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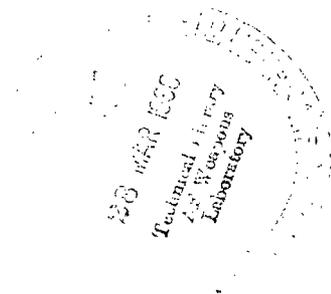
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NATIONAL AERONAUTICS AND SPACE ADMINISTRATION

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SUMMARY

An explicit closed form solution is given for the equations of a nonlinear theory for deep membranes of revolution. In the theory, it is assumed that strains are small and rotations are moderately small, that is, of the order of the square root of the strains. The solution is valid when the nonlinear behavior is confined to boundary-layer regions near edges. General symmetric surface loads can be conveniently included in the form of the solution. Results are obtained for the particular cases of a pressurized membrane and a rotating spherical membrane, both attached to hubs. Also, results are given for a cap of arbitrary shape attached to a cylindrical pressure vessel and for the arrest of a moving mass by a pressurized spherical membrane.

INTRODUCTION

In the design of space vehicles, membranelike structures (i.e., structures with negligible bending stiffness) continue to fine wide application. Large displacements of such structures must be considered, for instance, in the design of deployable space power configurations, radar antennas, and pressure vessels. Therefore, the solution of membrane shell problems, especially in the nonlinear deflection range, is a useful tool for establishing rational design procedures.

Several derivations of systems of equations appropriate for the analysis of a membrane of revolution in the nonlinear range are available. (See refs. 1, 2, and 3.) The analysis in the present paper is based on an approximate system of equations given by Sanders in reference 3, in which the strains are assumed to be small and the rotations are moderately small.

For satisfying the common boundary conditions imposed in practice, it is known that linear membrane theory is inadequate. In such cases a linear bending theory solution is often added to remedy this difficulty. This situation gives rise to the usual edge zone where bending action insures satisfaction of the necessary boundary conditions. However, for membranes or very thin shells of negligible bending stiffness this approach is not realistic, but a nonlinear membrane theory can yield reasonably accurate results and avoids the greater complications of nonlinear bending theory. The order of the system of equations

of nonlinear membrane theory, in contrast to linear membrane theory, is found to be high enough so that the usual boundary conditions can be satisfied. This result was first pointed out by Bromberg and Stoker in reference 4, who also noticed that nonlinear membrane behavior is confined to a narrow region near the boundary.

Nonlinear membrane theory has already been used successfully to solve particular membrane problems. By using an appropriate nonlinear theory, some deep membrane problems with specific shapes and loading conditions have been considered. (For example, see refs. 2 and 5.) Also, a linearized large deflection approach in the case of shallow membranes has been given in reference 6 in which shallow spherical and conical membranes internally pressurized and fixed at edges are considered.

The purpose of the present paper is to obtain a general closed form solution to the equations of nonlinear membrane theory. The solution has not appeared before and is valid for deep membranes of revolution of arbitrary shape under symmetric surface loads. For displacement boundary conditions, the constants of integration are given by closed form expressions. For stress boundary conditions, the constants of integration can be calculated in a straightforward manner as indicated in the particular problems treated.

Complete solutions are found for four particular problems. The first two involve a pressurized spherical membrane and a rotating spherical membrane, both attached to hubs. The other solutions concern a cap of arbitrary shape attached to a cylindrical pressure vessel and the arrest of a moving mass by a pressurized spherical membrane. The solutions obtained by the approach used are shown to be accurate to within small errors of a stated order of magnitude.

SYMBOLS

a	characteristic dimension of membrane
c, c_1, c_2, d, d_1	constants of integration
E	Young's modulus
$f_1, \ldots, f_7$	
g	functions of ξ which appear in general solutions
q, S	
H	horizontal component of edge load
h	thickness of membrane
k_E	energy parameter

M	rigid mass
N_ξ, N_θ	stress resultants
p	characteristic pressure loading of membrane
p_H, p_V	horizontal and vertical components of surface load
p_n, p_t	normal and tangential components of surface load
R	radius of spherical membrane
R_L	radius of cylinder
R_ξ, R_θ	principal radii of curvature
r	horizontal distance
s	arc length
t	time
u	horizontal displacement
V	vertical component of edge load
v	velocity
w	vertical displacement
w_1	vertical displacement at hub
$w_{1,max}$	maximum vertical displacement at hub (see eq. (B35))
X	distance along cylinder
z	vertical distance
α	factor for arc length along meridian
α_J	value of α at juncture of pressure vessel
ϵ	small parameter, $\dfrac{pa}{Eh}$
$\epsilon_\xi, \epsilon_\theta$	strains
θ	angle in circumferential direction

I

ν	Poisson's ratio
ξ	curvilinear coordinate along meridian
ξ_i	value of ξ at each edge of membrane; $i = 1,2$
ξ_J	value of ξ at juncture of pressure vessel
ξ_N	lower limit on certain integrals which appear in solutions
ρ	mass density of membrane material
φ	angle between axis of revolution and normal to middle surface
ψ	rotation of the normal to the middle surface in a meridional plane
$\psi_e, \psi_{e,max}$	boundary-layer rotation, and maximum value
ψ_L	rotation calculated from linear theory
Ω	frequency of rotation

Subscripts:

e	boundary layer
J	juncture
max	maximum

A prime over a symbol denotes differentiation with respect to ξ.

A bar over a symbol denotes a physical quantity. When a bar does not appear, the quantity is nondimensional. (See eqs. (17).) This rule does not apply to the physical quantities a, E, h, and p.

FUNDAMENTAL EQUATIONS

The geometry of the undeformed middle surface of a general membrane of revolution is shown in figure 1. Also shown is the notation for the membrane stress resultants along the meridional and circumferential directions $\bar{N}_\xi$ and $\bar{N}_\theta$, respectively. The positive directions of the displacements $\bar{u}$ and $\bar{w}$ and the rotation $\bar{\psi}$, the rotationally symmetric loads per unit area $\bar{p}_H$ and $\bar{p}_V$ $\left(\text{or } \bar{p}_n \text{ and } \bar{p}_t\right)$, and the edge stress resultants $\bar{H}$ and $\bar{V}$ are given in the figure.

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The middle surface is defined by the parametric equations

$$\bar{r} = \bar{r}(\xi) \qquad\qquad \bar{z} = \bar{z}(\xi) \tag{1}$$

The parameter ξ is the curvilinear coordinate in the meridional direction. Arc length on the undeformed middle surface is given by

$$ds^2 = \bar{\alpha}^2 d\xi^2 + \bar{r}^2 d\theta^2 \tag{2}$$

where

$$\bar{\alpha} = \left(\bar{r}'^2 + \bar{z}'^2\right)^{1/2} \qquad \bar{r}' = \bar{\alpha}\cos\varphi \qquad \bar{z}' = \bar{\alpha}\sin\varphi$$

and primes denote differentiation with respect to ξ. Lines of curvature coordinates are used, and the principal curvatures are given by

$$\frac{1}{\bar{R}_\xi} = \frac{\varphi'}{\bar{\alpha}} \qquad\qquad \frac{1}{\bar{R}_\theta} = \frac{\sin\varphi}{\bar{r}} \tag{3}$$

The governing equations are based on a nonlinear theory for thin shells given by Sanders in reference 3 in which small strains and moderately small rotations are assumed. The system of equations of the nonlinear membrane theory are derived by utilizing the principle of virtual work with appropriate strain-displacement relations given in reference 3. In this manner equilibrium equations with a consistent set of boundary conditions can be derived.

In terms of the notation used, the strain-displacement relations are

$$\bar{\epsilon}_\xi = \frac{1}{\bar{\alpha}}\left(\bar{u}'\cos\varphi + \bar{w}'\sin\varphi\right) + \frac{1}{2}\bar{\psi}^2 \tag{4}$$

$$\bar{\epsilon}_\theta = \frac{\bar{u}}{\bar{r}} \tag{5}$$

$$\bar{\psi} = \frac{1}{\bar{\alpha}}\left(\bar{w}'\cos\varphi - \bar{u}'\sin\varphi\right) \tag{6}$$

The stress-strain relations are

$$\bar{\epsilon}_\xi = \frac{1}{Eh}\left(\bar{N}_\xi - \nu\bar{N}_\theta\right) \tag{7}$$

$$\bar{\epsilon}_\theta = \frac{1}{Eh}\left(\bar{N}_\theta - \nu\bar{N}_\xi\right) \tag{8}$$

The principle of virtual work (which equates the internal virtual change in strain energy to the external virtual work of the loads $\bar{p}_H$ and $\bar{p}_V$ and the edge stresses $\bar{H}$ and $\bar{V}$) is expressed in the following form:

$$\int_{\xi_1}^{\xi_2}\int_0^{2\pi}\left(\bar{N}_\xi\,\delta\bar{\epsilon}_\xi + \bar{N}_\theta\,\delta\bar{\epsilon}_\theta\right)\bar{\alpha}\bar{r}\,d\theta\,d\xi = \int_{\xi_1}^{\xi_2}\int_0^{2\pi}\left(\bar{p}_H\,\delta\bar{u} + \bar{p}_V\,\delta\bar{w}\right)\bar{\alpha}\bar{r}\,d\theta\,d\xi$$

$$+ \int_0^{2\pi}\left(\bar{V}\,\delta\bar{w} + \bar{H}\,\delta\bar{u}\right)\Big]_{\xi_1}^{\xi_2}\bar{r}\,d\theta \tag{9}$$

If the relations in equations (4) to (8) are now used in equation (9), and the indicated variations are carried out, integration by parts gives

$$\int_0^{2\pi}\left\{\left[\bar{N}_\xi\left(\cos\varphi - \bar{\psi}\sin\varphi\right) - \bar{H}\right]\delta\bar{u} + \left[\bar{N}_\xi\left(\sin\varphi + \bar{\psi}\cos\varphi\right) - \bar{V}\right]\delta\bar{w}\right\}\Big]_{\xi_1}^{\xi_2}\bar{r}\,d\theta$$

$$+ \int_{\xi_1}^{\xi_2}\int_0^{2\pi}\left\{\left[\bar{r}\bar{N}_\xi\left(\cos\varphi - \bar{\psi}\sin\varphi\right)\right]' - \bar{\alpha}\bar{N}_\theta + \bar{\alpha}\bar{r}\bar{p}_H\right\}\delta\bar{u}\,d\theta\,d\xi$$

$$- \int_{\xi_1}^{\xi_2}\int_0^{2\pi}\left\{\left[\bar{r}\bar{N}_\xi\left(\sin\varphi + \bar{\psi}\cos\varphi\right)\right]' + \bar{\alpha}\bar{r}\bar{p}_V\right\}\delta\bar{w}\,d\theta\,d\xi = 0 \tag{10}$$

By equating to zero the coefficients of the variations of the displacements in equation (10), the following equilibrium equations and boundary conditions are obtained:

Equilibrium equations:

$$\left[\bar{r}\bar{N}_\xi\left(\sin\varphi + \bar{\psi}\cos\varphi\right)\right]' = -\bar{\alpha}\bar{r}\bar{p}_V \tag{11}$$

$$\left[\bar{r}\bar{N}_\xi\left(\cos\varphi - \bar{\psi}\sin\varphi\right)\right]' - \bar{\alpha}\bar{N}_\theta = -\bar{\alpha}\bar{r}\bar{p}_H \tag{12}$$

