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STATISTICAL MECHANICAL DESCRIPTION OF

PROCESSES IN COLLISIONLESS PLASMAS

MICHAEL S. MACRAKIS

NASA Electronics Research Center
Cambridge, Mass.

ABSTRACT

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By separating the microscopic equations of ~~ne~~ mo tion for the particle density and the fields in a plasma into ensemble average and fluctuating parts we have been able to obtain an unambiguous description of scattering of an electromagnetic wave from density fluctuations. The method described can be generalized to include all linear scattering processes in a plasma involving the interaction of transverse and longitudinal waves with fluctuations.

Author

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I. INTRODUCTION

In an earlier paper[1], the author had attempted an unambiguous formulation of the cross section describing fluctuation scattering and absorption in a space-dispersive collisionless plasma. In the process, an ansatz was invoked concerning the introduction of fluctuations in the Boltzmann-Vlassov (B-V) equation for a plasma. As is well known, the B-V equations are ensemble average equations for the particles and fields and as a consequence do not reflect the existence of fluctuations in a plasma; hence, the need for the aforementioned ansatz. In the present paper the same problem will be discussed in a more rigorous manner by introducing the fluctuations in the microscopic equations of motion for the particles in the plasma.

The need to introduce statistical mechanical methods for the systematic derivation of the macroscopic Maxwell's equations in a medium and of fluctuation scattering has been recognized many years ago[2,3]. It is however, relatively recently that Fixman[4] attempted the description of the problem by simultaneous use of microscopic and macroscopic equations for the electromagnetic field. The implicit assumptions in Fixman's work were that the microscopic equations truly represent the complete picture of the microscopic events in the interaction of fields with granulated matter and that the macroscopic equations truly represent the ensemble average properties of the same interaction. Simultaneous manipulation of the microscopic and ensemble average fields has yielded a sensible, if not completly, satisfactory, approach to the problem. More recently Mazur[5,6] has also discussed the statistical mechanics of the electromagnetic properties of matter and has given a molecular theory of light scattering that closely follows Fixman's steps. In what follows, we will develop the theory of fluctuation scattering in a collisionless plasma in a more natural way starting from the microscopic equations for particles and fields. It is believed that this theory can adequately and without ambiguity describe all linear scattering processes in a plasma (transverse waves from fluctuations, longitudinal waves from fluctuations, fluctuations from fluctuations) whose particle correlations higher than the second can be neglected.

II. THE MICROSCOPIC EQUATIONS FOR A PLASMA

For the microscopic description of the plasma, we follow the method initiated by Klimontovich[7]. He introduces the random function of the microscopic particle density for the μ-th species in a multicomponent plasma as

$$N_\mu(\bar{p}\bar{q}t) = \sum_{\lambda=1}^{N_\mu} \delta(\bar{p}_\mu - \bar{p}_{\mu\lambda}(t))\,\delta(\bar{q}_\mu - \bar{q}_{\mu\lambda}(t)) \tag{1}$$

to define the number of particles in phase space $d\bar{p}d\bar{q}$ around $\bar{p}$ and $\bar{q}$ at time t. $\bar{p}$ and $\bar{q}$ are canonical variables, N_μ is the number of particles of the μ-th species. From the Hamiltonian of the system (particles and fields) Klimontovich then derives the following exact equations of motion for $N_\mu(\bar{p}\bar{q}t)$ and of the electromagnetic field $\{\bar{A}(\bar{q}t), \phi(\bar{q}t)\}$ where $\bar{A}(\bar{q}t)$ is the vector potential and $\phi(\bar{q}t)$ the scalar potential

$$\left(\frac{\partial}{\partial t} + \frac{\bar{p}}{m_\mu}\nabla_q\right)N_\mu(\bar{p}\bar{q}t) - e_\mu\left(\nabla_{\bar{q}}\phi(\bar{q}t) + \frac{1}{c}\frac{\partial}{\partial t}\bar{A}(\bar{q}t)\right)\cdot\nabla_{\bar{p}} N_\mu(\bar{p}\bar{q}t) = 0 \tag{2}$$

$$\left(\nabla_q^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\bar{A}(\bar{q}t) - \frac{1}{c}\frac{\partial}{\partial t}\nabla_{\bar{q}}\phi(\bar{q}t) = -\frac{4\pi}{c}\sum_\mu \frac{e_\mu}{m_\mu}\int d\bar{p}\,\bar{p}\,N_\mu(\bar{p}\bar{q}t) \tag{3}$$

where

$$\phi(\bar{q}t) = \sum_{\nu} e_{\nu} \int \frac{N_{\nu}(\bar{p}'\bar{q}'t)}{|\bar{q}-\bar{q}'|} \, d\bar{p}' d\bar{q}' \tag{4}$$

As the discussion will not include an external magnetic field the term $-(\bar{p} \times \bar{H})/m_{\mu}c$ has been omitted in Eq. 2. It should be noted that there are as many equations 2 as there are species in the system under consideration.

