QUARTERLY PROGRESS REPORT

For Period

15 September through 31 December 1965

INVESTIGATION OF OPTIMIZATION OF ATTITUDE CONTROL SYSTEMS

For

Jet Propulsion Laboratory

4800 Oak Grove Drive

Pasadena, California

on

Contract No. 950670

School of Electrical Engineering

Purdue University

Lafayette, Indiana 47907

J. Y. S. LUH, Principal Investigator

This work was performed for the Jet Propulsion Laboratory, California Institute of Technology, sponsored by the National Aeronautics and Space Administration under Contract NAS7-100.

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PART A

GENERAL DISCUSSION

1. INTRODUCTION

This is the sixth quarterly report submitted in accordance with the provisions of Contract No. 950670, "Investigation of Optimization of Attitude Control Systems." It covers the period September 15, 1965 through December 31, 1965.

This report is in three parts. The first part summarizes the progress during the reporting period. The technical discussions are given in Parts B and C, in which the plans of future work are also included.

2. PROGRESS DURING REPORTING PERIOD

2.1 Coordination Meetings

A series of meetings were held at Purdue University on October 26, 1965. Those present at all the meetings included:

J. C. Nicklas, and A. E. Cherniack of Jet Propulsion Laboratory, and J. Y. S. Luh of Purdue University. J. C. Hancock, D. R. Anderson, T. J. Williams, G. E. O'Connor, and J. S. Shafran of Purdue University were present at some of the meetings.

The discussions brought out the following:

(a) Both J.P.L. and Purdue should seek the research areas which are of interest to both parties. Tentatively, three possible research areas were discussed; viz., antenna pointing, sensitive analysis, and soft landing problems.

(b) The third (also the last) annual report is due on or before July 30, 1967. The research fund for the third year period, however, is not guaranteed.

2.2 Technical Progress

In connection with the soft landing problems, an optimal control problem in bounded phase-coordinate and bounded control processes was studied. The general theory for a linear autonomous system was developed. Based on the theory, a method of determining the optimal control was derived. The complete technical discussion together with the future research plan is presented in Part B. This method was then applied successfully to the "time-optimal control of an unstable booster with actuator position and rate limits." The solution in detail is also included in Part B. The results, when evaluated with numerical data, are in agreement with those obtained by Friedland [20] and Tookey [21] who solved the problem by other methods including computer simulation.

The control problem of antenna pointing was also investigated. The problem was formulated as a stochastic optimal control problem in which the probability that the antenna pointing at the desired direction within an allowable tolerance is maximized. This scheme, when fully developed, has a potential application to the programmed pointing system to improve its accuracy. The technical discussion and the plan of future work are given in Part C.

The theoretical investigation of the sensitivity analysis was just begun. No significant results were achieved up to date and hence no detailed discussion will be presented in this report. The method of approach follows the idea given in the First Annual Report submitted to JPL in January 1965, pp. 79-87. Instead of using specific physical plants (such as the two plants chosen in that report), a general plant model is utilized. The control, however, will be limited to the simple linear feed-back type and possibly with an additional cubic term. The choice of such simple form of control function is motivated by the requirement of hardware simplicity.

3. PROFESSIONAL CONTRIBUTORS

Professional personnel contributing to progress during the reporting period are as follows:

J. Y. S. Luh, Principal Investigator

G. E. O'Connor, Staff Researcher

J. S. Shafran, Staff Researcher

PART B

OPTIMAL CONTROL IN BOUNDED PHASE-COORDINATE PROCESS

1. INTRODUCTION

Bounded phase-coordinate control problems arise naturally in many practical applications. The non-crush landing of a spacecraft is a trivial example. In many flight vehicles, engine deflection, angle of attack and bending moment contribute to the phase-coordinate constraints. For instance, if the controller input is engine gimbal rate, the engine displacement may be considered as a phase-coordinate of the dynamical system. Normally, the allowable engine displacements are small; an efficient use of the available control input often demands operating on the engine displacement limit. Unfortunately, such intuition is not always correct. The efficient use of the control input implies not only operating at the displacement limit, but also considering the displacement limit explicitly in the over-all design of the controller. The term "efficient use" can be specifically defined by a given minimization criterion, and problems of this type are called optimal control problems with phase-coordinate inequality constraints.

At the present time, available methods of solving this type of problem are time consuming, thus preventing the possibility of on-line operation using currently available facilities. This research will attempt to determine the optimal control as an explicit time function and thereby eliminate the difficulty of excessive computing time.

In the following sections, a discussion on the research problem is presented. Section 2 gives a brief survey on the known results for the bounded phase-coordinate control problems. It also motivates the problem for this research. Section 3 defines the problem that is under investigation. Section 4 discusses the method of solving the problem. In this Section 4, a summary is made for the general theory that is already known. Based on an analysis of the known theory, the problem is reformulated in such a manner as to lead to a method that determines the optimal controls as explicit time functions. This method is then applied to the time-optimal control of an unstable booster. A brief summary of the final results for this particular example is then presented. The derivations and discussions are given in detail in the Appendix. Section 5 outlines the plan of future work in a logical and sequential order.

2. HISTORICAL DEVELOPMENT

In recent years, considerable attention has been given to optimizing systems for which the state variables are bounded. The problem is not really new. As indicated by Bolza [1, pp. 125-126], Weierstrass formulated the analogous problems of calculus of variations with phase-coordinate inequality constraints in 1882, and developed the "corner" conditions for the two-dimensional Lagrange problems. The "corner" conditions deal with the discontinuities of the solution of the analogous problems. According to Bliss [2, pp. 43], the necessary and sufficient conditions for a minimum solution were studied subsequently by Carathéodory, Bolza, Dresden, Graves, Reid, Smiley, Bliss, and Underhill. Most of the studies were completed between 1904 and 1937.

In 1961, Berkovitz [3] reduced the general control problem with constraints to a problem of calculus of variations. In his discussion, a translation of necessary conditions for the problem of calculus of variations into the necessary conditions for the optimal control was established, including the application of Pontryagin's maximum principle [4]. His results, however, are not applicable to control problems with phase-coordinate inequality constraints that do not explicitly involve the control variable. In an independent study, Gamkrelidze [5, also Chapter VI of 4] treated the latter problem based entirely on the maximum principle. Berkovitz [6] then showed that Gamkrelidze's results could be achieved by solving the relevant problem of calculus of variations. Dreyfus [7] studied the same problem by means of the dynamic programming formulation. His results are in agreement with that of Berkovitz [8]. Among all the studies, sufficiency conditions were virtually ignored. For practical applications, even when solutions do exist, the necessary conditions derived by various authors are difficult to apply.

During 1961-1962, Chang derived a simpler necessary condition for a more restricted class of problems [9], and existence theorems based on the extension of Ascoli's Theorem [10]. For linear time-optimal control systems with a convex restraint set, the necessary condition is also the sufficient condition. An elegant proof of the necessity of the condition can be reduced from Neustadt's recent work [11] while a rigorous proof of the sufficiency is given by Russell [12, pp.26-30]. This condition is an improvement on Gamkrelidze's results. It establishes the fact that the normal vector appearing in the modified adjoint differential equation is always outward with respect to the set of attainability, and hence the necessary and sufficient condition is relatively easy to apply.

As to the computational aspects of the problem, there are essentially two classes of methods. One class is the direct method which includes the method of the gradient, steepest-descent or their equivalent. The direct method was studied by Dreyfus [7], Denham [13, 14] and Bryson [15] using the necessary conditions of the optimal control, and by Paiewonsky, et al. [16] using conditions both of the optimal control and from the calculus of variations. The other class is the indirect method which was discussed by Kahne [17], Ho and Brentani [18], and Nagata, et al. [19]. Because of the nature of the problem, each computational procedure requires either an iterative solution or a simulation on a sizable computer. Since a new computation is required for each different initial state, the possibility of on-line operation using currently available facilities is out of the question.

An ideal approach is to synthesize a so-called closed-loop optimal controller such that the control input is a function of the current state. This problem, however, is too difficult to solve. An alternative approach is to obtain the so-called open-loop optimal control as an explicit time function for each initial state. This problem, although not so difficult as the closed-loop optimal control problem, is complicated enough that no published results are known. In this research, we shall attempt to develop a new method to solve the optimal control problem with a bounded phase-coordinate. The main effort will be devoted to deriving an algorithm for expressing the open-loop control law as a time function. Once this goal is achieved, the closed-loop control law will then be attempted.

3. PROBLEM STATEMENT

The general problem of interest is stated as follows: Given a linear control process as described by the differential system

$$\dot{x} = A(t)\, x + B(t)\, u(t) \tag{1}$$

where x and $u(t)$ are the n-dimensional state vector and m-dimensional control vector, respectively; $A(t)$ and $B(t)$ are n by n and n by m matrices of measurable functions for t in some interval $[t_o, t_1]$. Let G be a closed convex subset of E^n, and Ω be a compact convex subset of E^m. Let the cost functional of control be

$$C(u) = g[x(t_1)] + \int_{t_o}^{t_1} [f^{\circ}(x,t) + h^{\circ}(u,t)]\, dt \tag{2}$$

where $f^{\circ}(x,t)$ and $h^{\circ}(u,t)$ are real-valued, non-negative, convex and continuously differentiable functions with respect to t, while g is convex and differentiable. The general problem of optimal control of bounded phase-coordinate systems is to choose an admissible control $u(t) \subset \Omega$ on the time interval $[t_o, t_1]$ which steers the system (1) from its given initial state, $x(t_o) = x_o$ at time t_o, to a point target in G at time t_1, such that the response $x(t) \subset G$ for all $t \in [t_o, t_1]$, and the cost functional is a minimum.

The general problem described above is difficult to solve. Instead, solutions of more restricted classes of problems will be attempted. As a first step, we will seek the method of obtaining analytical expression of the time optimal controller for autonomous processes. The processes are assumed normal so that the admissible extremal control is unique. Once this is completed, optimal control processes with integral quadratic cost criteria will be investigated.

4. TECHNICAL APPROACH

4.1 Background Results

The following is an outline of the technical approach of our first attempt on a time optimal controller for autonomous processes with a bounded phase-coordinate. Gamkrelidze [4, 5] and others have given necessary conditions that extremal controls must satisfy. These necessary conditions imply that an extremal control corresponds to a solution of a set of adjoint equations. The adjoint solution has certain jump discontinuities allowed, and hence depends on a number of parameters representing:

(a) the magnitudes of the number of jumps that appear in the adjoint solution, and

(b) the time lengths of the arcs of the corresponding trajectory which lie on ∂G, the boundary of the phase-coordinate restraint set G. The discontinuities are allowed at points where the trajectory (corresponding to an extremal control) enters upon or exits from an arc on ∂G.

These are the general results. They do not, however, indicate specifically at which points the trajectory enters upon the arc, and when the trajectory must exit from it. This research will attempt to answer these questions. The following analysis will lead to a method that determines extremal controls as explicit time functions. Then these functions can be represented in terms of adjoint solutions. A sufficiency condition given by Russell [12] shows that the solutions so obtained are optimal controls.

4.2 Analysis

For a linear autonomous process, the calculation of trajectories by the "backing out of the target" procedure is valid. In so doing, a set of attainability $K(t)$ can be found for every fixed time t. If t is small enough such that each set of attainability $K(t)$ is within the interior of G, then it is known that $K(t)$ is compact, convex and continuous in t. Moreover the transversality condition applies at $\partial K(t)$, the boundary of $K(t)$; and for each point on the boundary, there is a corresponding unique and admissible extremal control. When t is large, some segments of $K(t)$ may coincide with ∂G. Since G is convex by hypothesis, then $K(t)$ is again convex; and Ressell [12, pp. 22-53] showed that:

(a) at the boundary of $K(t)$, the transversality condition is still valid if the corresponding adjoint system is modified, and,

(b) corresponding to each point on the boundary, there is a unique and admissible extremal control.

Thus, by (a), for every unit vector η in E^n there is a state vector x corresponding to a point on $\partial K(t)$ for a fixed t, such that the projection P of x onto η is a maximum. By (a) and (b), the corresponding unique and admissible extremal control, which maximizes the projection, steers the linear, normal, autonomous process from the origin to a furthest point x in a fixed time t.

This is equivalent to the case that, with the time sense reversed, the same extremal control will steer the system from x to the origin in a fixed time t where t is minimal. Russell's sufficiency condition [12] shows that the unit vector η is the adjoint vector at time t, and the extremal control so obtained is the time optimal control.

Thus, the problem of determining a time optimal controller is now reduced to obtaining an admissible extremal control that maximizes the projection of a state vector x at a fixed time t (in the sense of "backing out of the target") onto a unit vector. In so doing, an extremal control can be found for every fixed finite time t and for every unit vector. This method of approach makes it possible to determine the extremal controls as explicit time functions. Once this is completed, the state vector $x(t)$ can be computed from the variation of parameters formula with the corresponding extremal control.

As a last step, take the limit of $x(t)$ as time t appraoches infinity. If all the state variables approach $\pm\infty$ as t approaches ∞, then the controlled autonomous process is completely controllable. If, however, some state variables approach finite limiting values, then these $x(t)$ form an uncontrollable region. It is quite natural that a linear process is completely controllable if the phase-coordinate is not bounded; but an uncontrollable region may exist for the same process if the phase-coordinate is bounded.

4.3 Summary of Preliminary Results

The above method of approach was applied successfully to "the time optimal control of an unstable booster with actuator position and rate limits". The optimal control function and the unstable region were obtained. The results, when evaluated with numerical data, are in agreement with those obtained by Friedland [20] and Toohey [21] who solved the problem by other methods including computer simulation. The derivation of the optimal control function and the unstable region via the method outlined in this report is given in the Appendix as an illustrative example.

5. PLAN OF FUTURE WORK

A preliminary study of a harmonic oscillator with bounded amplitude and bounded rate control is required. This study, although it does not solve the proposed problem, will give the information on the form of the time-optimal control function for a system having a pair of purely imaginary characteristic roots. It is anticipated that this problem will be more complicated in comparison with the unstable booster problem discussed in the Appendix. Because of the nature of oscillation, the control variable will enter upon and exit from its bound as often as the time duration permits.

The next step will be a study of an underdamped oscillatory plant with bounded amplitude and rate control. The investigation will yield the nature of the time-optimal control function for a process with a pair of complex conjugate characteristic roots.

At this point, the research can be divided into three parallel paths: (a) extend the study to the same processes but with integral quadratic cost criteria, (b) independently simulate the same problems on a computer and compare the data so obtained against those from analytical results, and (c) study the same time-optimal control problems analytically except that one of the state variables be also bounded (so far the bound is only applied to the augmented state variable, viz. $u = x_3$). Gamkrelidze's [4, 5] necessary conditions imply that the adjoint solution has certain jump discontinuities. However, his results do not indicate how many discontinuities will occur. In our preliminary study with only one bounded state variable (see Appendix for the unstable booster), there is at most one discontinuity which can be arranged at either the beginning or the end of the time interval. This is also true for an harmonic oscillator (a brief investigation on this problem has been completed). It is therefore conjectured that the number of jump discontinuities in the adjoint solution is the same as the number of bounded state variables. This conjective remains to be shown in the above study (part (c)).

Next, the investigation of a bounded phase-coordinate problem having one real and a pair of complex conjugate characteristic roots will be started. It is intended to develop an algorithm for the time-optimal control problem first, and then an algorithm for the problems with integral quadratic cost criteria. These algorithms will be programmed on a computer, and the results evaluated.

The simulation will again be carried out in the following order: (a) construct analog simulation of plant and controllers, (b) develop block diagrams of controllers suitable for future mechanization, (c) develop simulation, analog and/or digital, suitable for testing of practical control systems, (d) compare with the results from analytical expressions, and(e) test various ideas for simplifying and approximating the controller.