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Boundary conditions: At an edge $\left(\xi_1 \text{ or } \xi_2\right)$ prescribe

$$\bar{u} \quad \text{or} \quad \overline{N}_\xi\left(\cos \varphi - \overline{\psi} \sin \varphi\right) = \overline{H} \tag{13}$$

$$\bar{w} \quad \text{or} \quad \overline{N}_\xi\left(\sin \varphi + \overline{\psi} \cos \varphi\right) = \overline{V} \tag{14}$$

The surface loading may be divided into two general classes. In one type of loading, the directions of the load components $\bar{p}_V$ and $\bar{p}_H$ are always parallel and perpendicular, respectively, to the axis of revolution of the membrane; the magnitude does not change with deformation. For instance, in the case of a membrane spinning about the axis, $\bar{p}_V = 0$, and $\bar{p}_H$ is given by the centrifugal forces. In the second general loading situation, the loads are fixed in magnitude but are always normal $\bar{p}_n$ and tangential $\bar{p}_t$ to the middle surface during the deformation. In this case, for the moderately small rotations considered, the components perpendicular and parallel to the axis of revolution are given by

$$\bar{p}_H = \bar{p}_t\left(\cos \varphi - \overline{\psi} \sin \varphi\right) + \bar{p}_n\left(\sin \varphi + \overline{\psi} \cos \varphi\right) \tag{15}$$

$$\bar{p}_V = \bar{p}_t\left(\sin \varphi + \overline{\psi} \cos \varphi\right) - \bar{p}_n\left(\cos \varphi - \overline{\psi} \sin \varphi\right) \tag{16}$$

For example, for a pressurized membrane, $\bar{p}_t = 0$ and $\bar{p}_n$ equals the internal pressure. In the present theory, it is noted further that the terms with $\overline{\psi}$ in equations (15) and (16) can be dropped and still be within the accuracy of the present solution.

It is convenient to introduce the following nondimensional variables and parameters where bars denote physical quantities:

$$\left.\begin{aligned}
\left(N_\xi, N_\theta, V, H\right) &= \frac{1}{h \epsilon E}\left(\overline{N}_\xi, \overline{N}_\theta, \overline{V}, \overline{H}\right) \\[6pt]
(u, w) &= \frac{1}{\epsilon a}\left(\bar{u}, \bar{w}\right) \\[6pt]
\left(\epsilon_\xi, \epsilon_\theta, \psi\right) &= \frac{1}{\epsilon}\left(\bar{\epsilon}_\xi, \bar{\epsilon}_\theta, \overline{\psi}\right) \\[6pt]
\left(r, \alpha, R_\xi, R_\theta\right) &= \frac{1}{a}\left(\bar{r}, \bar{\alpha}, \overline{R}_\xi, \overline{R}_\theta\right) \\[6pt]
\left(p_H, p_V, p_t, p_n\right) &= \frac{1}{p}\left(\bar{p}_H, \bar{p}_V, \bar{p}_t, \bar{p}_n\right)
\end{aligned}\right\} \tag{17}$$

where $\epsilon = \dfrac{pa}{Eh}$ is a small parameter which is of the order of magnitude of the strains. The quantity p is some characteristic loading which in a given case shall be selected so that it has units of force per unit area. The quantity a is some characteristic dimension of the membrane.

In terms of these nondimensional variables the governing equations are

$$\left[rN_\xi\left(\sin\,\varphi\,+\,\epsilon\psi\,\cos\,\varphi\right)\right]' = -\alpha r p_V \qquad (18)$$

$$\left[rN_\xi(\cos\,\varphi\,-\,\epsilon\psi\,\sin\,\varphi)\right]' - \alpha N_\theta = -\alpha r p_H \qquad (19)$$

$$\epsilon_\xi = \frac{1}{\alpha}(u'\,\cos\,\varphi\,+\,w'\,\sin\,\varphi) + \frac{1}{2}\,\epsilon\psi^2 \qquad (20)$$

$$\epsilon_\theta = \frac{u}{r} \qquad (21)$$

$$\psi = \frac{1}{\alpha}(w'\,\cos\,\varphi\,-\,u'\,\sin\,\varphi) \qquad (22)$$

$$\epsilon_\xi = N_\xi\,-\,\nu N_\theta \qquad\qquad \epsilon_\theta = N_\theta\,-\,\nu N_\xi \qquad (23)$$

From equations (20), (21), and (22) the following compatibility equation may be obtained:

$$\epsilon_\xi\,\cos\,\varphi\,-\,\psi\,\sin\,\varphi = \frac{\left(r\epsilon_\theta\right)'}{\alpha} + \frac{1}{2}\,\epsilon\psi^2\cos\,\varphi \qquad (24)$$

This equation will be used in the next section to obtain a solution for the rotation ψ.

General Solution of the Equations

A closed form solution of the governing equations (eqs. (18) to (23)), which is accurate to within errors of order of magnitude $O\left(\epsilon^{1/2}\right)$ with respect to unity, is obtained by using boundary-layer methods and asymptotic integration. (See, for example, refs. 7, 8, and 9.) At the start, assumptions which involve orders of magnitude are made which are verified later by the solution. Near an edge where boundary or continuity conditions are prescribed, it is assumed that a boundary layer exists. Within the boundary layer it is further assumed that

the rotation $\psi = O\left(\epsilon^{-1/2}\right)$, and $\psi' = O\left(\epsilon^{-1/2}\right)\psi = O\left(\epsilon^{-1}\right)$, $\displaystyle\int \psi\,d\xi = O(1)$, and

$\left(rN_\xi\right)' = O(1)$. The stresses N_ξ and N_θ are assumed to be $O(1)$ everywhere. Outside the boundary layer, $\psi = O(1)$.

The solution is to be obtained first by solving the equilibrium equations (eqs. (18) and (19)) for N_ξ and N_θ in terms of the rotation ψ, and then by substituting these results via equations (23) into the compatibility equation (eq. (24)) to obtain a single equation for ψ. Equation (18) is integrated to give

$$rN_\xi(\sin\varphi + \epsilon\psi\cos\varphi) = c - g(\xi) \tag{25}$$

where c is a constant of integration and

$$g(\xi) = \int^\xi \alpha r p_V \, d\xi \tag{26}$$

Since $\epsilon\psi$ is small, equation (25) is used to write N_ξ as

$$N_\xi = \frac{c - g(\xi)}{r\sin\varphi}(1 - \epsilon\psi\cot\varphi) + O(\epsilon) \tag{27}$$

This step limits this method to values of φ that are not too small (i.e., those for deep membranes) so that $\cot\varphi$ does not become too large. It is also noted that the term with $\epsilon\psi$ in equation (27) is retained because it will be differentiated later, and therefore its order of magnitude will be increased.

Now, in order to calculate N_θ, it is necessary to differentiate rN_ξ; this term is given by

$$\left(rN_\xi\right)' = -(c - g)\left(\frac{\alpha\cos\varphi}{R_\xi\sin^2\varphi} + \frac{\epsilon\psi'\cos\varphi}{\sin^2\varphi}\right) - \frac{\alpha r p_V}{\sin\varphi} + O\left(\epsilon^{1/2}\right) \tag{28}$$

The term N_θ is obtained by using equations (19), (27), and (28) and is written as

$$N_\theta = \frac{-(c - g)}{\alpha\sin^2\varphi}\left(\frac{\alpha}{R_\xi} + \epsilon\psi'\right) - r p_V\cot\varphi + r p_H + O\left(\epsilon^{1/2}\right) \tag{29}$$

The stress-strain equations (eqs. (23)) are used next to write the compatibility equation (eq. (24)) in the form

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$$\left(N_\xi - \nu N_\theta\right)\cos\varphi - \psi\sin\varphi = \frac{\left(rN_\theta\right)'}{\alpha} - \frac{\nu}{\alpha}\left(rN_\xi\right)' + \frac{1}{2}\epsilon\psi^2\cos\varphi \tag{30}$$

If the expressions for the stress resultants are now substituted into equation (30), the following equation for the rotation ψ is obtained:

$$\frac{(c-g)\cos\varphi}{r\sin\varphi} + \frac{(c-g)\cos\varphi}{\alpha\sin^2\varphi}\left(\frac{\alpha}{R_\xi} + \epsilon\psi'\right) + rp_V\cos\varphi\cot\varphi - rp_H\cos\varphi$$

$$- \frac{r}{\alpha}\left[\left(rp_H - rp_V\cot\varphi\right)' - \frac{(c-g)}{R_\xi\sin^2\varphi}\left(\frac{R_\xi}{\alpha}\epsilon\psi'\right)' - \left(\frac{c-g}{R_\xi\sin^2\varphi}\right)'\left(1 + \frac{R_\xi}{\alpha}\epsilon\psi'\right)\right]$$

$$- \nu r\left(p_V\sin\varphi + p_H\cos\varphi\right) = \psi\sin\varphi + \frac{1}{2}\epsilon\psi^2\cos\varphi + 0\left(\epsilon^{1/2}\right) \tag{31}$$

The solution for ψ in equation (31) can be written as

$$\psi = \psi_e + \psi_L \tag{32}$$

The term ψ_L is the result of linear membrane theory and is of the order of magnitude unity. The solution for ψ_L can be obtained readily from equations (18) to (23) by setting $\epsilon = 0$. The first term in equation (32), namely ψ_e, is of the order of magnitude $0\left(\epsilon^{-1/2}\right)$ and represents a homogeneous edge solution which decays rapidly away from an edge.

When equation (32) is substituted into equation (31) various terms of linear membrane theory are canceled identically by the linear expression for ψ_L $\left(\text{i.e.,}\ \psi_L\ \text{is subtracted out at this stage}\right)$ so that what remains is given by

$$\psi_e\sin\varphi = \frac{(c-g)\cos\varphi}{\alpha\sin^2\varphi}\epsilon\psi_e' + \frac{r}{\alpha}\frac{(c-g)}{R_\xi\sin^2\varphi}\left(\frac{R_\xi}{\alpha}\epsilon\psi_e'\right)' + \frac{1}{2}\epsilon\psi_e^2\cos\varphi$$

$$+ \left(\frac{c-g}{R_\xi\sin^2\varphi}\right)'\frac{R_\xi}{\alpha}\epsilon\psi_e' + 0\left(\epsilon^{1/2}\right) \tag{33}$$

The assumption is now made that the membrane geometry is sufficiently smooth so that differentiation does not increase the order of magnitude of the quantities α, r, and R_ξ. Furthermore, if previous boundary-layer assumptions are invoked and only large terms of order $0\left(\epsilon^{-1/2}\right)$ are retained in equation (33) $\left(\text{which has a relative error of } 0\left(\epsilon^{1/2}\right) \text{ compared with unity}\right)$, the following

boundary-layer equation for ψ_e is obtained:

$$\psi_e" + \frac{1}{\epsilon} q(\xi,c)\psi_e = 0 \tag{34}$$

where

$$q(\xi,c) = \frac{\alpha^2 \sin^3\varphi}{r(c - g)}$$

The solution of equation (34) can be found by noting first that it is a differential equation which contains a large parameter, namely, $1/\epsilon$. Such equations are discussed in reference 8. Accordingly, it is convenient to make the following transformations for dependent and independent variables. Let

$$\beta = q^{1/4}\psi_e \qquad\qquad dx = q^{1/2}d\xi \tag{35}$$

Equation (34) becomes

$$\frac{d^2\beta}{dx^2} - \left[\frac{1}{\epsilon} + \rho_1(x)\right]\beta = 0 \tag{36}$$

where

$$\rho_1(x) = \frac{1}{4} \frac{q"}{q^2} - \frac{5}{16} \frac{(q')^2}{q^3}$$

The quantity $\rho_1(x)$ is continuous and order of magnitude unity $\left(\text{this holds for}\right.$ smooth loading so that $q" = \dfrac{d^2q}{d\xi^2} = O(1)\left.\right)$ and can therefore be neglected with respect to $1/\epsilon$. This approximation is well within the accuracy of the present solution. The solution for β, then, is clearly of the exponential type and taking the solution, which decays away from the edge ξ_1, gives the following boundary-layer solution for ψ_e:

$$\psi_e = dq^{-1/4}(\xi)\exp\left(- \frac{1}{\sqrt{\epsilon}} \left| \int_{\xi_1}^{\xi} q^{1/2}(\xi)d\xi \right|\right) \tag{37}$$

The assumption that ψ_e is $O\left(\epsilon^{-1/2}\right)$ implies that the constant of integration d in equation (37) is $O\left(\epsilon^{-1/2}\right)$.