Our objective is to begin with the foregoing equations and derive macroscopic (ensemble average) equations for the electromagnetic field that will describe the index of refraction and fluctuation scattering processes. We make the following assumptions[7]:

(a) The microscopic quantity $N_{\mu}(\bar{p}\bar{q}t)$ can be written as a sum of its ensemble average value $N_{\mu}^{(0)}(\bar{p}\bar{q}t) = \langle N_{\mu}(\bar{p}\bar{q}t) \rangle$ and a fluctuating part $N_{\mu}^{(1)}(\bar{p}\bar{q}t) = \delta N_{\mu}(\bar{p}\bar{q}t)$ having a zero ensemble average:

$$\begin{gathered} N_{\mu}(\bar{p}\bar{q}t) = N_{\mu}^{(0)}(\bar{p}\bar{q}t) + N_{\mu}^{(1)}(\bar{p}\bar{q}t) \\ \langle N_{\mu}(\bar{p}\bar{q}t) \rangle = N_{\mu}^{(0)}(\bar{p}\bar{q}t), \quad \langle N_{\mu}^{(1)}(\bar{p}\bar{q}t) \rangle = 0 \end{gathered} \tag{5}$$

(b) The microscopic electromagnetic field can be similarly separated in ensemble average and fluctuating parts

$$\begin{aligned} A(\bar{q}t) &= A^{(0)}(\bar{q}t) + A^{(1)}(\bar{q}t) \\ \varphi(\bar{q}t) &= \phi^{(0)}(\bar{q}t) + \phi^{(1)}(\bar{q}t) \end{aligned} \tag{6}$$

with

$$\langle A(\bar{q}t)\rangle = \bar{A}^{(0)}(\bar{q}t)\,, \quad \langle A^{(1)}(\bar{q}t)\rangle = 0$$
$$\langle \phi(\bar{q}t)\rangle = \phi^{(0)}(\bar{q}t)\,, \quad \langle \phi^{(1)}(\bar{q}t)\rangle = 0\,.$$

The separation of the fields in ensemble average and fluctuating parts is significant. The ensemble average parts describe the average properties in a plasma such as propagation, index of refraction, etc., i.e., the macroscopic properties. In this description the plasma has lost the granularity implied by Eq.-1; it is a continuum. The fluctuating parts describe the [illegible]tering due to the incomplete extinction of the microscopic propagating fields and, on the average, are zero. This statement corresponds to Yvon's[2] prescription for the evaluation of fluctuation scattering. It should be noted, that when only a transverse propagating ensemble average wave exists, $\phi^{(0)}(\bar{q}t)$ itself is zero.

We now introduce (5) and (6) in equations (2) and (3). For equation (3) we then have

$$\left(\nabla_{\bar{q}}^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\bar{A}^{(0)}(\bar{q}t) - \frac{1}{c}\frac{\partial}{\partial t}\nabla_{q}\phi^{(0)}(\bar{q}t) + \frac{4\pi}{c}\sum_{\mu}\frac{e_\mu}{m_\mu}\int d\bar{p}\,\bar{p}\,N_\mu^{(0)}(\bar{p}\bar{q}t)$$

$$\left(\nabla_{\bar{q}}^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\bar{A}^{(1)}(\bar{q}t) - \frac{1}{c}\frac{\partial}{\partial t}\nabla_{\bar{q}}\phi^{(1)}(\bar{q}t) + \frac{4\pi}{c}\sum_{\mu}\frac{e_\mu}{m_\mu}\int d\bar{p}\,\bar{p}\,N^{(1)}(\bar{p}\bar{q}t) = 0 \qquad (7)$$

On taking ensemble averages in Eq. (7) only terms in $\bar{A}^{(0)}, \phi^{(0)}, N_\mu^{(0)}$ survive; subtracting the resulting ensemble average equation from Eq. 7 we also obtain an equation for the fluctuating quantities $\bar{A}^{(1)}, \varphi^{(1)}, N_\mu^{(1)}$. We then have the system of equations

$$\left(\nabla_{\bar{q}}^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\bar{A}^{(\sigma)}(\bar{q}t) - \frac{1}{c}\frac{\partial}{\partial t}\nabla_{\bar{q}}\phi^{(\sigma)}(\bar{q}t) = -\frac{4\pi}{c}\sum \frac{e_\mu}{m_\mu}\int d\bar{p}\,\bar{p}\,N_\mu^{(\sigma)}(\bar{p}\bar{q}t) \tag{8}$$

$$\varphi^{(\sigma)}(\bar{q}t) = \sum e_\gamma \int \frac{N_\gamma^{(\sigma)}(\bar{p}'\bar{q}'t)}{|\bar{q}-\bar{q}'|}\, d\bar{p}'d\bar{q}' \tag{9}$$

where $\sigma = 0,1$. We now turn our attention to the equation of motion for N_μ. We introduce the operator

$$\mathcal{L}(\bar{p}\bar{q}t) \equiv \frac{\partial}{\partial t} + \frac{p}{m}\nabla_{\bar{q}}$$

and henceforth write

$$\bar{E}(qt) = -\nabla_{\bar{q}}\varphi(qt) - \frac{1}{c}\frac{\partial}{\partial t}\bar{A}(\bar{q}t).$$

