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D.P. Lindorff* and R.V. Monopoli**

SUMMARY

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A synthesis technique based on Liapunov's direct method is developed in relation to time variable nonlinear multivariable systems. It is shown that a simplified control law can be derived when the plant which may be interacting is caused to track a linear noninteracting model. The system design is shown to be stable in the sense of Lagrange, the ultimate bound on the error being within the designer's control. The synthesis technique is illustrated in relation to the attitude control of a rigid body. author

INTRODUCTION

Application of Liapunov's direct method to the synthesis of control systems containing a single-input single-output plant with nonlinearities and parameter variations has appeared in the literature.[1,2] The purpose of this paper is to extend the application of the method to multivariable systems. Although the synthesis procedure is fundamentally carried over directly to the multi-input multi-output plant, it is found that several new facets of the design problem appear as a consequence of the extension.

The nature of the problem may be described with reference to Fig. 1. It is desired that the q-input q-output plant should follow a q-input q-output model whose outputs are defined to be the desired outputs of the plant. Although the model is defined by a set of linear equations with constant coefficients, the plant is permitted to be characterized by nonlinear equations, in which parameter variations and disturbances may be present, but need be specified only in terms of the bounds on their variations. To force the plant to track the model, a nonlinear control law is implemented which causes the error between the state vectors of the model and the plant to lie eventually within some region about the origin of the error state space (a Lagrange type stability[3]). It will be seen that the size of this region lies within the designer's control.

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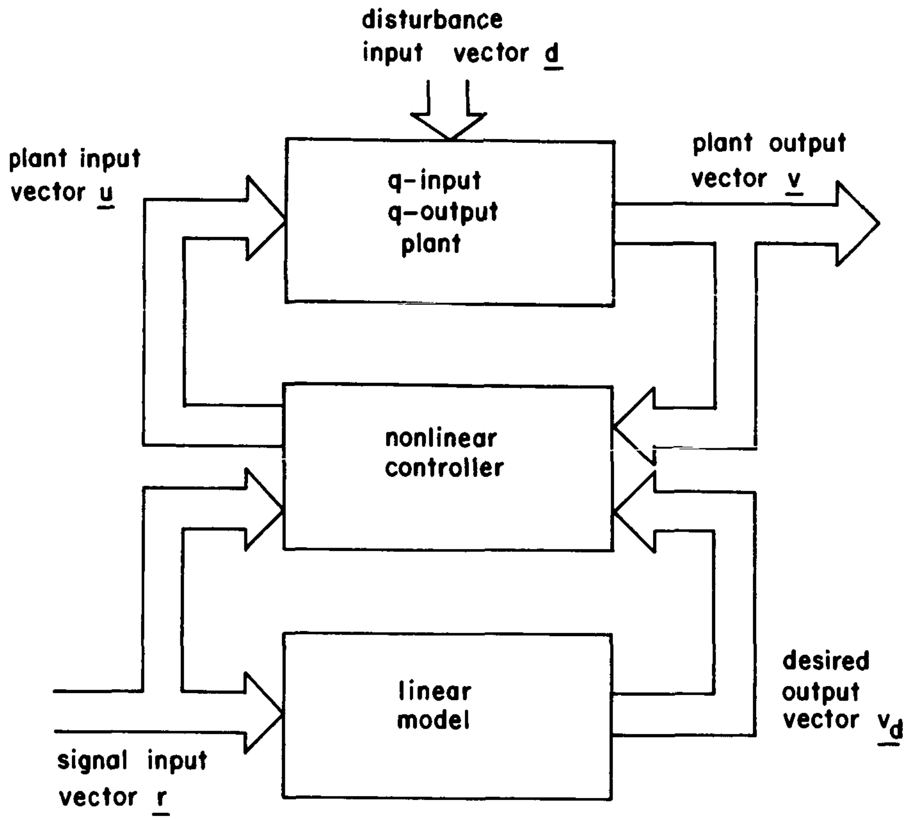

Fig. 1 -- General description of the multivariable system

Although the model, as well as the plant, may be interacting, it is shown that certain simplifications in the control law can be achieved by defining a noninteracting model. By causing the plant to follow a noninteracting model, the plant may in turn be caused to respond very nearly as a noninteracting system.

As in any synthesis procedure, care must be taken to see that the controlled inputs to the plant remain bounded in response to bounded system inputs. In the case of linear, time-invariant multivariable systems, boundedness is assured by prohibiting the cancellation of nonminimum-phase terms as part of the design procedure.[4] In the case of nonlinear, time-varying plants, there is unfortunately no such straightforward approach which can be incorporated in the synthesis procedure to insure boundedness of the plant inputs. Hence, for the class of plants to be considered, it appears that computer simulation is required to determine that plant inputs do in fact remain bounded.

The synthesis technique is applied to a three dimensional attitude control problem in which nonlinear interaction and uncertainties in moments of inertia are permitted. Results of simulation studies are presented in which the nonlinear interacting plant is required to follow a linear noninteracting model.

DEFINITIONS AND GENERAL MATHEMATICAL FORMULATION

The plant to be controlled is to be defined by the vector equations

$$\dot{\underline{x}} = \underline{g}\,(\underline{x},\ \underline{d},\ t) + D(t)\ \underline{u} \tag{1a}$$

$$\underline{v} = C\ \underline{x} \tag{1b}$$

where

$\underline{v}$ is a q dimensional plant-output vector,

$\underline{u}$ is a q dimensional plant-input vector,

$\underline{x}$ is an n dimensional state vector, composed of q subvectors, where $q \geq n$,

C is a $q \times n$ constant output matrix,

D(t) is an $n \times q$ time-variable plant-input matrix,

$\underline{g}$ is an n dimensional vector function satisfying conditions for existence and uniqueness of solution of (1a) under force-free conditions,

$\underline{d}$ is a q dimensional disturbance vector.

It is to be required that (1a) can be put into such form that the state variables of the plant are made up of the q outputs and the appropriate number of derivatives of each of the outputs. Each of the subvectors of $\underline{x}$ will then consist of one output and the appropriate number of its derivatives, the elements being arranged in ascending order.

A control law is to be derived which causes the n^{th} order plant to track an n^{th} order model defined by the vector differential equations

$$\dot{\underline{x}}_d = A\ \underline{x}_d + B\ \underline{r} \tag{2a}$$

$$\underline{v}_d = C\ \underline{x}_d \tag{2b}$$

where

$\underline{x}_d$ is an n dimensional (desired) state vector, composed of q subvectors, each of whose elements are defined and ordered in a manner similar to the elements of $\underline{x}$ above,

$\underline{v}_d$ is a q dimensional (desired) output vector,

$\underline{r}$ is a q dimensional signal-input vector,

A is an n x n constant stability matrix,

B is an n x q constant model-input matrix,

C is the q x n output matrix as above.