Finally, the same steps of investigation will be applied to the same class of control problems for linear time-varying processes. If data are available for practical systems, these systems will first be approximated by third-order systems, then computed and simulated by the methods developed in this research. A careful check of these results will determine the relative merit of this research.

6. APPENDIX - OPTIMAL CONTROL OF AN UNSTABLE BOOSTER

Friedland [20] and Toohey [21] studied an optimal autopilot design problem of an unstable booster with actuator position and rate limits. Their simplified plant transfer function consisted of three poles in the frequency domain: one at origin and two on the real axis with equal magnitude but opposite signs. They simplified the problem further by cancelling the pole at origin through physical design. Essentially, the simplified and normalized unstable booster is described by a second order differential system

$$\dot{\hat{x}} = \hat{A}\,\hat{x} + \hat{b}\,u(t) \tag{3}$$

where $\hat{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ \dot{x}_1 \end{bmatrix}$, $\hat{A} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $\hat{b} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

The problem is:

(a) to determine the maximum controllable region (in E^2) in which every point can be steered to the origin by a scalar control $u(t)$ subject to the constraints $|u(t)| \leq 1$ and $|\dot{u}(t)| \leq D$ on $[0, t_1]$, and

(b) to find a time-optimal control function for each initial state in the controllable region.

Friedland and Toohey solved the problem by other methods including computer simulation. In the following, the same problem will be solved by the method outlined in the preceding section.

Booster Problem as Bounded Phase-coordinate Problem

First, augment the system by defining $x_3 = u(t)$, then the augmented system becomes

$$\dot{x} = A\,x + b\,v(t) \tag{4}$$

where

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ \dot{x}_1 \\ u(t) \end{bmatrix},\quad A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix},\quad b = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \text{ and } v(t) = \dot{u}(t).$$

This becomes a bounded phase-coordinate problem (in the sense $|x_3|' = |u| \leq 1$) in which the scalar variable $v(t)$ is required, subject to the constraint $|v(t)| \leq D$ on $[0, t_1]$, to steer system (4) from an initial state $x(0) = x_o$ to $x(t^*) = 0$ where $t^* = \min. t_1$.

Assume that this control process is possible; then proceed by the method of "backing out of the target $x = 0$", and write the system (4) with time sense reversed (by defining $\tau = -t$),

$$^{dx}/_{d\tau} = - A\ x(\tau) - b\ v(\tau) \tag{5}$$

with $x(0) = 0$. By the variation of parameters formula, the system (5) has a solution

$$x(\tau) = \begin{bmatrix} \int_o^\tau [1 - \cosh(\tau - s)]\ v(s)\ ds \\ \int_o^\tau \sinh\ (\tau - s)\ v(s)\ ds \\ -\int_o^\tau v(s)\ ds \end{bmatrix} \tag{6}$$

where $|v(s)| \leq D$ is admissible on $[0, \tau]$. The adjoint system for the system (5) is

$$^{d\psi}/_{d\tau} = -(-A)'\ \psi(\tau) = A'\ \psi(\tau)$$

in which A' = transpose of A. Gankrelidze [4, 5] showed that, in order to represent the extremal v as a multiple of the signum of an adjoint solution for the bounded phase-coordinate control problem, the adjoint system must be modified. Thus a "total adjoint vector" $p(\tau)$ must satisfy the relation

$$^{dp}/_{d\tau} = \begin{cases} A'\ p(\tau), & \text{if } |x_3(\tau)| < 1 \\ \tilde{A}'\ p(\tau), & \text{if } |x_3(\tau)| = 1 \end{cases} \tag{7}$$

in which

$$\tilde{A} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} .$$

In so doing, the necessary conditions for v to be extremal can be expressed as

$$v(\tau) = D \operatorname{sgn}[p(\tau)\,(-b)]$$

or $$-v(\tau) = D \operatorname{sgn}[p_3(\tau)] \tag{8}$$

where:

(a) $p(\tau)$ satisfies the system (7),

(b) $p_3(\tau) = 0$ if $|x_3(\tau)| = 1$,

(c) $p(\tau)$ is allowed certain jump discontinuities at endpoints of intervals where $|x_3(\tau)| = 1$ (for this problem, p_1 and p_2 are required to be continuous and jumps can occur only in p_3 since only x_3 is restrained), and

(d)
$$\operatorname{sgn} p_3 = \begin{cases} +1, & \text{if } p_3 > 0 \\ 0, & \text{if } p_3 = 0 \\ -1, & \text{if } p_3 < 0. \end{cases} \tag{9}$$

Thus, the solution of the system (7) can be written as

$$\begin{aligned} p_1(\tau) &= p_1(0) \cosh \tau + p_2(0) \sinh \tau, \\ p_2(\tau) &= p_1(0) \sinh \tau + p_2(0) \cosh \tau, \\ p_3(\tau) &= \begin{cases} p_1(0) \cosh \tau + p_2(0) \sinh \tau + k, & \text{if } |x_3(\tau)| < 1, \\ 0 & \text{, if } |x_3(\tau)| = 1, \end{cases} \end{aligned} \tag{10}$$

where the value of constant k in $p_3(\tau)$ depends upon the interval in which $|x_3(\tau)| < 1$, and upon $p_1(0)$ and $p_2(0)$.

The Extremal Controls

To determine extremal controls as explicit time functions, form the projection P as defined in the preceding section. Let the unit adjoint vector at time τ be

$$\eta = \begin{cases} \cos\theta \cos\phi \\ \sin\theta \cos\phi, \qquad -\pi \le \theta,\ \phi \le \pi, \\ \sin\theta \end{cases}$$

then, by equations (6) and the definition of $\underline{P}$,

$$P = \int_0^\tau g(s;\ \tau,\ \theta,\ \phi)\, v(s)\, ds$$

in which

$$g(s;\ \tau,\ \theta,\ \phi) = \cos\phi\,[\cos\theta - \cos\theta \cosh(\tau - s) + \sin\theta \sinh(\tau - s)] - \sin\phi, \tag{11}$$

and $|v(s)| \leq D$ is admissible on $[0, \tau]$. By the transversality condition at $x(\tau)$ on $\partial K(\tau)$, $v(s)$ is extremal on $[0,\tau]$ if it maximizes P. By equation (8), the only possible values for $v(s)$ are $\pm D$ and zero. The condition $|x_3| = |u| \leq 1$ determines the choice of either $\pm D$ or zero for $v(s)$ for the following reason. When $|x_3| < 1$, the system (4) is normal, and hence the value of v can only be either +D or -D. When $|x_3| = 1$, by equations (8), (9) and (10) the value of v is zero, which implies that x_3 must stay on its bound. This conclusion is in agreement with Chang's statement [22] that if the system is time-optimally controlled, then either u is extemal or $^{du}/d\tau$ is extremal.

For this particular problem, the function $g(s; \tau, \theta, \phi)$, which is given by equation (11), has a property that

$$g(s; \tau, \theta, \phi) = g(s; \tau, \pi+\theta, \pi-\phi);$$

hence, it suffices to consider only half of the range of θ and half of range of ϕ. For the convenience of discussion, choose $-\pi \leq \theta \leq 0$ and $-\pi/2 < \phi < \pi/2$. Then P becomes

$$P = \cos \phi \int_0^\tau f(s; \tau, \theta, \phi) \, [-v(s)] \, ds$$

where

$$f(s; \tau, \theta, \phi) = \cos \theta \cosh(\tau-s) - \cos \theta - \sin \theta \sinh(\tau-s) + \tan \phi. \tag{12}$$

(a) For $^{-3\pi}/4 \leq \theta \leq 0$, $^{-\pi}/2 < \phi < {}^{\pi}/2$, and $0 \leq s < \tau < \infty$, the derivative $^{df}/ds < 0$ (=0 only when $s = \tau$ and $\theta = 0$). Hence, in this range of θ, $f(s; \tau, \theta, \phi)$ is monotone decreasing in s.

(b) For $-\pi \leq \theta < {}^{-3\pi}/4$, $^{-\pi}/2 < \phi < {}^{\pi}/2$, and $0 \leq s < \tau < \infty$, the function f has maximum value at $s_m = \tau - \tanh^{-1}(\tan \theta)$. However, for $\tan^{-1}(\tanh \tau) \leq \theta < {}^{-3\pi}/4$ where $\tan^{-1}(\tanh \tau) > -\pi$, the value of s_m is negative which is not in the range of interest $0 \leq s \leq \tau < \infty$.

Thus, for $^{-\pi}/2 < \phi < {}^{\pi}/2$ and $0 \leq s \leq \tau < \infty$, f is monotone decreasing in s if $-\pi < \tan^{-1}(\tanh \tau) \leq \theta < 0$; or f has a maximum at $s_m = \tau - \tanh^{-1}(\tan \theta)$ if $-\pi \leq \theta < \tan^{-1}(\tanh \tau) < {}^{-3\pi}/4$. Furthermore, for any real k, $f(s_m+k; \tau, \theta, \phi) = f(s_m-k; \tau, \theta, \phi)$ if $-\pi \leq \theta < {}^{-3\pi}/4$; hence, f is symmetric with respect to s_m.

Therefore, for a fixed τ, a fixed θ and a fixed ϕ, $f(s;\ \tau,\ \theta,\ \phi)$ can be sketched on the interval $0 \leq s \leq \tau$.

To determine the form of extremal $v(s)$ that maximizes P, the method given by Schmaedeke and Russell [23] can be used. For this particular problem, however, $v(s)$ can be obtained by inspection with a geometrical reasoning. Two typical cases are shown below, one corresponds to f being monotone decreasing in s and the other to f having a maximum at some $s_m > 0$.

In the case shown in Fig. 1, the ranges are $-3\pi/4 < \theta \leq 0$ and $1/D < \tau \leq 3/D$; hence, f is monotone decreasing in s. The form of extremal $v(s)$ is

$$-v(s) = \begin{cases} D \text{ for } 0 \leq s < 1/D \\ 0 \text{ for } 1/D \leq s \leq \tau \end{cases} \quad \text{if } \pi/2 > \phi > \phi_1; \tag{13}$$

$$-v(s) = \begin{cases} D \text{ for } 0 \leq s < 1/D \\ 0 \text{ for } 1/D \leq s < \tau - \ln(\alpha+\beta) \\ -D \text{ for } \tau - \ln(\alpha+\beta) \leq s \leq \tau \end{cases} \quad \text{if } \phi_1 > \phi > \phi_2; \tag{14}$$

$$-v(s) = \begin{cases} D \text{ for } 0 \leq s < \tau - \ln(\alpha+\beta) \\ -D \text{ for } \tau - \ln(\alpha+\beta) \leq s \leq \tau \end{cases} \quad \text{if } \phi_2 \geq \phi > \phi_3; \tag{15}$$

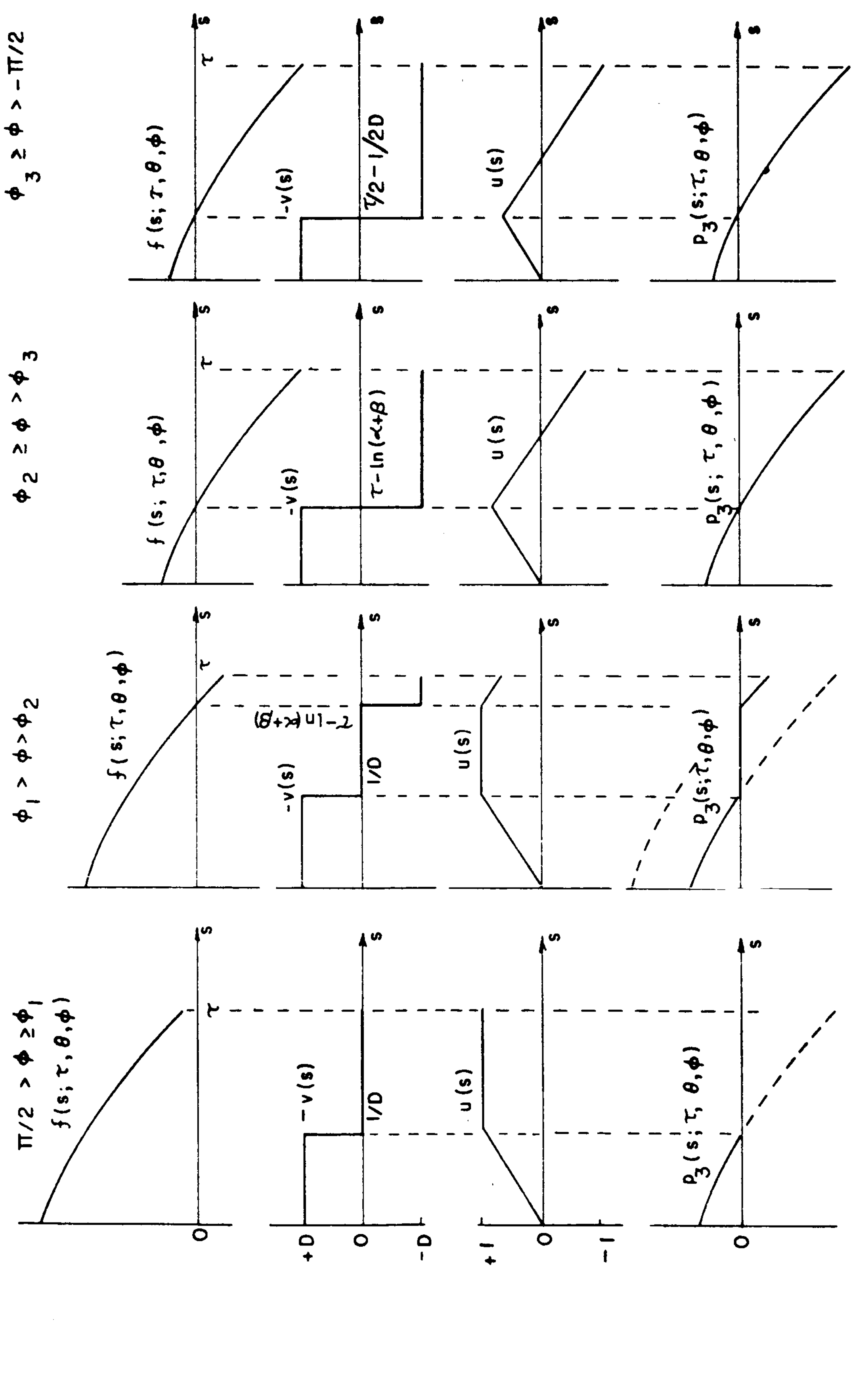
or

$$-v(s) = \begin{cases} D \text{ for } 0 \leq s < (\tau - 1/D)/2 \\ -D \text{ for } (\tau - 1/D)/2 \leq s \leq \tau \end{cases} \quad \text{if } \phi_3 \geq \phi > -\pi/2; \tag{16}$$

where

$$\begin{aligned}
\phi_1 &= 0 \\
\phi_2 &= -\tan^{-1} \{\cos\theta[\cosh(\tau - 1/D) - 1] - \sin\theta \sinh(\tau - 1/D)\}, \\
\phi_3 &= -\tan^{-1} \{\cos\theta[\cosh(\tau/2 - 1/2D) - 1] - \sin\theta \sinh(\tau/2 - 1/2D)\}, \\
\alpha &= (\cos\theta - \tan\phi)/(\cos\theta - \sin\theta), \text{ and} \\
\beta &= \sqrt{\alpha^2 - (\cos\theta + \sin\theta)/(\cos\theta - \sin\theta)}
\end{aligned}$$

By an inspection of the sketches in Fig. 1 with the basic requirement in mind that either $|v(s)| = D$ or $|u(s)| = 1$ on the entire interval $0 \leq s \leq \tau$, it is easy to show that any deviation from of $v(s)$ given above would decrease the value of $f(s;\ \tau,\ \theta,\ \phi)\ [-v(s)]$ and thereby would decrease P.