Eq. 2 then takes the form

$$\begin{aligned}
\mathcal{L}(\bar{p}\bar{q}t)N_\mu^{(0)}(\bar{p}\bar{q}t) &+ e_\mu \bar{E}^{(1)}(\bar{q}t)\cdot\nabla_{\bar{p}} N_\mu^{(0)}(\bar{p}\bar{q}t) \\
&+ e_\mu \bar{E}^{(0)}(\bar{q}t)\cdot\nabla_{\bar{p}} N_\mu^{(1)}(\bar{p}\bar{q}t) \\
&+ e_\mu \bar{E}^{(1)}(\bar{q}t)\cdot\nabla_{\bar{p}} N_\mu^{(1)}(\bar{p}\bar{q}t)
\end{aligned} \tag{10}$$

$$+ \quad \mathcal{L}(\bar{p}\bar{q}t)N_\mu^{(1)}(\bar{p}\bar{q}t) + e_\mu \bar{E}^{(1)}(\bar{q}t)\cdot\nabla_{\bar{p}} N_\mu^{(0)}(\bar{p}\bar{q}t) = 0$$

The first step here is to take the ensemble average of Eq. 10. Clearly, what remains of Eq. (10) is

$$\mathcal{L}(\bar{p}\bar{q}t)N_\mu^{(0)}(\bar{p}\bar{q}t) + e_\mu \bar{E}^{(1)}(\bar{q}t)\cdot\nabla_{\bar{p}} N_\mu^{(0)}(\bar{p}\bar{q}t) + e_\mu \langle \bar{E}^{(1)}(\bar{q}t)\cdot\nabla_p N_\mu^{(1)}(\bar{p}\bar{q}t)\rangle = 0 \quad (11)$$

Because of Eq. 8, $\langle\cdots\rangle$ of Eq. 11 represents quantities of the form $\langle N_\mu^{(1)}(\bar{p}'\bar{q}'t)\, N_\mu^{(1)}(\bar{p}\bar{q}t)\rangle = M_{\gamma\mu}(\bar{p},\bar{p}';\bar{q},\bar{q}';t)$ i.e., two-point, one-time correlations of the fluctuations. In the derivation of the celebrated Vlassov equation these terms are neglected thus breaking the coupling with the hierarchy of equations for the higher-order correlations. Parenthetically we note here that the solution of the Vlassov equation (Eq. 11 with $\langle\cdots\rangle = 0$) is obtained by writing $N_\mu^{(0)}(\bar{p}\bar{q}t) = N_\mu^{(0)}(\bar{p}) + N_{1\mu}^{(0)}(\bar{p}\bar{q}t)$ and by neglecting the non-linear terms $\nabla_p N_{1\mu}^{(0)}(\bar{p}\bar{q}t)$. These simplifications lead to the calculation of the space-time dispersive dielectric constants[8] $\epsilon_L(\bar{k}\omega)$, $\epsilon_T(\bar{k}\omega)$.

We now subtract Eq. 11 from Eq. 9 and obtain

$$\mathcal{L}(\bar{p}\bar{q}t)N_\mu^{(1)}(\bar{p}\bar{q}t) + e_\mu \bar{E}^{(1)}(\bar{q}t)\cdot\nabla_{\bar{p}} N_\mu^{(0)}(\bar{p}\bar{q}t) + e_\mu \bar{E}^{(0)}(\bar{q}t)\cdot\nabla_p N_\mu^{(1)}(\bar{p}\bar{q}t) + e_\mu \overline{\overline{\bar{E}^{(1)}(\bar{q}t)\cdot\nabla_{\bar{p}} N_\mu^{(1)}(\bar{p}\bar{q}t)}} = 0 \quad (12)$$

where the double bar implies that for a quantity F, $\overline{\overline{F}} = F - \langle F\rangle$.

In the absence of ensemble average fields in the system and on the assumption that two-point one-time correlations can be neglected (i.e. $\bar{\bar{F}}=0$) eq. 12 reduces to

$$\mathcal{L}(\bar{p}\bar{q}t)N^{(1)}_{\mu}(\bar{p}\bar{q}t)+e_{\mu}\bar{\mathcal{E}}^{(1)}(qt)\cdot\nabla_{p}N^{(0)}_{\mu}(\bar{p}\bar{q}t)=0 \tag{12a}$$

where $\bar{\mathcal{E}}^{(1)}=-\nabla\varphi^{(1)}-\frac{1}{c}\frac{\partial}{\partial t}\mathcal{A}^{(1)}$ and where $\varphi^{(1)},\mathcal{A}^{(1)}$ are the non-driven (naturally occuring) field fluctuations in the plasma. This equation will be useful later in the evaluation of correlations of fluctuations in $N^{(1)}_{\mu}$.

