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THE DOPPLER EQUATION IN RANGE AND RANGE RATE MEASUREMENT

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SUMMARY

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The Doppler equation and its use for range and range rate measurements is discussed, including special relativistic effects, the meaning of integrated Doppler, basic limitations in the electronic measuring equipment and, the dependence between range and range rate measurements.

The emphasis is on the two-way Doppler, which is of particular interest for range and range rate measurements. It is shown that the classical and relativistic two-way Doppler equations are identical. All derivations are based on time, propagation delay and phase rather than on frequency because the electronic equipment is only capable of measuring finite differences in time and phase. This approach also unifies the treatment of both classical and relativistic Doppler and the integration of Doppler frequency.

It is also shown, that range and range rate measurements are not independent measurements and that range rate measurements can be performed much more accurately than range measurements. The contributions of range measurements to orbit determination are therefore rather limited except for near earth phases of a mission.

Author

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THE DOPPLER EQUATION IN RANGE AND RANGE RATE MEASUREMENT

INTRODUCTION

Many papers have been written about the Doppler frequency shift of electromagnetic waves and the use of this frequency shift for determination of range and range rate. Then why write another paper? There are several reasons. Not all papers have come to the same conclusions, especially with regard to relativistic effects and the significance of higher order terms in the Doppler equation. It is the objective of this paper to clarify these matters. Another objective is to take into account the limitations in making physical measurements and to interpret the measurements correctly. We cannot measure a frequency instantaneously, a certain time is needed for the measurement. Of particular interest for range and range rate determination is the technique of integrating the Doppler frequency over a finite time. The correct interpretation of the integrated Doppler frequency is therefore as important as the correct derivation of the Doppler equation. It is shown in Chapter 4 that the integrated Doppler frequency is nothing but a measure of propagation delay. This fact is used in Chapter 5 to discuss the dependence between range and range rate measurements.

The emphasis in this paper is on time, propagation delay, and phase rather than on frequency. The importance of correct "time keeping" is stressed both in the derivation of the Doppler equation and in evaluating the integrated Doppler frequency. Emphasis is placed on the two-way Doppler as being the most important one for tracking of the Apollo vehicles with the Unified S-Band trackers.

Some of the derivations included in this paper can be found in the literature. Because of the tutorial nature of this paper they have been included in order to present a unified treatment of the Doppler frequency and its integration.

1. "CLASSIC" DERIVATION OF THE DOPPLER EQUATION

An electromagnetic wave of angular frequency ω_t is transmitted from transmitter Tr in Figure 1 at time t_1. The amplitude is assumed to be time independent and using complex notations, we can write the time dependence of the wave as $\exp\{j\varphi)$ where