π/2 > φ ≥ φ1
f(s; τ, θ, φ)
-v(s)
1/D
u(s)
p3(s; τ, θ, φ)
φ1 > φ > φ2
f(s; τ, θ, φ)
-v(s)
1/D
τ - ln(α+β)
u(s)
p3(s; τ, θ, φ)
φ2 ≥ φ > φ3
f(s; τ, θ, φ)
-v(s)
τ - ln(α+β)
u(s)
p3(s; τ, θ, φ)
φ3 ≥ φ > -π/2
f(s; τ, θ, φ)
-v(s)
τ/2 - 1/2D
u(s)
p3(s; τ, θ, φ)
τ
s
0
+D
-D
+1
-1

FIGURE I

For the case shown in Fig. 2, the ranges are

$-\pi < \tan^{-1}(\tanh 5/2D) < \theta \leq \tan^{-1}(\tanh 3/D) < {}^{-3\pi}/4$ and

$\tanh^{-1}(\tan\theta) + 1/2D < \tau < 4\tanh^{-1}(\tan\theta) - 7/D$; hence f has a maximum at

$s_m = \tau - \tanh^{-1}(\tan\theta)$. The form of extremal v(s) is

$$-v(s) = \begin{cases} -D \text{ for } 0 \leq s < (2s_m - 1/D)/3 \\ D \text{ for } (2s_m - 1/D)/3 \leq s < (4s_m + 1/D)/3 \\ 0 \text{ for } (4s_m + 1/D)/3 \leq s \leq \tau \end{cases} \text{ if } \pi/2 > \emptyset \geq \emptyset_1;$$

or

$$-v(s) = \begin{cases} -D \text{ for } 0 \leq s < (2s_m - 1/D)/3 \\ D \text{ for } (2s_m - 1/D)/3 \leq s < (4s_m + 1/D)/3 \\ 0 \text{ for } (4s_m + 1/D)/3 \leq s < \tau - \ln(\alpha-\beta) \\ -D \text{ for } \tau - \ln(\alpha-\beta) \leq s \leq \tau \end{cases} \text{ if } \emptyset_1 > \emptyset > \emptyset_4;$$

or

$$-v(s) = \begin{cases} -D \text{ for } 0 \leq s < (2s_m - 1/D)/3 \\ D \text{ for } (2s_m - 1/D)/3 \leq s < (4s_m + 1/D)/3 \\ 0 \text{ for } (4s_m + 1/D)/3 \leq s < \tau - \ln(\alpha-\beta) \\ -D \text{ for } \tau - \ln(\alpha-\beta) \leq s \leq \tau \end{cases} \text{ if } \emptyset_4 \geq \emptyset > {}^{-\pi}/2;$$

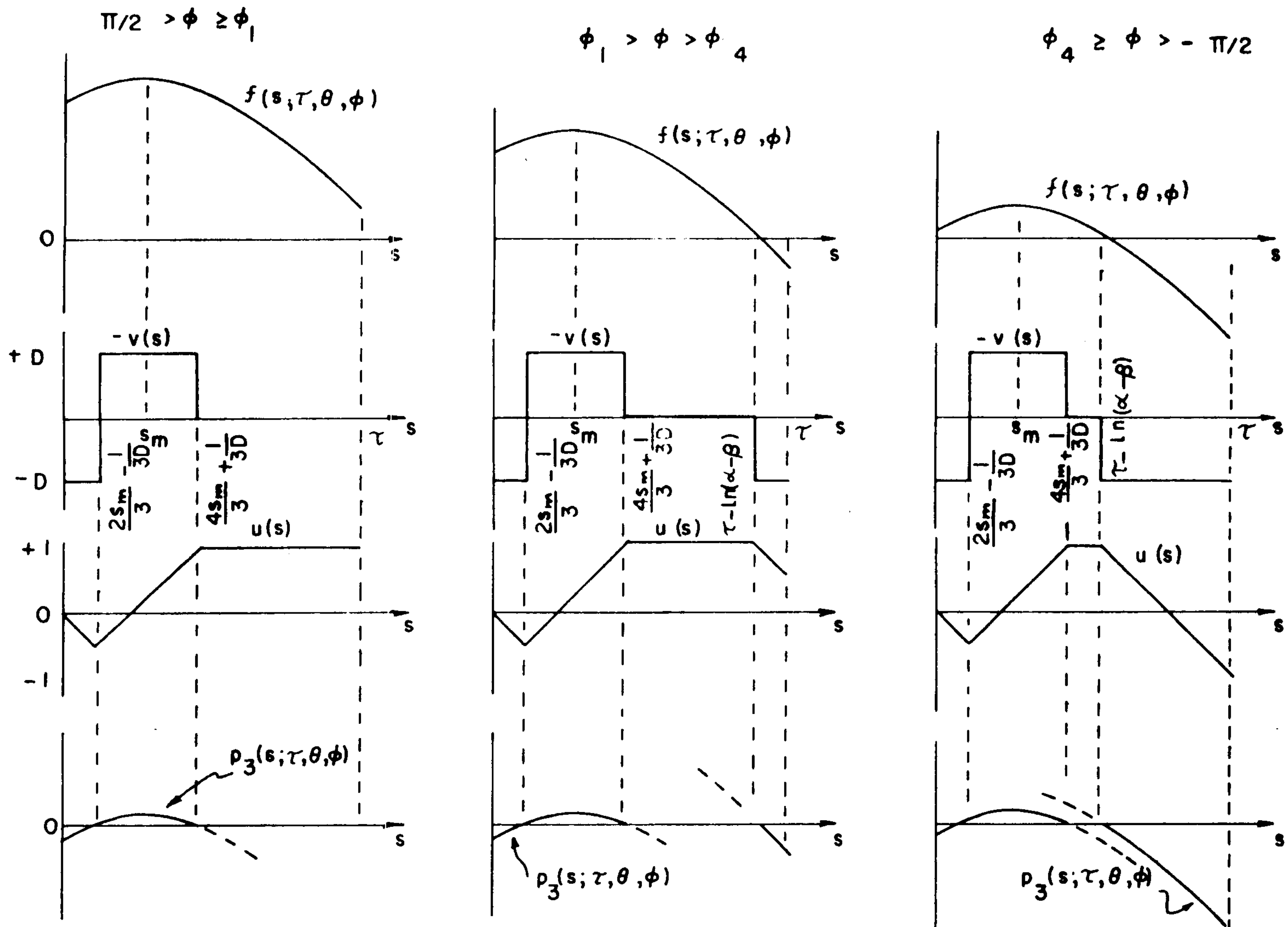
where $\emptyset_4 = -\tan^{-1}\{\cos\theta[\cosh 2/D - 1] - \sin\theta \sinh 2/D\}$, and all other parameters were defined previously. By an inspection of Fig. 2 with the same argument given in the previous case, the extremal v(s) must have the present form.

This procedure was carried out for all the possible cases. It was found that the extremal v(s) reaches zero and takes off from zero as many as four times. Denote the time s at which such events occur by τ_i, $i = 1,\ldots,4$, and let $\tau_o = 0$ and $\tau_5 = \tau$. Supposing the values of $x_3(s) = u(s)$ are such that

$$\begin{cases} |u(s)| < 1, \text{ if } \tau_{2i} \leq s < \tau_{2i+1}, \; i = 0,1,2; \\ |u(s)| = 1, \text{ if } \tau_{2j+1} \leq s < \tau_{2j+2}, \; j = 0,1 \; . \end{cases}$$

Then

$$\begin{cases} dp_3/ds = d\phi_3/ds \text{ for } \tau_{2i} \leq s < \tau_{2i+1}, \; i = 0,1,2; \\ p_3(s) = 0 \qquad \text{for } \tau_{2j+1} \leq s < \tau_{2j+2}, \; j = 0,1 \; . \end{cases}$$



$\pi/2 > \phi \geq \phi_1$
$\phi_1 > \phi > \phi_4$
$\phi_4 \geq \phi > -\pi/2$
$f(s;\tau,\theta,\phi)$
$-v(s)$
$u(s)$
$p_3(s;\tau,\theta,\phi)$
$+D$
$-D$
$+1$
0
-1
s
τ
$\frac{2s_m}{3} - \frac{1}{3D}$
s_m
$\frac{4s_m}{3} + \frac{1}{3D}$
$\tau - \ln(\alpha-\beta)$

FIGURE 2

It follows that choosing $p_3(s)$ to be continuous at τ_{2i+1} $(i=0,1,2)$ requires $p_3(\tau_{2i+1}) = 0$, and hence

$$p_3(s) = \psi_3(s) - \psi_3(\tau_{2i+1}) \text{ for } \tau_{2i} \leq s < \tau_{2i+1},\ i=0,1,2 .$$

With $p_3(s)$ so defined, the jump conditions have to be satisfied at τ_{2i}, $i=0,1,2$. Since η is the unit adjoint vector at $\partial K(\tau)$, therefore $p_3(\tau) = \psi_3(\tau) = \dot{\eta}_3 = \sin \phi$. Thus

$$\frac{p_3(s;\tau,\theta,\phi)}{\cos\phi} = \begin{cases} \cos\theta[\cosh(\tau-s)-\cosh(\tau-\tau_1)] - \sin\theta[\sinh(\tau-s) - \sinh(\tau-\tau_1)], & \text{if } 0 \leq s < \tau_1, \\ 0\ , & \text{if } \tau_1 \leq s < \tau_2, \\ \cos\theta[\cosh(\tau-s)-\cosh(\tau-\tau_3)] - \sin\theta[\sinh(\tau-s) - \sinh(\tau-\tau_3)], & \text{if } \tau_2 \leq s < \tau_3, \\ 0\ , & \text{if } \tau_3 \leq s < \tau_4, \\ \cos\theta[\cosh(\tau-s)-1]-\sin\theta\sinh(\tau-s)+\tan\phi+\delta(\tan\phi_o - \tan\phi), & \text{if } \tau_4 \leq s \leq \tau; \end{cases}$$

where

$$\delta = \begin{cases} 0, & \text{if } |x_3(\tau)| = |u(\tau)| < 1 \\ 1, & \text{if } |x_3(\tau)| = |u(\tau)| = 1. \end{cases}$$

Using the expression for $p_3(s;\ \tau,\ \theta,\ \phi)$, it has at most one jump discontinuity at $s = \tau$ (equivalently at $\partial K(\tau)$), and this happens only when $|x_3(\tau)| = |u(\tau)| = 1$. Furthermore, the explicit form of extremal $v(s)$ can be expressed as

$$\begin{aligned} -v(s) &= D\ \mathrm{sgn}[p_3(s;\ \tau,\ \theta,\ \phi)] \\ &= D\ \mathrm{sgn}\left[p_3(s;\ \tau,\ \theta,\ \phi)/\cos\phi\right] \end{aligned}$$

since $\cos\phi$ is positive on $-\pi/2 < \phi < \pi/2$. Finally, by Russell's sufficiency condition [12], the extremal $v(s)$ is also the time-optimal $v(s)$.

The function $p_3(s;\ \tau,\ \theta,\ \phi)$ for the two typical cases discussed previously are also sketched in Figs. 1 and 2. The formulas for parameters τ_i, $i = 1,\ldots,4$, δ, and ϕ_o are obtained for all possible cases in the ranges $-\pi \leq \theta \leq 0$, $-\pi/2 < \phi < \pi/2$ and $0 \leq t < \infty$. The results are listed in Tables I to VI.

Time-optimal Controls for the Booster

The state vector $x(\tau)$ can be readily computed from equations (6). Take a typical case as an example:

$-3\pi/4 < \theta \leq 0$, $1/D < \tau \leq 3/D$, $\emptyset_2 < \emptyset < \emptyset_1$ (see Fig. 1). For this case, the extremal $v(s)$ is given in equation (14), hence by intergration over $[0, \tau]$,

$$\begin{cases} x_1(\tau) = D \sinh(1/D-\tau) + D \sinh \tau - 1 + D \ln(\alpha+\beta) - [\alpha+\beta-1/(\alpha+\beta)]\, D/2 \\ x_2(\tau) = D \cosh(\tau-1/D) - D \cosh \tau - D + [\alpha+\beta - 1/(\alpha+\beta)]\, D/2 \\ x_3(\tau) = 1 - D \ln(\alpha+\beta). \end{cases}$$

Let $\alpha+\beta = e^{1/D}$ so that $x_3(\tau) = u(\tau) = 0$, then

$$\begin{cases} x_1(\tau) = D\, [\sinh\, (1/D - \tau) + \sinh \tau - \sinh 1/D] \\ x_2(\tau) = D\, [\cosh\, (\tau-1/D) - \cosh \tau - 1 + \sinh 1/D] \end{cases}$$

for $1/D < \tau \leq 3/D$. A further choice of $\tau = 2.5/D$ reduces the above to

$$\begin{cases} x_1(2.5/D) = D\, [-\sinh(1.5/D) + \sinh(2.5/D) - \sinh(1/D)] \\ x_2(2.5/D) = D\, [\cosh(1.5/D) - \cosh(2.5/D) - 1 + \sinh(1/D)] \\ x_3(2.5/D) = 0. \end{cases}$$

Using the results so obtained to solve the original booster problem stated in equation (3), reverse the time sense once again. Thus the extremal $v(s)$ now starts from $s = \tau$ and backs up to $s = 0$. Since $t = -\tau$, it follows that equation (14) is now replaced by

$$-v(s) = \begin{cases} D \text{ for } t \geq s > t - 1/D \\ 0 \text{ for } t - 1/D \geq s > \ln(\alpha+\beta) \\ -D \text{ for } \ln(\alpha+\beta) \geq s \geq 0. \end{cases}$$

Since $dx_3/dt = v(t)$ in equation (4) replaces $dx_3/d\tau = -v(\tau)$ in equation (5), hence $x_3 = u$ (shown in Fig. 1) now reverses its sign. Thus, the above example (now $t = 2.5/D$ instead) can be interpreted as follows: The control

$$u(s) = \begin{cases} Ds, \text{ if } 2.5/D \geq s > 1.5/D \\ -1, \text{ if } 1.5/D \geq s > 1/D \\ -Ds, \text{ if } 1/D \geq s \geq 0 \end{cases}$$

will steer the original booster control system (3) from the initial state

$$\begin{cases} x_1(0) = D[-\sinh(1.5/D) + \sinh(2.5/D) - \sinh(1/D)] \\ x_2(0) = D[\cosh\, (1.5/D) - \cosh(2.5/D) - 1 + \sinh(1/D)] \end{cases}$$

with $u(0) = 0$ to the origin with a minimum time $t^* = 2.5/D$ and $u(2.5/D) = 0$.

This example also illustrates the fact that the parameters θ and ϕ introduced in the adjoint vector η serve as an aid to derive the extremal $v(s)$ only, they disappear in the final solution of the time-optimal control problem.