Equations 8, 9, 11, and 12 are exact equations containing both the ensemble average and the fluctuations of particles and fields. They provide the essential quantities needed to describe propagation of transverse and longitudinal waves, scattering of fluctuations from fluctuations, etc. For the purposes of the present paper we shall make the following simplifications:

(a) The ensemble average propagating transverse electromagnetic field interacts only little with the heavy ions in the system. It is, therefore, sensible to keep one term in the summation of the right hand side of Eq. 8 representing the electron current.

(b) To obtain results amenable to calculation we will use the standard perturbation expansion $N^{(0)}_{\mu}=N^{(0)}_{\mu}(\bar{p})+N^{(0)}_{1\mu}$

with $N_\mu^{(0)}(\bar p) \underset{p\to\infty}{\to} 0$ and nonlinear terms such as $\nabla_{\bar p} N_{1\mu}^{(0)}(\bar p \bar q t)$ will be neglected.

III. THE HIERARCHY OF EQUATIONS FOR THE FIELDS

It becomes expedient at this point to transform the equations derived earlier from $\bar q t$ to $\bar k\omega$-space. With

$$f(\bar q t) = \int f(\bar k\omega) e^{i\bar k\cdot q - i\omega t} \frac{d\bar k\, d\omega}{(2\pi)^4}$$

we have for Eq. 8 (neglecting all particles except electrons)

$$\left(-k^2 + (\omega/c)^2\right)\bar A^{(\sigma)}(\bar k\omega) - (\bar k\omega/c)\phi^{(\sigma)}(\bar k\omega) = -\frac{4\pi}{c}\int d\bar p\, \bar p\, N_e^{(\sigma)}(\bar p \bar k\omega) \tag{13}$$

For *Eq. 9

$$\phi^{(\sigma)}(\bar k\omega) = \sum_\nu \frac{4\pi e_\nu}{k^2} N_\nu^{(\sigma)}(\bar k\omega), \quad \bar k \neq 0 \tag{14}$$

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*We have $\nabla^2(1/|\bar q-\bar q'|) = -4\pi\delta(\bar q-\bar q') = \nabla^2\psi(\bar q)$. ~~We~~ Clearly then after a Fourier transformation $-k^2\psi(\bar k) = -4\pi e^{-i\bar k\cdot\bar q'}$. However, $\int N_\nu^{(\sigma)}(q'\omega) e^{-i\bar k\cdot\bar q'} d\bar q' = N_\nu^{(\sigma)}(\bar k\omega)$; hence, the result of Eq. 14.

For Eq. 11 (with $\nabla_{\bar p} N_\mu^{(0)}(\bar p \bar q t) \sim \nabla_{\bar p} N_\mu^{(0)}(\bar p)$ and $\mathcal{L}(\bar p \bar k \omega) = -i\omega + i\bar k \cdot \bar p / m$):

$$\mathcal{L}(\bar p \bar k \omega) N_\mu^{(0)}(\bar p \bar k \omega) + e_\mu \bar E^{(0)}(\bar k \omega) \cdot \nabla_{\bar p} N_\mu^{(0)}(\bar p)$$

$$+ e_\mu \int \langle \bar E^{(1)}(\bar k' \omega') \cdot \nabla_{\bar p} N_\mu^{(1)}(\bar p, \bar k - \bar k', \omega - \omega') \rangle \frac{d\bar k' d\omega'}{(2\pi)^4} = 0 \tag{15}$$

and for Eq. 12 (again $\nabla_{\bar p} N_\mu^{(0)}(\bar p \bar q t) \sim \nabla_{\bar p} N_\mu^{(0)}(\bar p)$):

$$\mathcal{L}(\bar p \bar k \omega) N_\mu^{(0)}(\bar p \bar k \omega) + e_\mu \bar E^{(1)}(\bar k \omega) \cdot \nabla_{\bar p} N_\mu^{(0)}(\bar p)$$

$$+ e_\mu \int \bar E^{(0)}(\bar k' \omega') \cdot \nabla_{\bar p} N_\mu^{(1)}(\bar p, \bar k - \bar k', \omega - \omega') \frac{d\bar k' d\omega'}{(2\pi)^4}$$

$$+ e_\mu \int \overline{\overline{\bar E^{(1)}(\bar k' \omega') \cdot \nabla_{\bar p} N_\mu^{(1)}(\bar p, \bar k - \bar k', \omega - \omega')}} \frac{d\bar k' d\omega'}{(2\pi)^4} = 0 \tag{16}$$

