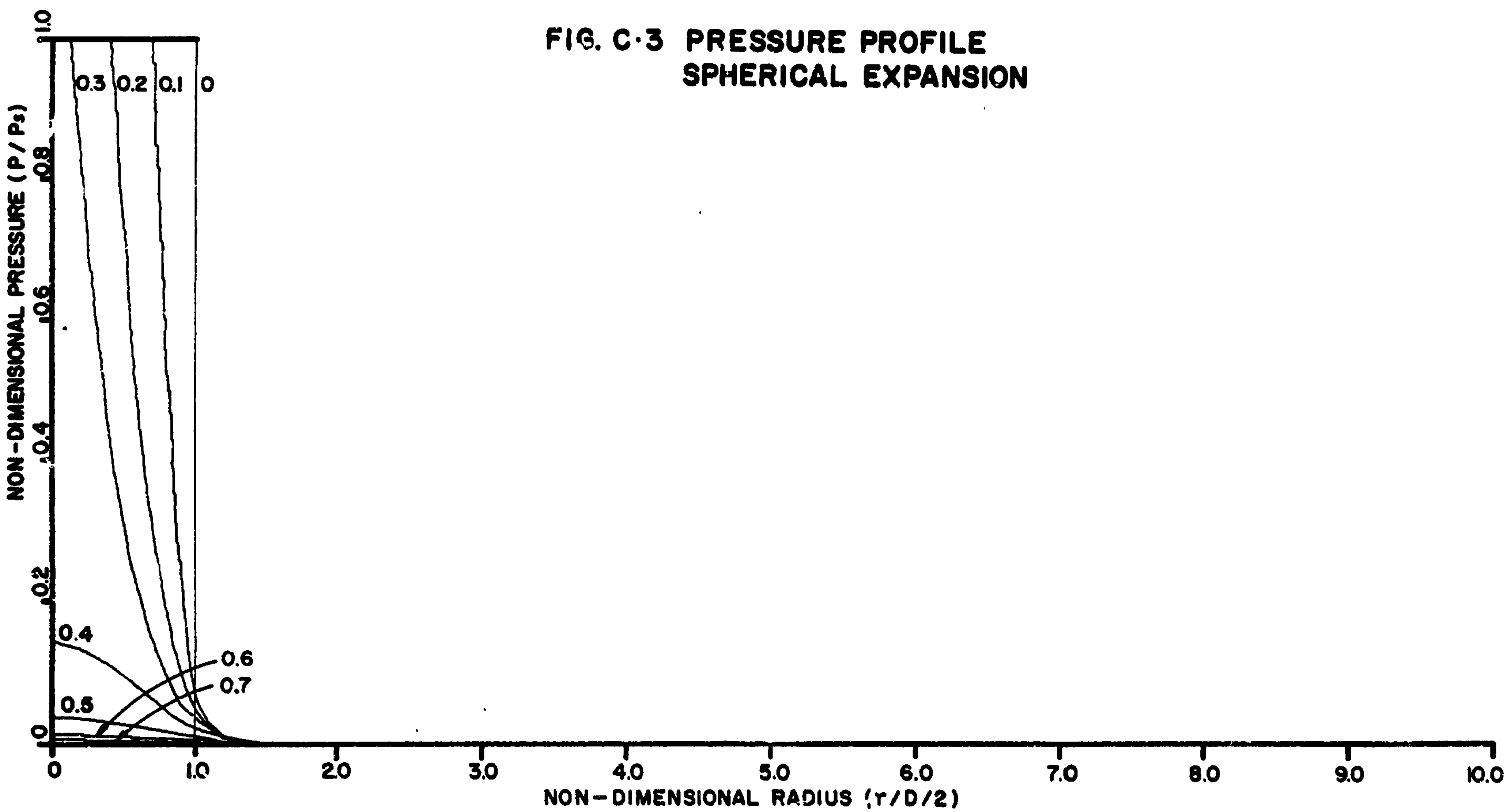


FIG. C·3 PRESSURE PROFILE
SPHERICAL EXPANSION

FIG. C·4 DENSITY PROFILE
SPHERICAL EXPANSION