Now if the membrane has two edges it is assumed that they are sufficiently far apart so that the boundary-layer solutions can be treated independently of each other. (Note that, in general, d can take on a different value for the boundary layer near the other edge ξ_2.) Once the rotation is calculated with equation (32), all other quantities that are expressed in terms of ψ can be determined. In particular the expressions for the stresses N_ξ and N_θ are given by equations (27) and (29).

The displacements u and w are now to be determined. For brevity the symbol ξ_i (i = 1 or 2) will represent the value for either of the edges ξ_1 or ξ_2. By using equations (21) and (23) together with the expressions for the stress resultants N_ξ and N_θ, the displacement u near edge ξ_i can be determined:

$$u = -\frac{rf_1(\xi)}{q} - rS(\xi) + \frac{\sqrt{\epsilon}d}{q^{3/4}} rf_2(\xi)\exp\left(-\frac{1}{\sqrt{\epsilon}}\left|\int_{\xi_i}^{\xi} q^{1/2}d\xi\right|\right) + 0\left(\epsilon^{1/2}\right) \quad (38)$$

where

$$f_1(\xi) = \frac{\alpha^2\sin\varphi}{rR_\xi} + \frac{\nu\alpha^2\sin^2\varphi}{r^2}$$

$$f_2(\xi) = \frac{\alpha\sin\varphi}{r}$$

$$S(\xi) = rp_V\cot\varphi - rp_H$$

The first two terms in equation (38) represent the linear membrane solution for u, and the third term (a decaying exponential) is a boundary-layer contribution due to the rotation ψ_e.

From equations (20) and (22) and from previous relations the following expression for w' is obtained:

$$w' = \frac{\alpha(c - g)}{r}\left(1 + \frac{\nu r}{R_\xi\sin\varphi}\right) + \frac{\sqrt{\epsilon}d}{q^{3/4}}\frac{\nu}{\sin\varphi}\exp\left(-\frac{1}{\sqrt{\epsilon}}\left|\int_{\xi_i}^{\xi} q^{1/2}d\xi\right|\right)$$

$$+ \alpha\nu S\sin\varphi + \alpha\psi\cos\varphi - \frac{\alpha}{2}\epsilon\psi^2\sin\varphi + 0\left(\epsilon^{1/2}\right) \quad (39)$$

Equation (39) can be integrated to give

$$w = cf_3(\xi) - f_4(\xi) + \int_{\xi_N}^{\xi} \alpha\psi \cos\varphi \, d\xi + 0\left(\epsilon^{1/2}\right) \qquad (40)$$

where

$$f_3(\xi) = \int_{\xi_N}^{\xi} \left(\frac{\alpha}{r} + \frac{\nu\alpha}{R_\xi \sin\varphi}\right) d\xi$$

$$f_4(\xi) = \int_{\xi_N}^{\xi} \left(\frac{\alpha g}{r} + \frac{\nu\alpha g}{R_\xi \sin\varphi} - \alpha\nu S \sin\varphi\right) d\xi$$

The constant of integration in equation (40) has been incorporated into the lower limit ξ_N of the integrals which appear. If the expression for the rotation ψ (given by eqs. (32) and (37)) is used in equation (40), the displacement w near the edge ξ_i can be written as

$$w = cf_3(\xi) + f_5(\xi) - \frac{\sqrt{\epsilon}d}{q^{3/4}(\xi_i)} f_6(\xi_i) \exp\left(-\sqrt{\frac{q(\xi_i)}{\epsilon}}\, |\xi - \xi_i|\right) + 0\left(\epsilon^{1/2}\right) \quad (41)$$

where

$$f_5(\xi) = \int_{\xi_N}^{\xi} \alpha \cos\varphi\psi_L \, d\xi - f_4(\xi)$$

$$f_6(\xi) = \alpha \cos\varphi$$

The first two terms in equation (41) are the result of linear membrane theory. In arriving at equation (41), the facts that the boundary layer is exponentially decaying and that $q(\xi)$ and $f_6(\xi)$ are smooth functions (i.e., $q'(\xi)$ and $f_6'(\xi)$ are $0(1)$) have been used to evaluate approximately (within the accuracy of the present solution) the following integral:

$$\int_{\xi_N}^{\xi} f_6(\xi)\psi_e \, d\xi \approx \frac{f_6(\xi_i)d}{q^{1/4}(\xi_i)} \int_{\infty}^{\xi} \exp\left(-\sqrt{\frac{q(\xi_i)}{\epsilon}}|\xi - \xi_i|\right)$$

$$= -\frac{f_6(\xi_i)d\sqrt{\epsilon}}{q^{3/4}(\xi_i)} \exp\left(-\sqrt{\frac{q(\xi_i)}{\epsilon}}\, |\xi - \xi_i|\right) \qquad (42)$$

The lower limit ξ_N of the integral in equation (42) is replaced by ∞ with negligible error because of the rapid decay of the integrand and because ξ_N has a value which is outside the boundary layer (given in the section on boundary conditions).

Boundary Conditions

If the boundary conditions are prescribed with respect to displacements, and if the membrane is attached to the hubs at each edge ξ_i ($i = 1,2$), the displacements must vanish there. If the two hubs are equal in size and the membrane shape and loading are symmetrical about a midpoint between the hubs, then the lower limit ξ_N in the integrals associated with equation (41) is taken to be the value of ξ at this midpoint. From symmetry, it is clear that $w(\xi_N) = 0$.

The constants of integration c and d are now evaluated explicitly for the class of problems where boundary conditions are prescribed on displacements. If the membrane of revolution is attached to hubs, the boundary conditions at a hub, say at $\xi = \xi_1$ (the hub at ξ_2 can be treated similarly and independently of ξ_1), are given by

$$u(\xi_1) = 0 \tag{43}$$

$$w(\xi_1) = 0 \tag{44}$$

The solutions for the constants of integration are given for the case of symmetry between hubs since they will be used for the particular problems treated in appendix A. The extension to the case of no symmetry between hubs, although not considered in this report, offers no new difficulty. In the symmetrical case it is sufficient to consider only one hub at $\xi = \xi_1$. The two boundary conditions (eqs. (43) and (44)) yield two equations for the constants c and d. From equations (38) and (41) the following two conditions are obtained:

$$-\frac{f_1(\xi_1)}{q(\xi_1,c)} + \frac{\sqrt{\epsilon}d f_2(\xi_1)}{q^{3/4}(\xi_1,c)} - S(\xi_1) = 0 \tag{45}$$

and

$$c f_3(\xi_1) + f_5(\xi_1) - \frac{\sqrt{\epsilon}d}{q^{3/4}(\xi_1,c)} f_6(\xi_1) = 0 \tag{46}$$

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When equations (45) and (46) are solved, c and d are given by

$$c = \left[\frac{\left(Sf_6 - f_5f_2 \right)rf_2^2 \sin \varphi - gf_1f_6}{rf_3f_2^3 \sin \varphi - f_1f_6} \right]_{\xi=\xi_1} \tag{47}$$

$$d = \frac{1}{\sqrt{\epsilon}} \left[\frac{q^{3/4}\left(cf_3 + f_5 \right)}{f_6} \right]_{\xi=\xi_1} \tag{48}$$

Note that the order of magnitude of d is $O\left(\epsilon^{-1/2}\right)$ for smooth loading. Equation (37) then shows that the assumed order of magnitude of ψ_e is confirmed. With c and d thus determined by equations (47) and (48) all relevant quantities may be calculated. Stresses and displacements are calculated in appendix A for particular problems by specializing the foregoing general results.

If the boundary conditions are prescribed with respect to stresses, the following equations must apply at an edge:

$$N_\xi (\cos \varphi - \epsilon\psi \sin \varphi) = H \tag{49}$$

$$N_\xi (\sin \varphi + \epsilon\psi \cos \varphi) = V \tag{50}$$

These equations are the nondimensional form of equations (13) and (14). The stress resultants given by H and V cannot be prescribed independently of each other but must be related by the following expression which is obtained from equations (49) and (50):

$$V \cos \varphi - H \sin \varphi = O\left(\epsilon^{1/2}\right) \tag{51}$$

Equation (51) is necessary in order to insure that rotations are not greater in magnitude than the order of the square root of the strains, since this is the basis for the present theory. However, for the physical problems involving deep membranes which actually occur in practice, the behavior implied by equation (51) is the usual case. This is clarified by the cases treated in appendix B, which presents more specific indications of actual stress boundary conditions.

Particular problems are solved in the appendixes which illustrate the use of the general results obtained. Stresses and displacements are calculated in appendix A for the special cases of a spherical membrane under internal pressure and of a rotating spherical membrane, both attached to hubs. These calculations are performed by specializing the general formulas in the text. The results of these calculations, which give stresses and displacements, are shown in figures 2 to 4.

In figure 2 the maximum physical rotation $\overline{\psi}_{e,max}$ at the hub is plotted against hub angle ξ_1 for several values of the small parameter $\epsilon = \dfrac{pR}{Eh}$. In the parameter ϵ, R is the radius of the membrane, whereas the quantity p represents the internal pressure for the pressurized membrane and the centrifugal force per unit area $\left(p = \rho h \Omega^2 R\right)$ for the rotating membrane. In the case of the pressurized membrane, $\overline{\psi}_{e,max}$ is independent of the hub angle ξ_1 whereas in the case of the rotating membrane it decreases with ξ_1. Figures 3 and 4 show the nondimensional stress resultants N_ξ and N_θ and the displacement w in each case, which occur near a hub having a hub angle $\xi_1 = 45^\circ$. The differences between linear and nonlinear membrane theory are clearly indicated by the boundary-layer regions near the hub. The curves for the displacement w show that linear theory does not allow satisfaction of the boundary condition at the hub $\left(\text{i.e., } w(\xi_1) = 0\right)$ so that nonlinear membrane boundary layers are needed here to accomplish this condition.