We can now solve for $N_\mu^{(0)}$ and $N_\mu^{(1)}$ and introduce the results in Eq. 13. The contributions of $\bar E^{(0)}(\bar k \omega) \cdot \nabla_{\bar p} N_\mu^{(0)}(\bar p)$ from Eq. 15 and of $\bar E^{(1)}(\bar k \omega) \cdot \nabla_{\bar p} N_\mu^{(0)}(\bar p)$ from Eq. 16 convert the free space operators of the left hand side of Eq. 13 to operators acting on the ensemble average fields in a space-dispersive medium. Explicitely, and introducing the transverse and longitudinal dielectric constants $\epsilon_t(\bar k \omega)$ and $\epsilon_L(\bar k \omega)$, we obtain

$$\mathcal{K}_T(\bar{k}\omega)\bar{A}^{(0)}(\bar{k}\omega) + \bar{\mathcal{K}}_L(\bar{k}\omega)\phi^{(0)}(\bar{k}\omega) =$$

$$-\frac{4\pi}{c}\frac{e^2}{m_e}\int\frac{d\bar{k}'d\omega'}{(2\pi)^4}\frac{d\bar{p}\,\bar{p}}{\mathcal{L}(\bar{p}\bar{k}\omega)}\left\langle \bar{E}^{(1)}(\bar{k}'\omega')\cdot\nabla_{\bar{p}}N_\mu^{(1)}(\bar{p},\bar{k}-\bar{k}',\omega-\omega')\right\rangle \tag{17}$$

and similarly

$$\mathcal{K}_T(\bar{k}\omega)\bar{A}^{(1)}(\bar{k}\omega) + \bar{\mathcal{K}}_L(\bar{k}\omega)\phi^{(1)}(\bar{k}\omega) =$$

$$-\left(\frac{4\pi}{c}\right)\frac{e^2}{m_e}\int\frac{d\bar{k}'d\omega'}{(2\pi)^4}\frac{d\bar{p}\,\bar{p}}{\mathcal{L}(\bar{p}\bar{k}\omega)}\Big[\bar{E}^{(0)}(\bar{k}'\omega')\cdot\nabla_{\bar{p}}N_\mu^{(1)}(\bar{p},\bar{k}-\bar{k}',\omega-\omega') + \overline{\overline{E^{(1)}(\bar{k}'\omega')\cdot\nabla_{\bar{p}}N_\mu^{(1)}(\bar{p},\bar{k}-\bar{k}',\omega-\omega')}}\Big] \tag{18}$$

where

~~and~~ $\mathcal{K}_T(\bar{k}\omega) = k^2 - (\omega/c)^2\epsilon_T(\bar{k}\omega)$, $\bar{\mathcal{K}}_L(\bar{k}\omega) = \bar{k}(\omega/c)\epsilon_L(\bar{k}\omega)$

and

$$\epsilon_T(\bar{k}\omega) = 1 + \sum_\nu \frac{4\pi c_\nu^2}{k^2}\int p_{\perp\nu}\,\bar{k}\cdot\nabla_{\bar{p}}N^{(0)}(\bar{p})\frac{d\bar{p}_{\nu\perp}}{\omega - k\cdot p_\nu/m + i\Delta}$$

capital T

$$\epsilon_L(\bar{k}\omega) = 1 + \sum_\nu \frac{4\pi c_\nu^2}{k^2}\int \bar{k}\cdot\nabla_{\bar{p}}N^{(0)}(p_\nu)\frac{d\bar{p}_\nu}{\omega - k\cdot p_\nu/m + i\Delta}$$

capital L

Equations 17 and 18 form the desired set of coupled equations for the ensemble average $(\bar{A}^{(0)}, \phi^{(0)})$ and the fluctuating $(\bar{A}^{(1)}, \phi^{(1)})$ fields. Except for the linearization, these equations are exact equations and by themselves form a hierarchy of equations for higher order correlations of particles and fields. All correlations above the second will henceforth be neglected by omitting the $\bar{\bar{F}}$ term in Eq. 18.

To fix ideas on how to proceed from this point we will assume that the only ensemble average field in the system is a transverse wave such as would be excited by an incident electromagnetic wave. Then $\phi^{(o)}=0$ so that

$$\mathcal{K}_T(\bar{k}\omega)\bar{A}^{(o)}(\bar{k}\omega) =$$

$$+\left(\frac{4\pi}{c}\right)\frac{e^2}{m_e} i\int\frac{d\bar{k}'d\omega'}{(2\pi)^4}\frac{d\bar{p}\,\bar{p}}{\mathcal{L}(\bar{p}\bar{k}\omega)}\left\langle\left(\phi^{(1)}(\bar{k}'\omega')\bar{k}' - (\omega'/c)\bar{A}^{(1)}(\bar{k}'\omega')\right)\cdot\nabla_{\bar{p}} N_\mu^{(1)}(\bar{p},\bar{k}-\bar{k}',\omega\omega')\right\rangle \tag{19}$$