Maximum Controllable Region

The maximum controllable region is determined by examining the values of $x(\tau)$ as $\tau \to \infty$. Among the total of twenty different cases for large τ in Tables I - VI, the boundary of the region for $u = 1$ can be determined from the cases of (a) $\pi/2 > \phi \geq \phi_1$, $3/D < \tau < \infty$ in Table I, and (b) $\pi/2 > \phi \geq \phi_4$, $3/D < \tau < \infty$ in Table VI as follows:

(a) By equations (6), this case yields

$$\begin{cases} x_1(\tau) = D \sinh(1/D - \tau) + D \sinh \tau - 1 \\ x_2(\tau) = D \cosh(\tau - 1/D) - D \cosh \tau \\ x_3(\tau) = 1 \quad . \end{cases}$$

Thus $\dfrac{x_1 + 1}{x_2} \to -1$ as $\tau \to \infty$ which gives a equation

$$x_1 + x_2 = -1 \qquad \text{for } u = 1. \tag{17}$$

(b) This case yields

$$\begin{cases} x_1(\tau) = -D \sinh(1/D - \tau) - D \sinh \tau + 1 + D \sinh (2/D) - 2 \\ x_2(\tau) = -D \cosh (\tau - 1/D) + D \cosh \tau + D - D \cosh(2/D) \\ x_3(\tau) = -1 + 2 = 1 \quad . \end{cases}$$

Thus $\dfrac{x_1 + 1 - D \sinh (2/D)}{x_2 - D + D \cosh (2/D)} \to -1$ as $\tau \to \infty$, or

$$x_1 + x_2 = -1 + D\,[1 - \exp(-2/D)] \text{ for } u = 1 \; . \tag{18}$$

The boundary of the region for $u = -1$ can be obtained from other cases, such as the case of $\phi_4 \geq \phi > -\pi/2$, $3/D < \tau < \infty$ in Table I. However, since

$$g(s;\, \tau,\theta,\phi) = g(s;\, \tau,\, \pi+\theta,\, \pi-\phi), \tag{19}$$

known relations will hold if all the signs of x_1, x_2 and $u(=x_3)$ are changed simultaneously. Therefore, corresponding to equations (17) and (18), the boundary for $u = -1$ is given by

$$\begin{cases} -x_1 - x_2 = -1 & \text{for } u = -1 \ , \quad (20) \\ -x_1 - x_2 = -1 + D[I - \exp(-2/D)] & \text{for } u = -1 \ . \quad (21) \end{cases}$$

The boundary of the region for $-1 \leq u \leq 1$ can be found from the case of $\phi_1 > \phi > \phi_4$, $3/D < \tau < \infty$ in Table I, which yields

$$\begin{cases} x_1(\tau) = D \sinh(1/D - \tau) + D \sinh \tau + D[1/(\alpha+\beta) - (\alpha+\beta)]/2 + D \ln(\alpha+\beta) - x_3(\tau) \\ x_2(\tau) = D \cosh(\tau - 1/D) - D \sinh \tau - D + D[1/(\alpha+\beta) - (\alpha+\beta)]/2 \\ x_3(\tau) = 1 - D \ln(\alpha+\beta) \ . \end{cases}$$

Since $u = x_3$ and $\alpha+\beta = \exp[(1-u)/D]$, hence the limit as $\tau \to \infty$ yields

$$x_1 + x_2 = -u - D\{1-\exp[-(1-u)/D]\} \text{ for } u = 1 - D \ln(\alpha+\beta) \qquad (22)$$

which reduces to equation (17) if $u = 1$, and to (21) if $u = -1$. By the property of equation (19) and the same argument, the other boundary equation for $-1 \leq u \leq 1$ can be deduced from (22) as

$$-x_1 - x_2 = u - D\{1 - \exp[-(1+u)/D]\} \text{ for } -u = 1 - D \ln(\alpha+\beta) \ . \qquad (23)$$

Equation (23) reduces to (18) if $u = 1$, and to (20) is $u = -1$. Consequently equations (22) and (23) determines the maximum controllable region (Fig. 3) for $-1 \leq u \leq 1$. Figure 4 shows the regions for $1/D = 0.709$, which agree with those given in Friedland's paper [20] when a scalar factor of 0.709 for x_1 and x_2 axes are considered.

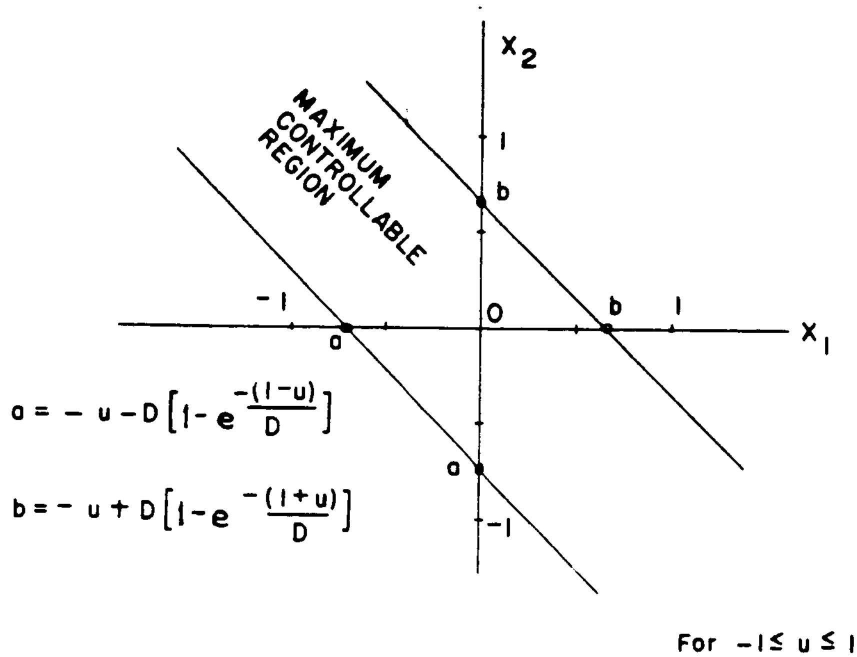


FIGURE 3

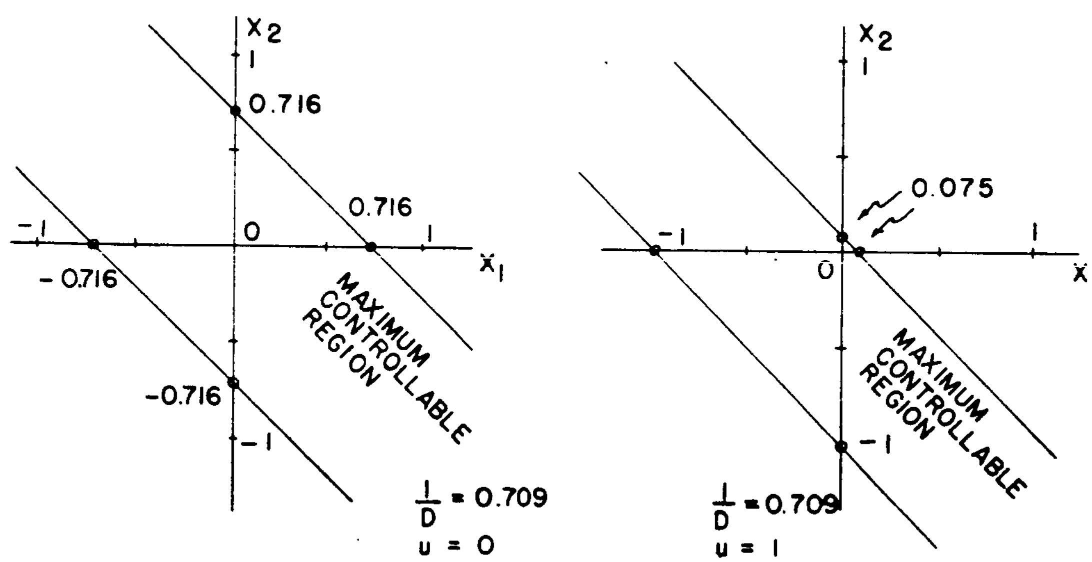


FIGURE 4

TABLE I

<table>
<tr><td>(I) $0 \geq \theta \geq -\frac{3\pi}{4}$
ϕ / τ_i's δ & ϕ_0 / τ</td><td>$\frac{\pi}{2} > \phi \geqslant \phi_1$
$\phi_1 = 0$</td><td>$\phi_1 > \phi > \phi_2$
$\phi_2 = -\tan^{-1}\{\cos\theta[\cosh(\tau - \frac{1}{D}) - 1] - \sin\theta \sinh(\tau - \frac{1}{D})\}$</td><td>$\phi_2 \geq \phi > \phi_3$
$\phi_3 = -\tan^{-1}\{\cos\theta[\cosh(\frac{\tau}{2} + \frac{1}{2D}) - 1] - \sin\theta \sinh(\frac{\tau}{2} + \frac{1}{2D})\}$</td><td>$\phi_3 \geqslant \phi > -\frac{\pi}{2}$</td></tr>
<tr><td>$0 \leq \tau \leq \frac{1}{D}$
f is monotone</td><td colspan="4">$\tau_1 = \tau_2 = \tau_3 = \tau_4 = 0 \quad , \quad \delta = 0$</td></tr>
<tr><td>$\frac{1}{D} < \tau \leq \frac{3}{D}$
f is monotone</td><td>$\tau_1 = \tau_2 = 0$
$\tau_3 = \frac{1}{D}$
$\tau_4 = \tau$
$\delta = 1 , \ \phi_0 = \phi_1$</td><td>$\tau_1 = \tau_2 = 0$
$\tau_3 = \frac{1}{D}$
$\tau_4 = \begin{cases} \tau - \ln[\alpha + \beta], \text{ if } 0 \leq \theta < \frac{3}{4}\pi \\ \tau + \ln[1 + \sqrt{2} \tan\phi], \text{ if } \theta = \frac{3}{4}\pi \end{cases}$
$\delta = 0$</td><td>$\tau_1 = \tau_2 = \tau_3 = \tau_4 = 0$
$\delta = 0$</td><td>$\tau_1 = \tau_2 = \tau_3 = \tau_4 = 0$
$\delta = 1 , \ \phi_0 = \phi_3$</td></tr>
<tr><td>$\frac{3}{D} < \tau < \infty$
f is monotone</td><td>,,</td><td colspan="2">$\tau_1 = \tau_2 = 0$
$\tau_3 = \frac{1}{D}$
$\tau_4 = \begin{cases} \tau - \ln[\alpha + \beta], \text{ if } 0 \leq \theta < \frac{3}{4}\pi \\ \tau + \ln[1 + \sqrt{2} \tan\phi], \text{ if } \theta = \frac{3}{4}\pi \end{cases}$
$\delta = 0$</td><td colspan="2">$\tau_1 = \tau_2 = 0$
$\tau_3 = \frac{1}{D}$
$\tau_4 = \tau - \frac{2}{D}$
$\delta = 1 , \ \phi_0 = \phi_4$</td></tr>
<tr><td>τ / τ_i's δ & ϕ_0 / ϕ
$0 \geqslant \theta \geqslant -\frac{3\pi}{4}$</td><td>$\frac{\pi}{2} > \phi \geqslant \phi_1$
$\phi_1 = 0$</td><td colspan="2">$\phi_1 > \phi > \phi_4$
$\phi_4 = -\tan^{-1}\{\cos\theta[\cosh\frac{2}{D} - 1] - \sin\theta \sinh\frac{2}{D}\}$</td><td colspan="2">$\phi_4 \geqslant \phi > -\frac{\pi}{2}$</td></tr>
</table>

TABLE II

$-\frac{3}{4}\pi > \theta > \tan^{-1}(\tanh\frac{3}{D}) > -\pi$; τ \ ϕ ; τ_i's δ & ϕ_o	$\frac{\pi}{2} > \phi \geq \phi_1$ $\phi_1 = 0$	$\phi_1 > \phi > \phi_2$ $\phi_2 = -\tan^{-1}\{\cos\theta[\cosh(\tau-\frac{1}{D})-1] - \sin\theta\sinh(\tau-\frac{1}{D})\}$	$\phi_2 \geq \phi > \phi_3$ $\phi_3 = -\tan^{-1}\{\cos\theta[\cosh(\frac{\tau}{2}+\frac{1}{2D})-1] - \sin\theta\sinh(\frac{\tau}{2}+\frac{1}{2D})\}$	$\phi_3 \geq \phi > -\frac{\pi}{2}$
$0 \leq \tau \leq \frac{1}{D}$ f is monotone	$\tau_1 = \tau_2 = \tau_3 = \tau_4 = 0$, $\delta = 0$			
$\frac{1}{D} < \tau \leq \frac{3}{D}$ f is monotone	$\tau_1 = \tau_2 = 0$ $\tau_3 = \frac{1}{D}$ $\tau_4 = \tau$ $\delta = 1$, $\phi_o = \phi_1$	$\tau_1 = \tau_2 = 0$ $\tau_3 = \frac{1}{D}$ $\tau_4 = \tau - \ln[\alpha - \beta]$ $\delta = 0$	$\tau_1 = \tau_2 = \tau_3 = \tau_4 = 0$ $\delta = 0$	$\tau_1 = \tau_2 = \tau_3 = \tau_4 = 0$ $\delta = 1$, $\phi_o = \phi_3$
$\frac{3}{D} < \tau \leq \tanh^{-1}(\tan\theta)$	, ,	$\tau_1 = \tau_2 = 0$ $\tau_3 = \frac{1}{D}$ $\tau_4 = \tau - \ln[\alpha - \beta]$ $\delta = 0$		$\tau_1 = \tau_2 = 0$ $\tau_3 = \frac{1}{D}$ $\tau_4 = \tau - \frac{2}{D}$ $\delta = 1$, $\phi_o = \phi_4$
$\tanh^{-1}(\tan\theta) < \tau \leq \tanh^{-1}(\tan\theta) + \frac{2}{D}$	$\tau_1 = \tau_2 = 0$ $\tau_3 = \frac{4}{3}[\tau - \tanh^{-1}(\tan\theta)] + \frac{1}{3D}$ $\tau_4 = \tau$ $\delta = 1$, $\phi_o = \phi_1$	$\tau_1 = \tau_2 = 0$ $\tau_3 = \frac{4}{3}[\tau - \tanh^{-1}(\tan\theta)] + \frac{1}{3D}$ $\tau_4 = \tau - \ln[\alpha - \beta]$ $\delta = 0$		$\tau_1 = \tau_2 = 0$ $\tau_3 = \frac{4}{3}[\tau - \tanh^{-1}(\tan\theta)] + \frac{1}{3D}$ $\tau_4 = \tau - \frac{2}{D}$ $\delta = 1$, $\phi_o = \phi_4$
$\tanh^{-1}(\tan\theta) + \frac{2}{D} < \tau < \infty$	$\tau_1 = {}^1/_D$ $\tau_2 = \tau - \tanh^{-1}(\tan\theta) - {}^1/_D$ $\tau_3 = \tau_2 + {}^2/_D$ $\tau_4 = \tau$ $\delta = 1$, $\phi_o = \phi_1$	$\tau_1 = {}^1/_D$ $\tau_2 = \tau - \tanh^{-1}(\tan\theta) - \frac{1}{D}$ $\tau_3 = \tau_2 + \frac{2}{D}$ $\tau_4 = \tau - \ln[\alpha - \beta]$ $\delta = 0$		$\tau_1 = {}^1/_D$ $\tau_2 = \tau - \tanh^{-1}(\tan\theta) - \frac{1}{D}$ $\tau_3 = \tau_2 + \frac{2}{D}$ $\tau_4 = \tau - {}^2/_D$ $\delta = 1$, $\phi_o = \phi_4$
τ ; τ_i's δ & ϕ_o \ ϕ ; $-\frac{3}{4}\pi > \theta > \tan^{-1}(\tanh\frac{3}{D}) > -\pi$	$\frac{\pi}{2} > \phi \geq \phi_1$ $\phi_1 = 0$	$\phi_1 > \phi > \phi_4$ $\phi_4 = -\tan^{-1}\{\cos\theta[\cosh\frac{2}{D} - 1] - \sin\theta\sinh\frac{2}{D}\}$		$\phi_4 \geq \phi > -\frac{\pi}{2}$