Although the stresses are positive for the pressurized spherical membrane, it is interesting to note that, in the case of the rotating spherical membrane, the circumferential stress N_θ becomes negative in a small region near the hub so that wrinkling can occur in that region.

In appendix B, two problems are given for which boundary conditions are prescribed on both stresses and displacements. In the first problem, juncture stresses are calculated for a cylindrical pressure vessel. Equations (52) (eqs. (B19) in appendix) give the result of computing the membrane stress resultants N_ξ and N_θ at the juncture of an arbitrarily shaped cap and a cylinder. The stresses are

$$N_\xi = \frac{1}{2} \qquad\qquad N_\theta = 1 - \frac{1}{4R_{\xi,J}} \qquad\qquad (52)$$

The results in equations (52) show that N_ξ is always positive, whereas N_θ can become negative when the ratio of the meridional radius of curvature of the cap at the juncture to the radius of the cylinder is less than 1/4 $\Big(\text{i.e.,}$ $R_{\xi,J} < \dfrac{1}{4}\Big)$. It is interesting to note that the result of linear membrane theory,

obtained by dropping $\epsilon\psi_e{}'$ in equation (53) (eq. (B5) in the appendix), indicates that N_θ becomes negative when $R_{\xi,J}$ is only less than $1/2$.

$$N_\theta = -\frac{r^2}{2\alpha\,\sin^2\varphi}\left(\frac{\alpha}{R_\xi} + \epsilon\psi_e{}'\right) + \frac{r}{\sin\varphi} \tag{53}$$

Also, in contrast to nonlinear membrane theory, linear theory yields discontinuous stresses at the juncture. These stresses are shown in figures 5 and 6 for the case of a spherical and a torispherical cap, respectively. The stress N_θ is plotted in these figures, with the boundary layers necessary to maintain continuous stresses.

The second problem in appendix B concerns the arrest of a moving rigid mass attached to a pressurized spherical membrane. This problem shows that the general results in the text offer a convenient means of obtaining the necessary relation between the unknown edge stress resultant and deflection at points of contact between the mass and membrane (i.e., relations similar to influence coefficients). Maximum stresses and displacements are calculated for a range of values of initial velocity of mass. The results are shown in figure 7 where the quantities are plotted against a kinetic energy parameter

$$k_E = \frac{\bar{M}\bar{v}^2}{\epsilon p R^3}\left(\frac{1}{4\pi\,\sin\xi_1}\right)$$

CONCLUDING REMARKS

Closed form analytical solutions have been obtained for the equations of a nonlinear membrane theory in the case of deep membranes of revolution under arbitrary but otherwise axially symmetric surface loads where the membrane geometry and loading are also assumed to be smoothly varying. The solutions are found to be accurate to within errors of the order of the square root of ϵ with respect to unity, where ϵ is a small parameter which has a magnitude characteristic of strains. The nonlinear behavior is represented by exponential type terms which decay rapidly away from the boundary.

Expressions for the constants of integration which appear in the solutions have been obtained in general form for the class of problems in which boundary conditions are prescribed on displacements. The general results can be specialized in a straightforward manner to handle specific problems. This specialization is accomplished in the particular cases of a pressurized membrane and a rotating spherical membrane, both attached to rigid hubs. The stresses and displacements for these two examples exhibit the expected boundary-layer behavior. Also, whereas the stresses are always positive in the pressurized membrane, the circumferential stress can become negative in the rotating membrane for certain values of the parameter ϵ, so that wrinkling can occur.

For problems in which boundary conditions are prescribed on stresses, it is found that, according to the present theory, the vertical and horizontal edge stress resultants cannot be prescribed independently of each other but must be related in a particular way. Two examples are given in which boundary conditions are prescribed on both stresses and displacements. The first concerns juncture stresses for a cylindrical pressure vessel for which boundary layers eliminate the usual stress discontinuities characteristic of linear membrane theory. Also, in this case, nonlinear theory predicts that negative circumferential stresses at the juncture can be avoided if the ratio of the meridional radius of curvature of the end cap to the radius of the cylinder at the juncture is greater than 1/4. (Linear membrane theory gives 1/2 for this ratio.) The second example involves the arrest of a moving rigid mass attached to a pressurized spherical membrane. For this example, the general solution provides a convenient linearized form for the necessary relation between edge stress and corresponding edge deflection (i.e., similar to influence coefficients). This allows a straightforward determination of maximum stresses and deflections for a range of initial velocities of the mass.

Langley Research Center,
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        Langley Station, Hampton, Va., November 22, 1965.

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PARTICULAR PROBLEMS WITH BOUNDARY CONDITIONS ON DISPLACEMENTS

Two examples are given for deep membrane problems in which boundary conditions are prescribed on displacements only (i.e., the membranes are attached to rigid hubs). The solutions are obtained by a straightforward specialization of the general results given in the text.

Pressurized Spherical Membrane Attached to Hubs

In the special case of a spherical membrane attached to a hub and of radius R and internal pressure p, the various quantities defined for the general problem are specialized. From equations (1), (2), (3), and (17) and from other relations the quantities are reduced as follows:

$$\left.\begin{array}{ccccc} R_\xi = R_\theta = 1 & \alpha = 1 & \varphi = \xi & r = \sin \xi & \xi_N = \dfrac{\pi}{2} \\[2em] a = R & \bar{p}_n = p & p_n = 1 & p_t = 0 & \epsilon = \dfrac{pR}{Eh} \end{array}\right\} \quad (A1)$$

From equations (15), (16), and (17)

$$p_H = \sin \xi + \epsilon\psi \cos \xi \tag{A2}$$

$$p_V = -(\cos \xi - \epsilon\psi \sin \xi) \tag{A3}$$

The solution for the rotation given by linear membrane theory ψ_L can be found from equations (18) to (23) by setting $\epsilon = 0$. In this case

$$\psi_L = 0 \tag{A4}$$

The various other relevant quantities which appear in the general solution given in the text may now be evaluated.

$$g(\xi) = -\frac{1}{2} \sin^2\xi$$

$$f_1(\xi) = 1 + \nu$$

$$f_2(\xi) = 1$$

$$f_3(\xi) = \int_{\pi/2}^{\xi} \frac{1 + \nu}{\sin \xi} \, d\xi = (1 + \nu)\log \tan \frac{\xi}{2}$$

$$f_4(\xi) = - \int_{\pi/2}^{\xi} \frac{1 - \nu}{2} \sin \xi \, d\xi = \frac{1 - \nu}{2} \cos \xi$$

$$f_5(\xi) = \frac{-(1 - \nu)}{2} \cos \xi$$

$$f_6(\xi) = \cos \xi$$

$$S(\xi) = 1$$

$$\left. \right\} \quad (A5)$$

The constants of integration c and d given by equations (47) and (48) can now be calculated by using equations (A5). The solution for c is

$$c = \frac{\cos \xi_1 \left(\dfrac{1 + \nu}{2} - 1 + \dfrac{1 - \nu}{2}\right)}{(1 + \nu)\left(\log \tan \dfrac{\xi_1}{2} - \cos \xi_1 \csc^2\xi_1\right)} = 0 \qquad (A6)$$

Then from the definition of $q(\xi)$ obtained from equation (34), $q(\xi) = 2$ so that the solution for d is

$$d = -\frac{1}{\sqrt{\epsilon}} \frac{1 - \nu}{(2)^{1/4}} \qquad (A7)$$

The maximum rotation at the hub $\xi = \xi_1$ is (see eq. (37))

$$\psi_{e,\max} = dq^{-1/4}(\xi_1) = -\frac{1 - \nu}{\sqrt{2\epsilon}} \qquad (A8)$$

The solutions for the stresses and displacements are determined from the general results given by equations (27), (29), (37), (38), and (41). For this problem they are

$$
\left.
\begin{aligned}
N_\xi &= \frac{1}{2} \\[2ex]
N_\theta &= \frac{1}{2} - \frac{1-\nu}{2}\, e^{-\sqrt{\frac{2}{\epsilon}}\,\left|\xi-\xi_1\right|} \\[2ex]
\psi &= \frac{-(1-\nu)}{\sqrt{2\epsilon}}\, e^{-\sqrt{\frac{2}{\epsilon}}\,\left|\xi-\xi_1\right|} \\[2ex]
u &= \frac{1-\nu}{2} \sin \xi \left(1 - e^{-\sqrt{\frac{2}{\epsilon}}\,\left|\xi-\xi_1\right|}\right) \\[2ex]
w &= \frac{-(1-\nu)}{2} \cos \xi + \frac{1-\nu}{2} \cos \xi_1\, e^{-\sqrt{\frac{2}{\epsilon}}\,\left|\xi-\xi_1\right|}
\end{aligned}
\right\}
\tag{A9}
$$

The solution given by equations (A9) is accurate to within errors of order $0\left(\epsilon^{1/2}\right)$ with respect to unity, as was indicated by the general results in the body of the paper. Also note that the linear membrane solution is exactly what remains when the exponential terms are dropped from equations (A9).

Rotating Spherical Membrane Attached to Hubs

For a rotating spherical membrane the radius of the membrane attached to a hub is also denoted by R. However, the characteristic loading quantity p, which is discussed in relation to equations (17), arises from centrifugal forces and is given as

$$
p = \rho h \Omega^2 R \tag{A10}
$$

where ρ is the mass density of membrane material and Ω is the frequency of rotation. The relevant quantities used for the rotating spherical membrane are given as

$$
\left.
\begin{aligned}
R_\xi = R_\theta = 1 \qquad \alpha = 1 \qquad \varphi = \xi \qquad r = \sin \xi \qquad \xi_N = \frac{\pi}{2} \\[2ex]
a = R \qquad p = \rho h \Omega^2 R \qquad p_H = r \qquad \epsilon = \frac{\rho \Omega^2 R^2}{E}
\end{aligned}
\right\}
\tag{A11}
$$

The linear solution for the rotation ψ_L is obtained from equations (18) to (23) by setting $\epsilon = 0$. The result is

$$\psi_L = -(3 + \nu)\sin \xi \cos \xi \tag{A12}$$

Under these conditions the functions of ξ are given by

$$g(\xi) = 0$$

$$f_1(\xi) = 1 + \nu$$

$$f_2(\xi) = 1$$

$$f_3(\xi) = \int_{\pi/2}^{\xi} \frac{1 + \nu}{\sin \xi} \, d\xi = (1 + \nu)\log \tan \frac{\xi}{2}$$

$$f_4(\xi) = \int_{\pi/2}^{\xi} \nu \sin^3\xi \, d\xi = -\nu \cos \xi + \nu \cos^3\xi \tag{A13}$$

$$f_5(\xi) = \cos^3\xi + \nu \cos \xi$$

$$f_6(\xi) = \cos \xi$$

$$S(\xi) = -\sin^2\xi$$

The constant c given by equation (47) then reduces to

$$c = \frac{\cos \xi_1}{\cos \xi_1 \csc^2\xi_1 - \log \tan \dfrac{\xi_1}{2}} \tag{A14}$$

For this case, from equation (34),

$$q\left(\xi_1, c\right) = \frac{\sin^2\xi_1}{c}$$

This result, together with equation (48), gives the following result for d:

$$d = q^{1/4}(\xi_1, c)\psi_{e,max} \tag{A15}$$

The expression for $\psi_{e,max}$ is now

$$\psi_{e,max} = \frac{c(1 + \nu)\csc \xi_1 - \sin^3\xi_1}{\sqrt{\epsilon c}} \tag{A16}$$