$$\mathcal{K}_T(\bar{k}\omega)\bar{A}^{(1)}(\bar{k}\omega) + \bar{\mathcal{K}}_L(\bar{k}\omega)\,\phi^{(1)}(\bar{k}\omega) =$$

$$-\left(\frac{4\pi}{c}\right)\frac{e^2}{m_e} i\int\frac{d\bar{k}'d\omega'}{(2\pi)^4}\frac{d\bar{p}\,\bar{p}}{\mathcal{L}(\bar{p}\bar{k}\omega)}(\omega'/c)\bar{A}^{(o)}(\bar{k}'\omega')\cdot\nabla_{\bar{p}} N_\mu^{(1)}(\bar{p},\bar{k}-\bar{k}',\omega-\omega') \tag{20}$$

Multiplying both terms of Eq. 20 by $\hat{k}$ and $\hat{k}\times(\hat{k}\times$ and using the fact that $\nabla\cdot A$ or $\bar{k}\cdot\bar{A}=0$ we can separate the transverse from the longitudinal components. We then find

$$\left\{\begin{matrix}\phi^{(1)}(\bar{k}\omega)\\ \bar{A}^{(1)}(\bar{k}\omega)\end{matrix}\right\} = -\left(\frac{4\pi}{c}\right)\frac{e^2}{m_e} i\int\frac{d\bar{k}'d\omega'}{(2\pi)^4}\left\{\begin{matrix}\Gamma_1(\bar{p},\bar{k},\omega)\\ \bar{\Gamma}_2(\bar{p},\bar{k},\omega)\end{matrix}\right\} A^{(1)}(\bar{k}'\omega')\cdot\nabla_{\bar{p}} N_\mu(\bar{p},\bar{k}-\bar{k}',\omega-\omega')\frac{d\bar{p}}{\mathcal{L}(\bar{p}\,\bar{k}\omega)} \tag{21}$$

where

$$\Gamma_1(\bar{p},\bar{k},\omega) = \frac{\omega'}{\omega}\frac{1}{\epsilon_L(\bar{k}\omega)}\frac{(\bar{k}\cdot\bar{p})}{k^2}$$

~~where~~ where (22)

$$\bar{\Gamma}_2(\bar{p},\bar{k},\omega) = -\frac{\omega'}{c}\,\frac{1}{k^2-(\omega/c)^2\epsilon_T(\bar{k}\omega)}\left(\bar{p}-\frac{\bar{k}(\bar{k}\cdot\bar{p})}{k^2}\right) \tag{23}$$

with expressions now available for $\bar{A}^{(0)}, \bar{A}^{(1)}$ and $\phi^{(1)}$ we can now proceed to evaluate the scattering cross sections.

IV. THE SCATTERING CROSS SECTION

As was indicated in Ref. 1 a possible method for the evaluation of scattering cross sections consists of introducing the values of $\bar{A}^{(1)}, \phi^{(1)}$ from Eq. 21 into Eq. 20 and then evaluating the changes this brings on the imaginary part of $\mathcal{K}_T(\bar{k}\omega)$. This would give a direct and unambiguous measure of the cross sections. A difficulty, however, occurs here because of the presence of terms having the form $\langle N^{(1)}(\bar{p}, \bar{k}-\bar{k}', \omega-\omega')\, N^{(1)}(\bar{p}', \bar{k}'-\bar{k}'', \omega'-\omega'')\rangle$, necessitating a rather intricate calculation. Although these terms have been evaluated in Ref. 7 we shall presently resort to some simplifications in the momentum space integrations which will hopefully make the results less opaque.

Take for instance the quantity appearing in Eq. 21:

$$I_1 = \int \frac{d\bar{p}}{\mathcal{L}(\bar{p}, \bar{k}, \omega)} (\bar{k}\cdot\bar{p})\bar{A}^{(0)}(\bar{k}'\omega')\cdot\nabla_{\bar{p}} N_\mu^{(1)}(\bar{p}, \bar{K}, \Omega) \;;\; \bar{K} = \bar{k}-\bar{k}', \Omega = \omega-\omega'$$

Using successive vector identities we find that*

$$I_1 = \int d\bar{p} \left[\nabla_{\bar{p}} \cdot \left(\frac{(\bar{k}\cdot\bar{p})\bar{A}^{(0)} N_\mu^{(1)}}{\mathcal{L}} \right) - N_\mu^{(1)} \bar{A}^{(0)} \nabla_{\bar{p}} \left(\frac{\bar{k}\cdot\bar{p}}{\mathcal{L}} \right) \right]$$

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* the shorthand notation used below is self explanatory.