TABLE III

(III) $-\frac{3}{4}\pi > \tan^{-1}(\tanh\frac{3}{D}) \geq \theta > \tan^{-1}(\tanh\frac{5}{2D}) > -\pi$; τ / τ_i's, δ & ϕ_0 / ϕ	$\frac{\pi}{2} > \phi \geq \phi_1$ $\phi_1 = 0$	$\phi_1 > \phi > \phi_4$ $\phi_4 = -\tan^{-1}\{\cos\theta[\cosh\frac{2}{D}-1] - \sin\theta\sinh\frac{2}{D}\}$ $\phi_1 > \phi > \phi_2$ $\phi_2 = -\tan^{-1}\{\cos\theta[\cosh(\tau-\frac{1}{D})-1] - \sin\theta\sinh(\tau-\frac{1}{D})\}$	$\phi_4 \geq \phi > -\frac{\pi}{2}$	$\phi_2 \geq \phi > \phi_5$, $\zeta = \tanh^{-1}(\tan\theta) + (\frac{\tau}{4} - \frac{1}{4D})$ $\phi_5 = -\tan^{-1}\{\cos\theta[\cosh\zeta - 1] - \sin\theta\sinh\zeta\}$ $\phi_2 \geq \phi > \phi_3$ $\phi_3 = -\tan^{-1}\{\cos\theta[\cosh(\frac{\tau}{2}+\frac{1}{2D})-1] - \sin\theta\sinh(\frac{\tau}{2}+\frac{1}{2D})\}$	$\phi_5 \geq \phi > -\frac{\pi}{2}$ $\phi_3 \geq \phi > -\frac{\pi}{2}$
$0 \leq \tau \leq \frac{1}{D}$ f is monotone	$\tau_1 = \tau_2 = \tau_3 = \tau_4 = 0$,		$\delta = 0$		
$\frac{1}{D} < \tau \leq \frac{4}{3}\tanh^{-1}(\tan\theta) - \frac{1}{3D}$ f is monotone on $\tau \in (\frac{1}{D}, \frac{5}{2D}]$	$\tau_1 = \tau_2 = 0$ $\tau_3 = 1/D$ $\tau_4 = \tau$ $\delta = 1$, $\phi_0 = \phi_1$	$\tau_1 = \tau_2 = 0$ $\tau_3 = 1/D$ $\tau_4 = \tau - \ln[\alpha - \beta]$ $\delta = 0$		$\tau_1 = \tau_2 = \tau_3 = \tau_4 = 0$ $\delta = 0$	$\tau_1 = \tau_2 = \tau_3 = \tau_4 = 0$ $\delta = 1$, $\phi_0 = \phi_3$
$\frac{4}{3}\tanh^{-1}(\tan\theta) - \frac{1}{3D} < \tau \leq \frac{3}{D}$	,,	,,		$\tau_1 = \tau_2 = \tau_3 = \tau_4 = 0$ $\delta = 0$	$\tau_1 = \tau_2 = \tau_3 = \tau_4 = 0$ $\delta = 1$, $\phi_0 = \phi_5$
$\frac{3}{D} < \tau \leq \tanh^{-1}(\tan\theta) + \frac{1}{2D}$	,,	$\tau_1 = \tau_2 = 0$ $\tau_3 = 1/D$ $\tau_4 = \tau - \ln[\alpha - \beta]$ $\delta = 0$	$\tau_1 = \tau_2 = 0$ $\tau_3 = 1/D$ $\tau_4 = \tau - 2/D$ $\delta = 1$, $\phi_0 = \phi_4$		
$\tanh^{-1}(\tan\theta) + \frac{1}{2D} < \tau < 4\tanh^{-1}(\tan\theta) - \frac{7}{D}$	$\tau_1 = \tau_2 = 0$ $\tau_3 = \frac{4}{3}[\tau - \tanh^{-1}(\tan\theta)] + \frac{1}{3D}$ $\tau_4 = \tau$ $\delta = 1$, $\phi_0 = \phi_1$	$\tau_1 = \tau_2 = 0$ $\tau_3 = \frac{4}{3}[\tau - \tanh^{-1}(\tan\theta)] + \frac{1}{3D}$ $\tau_4 = \tau - \ln(\alpha - \beta)$ $\delta = 0$	$\tau_1 = \tau_2 = 0$ $\tau_3 = \frac{4}{3}[t - \tanh^{-1}(-\tan\theta)] + \frac{1}{3D}$ $\tau_4 = t - 2/D$ $\delta = 1$, $\phi_0 = \phi_4$		
$4\tanh^{-1}(\tan\theta) - \frac{7}{D} \leq \tau \leq \frac{4}{3}\tanh^{-1}(\tan\theta) + \frac{1}{D}$	,,	$\tau_1 = \tau_2 = 0$ $\tau_3 = \frac{4}{3}[\tau - \tanh^{-1}(\tan\theta)] + \frac{1}{3D}$ $\tau_4 = \tau - \ln(\alpha - \beta)$ $\delta = 0$	$\tau_1 = \tau_2 = \tau_3 = \tau_4 = 0$ $\delta = 0$		$\tau_1 = \tau_2 = \tau_3 = \tau_4 = 0$ $\delta = 1$, $\phi_0 = \phi_5$
$\frac{4}{3}\tanh^{-1}(\tan\theta) + \frac{1}{D} < \tau \leq \tanh^{-1}(\tan\theta) + \frac{2}{D}$	,,	,,	$\tau_1 = \tau_2 = \tau_3 = \tau_4 = 0$ $\delta = 0$	$\tau_1 = \tau_2 = 0$ $\tau_3 = 1/D$ $\tau_4 = \tau - \ln[\alpha + \beta]$ $\delta = 0$	$\tau_1 = \tau_2 = 0$ $\tau_3 = 1/D$ $\tau_4 = \tau - \frac{4}{3}\tanh^{-1}(\tan\theta)$ $\delta = 1$, $\phi_0 = \phi_8$
$\tanh^{-1}(\tan\theta) + \frac{2}{D} < \tau < \infty$	$\tau_1 = 1/D$ $\tau_2 = \tau - \tanh^{-1}(\tan\theta) - \frac{1}{D}$ $\tau_3 = \tau_2 + 2/D$ $\tau_4 = \tau$, $\delta = 1$, $\phi_0 = \phi_1$	$\tau_1 = 1/D$ $\tau_2 = \tau - \tanh^{-1}(\tan\theta) - \frac{1}{D}$ $\tau_3 = \tau_2 + 2/D$ $\tau_4 = \tau - \ln[\alpha - \beta]$, $\delta = 0$	$\tau_1 = \tau_2 = 0$ $\tau_3 = 1/D$ $\tau_4 = \tau - \ln[\alpha + \beta]$ $\delta = 0$		,,
τ / τ_i's, δ & ϕ_0 / ϕ ; $-\frac{3}{4}\pi > \tan^{-1}(\tanh\frac{3}{D}) \geq \theta > \tan^{-1}(\tanh\frac{5}{2D}) > -\pi$	$\frac{\pi}{2} > \phi \geq \phi_1$ $\phi_1 = 0$	$\phi_1 > \phi > \phi_7$, $\rho = \tanh^{-1}(\tan\theta) + \frac{1}{D}$ $\phi_7 = -\tan^{-1}\{\cos\theta[\cosh\rho - 1] - \sin\theta\sinh\rho\}$ $\phi_1 > \phi > \phi_6$ $\phi_6 = -\tan^{-1}\{\cos\theta[\cosh(\frac{4}{3}\tanh^{-1}(\tan\theta) - (\frac{\tau}{3}+\frac{1}{3D})) - 1] - \sin\theta\sinh(\frac{4}{3}\tanh^{-1}(\tan\theta) - (\frac{\tau}{3}+\frac{1}{3D}))\}$	$\phi_7 \geq \phi > \phi_8$ $\phi_8 = -\tan^{-1}\{\cos\theta[\cosh(\frac{4}{3}\tanh^{-1}(\tan\theta)) - 1] - \sin\theta\sinh(\frac{4}{3}\tanh^{-1}(\tan\theta))\}$ $\phi_6 \geq \phi > \phi_2$ $\phi_2 = -\tan^{-1}\{\cos\theta[\cosh(\tau-\frac{1}{D}) - 1] - \sin\theta\sinh(\tau-\frac{1}{D})\}$	$\phi_2 \geq \phi > \phi_8$	$\phi_8 \geq \phi > -\frac{\pi}{2}$

TABLE IV

(IV) $-\frac{3}{4}\pi > \tan^{-1}(\tanh\frac{5}{2D}) \geq \theta > \tan^{-1}(\tanh\frac{1}{D}) > -\pi$ ϕ / τ_i's δ & ϕ_0 / τ	$\frac{\pi}{2} > \phi \geq \phi_1$ $\phi_1 = 0$	$\phi_1 > \phi > \phi_2$ $\phi_2 = -\tan^{-1}\{\cos\theta[\cosh(\tau-\frac{1}{D})-1] - \sin\theta\sinh(\tau-\frac{1}{D})\}$		$\phi_2 \geq \phi > \phi_5$ $\phi_5 = -\tan^{-1}\{\cos\theta[\cosh(\tanh^{-1}(\tan\theta)-(\frac{\tau}{4}-\frac{1}{4D}))-1] - \sin\theta\sinh(\tanh^{-1}(\tan\theta)-\frac{1}{4}(\tau-\frac{1}{D}))\}$ $\phi_2 \geq \phi > \phi_3$ $\phi_3 = -\tan^{-1}\{\cos\theta[\cosh(\frac{\tau}{2}+\frac{1}{2D})-1] - \sin\theta\sinh(\frac{\tau}{2}+\frac{1}{2D})\}$	$\phi_5 \geq \phi > -\frac{\pi}{2}$ $\phi_3 \geq \phi > -\frac{\pi}{2}$
$0 \leq \tau \leq \frac{1}{D}$ f is monotone	$\tau_1 = \tau_2 = \tau_3 = \tau_4 = 0$, $\delta = 0$				
$\frac{1}{D} < \tau \leq \frac{4}{3}\tanh^{-1}(\tan\theta) - \frac{1}{3D}$	$\tau_1 = \tau_2 = 0$ $\tau_3 = \frac{1}{D}$ $\tau_4 = \tau$ $\delta = 1$, $\phi_0 = \phi_1$	$\tau_1 = \tau_2 = 0$ $\tau_3 = \frac{1}{D}$ $\tau_4 = \tau - \ln[\alpha - \beta]$ $\delta = 0$		$\tau_1 = \tau_2 = \tau_3 = \tau_4 = 0$ $\delta = 0$	$\tau_1 = \tau_2 = \tau_3 = \tau_4 = 0$ $\delta = 1$, $\phi_0 = \phi_3$
$\frac{4}{3}\tanh^{-1}(\tan\theta) - \frac{1}{3D} < \tau$ $\leq \tanh^{-1}(\tan\theta) + \frac{1}{2D}$	,,	,,		$\tau_1 = \tau_2 = \tau_3 = \tau_4 = 0$ $\delta = 0$	$\tau_1 = \tau_2 = \tau_3 = \tau_4 = 0$ $\delta = 1$, $\phi_0 = \phi_5$
$\tanh^{-1}(\tan\theta) + \frac{1}{2D} < \tau$ $\leq \frac{4}{3}\tanh^{-1}(\tan\theta) + \frac{1}{D}$	$\tau_1 = \tau_2 = 0$ $\tau_3 = \frac{4}{3}[\tau - \tanh^{-1}(\tan\theta)] + \frac{1}{3D}$ $\tau_4 = \tau$ $\delta = 1$, $\phi_0 = \phi_1$	$\tau_1 = \tau_2 = 0$ $\tau_3 = \frac{4}{3}[\tau - \tanh^{-1}(\tan\theta)] + \frac{1}{3D}$ $\tau_4 = \tau - \ln[\alpha - \beta]$ $\delta = 0$	$\tau_1 = \tau_2 = \tau_3 = \tau_4 = 0$ $\delta = 0$		,,
$\frac{4}{3}\tanh^{-1}(\tan\theta) + \frac{1}{D} < \tau$ $\leq \tanh^{-1}(\tan\theta) + \frac{2}{D}$	,,	,,	$\tau_1 = \tau_2 = \tau_3 = \tau_4 = 0$ $\delta = 0$	$\tau_1 = \tau_2 = 0$ $\tau_3 = \frac{1}{D}$ $\tau_4 = \tau - \ln[\alpha + \beta]$ $\delta = 0$	$\tau_1 = \tau_2 = 0$ $\tau_3 = \frac{1}{D}$ $\tau_4 = \tau - \frac{4}{3}\tanh^{-1}(\tan\theta)$ $\delta = 1$, $\phi_0 = \phi_8$
$\tanh^{-1}(\tan\theta) + \frac{2}{D}$ $< \tau < \infty$	$\tau_1 = \frac{1}{D}$ $\tau_2 = \tau - \tanh^{-1}(\tan\theta) - \frac{1}{D}$ $\tau_3 = \tau_2 + \frac{2}{D}$ $\tau_4 = \tau$, $\delta = 1$, $\phi_0 = \phi_1$	$\tau_1 = \frac{1}{D}$ $\tau_2 = \tau - \tanh^{-1}(\tan\theta) - \frac{1}{D}$ $\tau_3 = \tau_2 + \frac{2}{D}$ $\tau_4 = \tau - \ln[\alpha - \beta]$, $\delta = 0$	$\tau_1 = \tau_2 = 0$ $\tau_3 = \frac{1}{D}$ $\tau_4 = \tau - \ln[\alpha + \beta]$ $\delta = 0$		,,
τ / τ_i's δ & ϕ_0 / ϕ $-\frac{3}{4}\pi > \tan^{-1}(\tanh\frac{5}{2D}) \geq \theta > \tan^{-1}(\tanh\frac{1}{D}) > -\pi$	$\frac{\pi}{2} > \phi \geq \phi_1$ $\phi_1 = 0$	$\phi_1 > \phi > \phi_7$, $\rho = \tanh^{-1}(\tan\theta) + 1/D$ $\phi_7 = -\tan^{-1}\{\cos\theta[\cosh\rho - 1] - \sin\theta\sinh\rho\}$ $\phi_1 > \phi > \phi_6$ $\phi_6 = \tan^{-1}\{\cos\theta[\cosh(\frac{4}{3}\tanh^{-1}(\tan\theta) - (\frac{\tau}{3}+\frac{1}{3D})) - 1] - \sin\theta\sinh(\frac{4}{3}\tanh^{-1}(\tan\theta) - (\frac{\tau}{3}+\frac{1}{3D}))\}$	$\phi_7 \geq \phi > \phi_8$ $\phi_8 = -\tan^{-1}\{\cos\theta[\cosh(\frac{2}{3}\tanh^{-1}(\tan\theta)) - 1] - \sin\theta\sinh(\frac{2}{3}\tanh^{-1}(\tan\theta))\}$ $\phi_6 \geq \phi > \phi_2$ $\phi_2 = -\tan^{-1}\{\cos\theta[\cosh(\tau - \frac{1}{D}) - 1] - \sin\theta\sinh(\tau - \frac{1}{D})\}$	$\phi_2 \geq \phi > \phi_8$	$\phi_8 \geq \phi > -\frac{\pi}{2}$