If the tracking error is defined by the error vector

$$\underline{e} = \underline{x}_d - \underline{x} \tag{3}$$

then a vector differential equation can be written in terms of the error variable. Thus, using (1a), (2a), and (3), it is possible to group terms so that

$$\dot{\underline{e}} = A\,\underline{e} - \underline{f}\,(\underline{x}, \underline{r}, \underline{u}, \underline{d}, \underline{t}) \tag{4}$$

with $\underline{f} = -A\underline{x} + \underline{g} + D\underline{u} - B\underline{r}$.

The control objective is to produce a vector function $\underline{u}(t)$ which causes (4) to have a stable equilibrium in the sense of Lagrange,[3] with equilibrium at $\underline{e} = \underline{0}$. In other words, the control law of $\underline{u}(t)$ is to be such that, for a sufficiently large value of time t, the system error will lie within some sufficiently small region about the origin of the error state space.

The approach to be used in designing the multivariable controller is similar to that which has been applied in relation to the single-input, single-output plant.[1,2] However, the multivariable feature introduces the need for considering constraints on certain matrix relationships which are not pertinent to the single-variable problem. Before treating these special features of the multivariable problem, a brief development of the general synthesis procedure will be presented.

Recalling that A has been defined in (2) as a constant stability matrix, it follows with $\underline{f} = \underline{0}$ that (4) has an equilibrium solution at $\underline{e} = \underline{0}$, and that this solution is asymptotically stable. It is therefore permissible to apply the theorem:[3]

With A as defined above, for any positive definite symmetric Q, there is a positive definite symmetric P which is a unique solution to the equation

$$P\,A + A^T\,P = -\,Q. \tag{5}$$

Accordingly, for the trivial solution of (4), namely with $\underline{f} = \underline{0}$, there is a Liapunov function defined by $V(\underline{e}) = \underline{e}^T\,P\,\underline{e}$ for which $\dot{V}(\underline{e}) = -\,\underline{e}^T\,Q\,\underline{e}$. If, however, the function $\underline{f}$ in (4) is so defined that $\underline{e} = \underline{0}$ continues to be the equilibrium solution, then $V(\underline{e})$ can be associated with (4) with $\underline{f}$ other than zero, in which case it is found that

$$\dot{V}(\underline{e}, \underline{f}) = - \underline{e}^T Q \underline{e} - 2 \underline{e}^T P \underline{f}. \tag{6}$$

It follows, if a $\underline{u}(t)$ can be found for $t \geq t_o$ so that $\underline{e}^T P \underline{f} \geq 0$ for $t \geq t_o$, that the equilibrium solution of (4) will continue to be asymptotically stable. This is tantamount to saying that the plant will track the model. Due to certain practical as well as purely theoretical considerations,[1,2] it is not realistic to impose the condition for all $\underline{e}$ that $\underline{e}^T P \underline{f} \geq 0$, with the result that the criterion for asymptotic stability must be abandoned, and a less stringent stability criterion be imposed whereby the error vector will ultimately lie within some finite region about the origin. This point will be elaborated upon in Description of the Multivariable Problem. For the present it is pointed out that this region can be made arbitrarily small as a part of the design procedure.

DESCRIPTION OF THE MULTIVARIABLE PROBLEM

The assumption is made at the outset that a linear stable model has been specified so that the q outputs represented by $\underline{v}_d$ in (2b) form the desired outputs of the plant. The factors entering into the choice of the model involve a comprehensive knowledge of the system environment, and will not be treated in this paper.

Writing the equations for the model in partitioned form, the n x n A matrix in (2a) may be represented by

$$A = \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1q} \\ A_{21} & A_{22} & & \\ \cdot & & \cdot & \cdot \\ \cdot & & \cdot & \cdot \\ \cdot & & \cdot & \cdot \\ A_{q1} & & \cdots & A_{qq} \end{bmatrix} \tag{7}$$

where the number of rows of each submatrix A_{ij} is equal to the number of state variables associated with the i^{th} output. From the statements under equations (1) and (2) relative to the choice and ordering of the state variables, it follows that each diagonal submatrix A_{ii} is in the canonic form

$$A_{ii} = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ \cdot & & & & & \\ \cdot & & & & & \\ \cdot & & & & & \\ 0 & 0 & 0 & \dots & 1 & 0 \\ 0 & 0 & 0 & \dots & 0 & 1 \\ a_1 & a_2 & a_3 & & a_{r-1} & a_r \end{bmatrix} \tag{8}$$

for which it is understood that there are r state variables associated with the i^{th} output. Specifically, the form of (8) has been obtained by defining the r state variables by an output and its successive r-1 derivatives. It will be seen that the canonic form in (8) is important to the results which are to be developed.

If each A_{ii} has the form in (8), then it follows that each offdiagonal submatrix A_{ij}, $j \neq i$, is in turn composed of r rows of which only the elements of the last row can be other than zero. For the special case in which the model is designed to have noninteracting outputs, each submatrix A_{ij}, $j \neq i$, is identically zero. These statements are readily verified by example, and will be exemplified in the Illustrative Example.

As a consequence of the assumed form for the partitioned A matrix, it follows that (4) can be written in partitioned form as

$$\begin{bmatrix} \dot{\underline{e}}_1 \\ \dot{\underline{e}}_2 \\ \cdot \\ \cdot \\ \cdot \\ \dot{\underline{e}}_q \end{bmatrix} = A \begin{bmatrix} \underline{e}_1 \\ \underline{e}_2 \\ \cdot \\ \cdot \\ \cdot \\ \underline{e}_q \end{bmatrix} - \begin{bmatrix} \underline{f}_1 \\ \underline{f}_2 \\ \cdot \\ \cdot \\ \cdot \\ \underline{f}_q \end{bmatrix} \tag{9}$$

where $\underline{e}$ and $\underline{f}$ have each been partitioned into a set of q subvectors in accordance with the form of $\underline{x}$ in (1). When written in explicit form it will be seen that only the last element of each subvector $\underline{f}_i$ can be nonzero, as a further and most important consequence of the choice of statevariables, as discussed previously.

With these general statements about the form of (7), (8), and (9), it is possible to develop equations for the control law which apply specifically to the multivariable problem. As stated in Definitions and General Mathematical Formulation, the objective is to control the sign of $\underline{e}^T P \, \underline{f}$ in (6). Toward this end, it is desirable to express $\underline{e}^T P \, \underline{f}$ in the expanded form

$$e^T P \, \underline{f} = \sum_{m=1}^{n} (p_{1m} e_1 = p_{2m} e_2 + \ldots + p_{nm} e_n) \, f_m \tag{10}$$

where in, as noted above, at most q of the elements of $\underline{f}$ are nonzero. In order that $\underline{e}^T P \, \underline{f} \geq 0$, it is sufficient to require that each of the n terms in the summation of (10) shall be nonnegative. Letting the i^{th} term of (10) be denoted by

$$\gamma_i f_i = (p_{1i} e_1 + p_{2i} e_2 + \ldots + p_{ni} e_n) \, f_i \tag{11}$$

it is in turn sufficient to require that f_i shall carry the sign of γ_i for each element of $\underline{f}$ which is not identically zero. Since there are q such elements of $\underline{f}$, and there are q elements of $\underline{u}(t)$, it is reasonable to suppose that the desired control function can be found. Conditions which guarantee that an appropriate $\underline{u}(t)$ can in fact be derived will now be specified.