Converting the integral over the divergence to a line integral and noting that for $\bar{p}\to\infty, N_\mu^{(1)}\to 0$ we readily obtain

$$I_1 = -\int d\bar{p}\, N_\mu^{(1)} \frac{\bar{A}^{(0)}\cdot\bar{k}}{\mathcal{L}} - \int d\bar{p}\, N_\mu^{(1)} (\bar{k}\cdot\bar{p})\bar{A}^{(0)}\cdot\nabla_p\left(\frac{1}{\mathcal{L}}\right)$$

Assuming further that $\omega \gg \bar{k}\cdot\bar{p}/m$ i.e. that $\mathcal{L}\to -i\omega$ we obtain

$$I_1 \underset{\omega\gg\bar{k}\cdot\bar{p}/m}{\Rightarrow} -\frac{i}{\omega}\left(\bar{A}^{(0)}(\bar{k}'\omega')\cdot\bar{k}\right)\int d\bar{p}\, N_\mu^{(1)}(\bar{p},\bar{K},\Omega) + \frac{i}{\omega^2}\left(\bar{A}^{(0)}(\bar{k}'\omega')\cdot\bar{k}\right)\bar{k}\cdot\int \frac{\bar{p}}{m} d\bar{p}\, N_\mu^{(1)}(\bar{p},\bar{K},\Omega)$$

$$= -\frac{i}{\omega^2}\left(\bar{A}^{(0)}(\bar{k}'\omega')\cdot\bar{k}\right)\left[\omega N_\mu^{(1)}(\bar{K}\Omega) - \bar{k}\cdot\bar{J}_\mu^{(1)}(\bar{K},\Omega)\right]$$

where $\bar{J}^{(1)}$ is the current fluctuation.

Note that the continuity equation would imply

$\Omega N_\mu^{(1)}(\bar{K},\Omega) - \bar{K}\cdot\bar{J}_\mu^{(1)}(\bar{K},\Omega)=0$ thus $I_1\neq 0$. In a similar fashion we find that

$$\bar{I}_2 = \int\frac{d\bar{p}}{\mathcal{L}(\bar{p}\bar{k}\omega)}\left(\bar{p} - \bar{k}\frac{(\bar{k}\cdot\bar{p})}{k^2}\right)\bar{A}^{(0)}(\bar{k}'\omega')\cdot\nabla_{\bar{p}} N_\mu^{(1)}(\bar{p},\bar{K},\Omega)$$

$$\underset{\omega\gg\bar{k}\cdot\bar{p}/m}{\Rightarrow} -\frac{i}{\omega}\left(\bar{A}^{(0)}(\bar{k}'\omega') - \bar{k}\frac{\bar{A}^{(0)}(\bar{k}'\omega')\cdot\bar{k}}{k^2}\right)N_\mu^{(1)}(\bar{K},\Omega)$$
$$+\frac{i}{\omega^2}\left(\bar{J}_\mu^{(1)}(\bar{K}\Omega) - \bar{k}\frac{\bar{J}_\mu^{(1)}(K\Omega)\cdot\bar{k}}{k^2}\right)\left(\bar{A}^{(0)}(k'\omega')\cdot\bar{k}\right)$$

Integrations in $\bar{p}$ of Eq. 19 can also be simplified. We find

$$I_3 = \int\frac{d\bar{p}\,\bar{p}}{\mathcal{L}(\bar{p}\bar{k}\omega)}\bar{k}'\cdot\nabla_p N_\mu^{(1)}(\bar{p},\bar{K},\Omega)$$

$$\underset{\omega\gg\bar{k}\cdot\bar{p}/m}{\Rightarrow} -\frac{i}{\omega}\bar{k}' N_\mu^{(1)}(\bar{K},\Omega) + \frac{i}{\omega^2}\left(\bar{k}'\cdot\bar{k}\right)\bar{J}_\mu^{(1)}(\bar{K},\Omega)$$

$$\bar{I}_4 = \int\frac{d\bar{p}\,\bar{p}}{\mathcal{L}(\bar{p}\bar{k}\omega)}\bar{A}^{(1)}(\bar{k}'\omega')\cdot\nabla_p N_\mu^{(1)}(\bar{p},\bar{K},\Omega)$$

$$\underset{\omega\gg\bar{k}\cdot\bar{p}/m}{\Rightarrow} -\frac{i}{\omega}\bar{A}^{(1)}(\bar{k}'\omega')N_\mu^{(1)}(\bar{K},\Omega) + \frac{i}{\omega^2}\left(\bar{A}^{(1)}(\bar{k}'\omega')\cdot\bar{k}\right)\bar{J}_\mu^{(1)}(\bar{K},\Omega)$$