TABLE V

(V) $-\frac{3}{4}\pi > \tan^{-1}(\tanh\frac{1}{D}) \geq \theta > \tan^{-1}(\tanh\frac{1}{2D}) > -\pi$ τ | τ_i's δ & ϕ_0 | ϕ	$\frac{\pi}{2} > \phi \geq \phi_1$ $\phi_1 = 0$	$\phi_1 > \phi > \phi_6$ $\phi_6 = -\tan^{-1}\{\cos\theta[\cosh(\frac{4}{3}\tanh^{-1}(\tan\theta)-(\frac{\tau}{3}+\frac{1}{3D}))-1] - \sin\theta\sinh(\frac{4}{3}\tanh^{-1}(\tan\theta)-(\frac{\tau}{3}+\frac{1}{3D}))\}$ $\phi_1 > \phi > \phi_2$ $\phi_2 = -\tan^{-1}\{\cos\theta[\cosh(\tau-\frac{1}{D})-1] - \sin\theta\sinh(\tau-\frac{1}{D})\}$	$\phi_6 \geq \phi > \phi_5$ $\phi_5 = -\tan^{-1}\{\cos\theta[\cosh(\tanh^{-1}(\tan\theta)-(\frac{\tau}{4}-\frac{1}{4D}))-1] - \sin\theta\sinh[\tanh^{-1}(\tan\theta)-\frac{1}{4}(\tau-\frac{1}{D})]\}$ $\phi_2 \geq \phi > \phi_5$	$\phi_5 \geq \phi > -\frac{\pi}{2}$
$0 \leq \tau \leq \frac{1}{D}$ f is monotone on $\tau \in [0, \frac{1}{2D}]$	$\tau_1 = \tau_2 = \tau_3 = \tau_4 = 0$, $\delta = 0$			
$\frac{1}{D} < \tau \leq \tanh^{-1}(\tan\theta) + \frac{1}{2D}$	$\tau_1 = \tau_2 = 0$ $\tau_3 = \frac{1}{D}$ $\tau_4 = \tau$ $\delta = 1$, $\phi_0 = \phi_1$	$\tau_1 = \tau_2 = 0$ $\tau_3 = \frac{1}{D}$ $\tau_4 = \tau - \ln[\alpha - \beta]$ $\delta = 0$	$\tau_1 = \tau_2 = \tau_3 = \tau_4 = 0$ $\delta = 0$	$\tau_1 = \tau_2 = \tau_3 = \tau_4 = 0$ $\delta = 1$, $\phi_0 = \phi_5$
$\tanh^{-1}(\tan\theta) + \frac{1}{2D} < \tau$ $< 4\tanh^{-1}(\tan\theta) - \frac{1}{D}$	$\tau_1 = \tau_2 = 0$ $\tau_3 = \frac{4}{3}[\tau - \tanh^{-1}(\tan\theta)] + \frac{1}{3D}$ $\tau_4 = \tau$ $\delta = 1$, $\phi_0 = \phi_1$	$\tau_1 = \tau_2 = 0$ $\tau_3 = \frac{4}{3}[\tau - \tanh^{-1}(\tan\theta)] + \frac{1}{3D}$ $\tau_4 = \tau - \ln[\alpha - \beta]$ $\delta = 0$	$\tau_1 = \tau_2 = \tau_3 = \tau_4 = 0$ $\delta = 0$	, ,
$4\tanh^{-1}(\tan\theta) - \frac{1}{D} \leq \tau$ $\leq \frac{4}{3}\tanh^{-1}(\tan\theta) + \frac{1}{D}$	$\tau_1 = \tau_2 = \tau_3 = \tau_4 = 0$ $\delta = 1$, $\phi_0 = \phi_3$	$\tau_1 = \tau_2 = \tau_3 = \tau_4 = 0$ $\delta = 0$		, ,
$\frac{4}{3}\tanh^{-1}(\tan\theta) + \frac{1}{D} < \tau$ $\leq \tanh^{-1}(\tan\theta) + \frac{2}{D}$	, ,	$\tau_1 = \tau_2 = \tau_3 = \tau_4 = 0$ $\delta = 0$	$\tau_1 = \tau_2 = 0$ $\tau_3 = 1/D$ $\tau_4 = \tau - \ln[\alpha + \beta]$ $\delta = 0$	$\tau_1 = \tau_2 = 0$ $\tau_3 = 1/D$ $\tau_4 = \tau - \frac{4}{3}\tanh^{-1}(\tan\theta)$ $\delta = 1$, $\phi_0 = \phi_8$
$\tanh^{-1}(\tan\theta) + \frac{2}{D}$ $< \tau < \infty$	$\tau_1 = \tau_2 = 0$ $\tau_3 = 1/D$ $\tau_4 = \tau - 2/D$ $\delta = 1$, $\phi_0 = \phi_4$	$\tau_1 = \tau_2 = 0$ $\tau_3 = 1/D$ $\tau_4 = \tau - \ln[\alpha + \beta]$ $\delta = 0$		, ,
τ | τ_i's δ & ϕ_0 | ϕ $-\frac{3}{4}\pi > \tan^{-1}(\tanh\frac{1}{D}) \geq \theta > \tan^{-1}(\tanh\frac{1}{2D}) > -\pi$	$\frac{\pi}{2} > \phi \geq \phi_4$ $\phi_4 = \tan^{-1}\{\cos\theta[1-\cosh\frac{2}{D}] + \sin\theta\sinh\frac{2}{D}\} \geq 0$ $\frac{\pi}{2} > \phi \geq \phi_3$ $\phi_3 = \tan^{-1}\{\cos\theta[1-\cosh(\frac{\tau}{2}+\frac{1}{2D})] + \sin\theta\sinh(\frac{\tau}{2}+\frac{1}{2D})\} \geq 0$	$\phi_4 > \phi > \phi_8$ $\phi_8 = -\tan^{-1}\{\cos\theta[\cosh(\frac{4}{3}\tanh^{-1}(\tan\theta))-1] - \sin\theta\sinh(\frac{4}{3}\tanh^{-1}(\tan\theta))\}$ $\phi_3 > \phi > \phi_2$ $\phi_2 = -\tan^{-1}\{\cos\theta[\cosh(\tau-\frac{1}{D})-1] - \sin\theta\sinh(\tau-\frac{1}{D})\}$ $\phi_3 > \phi > \phi_5$ $\phi_5 = -\tan^{-1}\{\cos\theta[\cosh(\tanh^{-1}(\tan\theta)-\frac{1}{4}(\tau-\frac{1}{D}))-1] - \sin\theta\sinh(\tanh^{-1}(\tan\theta)-\frac{1}{4}(\tau-\frac{1}{D}))\}$	$\phi_2 \geq \phi > \phi_8$	$\phi_8 \geq \phi > -\frac{\pi}{2}$ $\phi_5 \geq \phi > -\frac{\pi}{2}$

TABLE VI

(VI) $\frac{3}{4}\pi > \tan^{-1}\left(\tanh\frac{1}{2D}\right) \geqslant \theta > -\pi$ ϕ τ_i's δ & ϕ_o τ	$\frac{\pi}{2} > \phi \geqslant \phi_3$ $\phi_3 = \tan^{-1}\{\cos\theta[1-\cosh(\frac{\tau}{2}+\frac{1}{2D})] + \sin\theta\sinh(\frac{\tau}{2}+\frac{1}{2D})\}$ $\geqslant 0$	$\phi_3 > \phi > \phi_2$ $\phi_2 = -\tan^{-1}\{\cos\theta[\cosh(\tau-\frac{1}{D})-1] - \sin\theta\sinh(\tau-\frac{1}{D})\}$	$\phi_2 \geqslant \phi > \phi_8$ $\phi_8 = -\tan^{-1}\{\cos\theta[\cosh(\frac{2}{3}\tanh^{-1}(\tan\theta))-1] - \sin\theta\sinh(\frac{2}{3}\tanh^{-1}(\tan\theta))\}$	$\phi_8 \geqslant \phi > -\frac{\pi}{2}$
		$\phi_3 > \phi > \phi_5$ $\phi_5 = -\tan^{-1}\{\cos\theta[\cosh(\tanh^{-1}(\tan\theta)-(\frac{\tau}{4}-\frac{1}{4D}))-1] - \sin\theta\sinh(\tanh^{-1}(\tan\theta)-(\frac{\tau}{4}-\frac{1}{4D}))\}$		$\phi_5 \geqslant \phi > -\frac{\pi}{2}$
$0 \leqslant \tau \leqslant \frac{1}{D}$	$\tau_1 = \tau_2 = \tau_3 = \tau_4 = 0$,		$\delta = 0$	
$\frac{1}{D} < \tau \leqslant 2\tanh^{-1}(\tan\theta)+\frac{1}{D}$	$\tau_1 = \tau_2 = \tau_3 = \tau_4 = 0$ $\delta = 1$, $\phi_o = \phi_3$	$\tau_1 = \tau_2 = \tau_3 = \tau_4 = 0$ $\delta = 0$		$\tau_1 = \tau_2 = \tau_3 = \tau_4 = 0$ $\delta = 1$, $\phi_o = \phi_5$
$2\tanh^{-1}(\tan\theta)+\frac{1}{D} < \tau \leqslant \frac{3}{D}$	" "	$\tau_1 = \tau_2 = \tau_3 = \tau_4 = 0$ $\delta = 0$	$\tau_1 = \tau_2 = 0$ $\tau_3 = 1/D$ $\tau_4 = \tau - \ln[\alpha+\beta]$ $\delta = 0$	$\tau_1 = \tau_2 = 0$ $\tau_3 = 1/D$ $\tau_4 = \tau - \frac{4}{3}\tanh^{-1}(\tan\theta)$ $\delta = 1$, $\phi_o = \phi_8$
$\frac{3}{D} < \tau < \infty$	$\tau_1 = \tau_2 = 0$ $\tau_3 = 1/D$ $\tau_4 = \tau - 2/D$ $\delta = 1$, $\phi_o = \phi_4$	$\tau_1 = \tau_2 = 0$ $\tau_3 = 1/D$ $\tau_4 = \tau - \ln[\alpha+\beta]$ $\delta = 0$		" "
τ τ_i's δ & ϕ_o $-\frac{3}{4}\pi > \tan^{-1}\left(\tanh\frac{1}{2D}\right) \geqslant \theta > -\pi$ ϕ	$\frac{\pi}{2} > \phi \geqslant \phi_4$ $\phi_4 = \tan^{-1}\{\cos\theta[1-\cosh\frac{2}{D}] + \sin\theta\sinh\frac{2}{D}\}$ $\geqslant 0$	$\phi_4 > \phi > \phi_8$ $\phi_8 = -\tan^{-1}\{\cos\theta[\cosh(\frac{2}{3}\tanh^{-1}(\tan\theta))-1] - \sin\theta\sinh(\frac{2}{3}\tanh^{-1}(\tan\theta))\}$		$\phi_8 \geqslant \phi > -\frac{\pi}{2}$

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PART C

STOCHASTIC OPTIMAL CONTROL

1. INTRODUCTION

Stochastic control problems are concerned with the control of dynamical systems which are random in some sense. A programmed control of antenna pointing system for the spacecraft is one of many practical applications. As an example, a programmed antenna requires excessive preflight calibration to permit reduction of fixed errors and compensation of variable errors within the requirements of a prescribed geometry of the antenna. It is known that the relative accuracy of this approach, even with comprehensive compensation equipment, is questionable. A possible way of improving the accuracy is to formulate the problem as a stochastic optimal control problem such that the spacecraft and its antenna pointing system are acted upon by random perturbations. The problem then involves the determination of a control law which maximizes the probability of the antenna's pointing at the desired direction within a prescribed allowable tolerance.

In the following sections, a discussion on a method of stochastic optimal control is presented. Section 2 gives a brief summary of the work done to date in this area. Section 3 presents the formulation of the stochastic problem that is under investigation. The method of solving the problem is discussed in Section 4. Because of the complexity of the stochastic problem, the notation for the mathematical description is unavoidably tedious. Appendix I is written for the purpose of clarifying the definition of the notation which will be used repeatedly in the discussion of the problem. Appendix II explains the difficulties of the mathematical treatment of stochastic differential equations, and the equivalence of formulation between stochastic differential equations and integral equations. The stochastic pursuit problem solved by Mishchenko and Pontryagin [11] is summarized in Appendix III in which the notation is carefully selected to agree with the definitions given in Appendix I. Their results constitute a part of the solution to the stochastic control problem. Appendix IV discusses the computation of the transition density of the stochastic control process.

2. BRIEF REVIEW OF STOCHASTIC CONTROL PROBLEMS

A great deal of the work done to date in the field of stochastic optimal control has been an attempt to develope the subject along lines analagous to the deterministic theory by dealing with expectations of certain random variables, e.g.

integral performance indices. Kalman [1] for example, solves the following problem. Let a state vector x of dimension m be defined by

$$\frac{dx}{dt} = F(t)\ x + G(t)\ w(t) \tag{2.1}$$

where F(t) is an m by m matrix whose elements are continuous functions of t,

w(t) is a random vector of dimension $\ell \leq m$

G(t) is an m by ℓ matrix whose elements are continuous functions of t.

Let a vector z of dimension $k \leq m$ be defined by

$$z = H(t)\ x + v(t) \tag{2.2}$$

where H(t) is a k by m matrix whose elements are continuous in t,

v(t) is a random vector of dimension k.

Furthermore, w and v are sample functions of independent random processes with zero mean and covariance matrices of the form

$$\mathrm{cov}[w(t),\ w(\tau)] = Q(t)\ \delta(t-\tau) \tag{2.3}$$

$$\mathrm{cov}[v(t),\ v(\tau)] = R(t)\ \delta(t-\tau) \tag{2.4}$$

where Q is a time-varying, symmetric, nonnegative definite, continuously differentiable, ℓ by ℓ matrix,

R is a time-varying, symmetric, positive definite, continuously differentiable, k by k matrix.

Then Kalman [1] finds an estimate $\hat{x}(t_1|t)$ of $x(t_1)$ of the form

$$\hat{x}(t_1|t) = \int_{t_0}^{t} A(t_1, \tau)\, z(\tau)\, d\tau, \qquad (2.5)$$

where A is an m by k continuously differentiable matrix, which minimizes

$$E_{z(\tau),\ t_0 \le \tau \le t} \langle Bx,\ x(t_1) - \hat{x}(t_1|t) \rangle \qquad (2.6)$$

where $E_{z(\tau),\ t_0 \le \tau \le t}$ denotes expectation conditioned on $z(\tau)$, $t_0 \le \tau \le t$,

B is a specified m by m constant matrix, and $\langle , \rangle$ denotes inner product.

Kushner [2] considers a system defined by

$$\frac{dx}{dt} = f(x, u) + \xi \qquad (2.7)$$

where f is linear in x and u

x is an m- dimensional vector.

u is an $\ell \le m$ dimensional control

ξ is an m- dimensional random vector.

with a cost criterion

$$E \int_{0}^{T} g(x, u)\, dt \qquad (2.8)$$

where g is quadratic in x and u, and T is fixed. He gives a method of correction to optimal deterministic control when the effects of ξ are small. He extends this work [3], [4], [5], [6] to develop a stochastic maximum principle, complete with adjoint equations and a stochastic version of the Hamiltonian, for the minimization of

$$E \langle c, x(T) \rangle \qquad (2.9)$$

subject to $dt = f(x, u)\ dt + \sigma(x, u)\ dz \qquad (2.10)$

where c and x are m- dimensional vectors

u is an $\ell \leq m$ dimensional control

$\sigma(x, u)$ is a weighting matrix

z is a sample vector of a random process.

Later, Kushner [7] solves essentially the same problem by a technique involving a stochastic version of Lyapunov functions. In this paper, z is assumed to be a sample function of a Wiener process. In order for (2.10) to be meaningful, it is necessary to interpret it according to Ito (see [8], chapter VI, §3).

The theory of stochastic stability referred to in [7] is developed by Wonham [9] and Kushner [10]. Various types of stability were also defined and then discussed.

In most cases, the expectation of a random variable is not the most appealing performance index. As a matter of fact, such an index is used mainly for the mathematical convenience. It is true that even if the distribution of a random variable is not known, its variance places a bound on the probability of its displacement from its mean. This is, however, a rather crude bound. Only in special cases, notably the Gaussian case, does a knowledge of expectations give very precise information about error probabilities.

Pontryagin and Mischenko [11] have solved an entirely different problem. They considered a controlled point in the state space in pursuit of another point. The state variables of the pursued point are sample functions of a Markov process. The performance index is the probability of "capture" of the pursued point by the controlled point during a specified time interval. The pursued point is considered "captured" if the controlled point is brought within some specified spherical neighborhood of it.