By noting the assumed form of (1a), it is seen with reference to (4) that $\underline{f}$ can be separated into two parts, namely,

$$\underline{f}(\underline{x}, \underline{r}, \underline{d}, \underline{u}, t) = \underline{h}(\underline{x}, \underline{r}, \underline{d}, t) + D(t) \, \underline{u}(t). \tag{12}$$

If the q-elements of $\underline{f}$ which are not identically zero are now used to form a q-dimensional vector $\underline{f}^q$, then (12) can be written in reduced form as

$$\underline{f}^q(\underline{x}, \underline{r}, \underline{d}, \underline{u}, t) = \underline{h}^q(\underline{x}, \underline{r}, \underline{d}, t) + D^q(t)\underline{u}(t) \tag{13}$$

where $D^q(t)$ is a q x q matrix, with elements $d^q_{ij}(t)$ (It is assumed throughout that $d^q_{ii}(t) > 0$ for $i = 1, 2, \ldots q$). Finally, turning to the definition given to γ_i in (11), a q-element vector $\underline{\gamma}^q$ can be formed, so that (10) can be alternatively written as

$$\underline{e}^T P \, \underline{f} = (\underline{\gamma}^q)^T \, \underline{f}^q. \tag{14}$$

According to the design procedure, a solution for each u_i is sought, such that the expression in (10), or equivalently in (14), shall be non-negative. This can be assured by requiring that the following two conditions shall be satisfied, namely:

$$|u_i(t)| \geq \left| \frac{h_i^q(\underline{x}, \underline{r}, \underline{d}, t) - \sum_{\substack{j=1 \\ j \neq i}}^{q} d_{ij}^q(t)\, u_j(t)}{d_{ii}^q(t)} \right| \tag{15}$$

and

$$\text{sign } u_i(t) = \text{sign } \gamma_i^q(t) \tag{16}$$

where h_i^q is an element of $\underline{h}^q$ and γ_i^q is an element of $\underline{\gamma}^q$. An explicit expression for $|u_i(t)|$ depends upon the extent of the information which is known about the parameters and disturbances, and will be developed subsequently in Illustrative Example.

Examination of (15) for successive values of i reveals the possibility of a mathematical inconsistency arising in case there should be a dependence of $u_i(t)$ upon other plant inputs. For example, if each element of D^q were equal to unity, the inequality in (15) could not be assured for each i and all t. This mathematical difficulty can be removed by requiring that $D^q(t)$ must be in triangular form. The inequality in (15) can then be satisfied for each of the q inputs, if it is additionally required that $D^q(t)$ be of rank q for any time t. This is seen to be guaranteed by the previous assumption that $d_{ii}^q(t) > 0$ for each $i = 1, 2, \ldots q$.

As noted in [1,2], use of the sign function in defining $u_i(t)$ is unsuitable for two reasons, the one being based on purely theoretical considerations relating to the conditions for existence of solutions, the other being concerned with the high sensitivity of the sign function to noise which may be present in the measured variables. As a solution to the first of these problems, and an amelioration of the second, the function sign γ will be replaced by

$$\text{sat } k\gamma = \begin{cases} 1 \text{ for } \gamma > 1/k \\ k\gamma \text{ for } -1/k \leq \gamma \leq 1/k \\ -1 \text{ for } \gamma < -1/k \end{cases} \tag{17}$$

as part of the design procedure, wherein k, defined as a positive number, is to be treated as a design parameter.

With sign γ^q_i replaced by sat $k_i\gamma_i^q$ in the expression for $u_i(t)$, it is no longer assured that the magnitudes of the respective plant inputs will be large enough to maintain $\underline{e}^T P \underline{f} \geq 0$ as required in (6) to satisfy conditions for asymptotic stability. For this reason it is necessary to turn to an alternative definition of stability. It is shown in Appendix A for a class of nonlinear time variable plants that, when sat $k\gamma$ replaces sign γ, there is for large enough values of the sat gains a region about the origin $\underline{e} = \underline{0}$ outside of which $\dot{V}(\underline{e}) < 0$, and that this region can be made arbitrarily small by making the gain of each sat function sufficiently large. In practice the presence of measurement noise would be taken into consideration in establishing an upper limit on the magnitudes of the sat gains.

SIMPLIFICATION OF THE CONTROL LAW BY EMPLOYING A NONINTERACTING MODEL

The purpose of this section is to show that, if the model outputs are noninteracting, it is possible to simplify substantially the control law, notwithstanding that the plant may exhibit output interaction.

It is assumed that A is a stability matrix of a model in which the outputs are noninteracting. If A is now assumed to be written in the partitioned form of (7), then P and Q in (5) can be similarly partitioned. By using the statement that off-diagonal submatrices of A are zero ($A_{ij} = 0$, $i \neq j$) due to the preceding assumption of noninteraction, it follows that the following equations can be written in terms of submatrices:

$$P_{rn}A_{nn} + A^T_{rr}P_{rn} = - Q_{rn}. \tag{18}$$

Now if Q is shown to be of such a form that $Q_{rn} = 0$ for $r \neq n$, then (18) becomes

$$P_{rn}A_{nn} + A^T_{rr}P_{rn} = 0, \qquad r \neq n. \tag{19}$$

If the elements of P_{rn} are denoted by p_{ij}, the (19) can be recast into the form

$$M\underline{p} = \underline{0} \tag{20}$$

where the column matrix $\underline{p}$ consists of all of the elements of P_{rn} taken in any order. On the assumption that A is a stability matrix and that Q is a positive definite symmetric matrix, it has been stated that P is a positive definite symmetric matrix which is a unique solution to (5). However, if (20) is to provide a unique solution for the elements of P_{rn} as required by the preceding statement, it follows that $\underline{p} = \underline{0}$ must be a solution, i.e.determinant of M must be nonzero in order for $(2\overline{0})$ to have a unique solution.

An application of this result will be made in the example to be treated in the following section.