$$\omega_t = \frac{d\varphi}{dt_1} \tag{1-1}$$

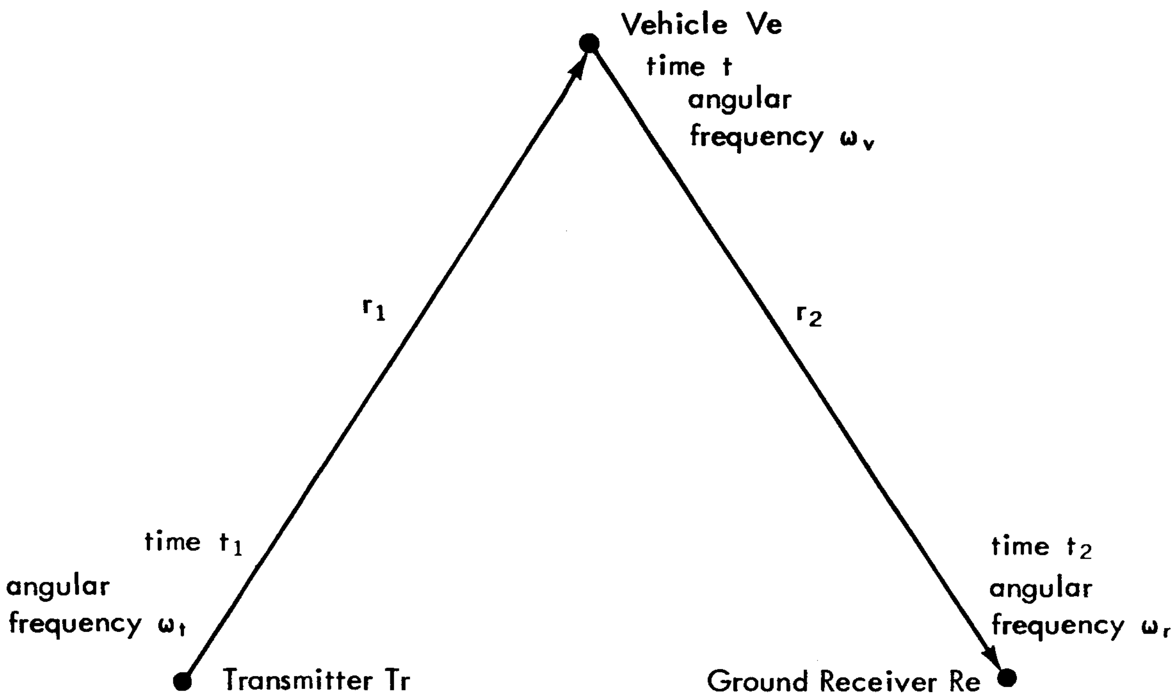

Figure 1–Two Way Doppler Geometri

The wave will reach the vehicle Ve in Figure 1 at a later time t, given by

$$t = t_1 + \frac{r_1}{c} \tag{1-2}$$

where r_1 = range between transmitter Tr and vehicle Ve

c = propagation velocity of the wave.

An observer at the vehicle Ve will observe an angular frequency ω_v

$$\omega_v = \frac{d\varphi}{dt} \tag{1-3}$$

The ratio between ω_v and ω_t is easily obtained from Equations (1-1) and (1-3)

$$\frac{\omega_v}{\omega_t} = \frac{dt_1}{dt} \tag{1-4}$$

We now choose a frame of reference, in which the transmitter Tr is stationary. In this frame r_1 is a function of vehicle position only and as the vehicle V_e receives the electromagnetic wave at time t, r_1 has to be considered to be a function of $\tilde{t}$. With this in mind, differentiation of Equation (1-2) with respect to t yields

$$\frac{d t_1}{dt} = 1 - \frac{1}{c} \frac{d r_1}{dt} \tag{1-5}$$

and thus

$$\frac{\omega_v}{\omega_t} = 1 - \frac{1}{c} \frac{d r_1}{dt} \tag{1-6}$$

which is the one-way Doppler equation for a moving observer.

The wave ω_v is now reflected or instantaneously retransmitted from the vehicle Ve to a ground receiver Re. It is assumed that the receiver Re is stationary relative to the transmitter Tr. The range r_2 (see Figure 1) is therefore a function of the time of retransmission t. The retransmitted wave will be received by the ground receiver Re at a time t_2 given by

$$t_2 = t + \frac{r_2}{c} = t_1 + \frac{r_1 + r_2}{c} \tag{1-7}$$

and the received angular frequency ω_r is

$$\omega_r = \frac{d\varphi}{dt_2} \tag{1-8}$$

and thus

$$\frac{\omega_r}{\omega_t} = \frac{dt_1}{dt_2} \tag{1-9}$$

Differentiation of Equation (1-7) yields, observing that r_1 and r_2 are functions of t

$$\frac{dt_1}{dt_2} = \frac{1 - \frac{1}{c}\frac{dr_1}{dt}}{1 + \frac{1}{c}\frac{dr_2}{dt}} \tag{1-10}$$

and with Equation (1-9) we obtain

$$\frac{\omega_r}{\omega_t} = \frac{1 - \frac{1}{c}\frac{dr_1}{dt}}{1 + \frac{1}{c}\frac{dr_2}{dt}} \tag{1-11}$$

The angular Doppler frequency ω_d is defined as

$$\omega_d = \omega_r - \omega_t \tag{1-12}$$

and using Equation (1-11)

$$\omega_d = -\frac{\frac{1}{c}\frac{dr_1}{dt} + \frac{1}{c}\frac{dr_2}{dt}}{1 + \frac{1}{c}\frac{dr_2}{dt}} \cdot \omega_t \tag{1-13}$$

which is the two-way Doppler equation.

Note that all derivatives in the Doppler equation reflect the vehicle time t rather than the time t_2 at the ground receiver.