It might be argued that this approach lacks appeal on at least two grounds:

1. How often is a control system asked to guide an object toward another object whose motion is Markov?
2. The solution presented by Pontryagin and Mischenko assume only a knowledge of the initial position of the pursued object. Is the assumption realistic?

Granting the validity of these points, the work done by Pontryagin is applicable, after some manipulation and additional development, to a more appealing problem. This problem is the subject of this research. For the purpose of convenient reference, the pursuit problem and its solution are described in more detail in Appendix III.

3. PROBLEM STATEMENT

The problem to be considered is that of bringing the state of a system under the influence of additive noise from an initial state to some spherical neighborhood of the state space origin within a time interval with a maximum probability. The formulation of the problem is based on the following reasons:

1. It is a logical modification of a well-known class of deterministic optimal control problems,
2. A broad class of engineering systems are subject to additive noise, and
3. The performance index of maximum probability is highly appealing.

The investigation will be aimed at the linear case. To be more precise, it is desired to determine the vector u which maximizes

$$\mathrm{Prob}[\,\|\, x\,(\tau)\,\| \le \epsilon\,] \text{ for some } 0 \le \tau \le T \qquad (3.1)$$

subject to

$$\frac{dx}{dt} = A(t)\, x + B(t)\, u + C(t)\, n;\; x(0) = x_0 \qquad (3.2)$$

where x is an m dimensional vector,

A is an m by m matrix,

u is an $\ell < m$ dimensional vector,

B is an m by ℓ matrix,

n is a $k \leq m$ dimensional sample vector of a random process,

C is an m by k matrix,

ϵ, T, and x_0 are given as part of the problem.

The determination of the restrictions on A, B, C, u, and n are part of the research problem.

4. TECHNICAL APPROACH

Method of Approach

The problem described in the preceding section will be investigated in the following manner:

(A) Compute the transition densities of a state vector defined by

$$\frac{dz}{dt} = -A(t)z - C(t)n \qquad z(0) = 0 \tag{4.1}$$

(B) Compute u to maximize

$$\text{Prob}\left[\|\, y(\tau) - z(\tau)\,\| < \epsilon\right]$$

subject to

$$\frac{dy}{dt} = A(t)y + B(t)u \qquad (4.2)$$

$$y(0) = x_0$$

The method of approach is motivated by the advantage of the superposition property of linear systems. A proper translation of the coordinate-system reduces the present problem to Mishchenko-Pontryagin's pursuit problem which is summarized in Appendix III. Thus, if the statistics of the z-process is in agreement with the hypotheses for the pursuit problem, then the known results can be used to complete the solution. The work to date has been concerned with establishing the conditions on the n-process which make the z-process Markov, and the computation of the z-process transition densities from the n-process statistics.

Preliminary Results

A preliminary result (see Appendix II) is that if the z-process is Markov, then it is also a process with independent increments, and the n-process is white noise. These properties (see [8],[12]) require that (4.1) must be written as a stochastic equation. Appendix IV shows how it may be possible to estimate the transition densities of z in terms of the statistics of n.

Plan of Future Work

The investigation will be divided roughly into two categories, viz., theoretical and computational.

The theoretical work includes the following items:

1. The amplification of the work in Appendices II and IV into rigorous arguments. In particular, the assumption of the convergence of sequence $\left\{p_{I^i_{\xi\eta}}\right\}$ discussed in Appendix IV must be justified.
2. An error estimate for $p_{I^i_{\xi\eta}} - p_{I_{\xi\eta}}$ will be developed.
3. The conditions on the noise that guarantee z-process satisfying the hypotheses of the pursuit problem [see Appendix III] will be determined.

The computational work includes:

1. The development of a computer program for the computation of
   a) z transition densities from noise densities;
   b) optimal controls from z transition densities;
   c) probability of "capture" under optimal control.
2. The use of this algorithm to investigate the relationships among
   a) noise statistics,
   b) optimal control signal,

c) system parameters,

d) optimal performance, and

e) initial conditions, time interval, and target neighborhood size.

APPENDIX I

The purpose of this appendix is to define notation. Let $(\Omega, \mathcal{B}, \mu)$ be a probability space. Let $Y_1, Y_2, \ldots$ be random vectors from Ω to E^m. That is,

$$Y_i = \begin{bmatrix} Y_i^1 \\ Y_i^2 \\ \vdots \\ Y_i^m \end{bmatrix} \tag{I.1}$$

and for every $\omega \in \Omega$, there is a $y \in E^m$ such that

$$y = \begin{bmatrix} y^1 \\ y^2 \\ \vdots \\ y^m \end{bmatrix} = Y_i(\omega) = \begin{bmatrix} Y_i^1(\omega) \\ \vdots \\ Y_i^m(\omega) \end{bmatrix} \tag{I.2}$$

Furthermore, for every Borel set B in E^m,

$$Y_i^{-1}(B) \in \mathcal{B} \tag{I.3}$$

The distribution function $P_{Y_i}(y)$, from E^m to the real line, is defined as follows: Let $A \in \mathcal{B}$ be the set of $\omega \in \Omega$ for which

$$Y_i(\omega) < y \tag{I.4}$$

or, in component-wise notation,

$$Y_i^j(\omega) < y^j,\ j = 1,\ \cdots,\ m \tag{I.5}$$

then

$$P_{Y_i}(y) = \mu\ (A) \tag{I.6}$$

The joint distribution function

$$P_{Y_i,\ Y_{i+1},\ldots,Y_{i+k}}(y_i,\ y_{i+1},\ldots,y_{i+k})$$

from the $E^m \times E^m \times \cdots \times E^m$ ($k + 1$ factors) product space to the real line is defined as follows. Let $A_i \in \mathcal{B}$ be the set of $\omega \in \Omega$ for which

$$Y_i(\omega) < y_i \tag{I.7}$$

Then $$P_{Y_i,Y_{i+1},\ldots,Y_{i+k}}(y_i,\ldots,y_{i+k}) = \mu\left[\bigcap_{j=i}^{i+k} A_j\right] \tag{I.8}$$

The definition of the conditional distribution function

$$P_{Y_1,Y_2,\ldots,Y_k|Y_{k+1},\ldots,Y_{k+\ell}}(y_1,y_2,\ldots y_k|y_{k+1},\ldots,y_{k+\ell}) \tag{I.9}$$

is defined as follows. Let Ai be the ω set for which

$$Yi(\omega) < y_i,\ i = 1,\ \ldots,\ k \tag{I.10}$$

Let Bi be a Borel set in E^m. Let

$$C_i = Y^{-1}_{k+i}(B_i) \qquad i + 1, \ldots, \ell \tag{I.11}$$

Let

$$C = \bigcap_{i=1}^{\ell} C_i \tag{I.12}$$

Let B be the set in $E^m \times E^m \times \ldots \times E^m$ (ℓ factor),

$$B = \prod_{i=1}^{\ell} B_i \tag{I.13}$$

Then the function of (I.9) is defined as that function satisfying

$$\int_B p_{Y_1,\ldots Y_k|Y_{k+1},\ldots,Y_{k+\ell}}(y_1,\ldots,y_k|y_{k+1},\ldots,y_{k+\ell})$$

$$dP_{Y_{k+1},\ldots,Y_{k+\ell}}(y_{k+1},\ldots,y_{k+\ell}) = \mu\ (A \cap C) \ \forall\ B \tag{I.14}$$

Finally, the conditional density

$$p_{Y_1,Y_2,\ldots,Y_k|Y_{k+1},\ldots,Y_{k+\ell}}(y_1,y_2,\ldots,y_k|y_{k+1},\ldots,y_{k+\ell}) \tag{I.15}$$

is defined as

$$\left\{\frac{\partial^k}{\partial w_1\ \partial w_2 \ldots \partial w_k}\left[P_{Y_1,\ldots Y_k|Y_{k+1},\ldots,Y_{k+\ell}}(w_1,\ldots,w_k|y_{k+1},\ldots,y_{k+\ell}\right]\right\}$$

evaluated at $w_i = y_i$,

$$i = 1,\ldots,k \tag{I.16}$$

APPENDIX II

This appendix discusses the circumstances under which the process defined by (II.3) is Markov. The fact that the point of definition is (II.3) rather (II.1) has much significance. As a matter of fact $\dot{z}$ does not even exist. This follows from the fact that the z process, as will be shown, is a process with independent increments [8], which creates a mathematical difficulty for (II.7). This difficulty, however, can be avoided by utilizing the concept of stochastic differential equations [8], [12]. The integral of (II.3) exists in a very slightly modified form as a stochastic integral [8]. This form of integration is compatible with the arguments contained in the remainder of the Appendix.

Consider a random vector $z(t)$ defined by

$$\dot{z}(t) = A(t)\ z(t) + C(t)\ n(t),\ z(0) = 0 \qquad \text{(II.1)}$$

where $n(t)$ is also a random vector with less than or equal to the number of elements of $Z(t)$,

$A(t)$ and $C(t)$ are matrices of suitable dimension

Let $\Phi(t,\tau)$ satisfy

$$\frac{d}{dt} \Phi(t,\tau) = A(t)\ \Phi(t,\tau),\ t \geq \tau \qquad \text{(II.2)}$$

$\Phi(\tau,\tau)$ = identy matrix with same rank as A.

Then

$$z(t) = \int_0^t \Phi(t,\tau)\ C(\tau)\ n(\tau)\ d\tau,\ t \geq 0 \qquad \text{(II.3)}$$

provided the integral exists. The question to be answered is: Supposing the z process is Markov. What does this say about the n process?

In order to facilitate the discussion, some notation will be introduced first. Let

$$I_{0,t_j} = \int_0^{t_j} \Phi(t_j,\tau)\ C(\tau)\ n(\tau)\ d\tau \qquad \text{(II.4)}$$

$$I_{t_j t_k} = \int_{t_j}^{t_k} \Phi(t_k,\tau)\ C(\tau)\ n(\tau)\ d\tau \qquad \text{(II.5)}$$

In the subscripting of t, the convention

$$j > k > 0 \Rightarrow t_j > t_k > 0 \qquad \text{(II.6)}$$

(e.g. $t_4 > t_3 > t_2 > t_1 > 0$)

will always be followed. An obvious identity which will be used repeatedly (tacitly) is

$$\forall\, t_j > t_k > t_\ell,\ I_{t_\ell t_j} = I_{t_\ell t_k} + I_{t_k t_j}. \tag{II.7}$$

If the z process is Markovian, it is necessary that

$$p_{I_{0t_3}|I_{0t_2},I_{0t_1}}(z_3|z_2,z_1) = p_{I_{0t_3}|I_{0t_2}}(z_3|z_2). \tag{II.8}$$

It will now be shown that equation (II.8) implies

$$p_{I_{t_2t_3},I_{t_1t_2},I_{0t}}(z_3-z_2, z_2-z_1, z_1) =$$

$$p_{I_{t_2t_3}}(z_3-z_2)p_{I_{t_1t_2}}(z_2-z_1)p_{I_{0t_1}}(z_1).$$

This will be done in two steps. A relevant lemma will be discussed first.

<u>Lemma</u> If $p_{I_{0t_3}|I_{0t_2},I_{0t_1}}(z_3|z_2,z_1) = p_{I_{0t_3}|I_{0t_2}}(z_3|z_2)$,

then $p_{I_{t_2t_3}|I_{0t_2}}(z_3-z_2|z_2) = p_{I_{t_2t_3}}(z_3-z_2)$

<u>Proof</u> Note that

$$p_{I_{0t_3}|I_{0t_2}}(z_3|z_2) = p_{I_{t_2t_3}|I_{0t_2}}(z_3-z_2|z_2) \tag{II.9}$$

$$p_{I_{0t_3}|I_{0t_2},I_{0t_1}}(z_3|z_2,z_1) = p_{I_{t_2t_3}|I_{t_1t_2},I_{0t_1}}(z_3-z_2|z_2-z_1,z_1) \tag{II.10}$$

the hypothesis, then is

$$p_{I_{t_2t_3}|I_{0t_2}}(z_3-z_2|z_2) = p_{I_{t_2t_3}|I_{t_1t_2},I_{0t_1}}(z_3-z_2|z_2-z_1,z_1) \tag{II.11}$$

The only quantity upon which both sides of (II.11) are dependent is z_3-z_2. Therefore,

$$p_{I_{t_2t_3}|I_{t_1t_2},I_{0t_1}}(z_3-z_2|z_2-z_1,z_1) = g(z_3-z_2) \tag{II.12}$$

Next, note that

$$p_{I_{t_2t_3}}(z_3-z_2) = \int d\nu_1 \int d\nu_2\, p_{I_{t_2t_3}|I_{t_1t_2},I_{0t_1}}(z_3-z_2|\nu_1,\nu_2)\, p_{I_{t_1t_2},I_{0t_1}}(\nu_1,\nu_2) \tag{II.13}$$

or, from (II.12),

$$p_{I_{t_2t_3}}(z_3-z_2) = \int d\nu_1 \int d\nu_2\, g(z_3-z_2)\, p(\nu_1,\nu_2)$$

or,

$$p_{I_{t_2t_3}}(z_3-z_2) = g(z_3-z_2) \tag{II.14}$$

From (II.11), (II.12), and (II.14)

$$p_{I_{t_2t_3}|I_{0t_2}}(z_3 - z_2 | Z_2) = p_{I_{t_2t_3}}(z_3 - z_2) \qquad \text{(II.15)}$$

Q.E.D.

<u>Theorem</u> If $p_{I_{0t_3}|I_{0t_2}, I_{0t_1}}(z_3 | z_2, z_1) = p_{I_{0t_3}|I_{0t_2}}(z_3 | z_2)$,

then $p_{I_{t_2t_3}, I_{t_1t_2}, I_{0t_1}}(z_3 - z_2, z_2 - z_1, Z_1) = p_{I_{t_2t_3}}(z_3 - z_2)\, p_{I_{t_1t_2}}(z_2 - z_1)\, p_{I_{0t_1}}(z_1)$

<u>Proof</u> By the Lemma,

$$p_{I_{t_2t_3}}(z_3 - z_2) = P_{I_{0t_3}|I_{0t_2}, I_{0t_1}}(z_2 | z_3, z_1) \qquad \text{(II.16)}$$

Now

$$p_{I_{0t_3}|I_{0t_2}, I_{0t_1}}(z_3 | z_2, z_1) = \frac{p_{I_{t_2t_3}, I_{t_1t_2}, I_{0t_1}}(z_3 - z_2, z_2 - z_1, z_1)}{p_{I_{t_1t_2}, I_{0t_1}}(z_2 - z_1, z_1)} \qquad \text{(II.17)}$$

Also from the Lemma,

$$p_{I_{t_1t_2}, I_{0t_1}}(z_2 - z_1, z_1) = p_{I_{t_1t_2}}(z_2 - z_1)\, p_{I_{0t_1}}(z_1), \qquad \text{(II.18)}$$

hence (II.16), (II.17), and (II.18) yield

$$p_{I_{t_2t_3}, I_{t_1t_2}, I_{0t_1}}(z_3 - z_2, z_2 - z_1, z_1) = p_{I_{t_2t_3}}(z_3 - z_2)\, p_{I_{t_1t_2}}(z_2 - z_1)\, p_{0t_1}(z_1)$$

Q.E.D.

This established the fact that any three successive increments are independent. A similar handling for m successive increments yields the same result. The z process, then, is one with independent increments.