ILLUSTRATIVE EXAMPLE

This example is concerned with controlling the rotational rates of a rigid body about its principal axes. As indicated in Fig. 2, the principal axes are to be denoted by s_1, s_2, s_3, and the angular rates about each of

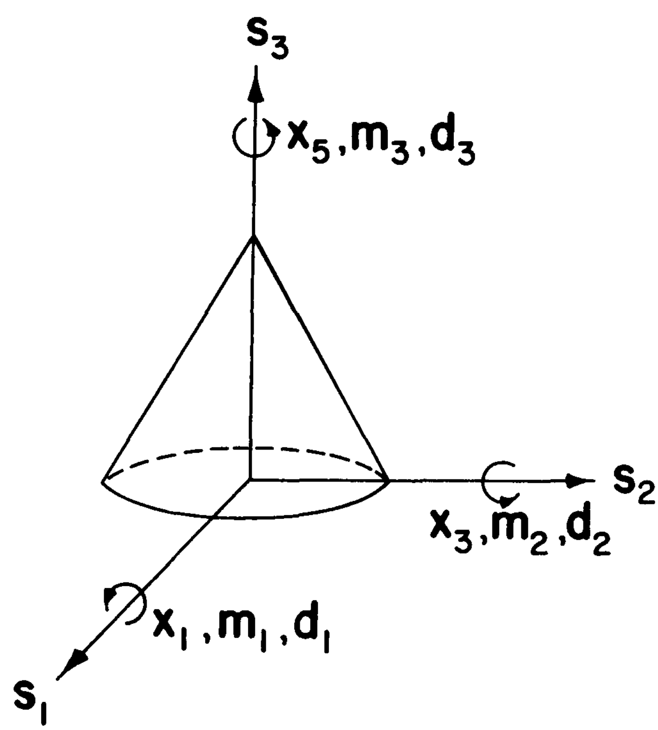

Fig. 2 -- Free body showing principal axes

these axes are to be denoted by x_1, x_3, x_5, respectively. Let it further be assumed that:

J_i is the moment of inertia about the s_i axis

d_i is the disturbance torque about the s_i axis

m_i is the controlled torque about the s_i axis

as shown in the figure. The Euler equations[5] describing the rotational rates in terms of x_1, x_3, x_5 can now be written as follows:*

$$
\begin{aligned}
\dot{x}_1 &= \frac{(J_2 - J_3)}{J_1} x_3 x_5 - \frac{d_1}{J_1} + \frac{m_1}{J_1} \\
\dot{x}_3 &= \frac{(J_3 - J_1)}{J_2} x_1 x_5 - \frac{d_2}{J_2} + \frac{m_2}{J_2} \\
\dot{x}_5 &= \frac{(J_1 - J_2)}{J_3} x_1 x_3 - \frac{d_3}{J_3} + \frac{m_3}{J_3} \; .
\end{aligned}
\tag{21}
$$

If the torque m_i is in turn assumed to be related to the controlled variable u_i in the transform domain by

$$
M_i(s) = \frac{U_i(s)}{\tau_i \, s + 1}, \quad i = 1, 2, 3, \tag{22}
$$

based on the assumption that a simple time constant is contained in each of the three torque controllers, then the plant dynamics are fully defined by (21) and (22).

The problem is now to derive a set of state equations in the form of (1). Toward this end it is convenient to identify a block diagram which relates to the motion of the body about one of its principal axes, such as s_1, as shown in Fig. 3. Since the dynamics of the plant in relation to the s_1 axis are characterized by a second order differential equation, it is necessary to assign two state variables to the process, in this case x_1 and x_2, as shown in the figure.

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* In order to write the equations in this form, it is assumed the effects due to rates of change of inertias can be neglected.

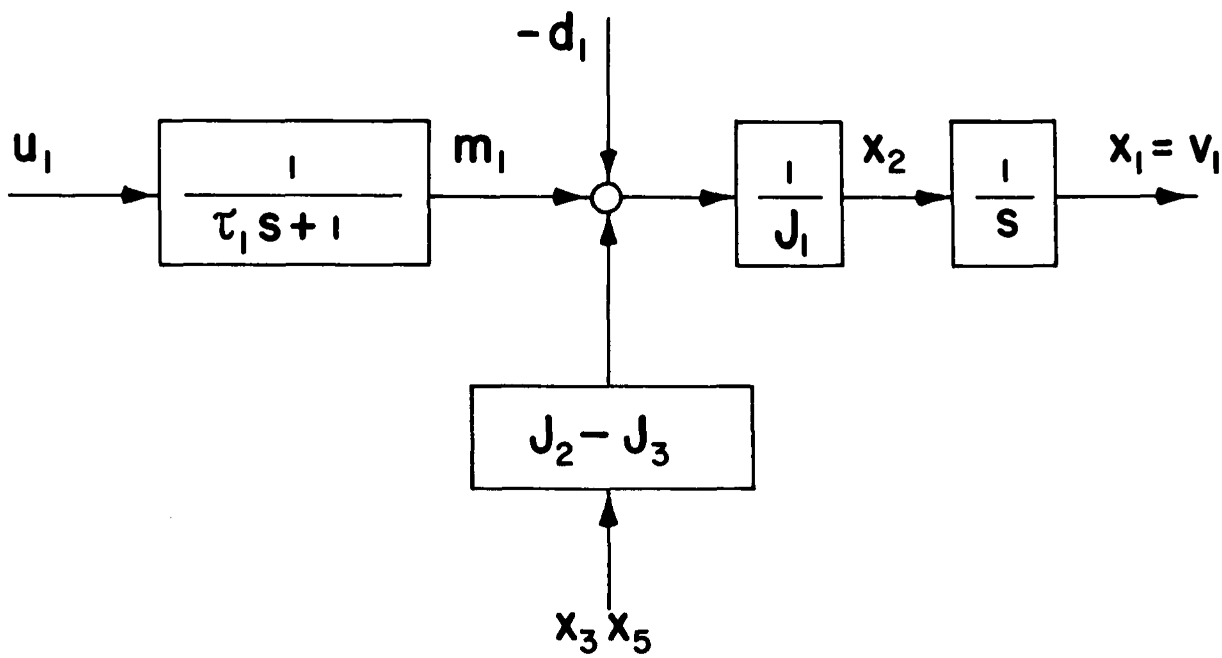

Fig. 3 -- Block diagram showing body dynamics about axis s_1

At this point, it might be noted that the synthesis procedure does not require that the parameters of the plant equations should be stationary with respect to time. However, if the plant parameters were rapidly varying, it is possible to show that bounds on certain derivatives of the parameter variations would normally have to be considered in formulating the control law.

In this way, it is possible to obtain a quantitative statement, with respect to the control law being implemented, as to what constitutes a "slowly varying parameter." This aspect of the problem will not be pursued further in this paper, but is mentioned to emphasize the fact that the presence of rapid parameter variations presents no conceptual difficulty in the synthesis procedure.

Since it is assumed in this example that J_1, J_2, and J_3 are slowly varying, it is permissable to state that the block diagram in Fig. 4 is equivalent to that of Fig. 3 with respect to the state variables, on the assumption that τ_1 is a known constant. Similar diagrams pertain to the s_2 and s_3 axes, assuming τ_2 and τ_3 are known constants.