Equation (1-13) may also be expressed as a series.

$$\omega_d = -\left\{\frac{1}{c}\left(\frac{dr_1}{dt} + \frac{dr_2}{dt}\right) - \frac{1}{c^2}\left(\frac{dr_1\, dr_2}{dt^2} + \left(\frac{dr_2}{dt}\right)^2\right) + \ldots\right\}\omega_t \tag{1-13a}$$

Although the series expansion is in common use we prefer to use the exact form, as given by Equation (1-13), in this paper.

Another useful form of Equation (1-13) is obtained by solving for the range rate

$$\frac{dr_1}{dt} + \left(1 + \frac{\omega_d}{\omega_t}\right)\frac{dr_2}{dt} = c\,\frac{\omega_d}{\omega_t} \tag{1-13b}$$

Of special interest is the case where the transmitting antenna coincides with the receiving antenna. In this case $r_1 = r_2$ and the Doppler equation can be simplified to

$$\omega_d = -2 \frac{\frac{1}{c}\frac{dr}{dt}}{1 + \frac{1}{c}\frac{dr}{dt}} \cdot \omega_t \tag{1-14}$$

$$\omega_d = -2 \left\{ \frac{1}{c}\frac{dr}{dt} - \frac{1}{c^2}\frac{d^2r}{dt^2} + \cdots \right\} \omega_t \tag{1-14a}$$

and

$$\frac{dr}{dt} = \frac{c\frac{\omega_d}{\omega_t}}{2 + \frac{\omega_d}{\omega_t}} \tag{1-14b}$$

The two way Doppler equation can easily be generalized. In some applications the vehicle transponder multiplies the received frequency by a factor k before retransmission. The retransmitted angular frequency is then $k\omega_v$ and Equation (1-11) changes to

$$\frac{\omega_r}{k\omega_t} = \frac{1 - \frac{1}{c}\frac{dr_1}{dt}}{1 + \frac{1}{c}\frac{dr_2}{dt}}$$

The angular Doppler frequency for this case is defined as

$$\omega_d = \omega_r - k\omega_t$$

and thus

$$\omega_d = -\frac{\frac{1}{c}\frac{dr_1}{dt} + \frac{1}{c}\frac{dr_2}{dt}}{1 + \frac{1}{c}\frac{dr_2}{dt}} \cdot k\omega_t \tag{1-15}$$

So far we have assumed that the propagation velocity is constant and that the propagation path is a straight line. The Doppler equation can be derived under much more general conditions. Let the delay be τ_1 for the propagation from the transmitter Tr to the vehicle Ve and τ_2 from Ve to the receiver Re. Equation (1-7) can then be written, see Figure 2.

$$t_2 = t + \tau_2 = t_1 + \tau_1 + \tau_2 \tag{1-16}$$

where τ_1 and τ_2 are functions of t. Equation (1-9)