Next, apply the law of the mean to $I_{t_j t_{j+1}}$ and $I_{t_{j+2}, t_{j+3}}$:

$$I_{t_j t_{j+1}} = \Phi(t_{j+1}, \xi)\ C(\xi)\ n(\xi)(t_{j+1} - t_j), \text{ for some } \xi,\ t_j \leq \xi \leq t_{j+1}$$

$$I_{t_{j+2} t_{j+3}} = \Phi(t_{j+3}, \nu)\ C(\nu)\ n(\nu)(t_{j+3} - t_{j+2}), \text{ for some}$$

$$\nu,\ t_{j+2} \leq \nu \leq t_{j+3}$$

Since the z process is one with independent increments, $I_{t_j t_{j+1}}$ and $I_{t_{j+2} t_{j+3}}$ and independent for every choice of $0 < t_j, < t_{j+1} < t_{j+2} < t_{j+3}$. Then $n(\xi)$ is dependent of $n(\nu)$ for every $0 < \xi < \nu$. This means the n process being "white."

APPENDIX III

The pursuit problem [11] can be stated as follows. Let x be the m- dimensional state vector of a system defined by

$$\frac{dx}{dt} = f(x,\ t,\ u) \qquad \text{(III.1)}$$

$$x(0) = x_0$$

where u is the control vector.

Let z be the state vector of a randomly moving point. Given that z is a sample function of a Markov process with transition density

$$p(\sigma,\ \eta,\ \tau,\ \zeta) = p_{z(\tau)|z(\sigma)}(\zeta|\eta) \qquad \text{(III.2)}$$

where the right-hand side is as defined in Appendix I.

Further conditions imposed on the process are:

$$\forall\ \delta > 0,\ \lim_{\Delta\sigma\to 0} \frac{1}{\Delta\sigma} \int_{|\zeta-\eta| \geq \delta} p(\sigma - \Delta\sigma,\ \eta,\ \sigma,\ \zeta)\ d\zeta = 0 \qquad \text{(III.3)}$$

$$\frac{\partial^2}{\partial\eta_i\partial\eta_j} p(\sigma,\ \eta,\ \tau,\ \zeta) \text{ exists and is continuous for every} \qquad \text{(III.4)}$$

$$\sigma,\ \eta,\ \tau,\ > \sigma,\ \zeta,$$

$$i = 1,\ 2,\ \ldots,\ m.$$

$$\forall\ \delta > 0,\ \lim_{\Delta\sigma\to 0} \frac{1}{\Delta\sigma} \int_{|\zeta-\eta| \geq \delta} (\zeta_i - \eta_i)\ p(\sigma - \Delta\sigma,\ \eta,\ \sigma,\ \zeta)\ d\zeta = b_i(\sigma,\ \eta) \qquad \text{(III.5)}$$

$$i = 1,\ 2,\ \ldots,\ m.$$

$$\forall\, \delta > 0, \lim_{\Delta\sigma \to 0} \frac{1}{\Delta\sigma} \int_{|\zeta - \eta| \geq \delta} (\zeta_i - \eta_i)(\zeta_j - \eta_j) p(\sigma - \Delta\sigma, \eta, \sigma, \zeta)\, d\zeta = a_{ij}(\sigma, \eta)$$

$$i, j = 1, \ldots, m. \qquad \text{(III.6)}$$

$$z(0) = z_0 \text{ is known} \qquad \text{(III.7)}$$

$a_{ij}(\sigma, \eta)$, $b_i(\sigma, \eta)$ are continuous for $\sigma > 0$ (III.8)

The eigenvalues of $[a_{ij}]$ are positive and bounded, (III.9)

and

$$b_i(\sigma, \eta) = O(e^{|\eta|}) \qquad \text{(III.10)}$$

The problem is to find u that maximizes the probability that

$$|\,|\, z(\tau) - x(\tau) \,|\,| < \epsilon \qquad \text{(III.11)}$$

for a given $\epsilon > 0$ and for some $\tau \epsilon [0, T]$ where T is given.

The problem is solved as follows. The functional $\phi_u(\sigma, y, \tau)$ is defined as the probability that the randomly moving point is captured between times σ and τ given that $z(\sigma) = y$, and that the control function is u. If the functional ϕ_u were available, it would be straightforward to apply the maximum principle, and thus solve the problem. The following is an outline of Pontryagin's approximation to $\phi_u(\sigma, y, \tau)$.

The first step is to show that $\phi_u(\sigma, y, \tau)$ is a solution to

$$\frac{\partial\psi}{\partial\sigma} + \sum_{i,j} a_{ij} \frac{\partial^2 \psi}{\partial y_i \partial y_i} + \sum_i b_i \frac{\partial\psi}{\partial y_i} = 0 \qquad \text{(III.12)}$$

subject to the boundary conditions

$$\psi(\tau, y, \tau) = 0$$

$$\psi(\sigma, y, \tau) \Big|_{S\sigma} = 1 \qquad \text{(III.13)}$$

where $S\sigma$ = surface defined by $\| x(\sigma) - z(\sigma) \| = \epsilon$

A solution is then obtained in the form

$$\psi(0, Z_0, T) = \epsilon^{m-2}\Gamma(0, Z_0, T) + o(\epsilon^{n-2}) \qquad \text{(III.14)}$$

where m is the dimension of the state space, and Γ, for the case where a_{ij} and b_i are independent of σ or y, is given by

$$\Gamma(0, Z_0, T) = \Gamma_0(0, Z_0, T) + \Gamma_1(0, Z_0, T) \qquad \text{(III.15)}$$

where

$$\Gamma_0(S, y, \tau) = \frac{\alpha}{[(y-x_0)'[a_{ij}](y-x_0)]^{(m-2)/2}} \qquad \text{(III.16)}$$

$$- \int d\eta \, \frac{\alpha \sqrt{\prod_{i}^{N} \lambda_i} \, \exp\left\{ \frac{\sqrt{(\eta-y+x_0)'[a_{ij}](y-x+x_0)}}{4(\tau-S)} \right\}}{[2\pi(\tau-S)]^{m/2} \left[\eta'[a_{ij}]\eta\right]^{(m-2)/2}}$$

$$\Gamma_1(0, Z_0, T) = \int_0^T ds \int p_Z(0, Z_0, s, y)$$

$$\sum_{i=1}^{m} \left\{ b_i - f_i[x(s), u(s)] \right\} \frac{\partial \Gamma_0(s, y, T)}{\partial y_i} \, dy \tag{III.17}$$

λ_i are the eigenvalues of $[a_{ij}]$ (III.18)

$$\alpha = \frac{\int_S \nu_0(\bar{\eta})\, dS}{\int_S \frac{\nu_0(\bar{\eta})}{r^{m-2}(\eta)}\, dS} \tag{III.19}$$

$\nu_0(\eta)$ = eigenfunction satisfying (III.20)

$$\nu(\eta) = \int \frac{1}{2\pi} \frac{\cos \varphi}{\rho^{m-1}(\eta_1, \eta)} \nu(\eta_1)\, dS \tag{III.21}$$

S = a continuous closed surface defined by

$$\sum \lambda_i^2 (\eta_1^i)^2 = 1 \tag{III.22}$$

φ = angle between the radius vector ρ from η_1 to η

and the normal to S at η_1. (III.23)

The case for a_{ij} and b_i which are dependent upon σ and y is also solved, with results in much the same form, but slightly more complicated.

APPENDIX IV

This appendix discusses the computation of the transition densities of a process defined by a stochastic differential equation (IV.1). It is interpreted that the integrals of the two members of (IV.1) are equal. It turns out [8] that, in the usual stieltjes sense, the integral

$$\int C(t)\ dn,$$

where $C(t)$ is an m by ℓ continuous martix,

n is a sample vector (with dimension $\ell < m$) of a random process with independent and orthogonal increments.

does not exist (with probability 1) because the sample functions of processes with independent increments are of unbounded variation with probability 1. This integral can be redefined as a stochastic integral [8] so that it does exist. Under this definition, the limit of the sequence of Stieltjes sums exists in an ℓ.i.m. sense.

Consider the stochastic differential equation

$$dz = A(t)\ z\ dt + C(t)\ dn \tag{IV.1}$$

$$z(o) = 0$$

where z is an m- dimensional vector,

$A(t)$ is an m by m continuous matrix,

$C(t)$ and n are defined previously.

The solution is known as ([8], [12])

$$z(t) = \int_{o}^{t} \Phi(t, \tau)\ C(\tau)\ dn(\tau) \tag{IV.2}$$

where $\Phi(t, \tau)$ is the m by m continuous matrix satisfying

$$\frac{d}{dt}\ \Phi(t, \tau) = A(t)\ \Phi(t, \tau), \tag{IV.3}$$

$$\Phi(\tau, \tau) = \text{identity matrix}$$

and the integral in (IV.2) is a stochastic integral.

To facilitate the discussion let

$$I_{\xi\eta} = \int_{\xi}^{\eta} \Phi(\eta, \tau)\ C(\tau)\ dn(\tau) \tag{IV.4}$$

and

$$I^{i}_{\xi\eta} = \sum_{k=o}^{i-1} \Phi(\eta, \tau_k)\ C(\tau_k)\ [n(\tau_{k+1}) - n(\tau_k)], \tag{IV.5}$$

where

$$\tau_i = \eta \text{ and } \tau_0 = \xi,$$

be random variables.

The transition density

$$p_{Z(\tau_2) \mid Z(\tau_1)}(z_2 \mid z_1) = p_{I_{\tau_1 \tau_2} \mid I_0, \tau_1}(z_2 - z_1 \mid z_1) \quad \text{(IV.6)}$$

where the p's are defined in Appendix I.

The sequence $\left\{I^{i}_{\xi\eta}\right\}$ converges to $I_{\xi\eta}$ in an ℓ.i.m. sense as described by Doob [8]. A question arises as to the conditions upon which the convergence of $p_{I^i_{\xi\eta}} \to p_{I_{\xi\eta}}$ in a suitable sense as $i \to \infty$. Once the convergence is established, then, (IV.6) implies that the z- process transition densities can be approximated by the conditional densities $p_{I^i_{\tau_1\tau_2} \mid I^i_{o,\ \tau_1}}$. The investigation of the convergence problem will be deferred for the future study. The computation of the conditional densities, however, will be discussed in the following.

In order to facilitate the discussion, the problem will be restated in the following notation. Let Y_{kq} be a random vector of dimension m defined by

$$Y_{kq} = \Phi(\tau_q, \tau_k)\ C(\tau_k)[n(\tau_{k+1}) - n(\tau_k)] \quad \text{(IV.7)}$$

and S_{qr} be a random vector defined as

$$S_{qr} = \sum_{k=q}^{r} Y_{kq} \quad \text{(IV.8)}$$

To express the conditional density

$$p_{S_{qr}|S_{0,\,q-1}}(s_{qr}|s_{0,\,q-1})$$

in terms of n statistics, the following two steps are required:

(1) The Y_{kq} statistics will be written in terms of n statistics, and

(2) the desired S distributions will be written in terms of Y_{kq} statistics.

For the first of these two steps, consider the dimension of the elements of (IV.7):

Y_{kq} is an m- dimensional vector,

$n(\tau_{k+1}) - n(\tau_k)$ is an $\ell \leq m$ dimensional vector,

$\Phi(\tau_{q}, \tau_k)\ C(\tau_k)$ is an m by ℓ matrix, and is assumed to have rank ℓ.

To facilitate the discussion, let

$$\Delta n_k = n(\tau_{k+1}) - n(\tau_k) \tag{IV.9}$$

$$D_{qk} = \Phi(\tau_q \tau_k)\ C(\tau_k) \tag{IV.10}$$

Also, superscripts will be used to denote vector elements, e.g. the $i^{\underline{th}}$ element of Δn_k is Δn_k^i. Thus (IV.7) becomes

$$Y_{kq} = D_{qk} \Delta n_k \tag{IV.11}$$

From the dimensional considerations stated earlier, (IV.11) represents a mapping of E^{ℓ} into a subspace, Ψ, of E^m. The next step is to construct a suitable "coordinate" system as follows. Let $v_{1kq}, \ldots, v_{\ell kq}$ be an orthonormal basis for Ψ. Let $v_{1kq}, \ldots, v_{\ell kq}, \ldots, v_{mkq}$ be an orthonormal basis for E^m. Let $V_{\ell kq}$ be the m x ℓ matrix whose columns are $v_{1kq}, \ldots, v_{\ell kq}$, and let V_{mkq} be the m by m matrix whose columns are $v_{1kq}, \ldots, v_{mkq}$. For every Y_{kq}, there is a unique m- dimensional vector α such that

$$Y_{kq} = V_{mkq} \, \alpha_{Y_{kq}} ; \qquad \text{(IV.12)}$$

this is true because the columns of V_{mkq} form a basis for E^m. Moreover, since the first ℓ columns of V_{mkq} form a basis for Ψ, then

$$\alpha^{i}_{Y_{kq}} = 0, \; i = \ell + 1, \; ;;;, \; m, \qquad \text{(IV.13)}$$

if $Y_{kq} \in \Psi$. Thus $Y_{kq} \in \Psi$ is equivalent to

$$\left(V^{-1}_{mkq} \, Y_{kq}\right)^{i} = 0, \; i = \ell + 1, \; \ldots, \; m. \qquad \text{(IV.14)}$$

From (IV.11),

$$\Delta n_k = \left(V^{T}_{\ell kq} \, D_{qk}\right)^{-1} V^{T}_{\ell kq} Y_{kq} \qquad \text{(IV.15)}$$

where T denotes transpose. From (IV.14) and (IV.15),

$$p_{Y_{kq}}(y) = p_{\Delta n_k}\left[(V^T_{\ell kq}D_{qk})^{-1}V^T_{\ell kq}y\right] \quad \delta\left[(V^{-1}_{mkq}Y_{kq})^i,\ i = \ell + 1,\ \ldots.\ m\right] \tag{IV.16}$$

where δ is the Dirac delta. This completes the first of the two steps.

For the computation of $p_{S_{qr}|S_{0,\ q-1}}$, it is noted that the conditioning variable is a linear combination of those Δn's which do not appear in S_{qr}. Since the Δn's are independent, hence

$$p_{S_{qr}|S_{0,q-1}}(s_{qr}|s_{0,\ q-1}) = p_{S_{qr}}(s_{qr}) \tag{IV.17}$$

From (IV.17) and (IV.8),

$$p_{S_{qr}|S_{0,\ q-1}}(s_{qr}|s_{0,\ q-1}) =$$

$$\int ds_{q,\ r-1}p_{S_{q,\ r-1},\ Y_{rq}}(s_{q,\ r-1},\ s_{qr} - s_{q,\ r-1}) \tag{IV.18}$$

Since the Δn's are independent,

$$p_{S_{q,\ r-1},Y_{rq}}(s_{q,\ r-1},\ s_{qr} - s_{q,\ r-1}) =$$

$$p_{S_{q,\ r-1}}(s_{q,\ r-1})p_{Y_{rq}}(s_{qr} - s_{q,\ r-1}) \tag{IV.19}$$

Thus (IV.18) becomes

$$p_{S_{qr}|S_{0,\,q-1}}(s_{qr}|s_{0,\,q-1}) =$$

$$\int ds_{q,\,r-1} p_{Y_{rq}}(s_{qr} - s_{q,\,r-1}) p_{S_{q,\,r-1}}(s_{q,\,r-1}) \qquad \text{(IV.20)}$$

By applying the same procedure repeatedly, one obtains

$$p_{S_{qr}|S_{0,\,q-1}}(s_{qr}|s_{0,\,q-1}) =$$

$$\int ds_{q,\,r-1} p_{Y_{rq}}(s_{qr} - s_{q,\,r-1}) \int ds_{q,\,r-2} p_{Y_{r-1,\,q}}(s_{q,\,r-1} - s_{q,\,r-2})$$

$$--- \int ds_{qq} p_{Y_{q,\,q}}(s_{qq}) \qquad \text{(IV.21)}$$

which gives the conditional density in terms of Y_{kq} statistics.

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