It now becomes possible to write the following set of plant state equations in the form of (1a):

$$
\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= -\frac{x_2}{\tau_1} + \frac{(J_2 - J_3)}{J_1}\left(x_4 x_6 + \frac{x_3 x_5}{\tau_1}\right) - \frac{1}{J_1}\left(\dot{d}_1 + \frac{d_1}{\tau_1}\right) + \frac{u_1}{J_1 \tau_1} \\
\dot{x}_3 &= x_4 \\
\dot{x}_4 &= -\frac{x_4}{\tau_2} + \frac{(J_3 - J_1)}{J_2}\left(x_2 x_6 + \frac{x_1 x_5}{\tau_2}\right) - \frac{1}{J_2}\left(\dot{d}_2 + \frac{d_2}{\tau_2}\right) + \frac{u_2}{J_2 \tau_2} \\
\dot{x}_5 &= x_6 \\
\dot{x}_6 &= -\frac{x_6}{\tau_3} + \frac{(J_1 - J_2)}{J_3}\left(x_2 x_4 - \frac{x_1 x_3}{\tau_3}\right) - \frac{1}{J_3}\left(\dot{d}_3 + \frac{d_3}{\tau_3}\right) + \frac{u_3}{J_3 \tau_3} .
\end{aligned}
\tag{23}
$$

By comparing (23) with (1a), the identification of $\underline{g}(\underline{x}, \underline{d}, t)$ and $D(t)\underline{u}$ is readily apparent. If the three outputs of the plant are defined as $v_1 = x_1$, $v_2 = x_3$, $v_3 = x_5$, then the C matrix in (1b) is given by

$$
C = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \tag{24}
$$

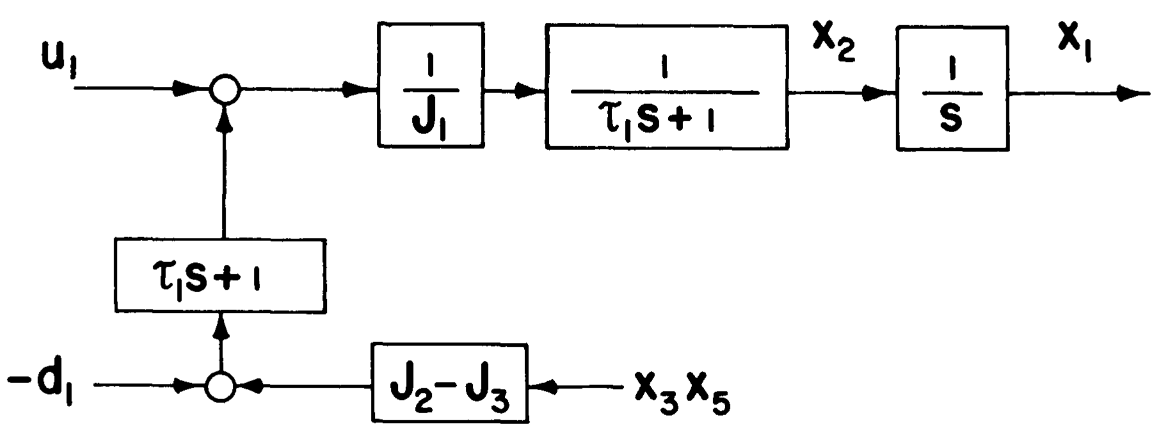

Fig. 4 -- Equivalent representation of Fig. 3 with respect to the variables x_1, x_2

The equations corresponding to (2) can be derived from a block diagram of the model, the outputs of which are assumed to be the desired outputs of the plant. If it is assumed that noninteracting control of v_1, v_2, and v_3 is desired, then an appropriate representation of the model might be that which is shown in Fig. 5 wherein the state variables comprising $\underline{x}_d$, and the

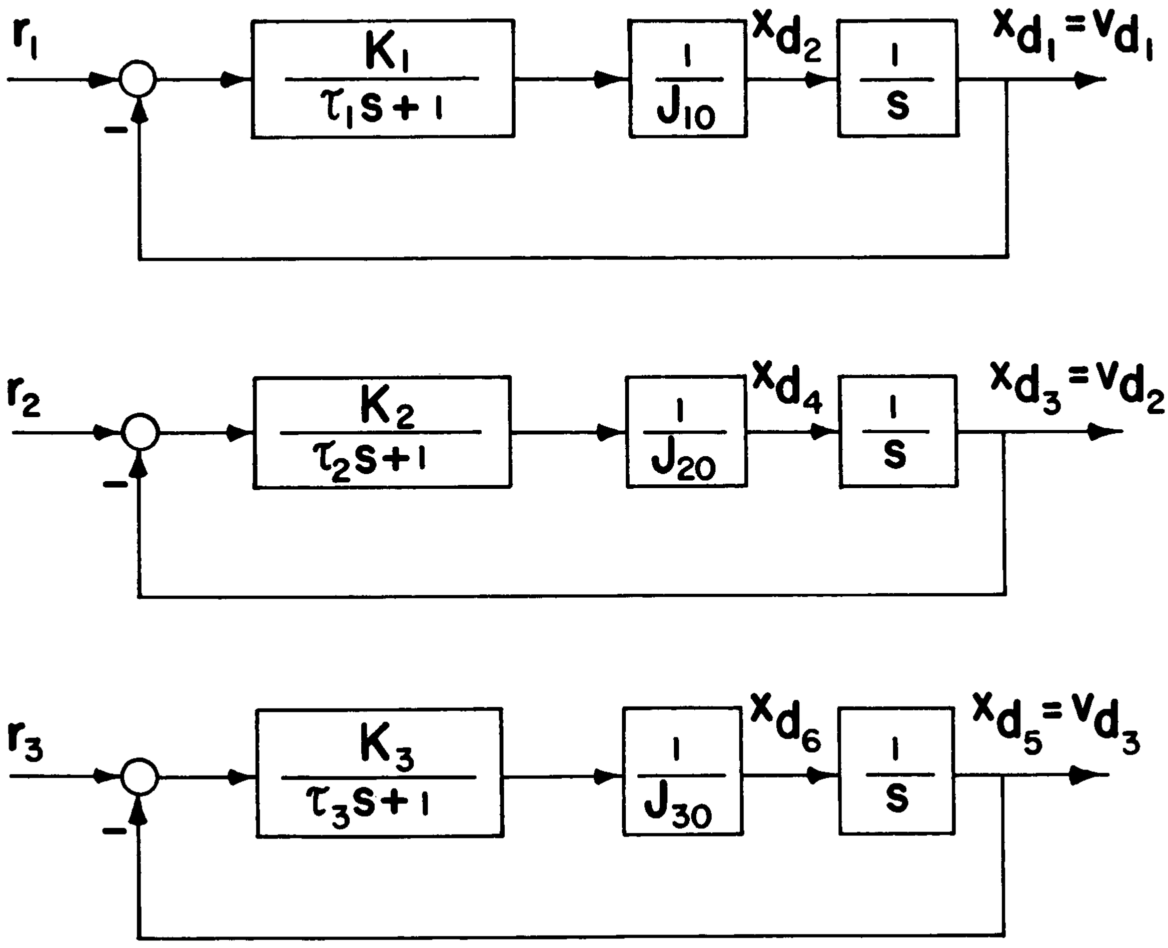

Fig. 5 -- Block diagram of model

component outputs comprising $\underline{v}_d$ have been identified. The state equations corresponding to (2a) can now be written as follows:

$$\begin{bmatrix} \dot{x}_{d_1} \\ \dot{x}_{d_2} \\ \dot{x}_{d_3} \\ \dot{x}_{d_4} \\ \dot{x}_{d_5} \\ \dot{x}_{d_6} \end{bmatrix} = \left[\begin{array}{cc|cc|cc} 0 & 1 & 0 & 0 & 0 & 0 \\ -\frac{K_1}{J_{10}\tau_1} & -\frac{1}{\tau_1} & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -\frac{K_2}{J_{20}\tau_2} & -\frac{1}{\tau_2} & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -\frac{K_3}{J_{30}\tau_3} & -\frac{1}{\tau_3} \end{array}\right] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{K_1 r_1}{J_{10}\tau_1} \\ 0 \\ \frac{K_2 r_2}{J_{20}\tau_2} \\ 0 \\ \frac{K_3 r_3}{J_{30}\tau_3} \end{bmatrix} \tag{25}$$

where A in (2a) is identified with the square matrix, the form of which is seen to agree with that of (7) and (8). To identify an explicit form for B serves no purpose in this discussion since only the column vector $B\underline{r}$ is needed in the analysis. It is easily shown in this example that the expression for the C matrix in (2b) is given by (24). Therefore the dynamics of the model are fully defined by (24) and (25).

In order to develop the control laws for u_1, u_2, and u_3, it is necessary to obtain an explicit expression for the elements of $\underline{f}$ in (4). Noting from the section Description of the Multivariable Problem, that $f_1 = f_3 = f_5 = 0$, the results of interest are

$$\begin{aligned} f_2 &= -a_{21}x_1 - a_{22}x_2 + g_2 - \frac{K_1 r_1}{J_{10}\tau_1} + \frac{u_1}{J_1\tau_1} \\ f_4 &= -a_{43}x_3 - a_{44}x_4 + g_4 - \frac{K_2 r_2}{J_{20}\tau_2} + \frac{u_2}{J_2\tau_2} \\ f_6 &= -a_{65}x_5 - a_{66}x_6 + g_6 - \frac{K_3 r_3}{J_3\tau_3} + \frac{u_3}{J_3\tau_3} \end{aligned} \tag{26}$$

where the a_{ij} coefficients relate to the elements of the square matrix (A) in (25), and g_2, g_4, g_6 are the elements of $\underline{g}$ appearing in (23). In the remaining discussion, the derivation of a control law for u_1 will suffice to illustrate the method, since the equation for u_2 and u_3 are similar. Hence f_2 in (26) is the term which will be discussed in detail. It will be advantageous to express f_2 in the form

$$f_2 = \frac{1}{J_1\tau_1}\left(- a_{21}J_1\tau_1x_1 - a_{22}J_1\tau_1x_2 + J_1\tau_1g_2 - \frac{J_1K_1r_1}{J_{10}} + u_1 \right) \tag{27}$$

It is convenient at this point to introduce the notation $J_i \equiv J_{i0} + J_{i1}$, wherein the nominal value of the inertia about the s_i axis is given by J_{i0}, its deviation being denoted by J_{i1}. It is also useful to separate u_1 in two parts, namely, $u_1 = u_{10} + u_{11}$. The terms in f_2 will now be grouped into several parts, so that (27) may be expressed as

$$f_2 = \frac{1}{J_1\tau_1}\left(w_0 + w_1 + u_{10} + u_{11}\right) \tag{28}$$

where w_0 contains terms with nominal values of the parameters, and w_1 contains the terms with parameter deviations.

By introducing the appropriate feedback around the plant, it is possible to satisfy the equation $u_{10} = - w_0$ which is explicitly given by

$$u_{10} \equiv K_1(r_1 - x_1) - (J_{20} - J_{30})(\tau_1x_4x_6 + x_3x_5)\,. \tag{29}$$

Let us now assume that the plant parameters have assumed their nominal values, in which case $w_1 = 0$. If in addition we let $u_{11} = 0$ and assume that (29) has been satisfied, it is seen that $f_2 = 0$. With similar conditions imposed on f_4 and f_6, it follows from (6) that the equilibrium at $\underline{e} = \underline{0}$ is asymptotically stable.

The problem that remains is to implement u_{11} so that (6) continues to be negative definite, even though w_1 takes on nonzero values. According to (15) and (16), this will be accomplished if (noting that u_{11} is used in place of u_i)

$$|u_{11}| \geq |w_1| \tag{30}$$

and

$$\text{sign } u_{11} = \text{sign } \gamma_1 \tag{31}$$

where

$$w_1 = (J_{31} - J_{21})(\tau_1 x_4 x_6 + x_3 x_5) + \tau_1 \dot{d}_1 + d_1 + \frac{J_{11}K_1}{J_{10}}(r_1 - x_1) \tag{32}$$

and from (11)

$$\gamma_1 = p_{21}e_1 + p_{22}e_2. \tag{33}$$

That γ_1 contains only two terms instead of six stems from the fact that the model is noninteracting, as discussed in the section Simplification of the Control Law by Employing a Noninteracting Model.

In order to satisfy (30), use is made of the fact that bounds on the deviations of parameter values and disturgances are known, these bounds being denoted by $|.|_m$. It then follows that (30) will be satisfied if

$$|u_{11}| = |J_{31} - J_{21}|_m |\tau_1 x_4 x_6 + x_3 x_5| + \tau_1 |\dot{d}_1|_m + |d_1|_m$$
$$+ \frac{|J_{11}|_m K_1}{J_{10}} |r_1 - x_1| . \tag{34}$$

Finally, according to the discussion leading to (17), it is necessary to modify (31) so that

$$u_{11} = |u_{11}| \text{ sat } k_1\gamma_1 \tag{35}$$

where k_1 is a design parameter. It is shown in Appendix A that, in the absence of noise, a bound on the tracking error varies inversely with k_1. However, if noise is present, the noise level at the input to the plant is proportional to k_1. Therefore an optimum choice of k_1 must be found in relation to a real system environment.