The stresses and displacements are then given by

$$N_\xi = \frac{c}{\sin^2\xi}$$

$$N_\theta = -\frac{c}{\sin^2\xi}\left(1 - \sqrt{\frac{\epsilon}{c}}\sin \xi_1 \sin \xi \ \psi_{e,max} \exp\frac{|\cos \xi_1 - \cos \xi|}{-\sqrt{\epsilon c}}\right) + \sin^2\xi$$

$$\psi = \psi_{e,max} \exp\frac{|\cos \xi_1 - \cos \xi|}{-\sqrt{\epsilon c}} - (3 + \nu)\sin \xi \cos \xi$$

$$u = \frac{-c(1 + \nu)}{\sin \xi} + \sqrt{\frac{\epsilon c \sin \xi_1}{\sin \xi}} \ \psi_{e,max} \exp\frac{|\cos \xi_1 - \cos \xi|}{-\sqrt{\epsilon c}} + \sin^3\xi$$

$$w = c(1 + \nu)\log \tan \frac{\xi}{2} + \cos^3\xi + \nu \cos \xi$$

$$\qquad - \sqrt{\epsilon c} \cot \xi_1 \psi_{e,max} \exp\left(-\frac{\sin \xi_1}{\sqrt{\epsilon c}}|\xi - \xi_1|\right)$$

$$\left.\begin{array}{c} \\ \\ \\ \\ \\ \\ \\ \\ \\ \end{array}\right\} \tag{A17}$$

The form of these equations and the numerical results based on these equations are essentially the same as those given in reference 2.

APPENDIX B

PARTICULAR PROBLEMS WITH BOUNDARY CONDITIONS

ON BOTH STRESSES AND DISPLACEMENTS

End Cap Attached to a Cylinder Under Internal Pressure

Stresses and displacements are calculated at the juncture of an end cap of arbitrary but rotationally symmetric shape and a cylinder under internal pressure. Solutions are first obtained separately for the cap and cylinder by using the general results found previously. Then the constants of integration are evaluated by using suitable matching conditions at the juncture. For this problem a boundary layer is expected at the juncture $\left(\text{i.e.,} \quad \varphi = \frac{\pi}{2}, \text{ fig. 5}\right)$.

Cap solution.- The internal pressure p is taken as the characteristic loading quantity of equations (17). For the cap solution then, $p_n = 1$ and $p_t = 0$ so that the horizontal and vertical loads are

$$p_H = \sin \varphi + \epsilon \psi \cos \varphi \tag{B1}$$

$$p_V = -(\cos \varphi - \epsilon \psi \sin \varphi) \tag{B2}$$

Also, for the cap solution, the quantity a used in equations (17) is taken as the radius of the cylinder R_L so that the small parameter is given by

$$\epsilon = \frac{pR_L}{Eh}$$

Now, from equation (26) and with the relation $r' = \alpha \cos \varphi$

$$g(\xi) = -\frac{r^2}{2} \tag{B3}$$

Equation (27) is next used to obtain an expression for N_ξ. The constant of integration c is taken to equal zero so that the solution reduces to the linear membrane solution outside of the boundary layer. The stress resultant N_ξ is then given by

$$N_\xi = \frac{r}{2 \sin \varphi} \tag{B4}$$

From equation (29) N_θ is obtained:

$$N_\theta = - \frac{r^2}{2\alpha \sin^2\varphi}\left(\frac{\alpha}{R_\xi} + \epsilon\psi_e{'}\right) + \frac{r}{\sin \varphi} \tag{B5}$$

The boundary-layer solution for the rotation ψ_e is given by equation (37) and in this case becomes

$$\psi_e = dq^{-1/4}(\xi)\exp\left(- \frac{1}{\sqrt{\epsilon}}\left|\int_\xi^{\xi_J} q^{1/2}(\xi)d\xi\right|\right) \tag{B6}$$

where

$$q(\xi) = \frac{2\alpha^2\sin^3\varphi}{r^3}$$

and ξ_J is the value of ξ at the juncture. The displacements can be obtained with the help of equations (38) and (41) and are given by

$$u = - \frac{r^2}{2\alpha \sin^2\varphi}\left(\frac{\alpha}{R_\xi} + \epsilon\psi_e{'}\right) + \frac{r}{\sin \varphi}\left(1 - \frac{\nu}{2}\right) \tag{B7}$$

$$w = f_7(\xi) - \frac{\sqrt{\epsilon}d}{q^{3/4}(\xi_J)} f_6(\xi_J)\exp\left(-\sqrt{\frac{q(\xi_J)}{\epsilon}} \left|\xi_J - \xi\right|\right) + c_1 \tag{B8}$$

where

$$f_7(\xi) = \int_0^\xi \left(\alpha \cos \varphi\psi_L + \frac{\alpha r}{2} + \frac{\nu\alpha r^2}{2R_\xi \sin \varphi} - \alpha\nu r\right)d\xi \tag{B9}$$

Cylinder solution.- For the cylinder solution, $\alpha = 1$, $r = 1$, and $\varphi = \frac{\pi}{2}$. Note also, for this solution, that the coordinate ξ is the ratio of the distance along the cylinder to the radius R_L. The corresponding solutions for stresses and displacements are obtained and, to an accuracy with an error of $O\left(\epsilon^{1/2}\right)$ compared to unity, give

$$N_\xi = c_2 \tag{B10}$$

$$N_\theta = 1 - c_2 \epsilon \psi_e' \qquad (B11)$$

$$u = 1 - c_2 \epsilon \psi_e' - \nu c_2 \qquad (B12)$$

$$w = \left(c_2 - \nu\right)\xi \qquad (B13)$$

$$\psi_e = d_1 \exp\left(-\frac{1}{\sqrt{\epsilon c_2}}\left|\xi - \xi_J\right|\right) \qquad (B14)$$

The constants of integration are now evaluated by requiring that the displacements are continuous and the forces are in equilibrium at the juncture of the cap and cylinder. The general form for the boundary conditions is given by equations (13) and (14), so that the matching conditions are as follows. At the juncture, where $\varphi = \frac{\pi}{2}$, these conditions require that

$$u_{cyl} = u_{cap} \qquad (B15a)$$

$$w_{cyl} = w_{cap} \qquad (B15b)$$

$$\left(\psi N_\xi\right)_{cyl} = \left(\psi N_\xi\right)_{cap} \qquad (B16a)$$

$$N_{\xi_{cyl}} = N_{\xi_{cap}} \qquad (B16b)$$

The subscripts cyl and cap refer to the cylinder and cap. The constant c_1 appears only in equation (B15b), so that the other three conditions can be solved directly for d, c_2, and d_1. Equations (B16a) and (B16b) yield

$$\left.\begin{aligned} c_2 &= \frac{1}{2} \\[2mm] d_1 &= \frac{d}{2^{1/4}\sqrt{\alpha_J}} \end{aligned}\right\} \qquad (B17)$$

Equation (B15a) is then used to get

$$d = -\frac{1}{2^{1/4}R_{\xi,J}}\sqrt{\frac{\alpha_J}{4\epsilon}} \qquad (B18)$$

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where $R_{\xi,J}$ and α_J denote values of R_ξ and α at the juncture. If the stresses are now computed at the cap-cylinder juncture, it is found that N_ξ and N_θ are given by

$$N_\xi = \frac{1}{2} \qquad\qquad N_\theta = 1 - \frac{1}{4R_{\xi,J}} \qquad\qquad (B19)$$

Arrest of a Moving Mass by a Pressurized Spherical Membrane

The fact that masses may be attached to pressurized spherical membranelike structures during deployment of space vehicle packages makes it of interest to calculate the maximum stresses which would occur in the arrest of such masses. The general solutions in this paper offer a convenient means for calculating these stresses. In the present problem it is assumed that, in the deployment operation, no extensional strains are suffered by the folded membranelike material as it unfolds until the spherical shape is completely formed; it is also assumed that the velocity of the moving mass is known (from other considerations) at this instant. From then on membrane strains exist and contribute to the arrest of the moving mass.

For the pressurized spherical membrane (see fig. 8),

$$\left.\begin{array}{c} R_\xi = R_\theta = 1 \qquad \alpha = 1 \qquad \varphi = \xi \qquad r = \sin\xi \qquad \xi_N = \frac{\pi}{2} \\[2em] a = R \qquad \bar{p}_n = p \qquad p_t = 0 \qquad \epsilon = \frac{pR}{Eh} \end{array}\right\} \qquad (B20)$$

From equations (15), (16), and (17)

$$p_V = -(\cos\xi - \epsilon\psi\sin\xi) \qquad\qquad (B21)$$

$$p_H = \sin\xi + \epsilon\psi\cos\xi \qquad\qquad (B22)$$

From equation (26)

$$g(\xi) = -\frac{\sin^2\xi}{2} \qquad\qquad (B23)$$

The edge of the membrane which is attached to the rigid mass $\bar{M}$ will suffer no displacement $\bar{u}$ while the mass is moving (fig. 8). At the same time there will be a vertical edge force $\bar{V}$ because of the deceleration of the mass, so that the appropriate boundary conditions at $\xi = \xi_1$, in dimensionless form, are

$$u\left(\xi_1\right) = 0 \tag{B24}$$

$$N_\xi\left(\sin \xi_1 + \epsilon\psi \cos \xi_1\right) = V \tag{B25}$$

Equation (B23) and equation (27) yield

$$N_\xi = \frac{c + \dfrac{\sin^2\xi}{2}}{\sin^2\xi}(1 - \epsilon\psi \cot \xi) + O(\epsilon) \tag{B26}$$

If this expression for N_ξ is substituted into equation (B25), the terms $O\left(\epsilon^{1/2}\right)$ cancel identically so that the following relation is obtained:

$$V = \frac{c}{\sin \xi_1} + \frac{1}{2} \sin \xi_1 + O(\epsilon) \tag{B27}$$

Next, an expression for N_θ is obtained from equation (29):

$$N_\theta = 1 - \frac{\left(c + \dfrac{1}{2} \sin^2\xi\right)}{\sin^2\xi}\left(1 - \sqrt{\epsilon}dq^{1/4} \exp \frac{1}{-\sqrt{\epsilon}} \left|\int_{\xi_1}^{\xi} q^{1/2}d\xi\right|\right) \tag{B28}$$

where

$$q = \frac{\sin^2\xi}{c + \dfrac{\sin^2\xi}{2}}$$

If the boundary condition (eq. (B24)) is now applied and terms of order $O\left(\epsilon^{1/2}\right)$ are neglected since they are small compared to unity, the following equation is obtained:

$$u\left(\xi_1\right) = -\left(\frac{c}{\sin^2\xi_1} + \frac{1}{2}\right)\left[1 + v - \sqrt{\epsilon}dq^{1/4}\left(\xi_1\right)\right] = 0 \tag{B29}$$

Equation (B29) can be used to calculate the constant d:

$$d = \frac{1 + v}{\sqrt{\epsilon}}\left(\frac{c + \dfrac{1}{2} \sin^2\xi_1}{\sin^2\xi_1}\right)^{1/4} \tag{B30}$$