$$\frac{\omega_r}{\omega_t} = \frac{dt_1}{dt_2} \tag{1-9}$$

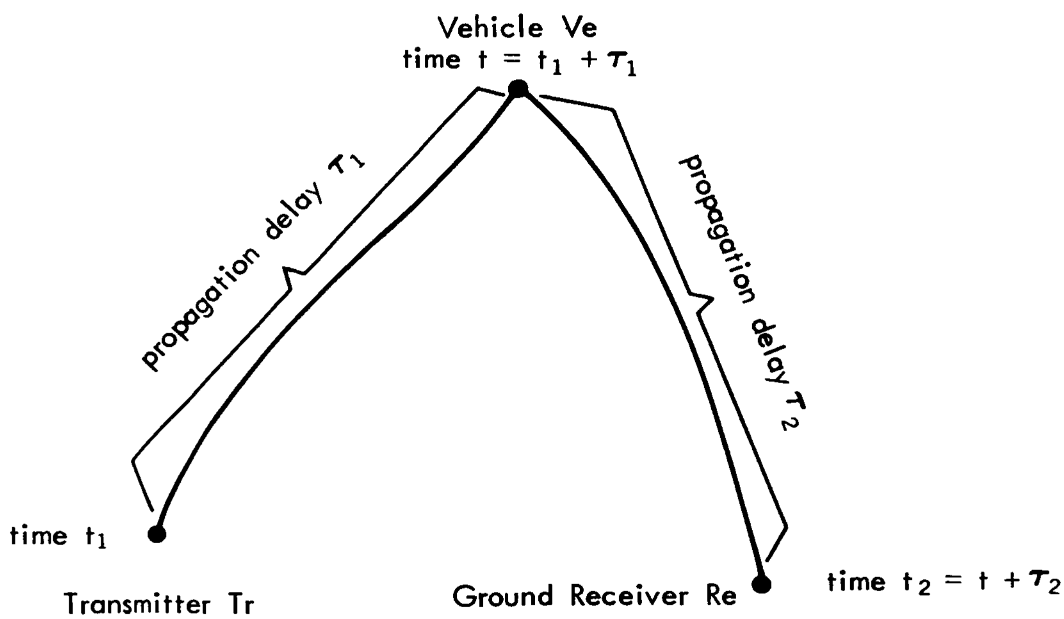

Figure 2–Propagation Delays

holds true for arbitrary transmission media. Differentiating (1-16) we obtain

$$\frac{dt_1}{dt_2} = \frac{1 - \frac{d\tau_1}{dt}}{1 + \frac{d\tau_2}{dt}} \tag{1-17}$$

Combining Equation (1-9) and (1-17) with (1-12) results in

$$\omega_d = - \frac{\frac{d\tau_1}{dt} + \frac{d\tau_2}{dt}}{1 + \frac{1}{c}\frac{d\tau_2}{dt}}\, \omega_t \tag{1-18}$$

which is the two-way Doppler equation for a general propagation media.

This equation clearly shows that the Doppler frequency is caused by the time derivatives of the propagation delay and is independent of whether the propagation delay varies because of variation in range or variation in the propagation media.

For the special case

$$\tau_1 = \frac{r_1}{c} \quad \tau_2 = \frac{r_2}{c}$$

Equation (1-18) reduces to (1-13). The problem in the general case is, of course, to establish what functions τ_1 and τ_2 are of t. Other generalizations such as transponder frequency multiplication and transponder time delay are easily incorporated into Equation (1-18).

So far, the Doppler equations have been expressed in space vehicle time t and the equations therefore reflect the range rate at time t. If we want to know the actual range rate at observation time t_2, a correction has to be applied. The range has to be expressed as a function of t_2. From

$$t = t_2 - \frac{r_2(t)}{c} \tag{1-19}$$

we obtain

$$r_2(t) = r_2\left(t_2 - \frac{r_2(t)}{c}\right)$$

and after expansion into a Taylor series

$$r_2(t) = r_2(t_2) - \frac{r_2(t)}{c}\,\frac{dr_2(t_2)}{dt_2} + \frac{1}{2}\left(\frac{r_2(t)}{c}\right)^2 \frac{d^2 r_2(t_2)}{(dt_2)^2} - \cdots \quad (1\text{-}20)$$

In taking the derivative and observing that t_2 is a function of t given by Equation (1-19) we find

$$\frac{dr_2(t)}{dt} = \frac{dr_2(t)}{dt_2} - \frac{r(t)}{c}\,\frac{d^2 r_2(t_2)}{(dt_2)^2} + \frac{1}{2}\left(\frac{r_2(t)}{c}\right)^2 \frac{d^3 r_2(t_2)}{(dt_2)^3} \quad (1\text{-}21)$$

With the aid of Equation (1-20) we can eliminate $r_2(t)$, resulting in

$$\frac{dr_2(t)}{dt} = \frac{dr_2(t_2)}{dt_2} - \frac{r_2(t_2)}{c}\,\frac{d^2 r_2(t_2)}{(dt_2)^2}$$

$$+ \frac{r_2(t_2)}{c^2}\left\{\frac{dr_2(t_2)}{dt_2}\,\frac{d^2 r_2(t_2)}{(dt_2)^2} + \frac{r_2(t_2)}{2}\,\frac{d^3 r_2(t_2)}{(dt_2)^3}\right\} \quad (1\text{-}22)$$

In the same way we obtain

$$\frac{dr_1(t)}{dt} = \frac{dr_1(t_2)}{dt_2} - \frac{r_2(t_2)}{c}\,\frac{d^2 r_1(t_2)}{(dt_2)^2}$$

$$+ \frac{r_2(t_2)}{c^2}\left\{\frac{dr_2(t_2)}{dt_2}\,\frac{d^2 r_1(t_2)}{(dt_2)^2} + \frac{r_2(t_2)}{2}\,\frac{d^3 r_1(t_2)}{(dt_2)^3}\right\} + \cdots \quad (1\text{-}23)$$