• Results of simulation studies are presented in which the control laws for u_1, u_2, and u_3 have been defined according to the preceding discussion. Prior to t = 0, the model inputs were set at $r_1 = r_2 = 0$, $r_3 = r_0$, so that the simulated plant represented a free body with a constant spin rate, r_0, about its s_3 axis. At t = 0, a pulse was applied to r_2, while maintaining $r_1 = 0$, $r_3 = r_0$. The response of the simulated system to this set of inputs was obtained for the following parameter adjustments:

1. (Fig. 6) -- Plant parameters have nominal values. In this case, the plant is essentially noninteracting as evidenced by the fact that $v_1(t)$ and $v_3(t)$ are held essentially constant.

2. (Fig. 7) -- Plant parameters have assumed other than nominal values, but with $u_{11}(t) = u_{21}(t) = u_{31}(t) = 0$. As is to be expected, the control law is no longer adequate, as evidenced by the variations which appear in $v_1(t)$ and $v_3(t)$.

3. (Fig. 8) -- Plant parameters have the same values as in Case 2, but the nonlinear control law is now allowed to operate. The plant response is caused to be essentially noninteracting as in Case 1.

It was observed that the errors can be reduced by increasing the magnitudes of the sat gains, as was to be expected. However, since noise was not present in the simulation studies, the sat gains were given arbitrary settings. It was observed that the three controlled inputs showed no tendency to grow without bounds.

CONCLUSIONS

The synthesis of multivariable systems has been recognized as a difficult task.[6] In particular, satisfactory design procedures for taking into account the effects of parameter variations have not been developed to the same degree of sophistication as in the case of single-variable systems. This is understandable when it is recognized that the design approach to multivariable systems has been based largely on diagonalization techniques with application to linear systems. It is particularly important, therefore, that those techniques be explored which may be used to broaden the category of multivariable systems that can be treated by systematic synthesis procedures. The method advocated in this paper is attractive in that it offers one such opportunity. It is required only that the desired response characteristics can be expressed in terms of a linear time-invariant model for which design techniques are well established.

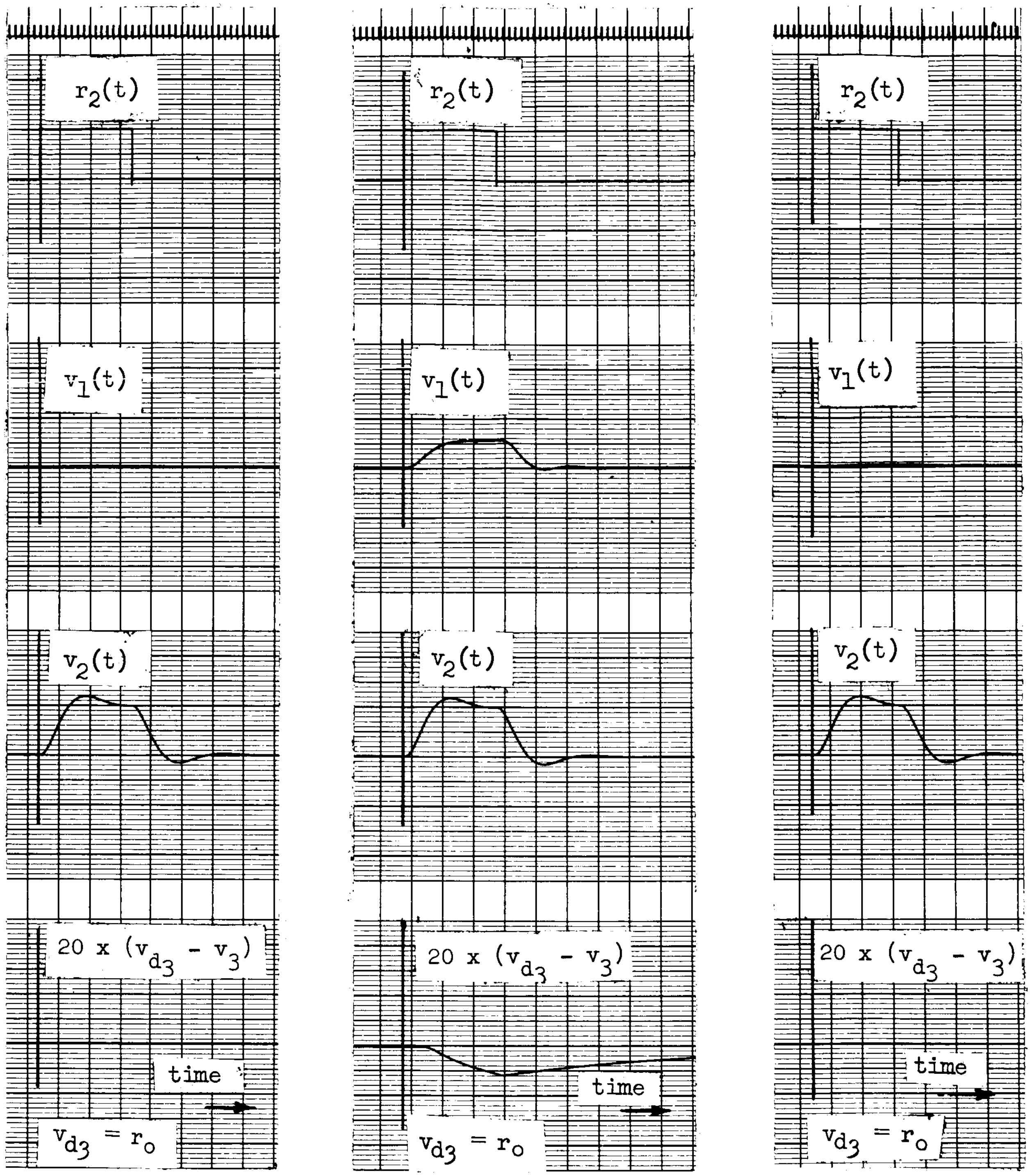

Fig. 6 -- Responses with plant parameters at their nominal values

Fig. 7 -- Responses with plant parameters at other than nominal values, but with nonlinear control law inoperative

Fig. 8 -- Responses with plant parameters as in Fig. 7, but with nonlinear control law operative

• As in any synthesis technique which depends upon Liapunov's direct method, the need for generating higher derivatives creates a potential drawback, due to the adverse effects of instrumentation noise which may represent a severe problem in the case of high-order plants. This as well as other practical engineering problems associated with an application of the method are thought to be deserving of continued study.