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An expression for the displacement w at $\xi = \xi_1$ will be needed. First, note that

$$
\left.\begin{aligned}
S(\xi) &= -1 \\[6pt]
f_3(\xi) &= (1 + \nu)\log\left|\tan\frac{\xi}{2}\right| \\[6pt]
f_4(\xi) &= \frac{1 - \nu}{2}\cos\xi
\end{aligned}\right\}
\tag{B31}
$$

Equation (41) evaluated at $\xi = \xi_1$ gives

$$
w\left(\xi_1\right) = A\left(\xi_1\right)V - B\left(\xi_1\right)
\tag{B32a}
$$

or

$$
V = \frac{w_1}{A} + \frac{B}{A}
\tag{B32b}
$$

where

$$
A\left(\xi_1\right) = (1 + \nu)\left(\sin\xi_1\log\left|\tan\frac{\xi_1}{2}\right| - \cot\xi_1\right)
$$

$$
B\left(\xi_1\right) = \frac{1}{2}\left[(1 - \nu)\cos\xi_1 + (1 + \nu)\sin^2\xi_1\log\left|\tan\frac{\xi_1}{2}\right|\right]
$$

Equations (B32) provide a relation between the as yet unknown V and the displacement w at $\xi = \xi_1$. Another relation is needed to determine the maximum deflection $w_{max}\left(\xi_1\right)$, which occurs during the arrest of the mass. This relation can be obtained by equating the work done by the edge resultant V to the kinetic energy of the moving mass at the instant the spherical shape is formed. When written in terms of physical quantities this relation is

$$
-2\pi R \sin\xi_1 \int_0^{\bar{w}_{1,max}} \bar{V}\, d\bar{w}_1 = \frac{1}{2}\bar{M}\bar{v}^2
\tag{B33}
$$

Equation (B33) can be written in terms of nondimensional quantities:

$$
-\int_0^{w_{1,max}} V\, dw_1 = k_E
\tag{B34}
$$

where the nondimensional parameter k_E $\left(k_E = \dfrac{1}{4\pi \sin \xi_1} \dfrac{\overline{M}\overline{v}^2}{\epsilon \rho R^3}\right)$ is seen to be proportional to the initial kinetic energy of the mass. The integration in equation (B34) can be performed directly if the relation between V and w_1 given by equations (B32) is used. The following result is then obtained:

$$\frac{w_{1,\max}^2}{A(\xi_1)} + \frac{B(\xi_1)}{A(\xi_1)} w_{1,\max} = -k_E \qquad (B35)$$

The maximum deflection $w_{1,\max}$ can be determined from equation (B35). Equations (B27), (B30), and (B32) are associated with this deflection and can be used to calculate c, d, and V, and therefore the maximum stresses. Maximum values of N_ξ, N_θ, w, and ψ_e at $\xi = \xi_1$ $\left(\xi_1 = 15^\circ\right)$ are plotted in figure 7 against the parameter k_E.