Substitution of Equations (1-22) and (1-23) into the Doppler equations will yield equations which express the Doppler shift as a function of range rate at observation time t_2. In particular, the two-way Doppler Equation (1-13) will transform into

$$\omega_d = -\frac{1}{c}\left\{\frac{d(r_1 + r_2)}{dt_2} - \frac{1}{c}\left(\frac{dr_1\, dr_2 + (dr_2)^2}{(dt_2)^2} + r_2 \frac{d^2 r_1 + d^2 r_2}{(dt_2)^2}\right)\right.$$

$$\left. + \frac{1}{c^2}\left(r_2 \frac{dr_1 d^2 r_2 + 2dr_2 d^2 r_1 + 3dr_2 d^2 r_2}{(dt_2)^3} + \left(\frac{dr_2}{dt_2}\right)^2 \frac{dr_1 + dr_2}{dt_2} + r_2^2 \frac{d^3 r_1 + d^3 r_2}{(dt_2)^3}\right) - \cdots\right\}\omega_t \tag{1-24}$$

where r_1 and r_2 now are functions of t_2. For the special case of the coinciding receiver and transmitter antenna this equation reduces to

$$\omega_d = -\frac{2}{c}\left\{\frac{dr}{dt_2} - \frac{1}{c}\left(\left(\frac{dr}{dt_2}\right)^2 + r\frac{d^2 r}{(dt_2)^2}\right) + \frac{1}{c^2}\left(3r\frac{dr d^2 r}{(dt_2)^3} + \left(\frac{dr}{dt_2}\right)^3 + r^2 \frac{d^3 r}{(dt_2)^3}\right) - \cdots\right\}\omega_t \tag{1-25}$$

It is obvious from the preceeding that the Doppler equations expressed in observation time t_2 are rather lengthy. Also, they are no longer exact and in the above derivation terms containing $1/c^3$ and higher orders have been neglected. It might be simpler in many applications to use the Doppler equations for time t and to use a time correction as given by Equation (1-19).

2. RELATIVISTIC DERIVATION OF THE DOPPLER EQUATION

Reviewing the classical derivation of the Doppler equations we find that the following assumptions were made:

a. The propagation velocity c was assumed to be constant both for transmission from the ground and from the vehicle.

b. The reflection or retransmission from the vehicle is instantaneous.

c. Transmission and reception occurs in the same reference frame.

It therefore seems very likely that no relativistic effects should occur in the two-way Doppler. A relativistic derivation is given here in order to verify this assumption. The derivation is based on the Lorentz transformation and thus takes the effects of special relativity into account.

We choose two reference frames S and S′ so that the transmitter Tr and the receiver Re are stationary in the S frame and the vehicle Ve is stationary in the S′ frame. Without loss of generality we can orient the frames so that their x- and y-axes are in the same plane and their x-axes are parallel to the velocity vector $\bar{v}$ of the vehicle. In addition to this, we let the x-axes coincide as shown in Figure 3.