Derivation of a bound on the solution to the error equation (4)

It is to be shown that the solution to (4) can be ultimately bounded[3] if the sat functions are used in place of the sign functions in generating the control law, and that this bound can be made arbitrarily small by increasing the sat gains. The solution is said to be ultimately bounded with bound R_1 if for any $\underline{e}(0)$

$$\lim_{t \to \infty} \|\underline{e}(t)\| < R_1 . \tag{A1}$$

where $\|\underline{e}\|$ is the Euclidian norm defined by $\|\underline{e}\| = (\sum_{i=1}^{n} e_i^2)^{1/2}$.
To prove boundedness according to (A1) it is sufficient to show that there is a region about the origin outside of which $V > 0$, $\dot{V} < 0$.

In the following derivation, it is assumed that in (1a) the function $g(\underline{x}, \underline{d}, t)$ may contain time variable coefficients which enter the equation either as multipliers of linear terms in the state variables, such as $\phi(t)\, x_i(t)$, or as multipliers of product terms in the state variables, such as $\psi(t)\, x_i(t)\, x_j(t)$. Using (6), (10), (11), and (13), and arbitrarily setting $Q = I$ for simplicity, we obtain

$$\dot{V} = - \underline{e}^T \underline{e} - 2(\underline{\gamma}^q)^T\, \underline{f}^q \tag{A2}$$

where according to (13)

$$\underline{f}^q\, (\underline{x}, \underline{r}, \underline{d}, \underline{u}, t) = \underline{h}^q\, (\underline{x}, \underline{r}, \underline{d}, t) + D^q(t)\, \underline{u}(t) . \tag{A3}$$

It is now assumed that each element of $\underline{u}(t)$ is specified as in (15), except that for each $u_i(t)$, sign γ_i^q is replaced by sat $k_i\, \gamma_i^q$ in accordance with (16). It follows that $\dot{V} \leq 0$ if for each value of $i = 1, 2, \dots q$, $|\gamma_i^q| \geq 1/k_i$. It is only necessary, therefore, to examine the sign of $\dot{V}$ for the condition that $|\gamma_i^q| < 1/k_i$.

From (A2) and (A3), we obtain

$$\dot{V} = - \underline{e}^T \underline{e} - 2(\underline{\gamma}^q)^T \underline{h}^q - 2(\underline{\gamma}^q)^T\, D^q\, \underline{u} . \tag{A4}$$

Since the term $2(\underline{\gamma}^q)^T D^q \dot{\underline{u}}$ is always nonnegative, by neglecting this term it is seen that $\dot{V}$ satisfies the inequality

$$\dot{V} \leq - \underline{e}^T \underline{e} - 2(\underline{\gamma}^q)^T \underline{h}^q . \tag{A5}$$

(A5) can in turn be expressed in the explicit form

$$\dot{V} \leq - \sum_{i=1}^{n} e_i^2 - 2 \sum_{i=1}^{q} \gamma_i^q h_i^q. \tag{A6}$$

A more conservative inequality derived from (A6) is given by

$$\dot{V} \leq - \sum_{i=1}^{n} e_i^2 + 2 \sum_{i=1}^{q} |\gamma_i^q| \, |h_i^q|. \tag{A7}$$

Here h_i^q has the general form

$$\begin{aligned} h_q^i(\underline{x}, \underline{r}, \underline{d}, t) = \sum_{j=1}^{q} b_{ij}^q r_i(t) + \sum_{j=1}^{q} d_{ij}^q d_i + \sum_{j=1}^{n} \phi_{ij}(t) x_i(t) \\ + \sum_{j=1}^{n} \sum_{j=1}^{n} \psi_{kj}(t) x_i(t) x_j(t). \end{aligned} \tag{A8}$$

In order to obtain an upper bound on h_i^q, it is expedient to substitute $x_i = x_{di} - e_i$ into (A8) so as to eliminate its explicit functional dependence on $\underline{x}$. It follows that a group of terms in $h_i^q(\underline{x}_d, \underline{e}, \underline{r}, \underline{d}, t)$ may appear which involves components of $\underline{e}$ such as e_i, $e_i e_j$, while another group of terms may appear which do not contain components of $\underline{e}$, such as x_{di}, $x_{di} x_{dj}$. By having assumed a stable model, i.e. A is a stability matrix, it follows, if model inputs are bounded, that $\|\underline{x}_d\|$ is also bounded. Thus for bounded inputs and disturbances, all terms in $h_i^q(\underline{x}_d, \underline{e}, \underline{r}, \underline{d}, t)$ are known to be bounded, with the possible exception of those containing components of $\underline{e}$. From the preceding statements, and a knowledge of the bounds on the parameter variations, an inequality can be derived from (A7) of the form

$$h_i^q \leq N_i \left(\left| \sum_{i=1}^{n} \sum_{j=1}^{n} e_i e_j \right| + \left| \sum_{i=1}^{n} e_i \right| \right) + M_i \tag{A9}$$

where N_i and M_i are finite positive numbers. A more conservative inequality can now be written in the form

$$h_i^q \leq N_i \left(\sum_{i=1}^{n} \sum_{j=1}^{n} |e_i|\,|e_j| + \sum_{i=1}^{n} |e_i| \right) + M_i \,. \tag{A10}$$

It is convenient at this point to define a hyperspherical surface of radius R by $R = ||\underline{e}||$ where $||\underline{e}|| = (\sum_{i=1}^{n} e_i^2)^{1/2}$. In addition use will be made of the inequalities

$$\sum_{i=1}^{n} |e_i| \leq (n \sum_{i=1}^{n} e_i^2)^{1/2} \tag{A11}$$

and

$$\sum_{i=1}^{n} \sum_{j=1}^{n} |e_i||e_j| \leq n \sum_{i=1}^{n} e_i^2 \,.$$

It follows, for any value of $\underline{e}$, with $R = ||\underline{e}||$, that

$$h_i^q \leq N_i \,(nR^2 + n^{1/2} R) + M_i \,. \tag{A12}$$

From the statement following (A3) it is noted that the sign of $\dot{V}$ can be positive only when $|\gamma_i^q| < 1/k_i$ for at least one value of i. Thus if each of the terms $|\gamma_i^q|$ in (A7) is replaced by 1/k, where k is taken to represent a lower bound on the sat gains, then from (A7) and (A12), a more conservative expression for $\dot{V}$ becomes

$$\dot{V} \leq - R^2 \left(1 - \frac{\alpha_1}{k}\right) + \frac{\alpha_2 R}{k} + \frac{\alpha_3}{k} \tag{A13}$$

where α_1, α_2, and α_3, are finite positive constants. It follows, for a large enough value of k, that $\dot{V} < 0$ everywhere outside of some hyperspherical region of radius R_1, and that this region can be made arbitrarily small by choosing k as a large enough lower bound on the sat gains.

It is concluded that, for a large enough value of k, the solution is ultimately bounded since, for any $\underline{e}(0)$, (A1) is satisfied. It should be noted that, in the absence of nonlinear terms such as $x_i x_j$ in the defining equations of the plant, the term α_1 in (A13) does not appear. In this case, the solution is ultimately bounded for any $k > 0$.

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