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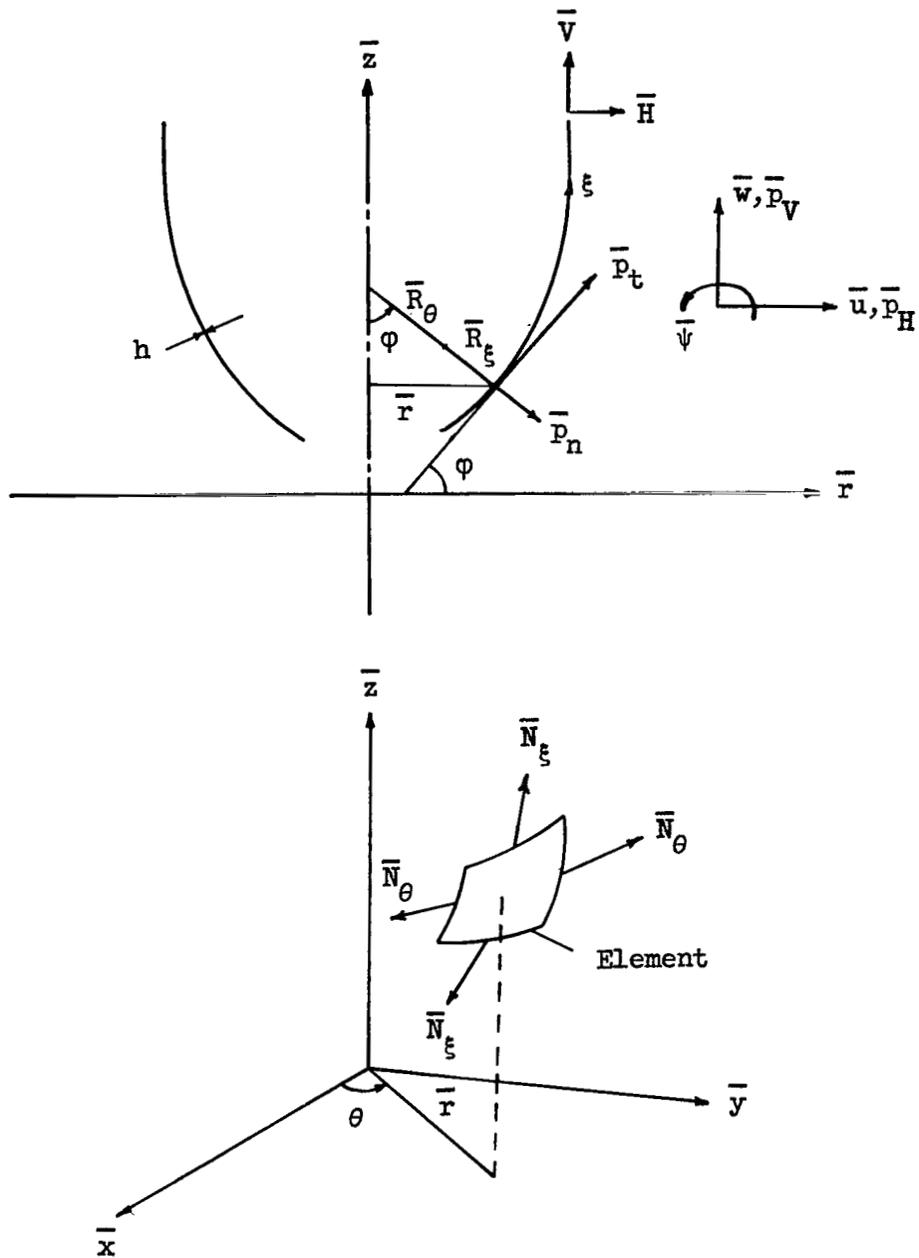
Figure 1.- Geometry and notation.
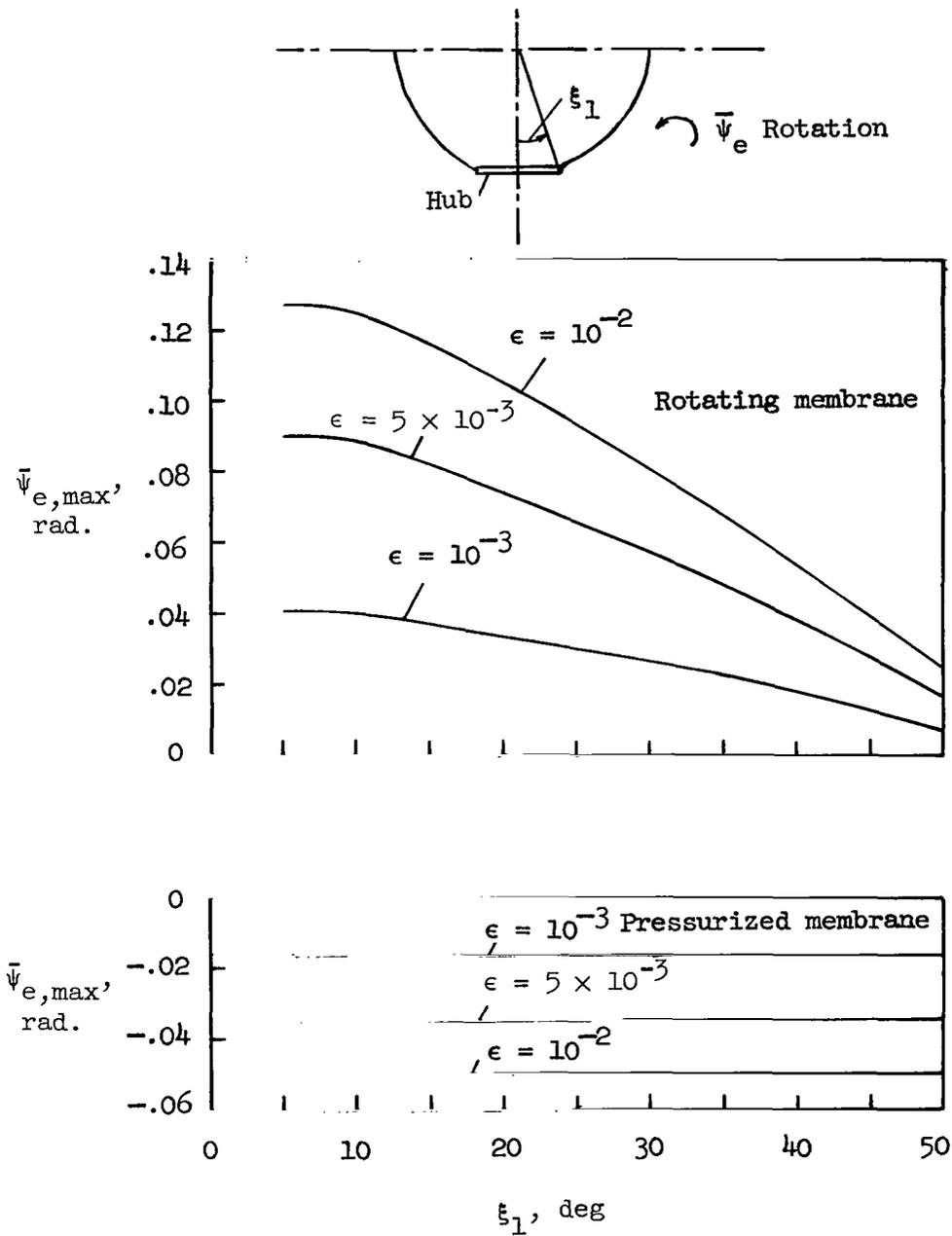
Figure 2.- Maximum rotation $\overline{\psi}_{e,max}$ at hub $\xi = \xi_1$ $(\nu = 0.3)$ for rotating and pressurized spherical membranes.
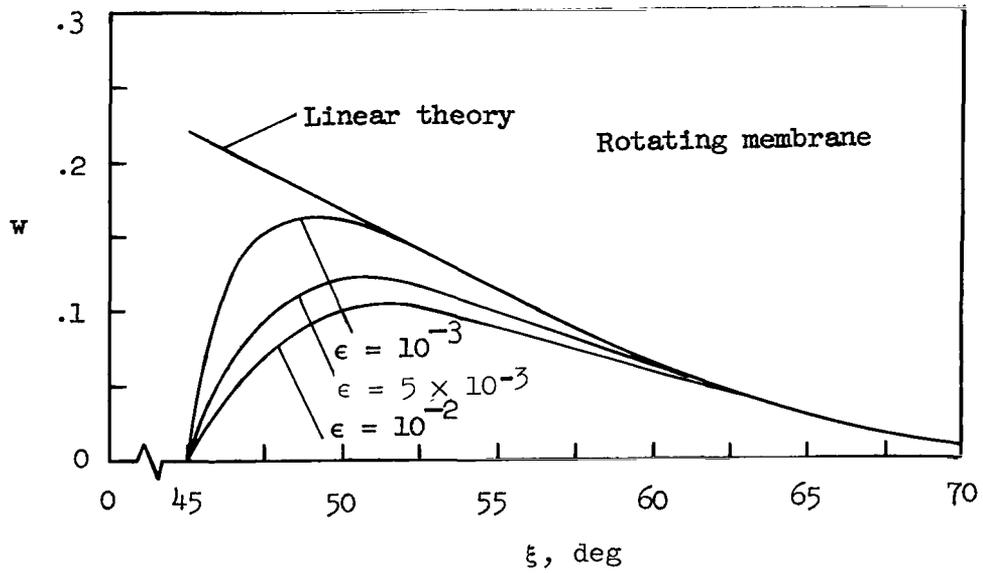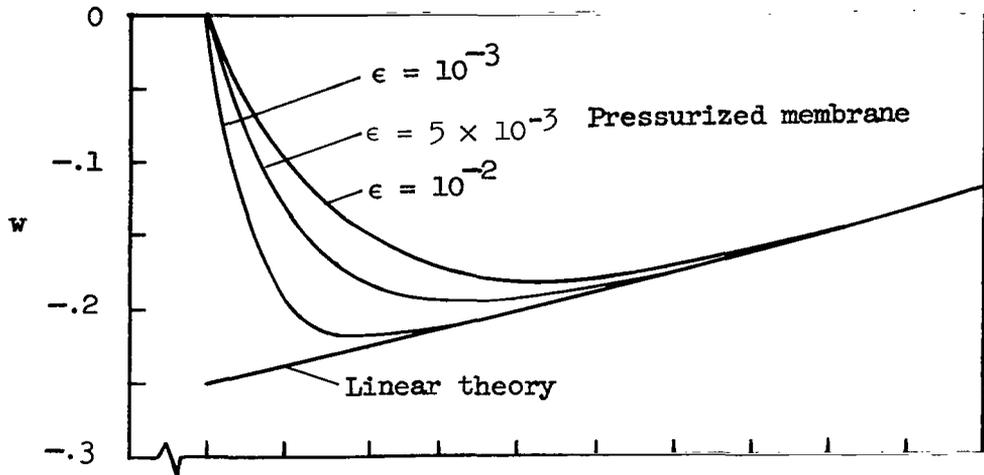
Figure 3.- Displacement w near hub $\left(\xi_1 = 45^\circ\right)$ for pressurized and rotating spherical membranes $(\nu = 0.3)$.
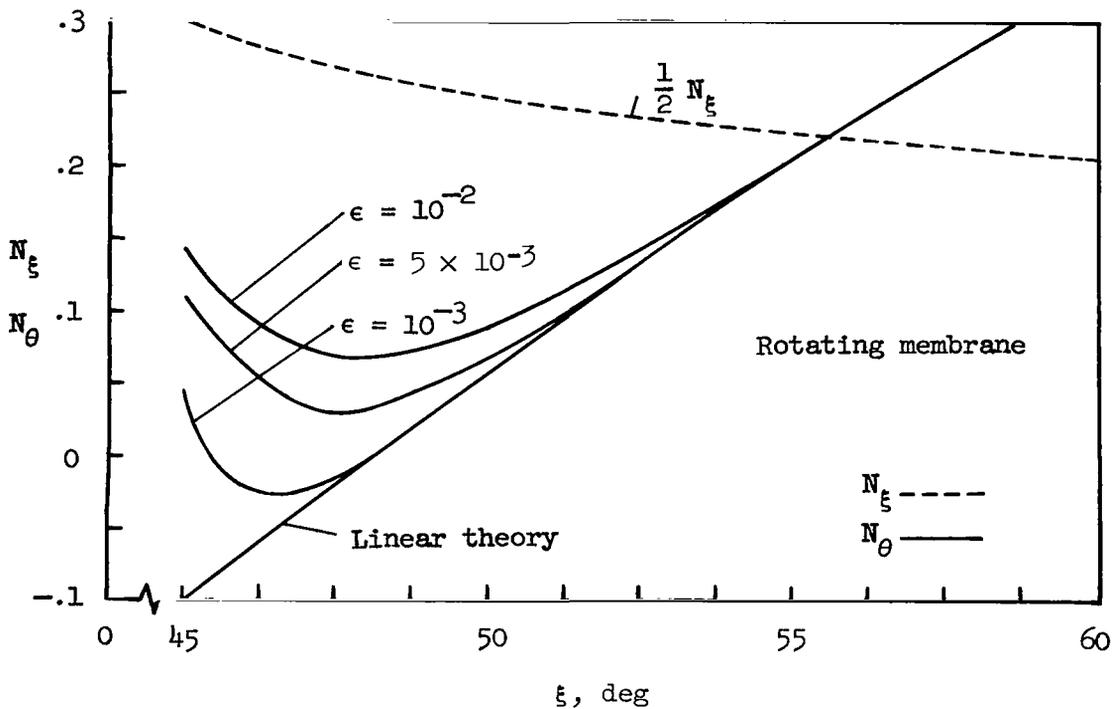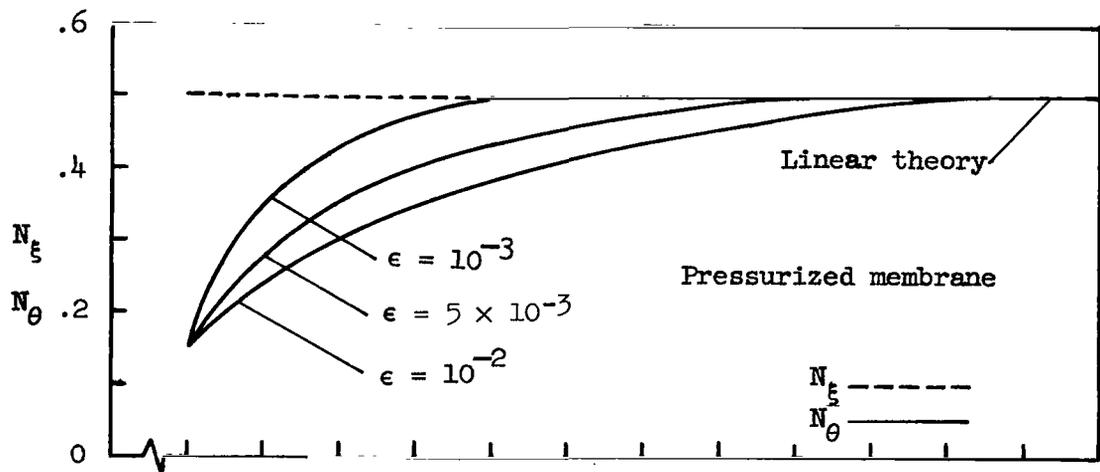
Figure 4.- Stresses N_ξ and N_θ near hub $\left(\xi_1 = 45^\circ\right)$ for pressurized and rotating spherical membranes $(\nu = 0.3)$.
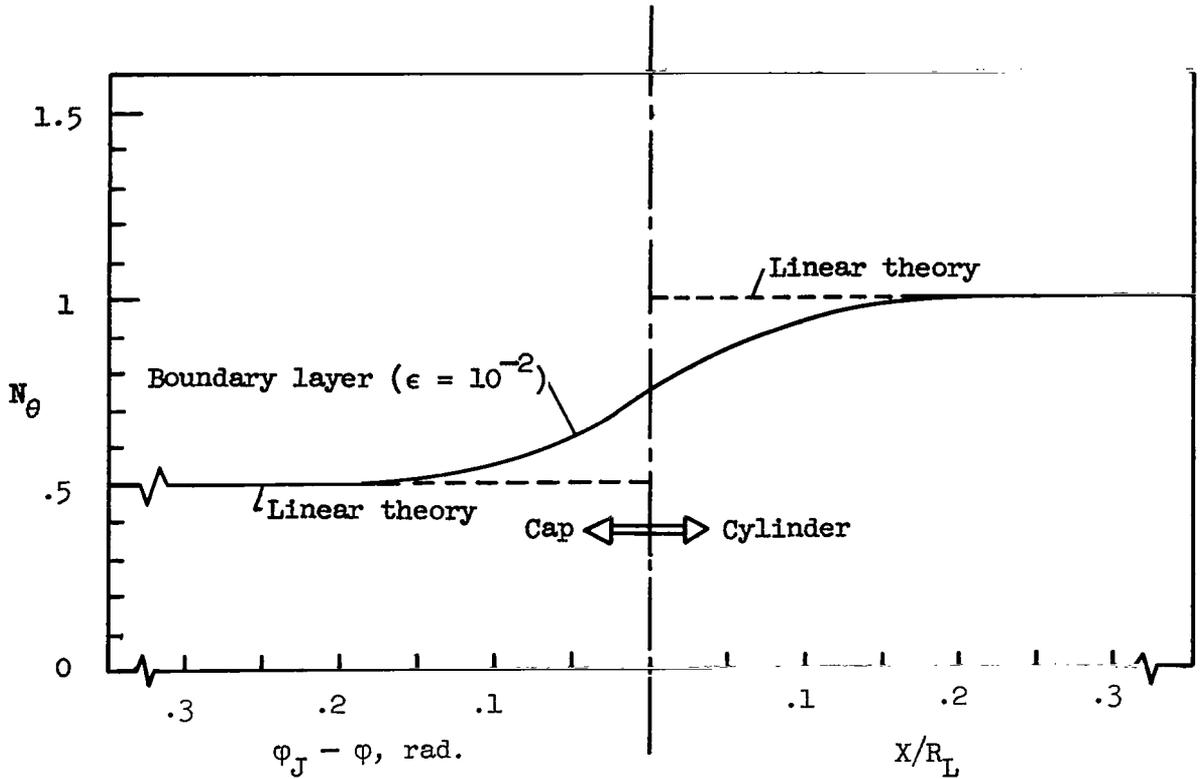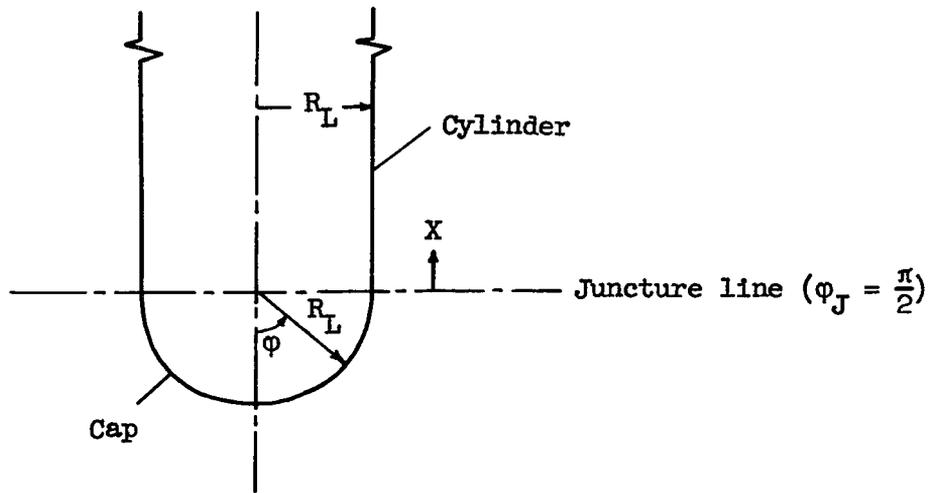