It now becomes obvious that inserting $\phi^{(1)}, \bar{A}^{(1)}$ in Eq. 19 leads to terms

$$<N^{(1)}(\bar{k},\Omega)N^{(1)}(\bar{k}',\Omega')>, <J^{(1)}(\bar{k}\Omega)\, J^{(1)}(\bar{k}',\Omega')>, <N^{(1)}(\bar{k}\Omega)\bar{J}^{(1)}(k',\Omega')>$$

etc. Restricting ourselves only to density-density correlations we then find for Eq. 19 when $\omega \gg \bar{k}\cdot\bar{p}/m$

$$\mathcal{K}_T(\bar{k}\omega)\, \bar{A}^{(0)}(\bar{k}\omega) = -\left(\frac{4\pi}{c}\right)^2\left(\frac{e^2}{m_e}\right)^2 \int \frac{d\bar{k}'d\omega'}{(2\pi)^4}\frac{d\bar{k}''d\omega''}{(2\pi)^4} < N_\mu^{(1)}(\bar{k}-\bar{k}',\omega-\omega')N_\mu^{(1)}(\bar{k}'-\bar{k}'',\omega'-\omega'') >$$

$$\times\left[\frac{\omega''}{\omega\omega'^2}\,\frac{\bar{A}^{(0)}(\bar{k}''\omega'')\cdot\bar{k}'\,\bar{k}''}{k'^2\,\epsilon_L(\bar{k}'\omega')} + \frac{\omega''}{c^2\omega}\,\frac{\bar{A}^{(0)}(\bar{k}''\omega'') - \bar{k}'\,\dfrac{\bar{A}^{(0)}(\bar{k}''\omega'')\cdot\bar{k}'}{k'^2}}{k'^2-(\omega'/c)^2\epsilon_T(\bar{k}'\omega')}\right] \tag{24}$$

However (1)

$$<N_\mu^{(1)}(\bar{k}-\bar{k}',\omega-\omega')N_\mu^{(1)}(\bar{k}'-\bar{k}'',\omega'-\omega'')> = (2\pi)^4\delta(\bar{k}-\bar{k}'')\,\delta(\omega-\omega')<|N_\mu(\bar{k}-\bar{k}',\omega-\omega')|^2> \tag{25}$$

so that finally

$$\left[\mathcal{K}_T(\bar{k}\omega) + \Delta\mathcal{K}_T(\bar{k}\omega)\right]A^{(0)}(\bar{k}\omega) = 0 \tag{26}$$

where

$$\Delta\mathcal{K}_T(\bar{k}\omega) = (4\pi)^2 r_e^2 \int\frac{d\bar{k}'d\omega'}{(2\pi)^4}\,\Gamma(\bar{k}'\omega')<|N_\mu^{(1)}(\bar{k}-\bar{k}',\omega-\omega')|^2> \tag{27}$$

$$\Gamma(k'\omega') = \frac{1-(\hat{a}\cdot k')^2 k'^{-2}}{k'^2-(\omega'/c)^2\epsilon_T(\bar{k}'\omega')} + \frac{c^2}{\omega'^2}\,\frac{(\hat{a}\cdot\bar{k}')^2}{k'^2\epsilon_L(\bar{k}'\omega')} \tag{28}$$

and $\hat{a}$ is a unit rector for the rector potential: $\bar{A}^{(0)} = \hat{a} A^{(0)}$.
The scattering cross section , can then be evaluated *from $\mathrm{Im}\{\Delta \mathcal{K}_T(\bar{k}\omega)\}$. The first term in Eq. 28 describes the scattering of a transverse wave on density fluctuations to produce a transverse wave; the second term describes the scattering of a transverse wave on fluctuations to produce a longitudinal wave. If all the terms in Eqs. 17 and 18 (except $\bar{\bar{F}}$) were kept, additional linear scattering processes could be described.

The cross section $\sigma(\bar{k}\omega)$, gives a generalized version of the Prins-Zernike[9] formula describing scattering from the covariance of density fluctuations, for it also includes "inelastic" scattering processes[10].

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* $\sigma(\bar{k}\omega) = 2 N_0^{-1}(\omega/c)\,\mathrm{Im}\{n(\bar{k}\omega)\} \sim N_0^{-1}(c/\omega)\,\mathrm{Im}\{\Delta \mathcal{K}_T(\bar{k}\omega)\}$

where $n^2(k\omega) \equiv \epsilon_r(k\omega)$ and where N_0 is the particle density. The evaluation of σ and the consequences of the second term of $\Gamma(\bar{k}'\omega')$ involving $\epsilon_L(\bar{k}'\omega')$ are discussed at some length in Ref. 1.

CONCLUSIONS

By separating the microscopic equations for particles and fields in a plasma into ensemble average and fluctuating parts we have been able to derive a generalized version of the Prins-Zernike formula for scattering of an electromagnetic wave from the covariance of density fluctuations in a plasma. The procedure leading to the desired results is shown schematically in Table I. The method developed can be extended to cover all linear scattering processes involving the interaction of transverse and longitudinal waves with fluctuations in a plasma. This method does not seem to be restricted to the description of small fluctuations, for nowhere did we have to resort to "small signal" analysis.

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