Figure 5.- Stress N_θ near juncture of a cylindrical pressure vessel and a hemispherical cap.
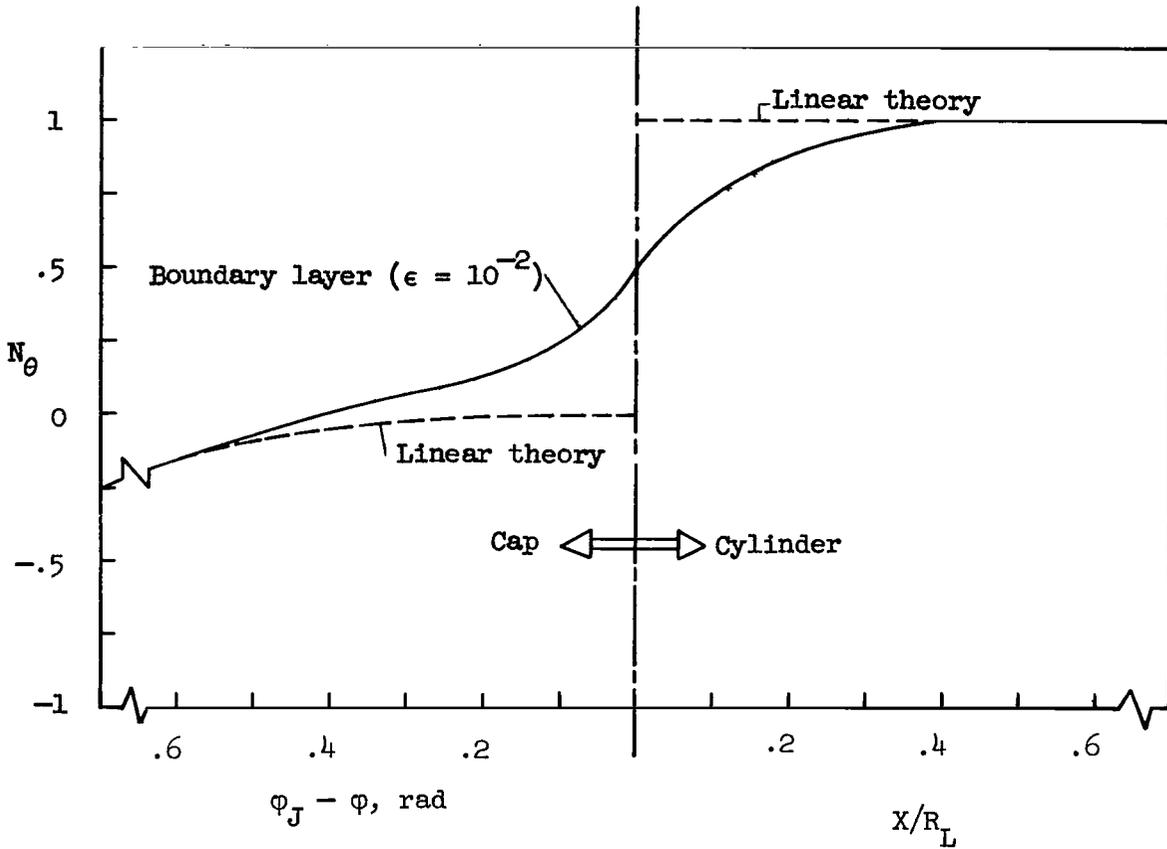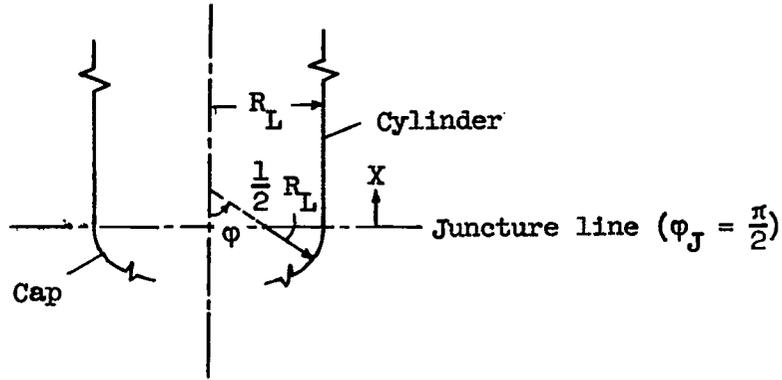
Figure 6.- Stress N_θ near juncture of a cylindrical pressure vessel and a torispherical cap.
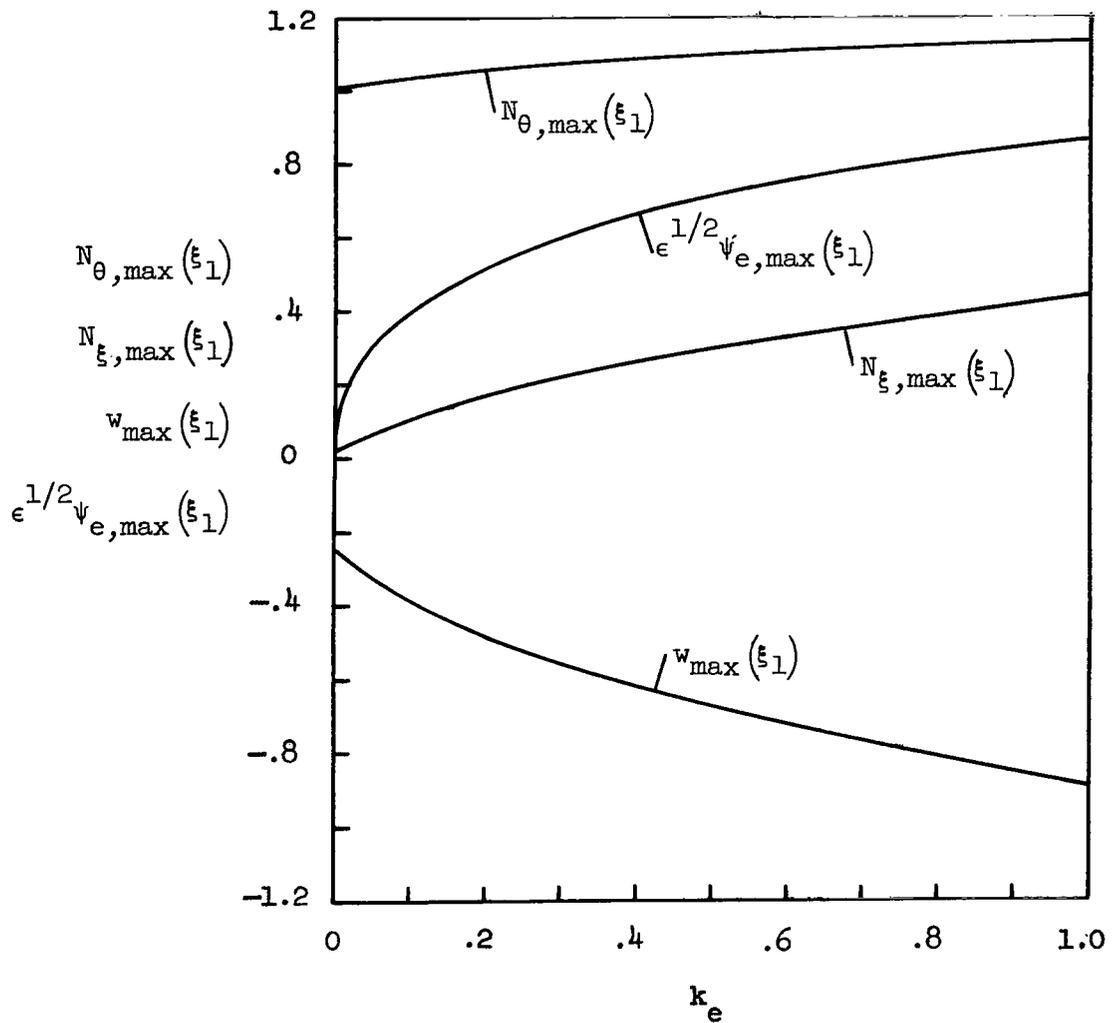
Figure 7.- Maximum stresses $N_{\theta,max}(\xi_1)$ and $N_{\xi,max}(\xi_1)$, displacement $w_{max}(\xi_1)$, and rotation quantity $\epsilon^{1/2}\psi_{e,max}(\xi_1)$ plotted against kinetic energy parameter k_E $\left(\xi_1 = 15^\circ;\ \nu = 0.3\right)$.
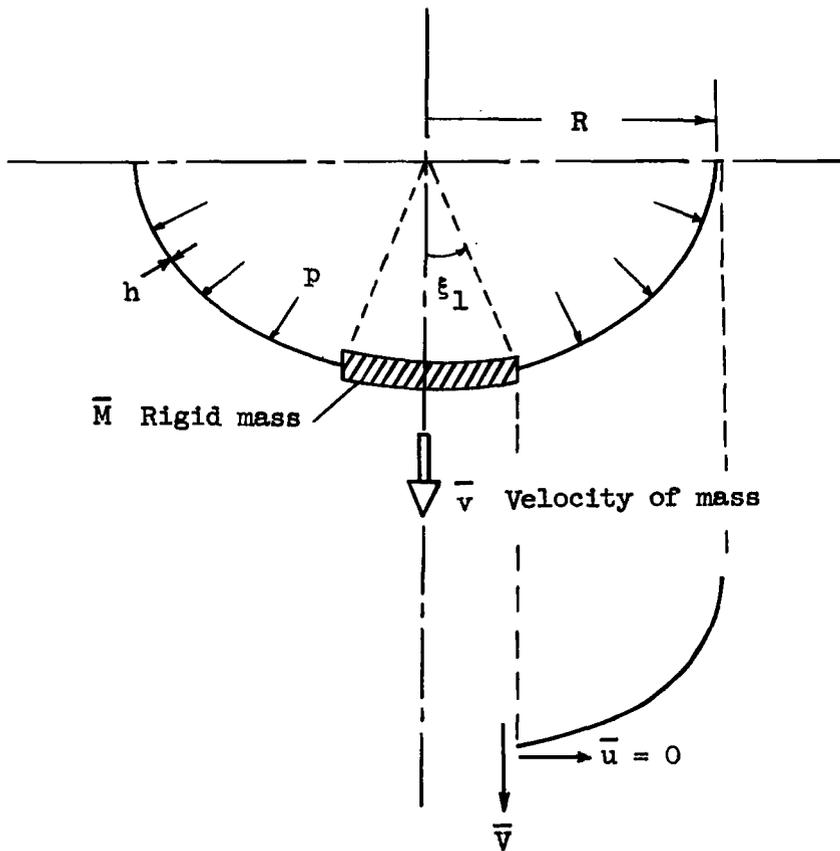
Figure 8.- Notation for arrest of mass $\overline{M}$ by pressurized spherical membrane.

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