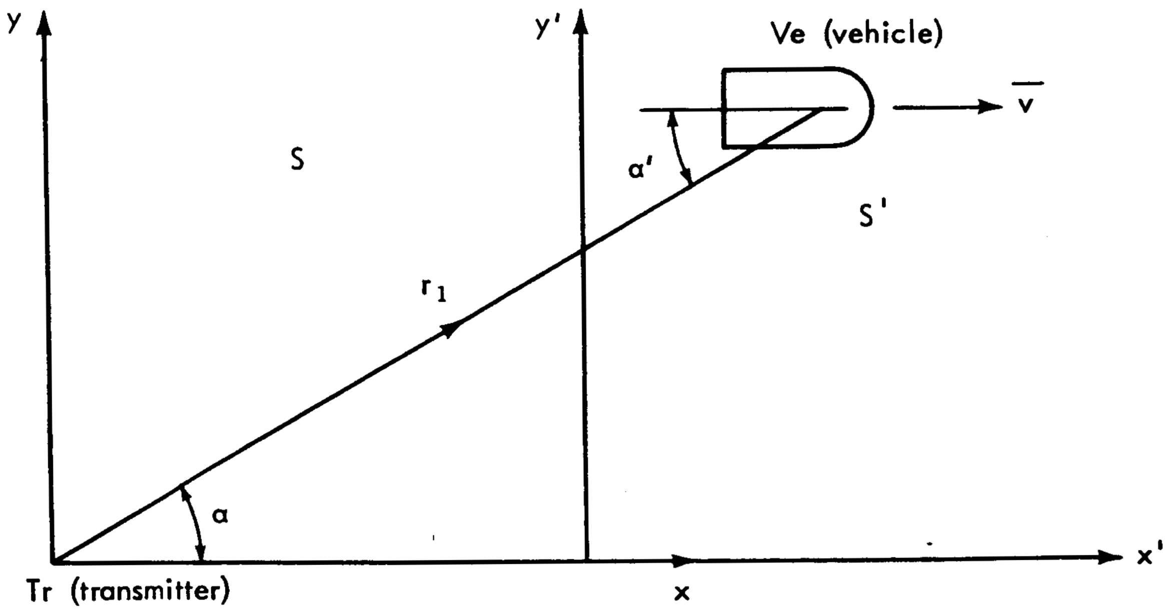

Figure 3–Reference Frames for the Lorentz Transformation. Transmission from the Ground Transmitter Tr.

From the S frame as electromagnetic wave which we, as in Chapter 1, write $\exp\{j\varphi\}$ is transmitted at time t_1 and reaches the vehicle at time t, given by Equation (1-2)

$$t = t_1 + \frac{r_1}{c} \tag{1-2}$$

From Figure 3 we see that

$$r_1 = x \cos \alpha + y \sin \alpha \tag{2-1}$$

and hence

$$t = t_1 + \frac{x \cos \alpha + y \sin \alpha}{c} \tag{2-2}$$

By means of the Lorentz transformation x, y and t from the S frame can be transformed into x′, y′ and t′ in the S′ frame, see reference [1]

$$x = \frac{x' + vt'}{\sqrt{1 - \frac{v^2}{c^2}}} \qquad y = y' \qquad z = z' = 0$$

$$t = \frac{t' + \frac{v}{c^2} x'}{\sqrt{1 - \frac{v^2}{c^2}}} \tag{2-3}$$

By applying Equation (2-3) to Equation (2-2) we can solve for the time t_1 of transmission (from the S frame) in terms of time t' of reception (in the S′ frame)

$$t_1 = \frac{t' + \frac{v}{c^2} x'}{\sqrt{1 - \frac{v^2}{c^2}}} - \frac{1}{c}\left(\frac{x' + vt'}{\sqrt{1 - \frac{v^2}{c^2}}} \cos \alpha + y' \sin \alpha\right) \tag{2-4}$$

The transmitted angular frequency is

$$\omega_t = \frac{d\varphi}{dt_1}$$

and the angular frequency received by the vehicle is

$$\omega_v = \frac{d\varphi}{dt'}$$

and thus

$$\frac{\omega_v}{\omega_t} = \frac{dt_1}{dt'} \tag{2-5}$$

From Equation (2-4) we obtain

$$\frac{\omega_v}{\omega_t} = \frac{dt_1}{dt'} = \frac{1 - \frac{v}{c}\cos\alpha}{\sqrt{1 - \frac{v^2}{c^2}}} \tag{2-6}$$

But

$$v \cos \alpha = \frac{dr_1}{dt}$$

and thus

$$\frac{\omega_v}{\omega_t} = \frac{1 - \frac{1}{c}\frac{dr_1}{dt}}{\sqrt{1 - \frac{v^2}{c^2}}} \tag{2-7}$$

Comparing Equation (2-7) with Equation (1-6) we see that there is a difference between the classic and relativistic one-way Doppler equation. Expanding Equation (2-7) into a series we find that the relative difference is $\frac{1}{2} \frac{v^2}{c^2}$ plus higher order terms.

The direction cosine in the S′ frame can be found from Equation (2-4) which reads after rearranging terms

$$t_1 = \frac{1 - \frac{v}{c}\cos\alpha}{\sqrt{1 - \frac{v^2}{c^2}}} \left\{ t' - \frac{1}{c}\left(\frac{\cos\alpha - \frac{v}{c}}{1 - \frac{v}{c}\cos\alpha} x' + \frac{\sqrt{1 - \frac{v^2}{c^2}}\sin\alpha}{1 - \frac{v}{c}\cos\alpha} y' \right) \right\} \tag{2-8}$$

and the direction cosine in the S′ frame is

$$\cos \alpha' = \frac{\cos \alpha - \frac{v}{c}}{1 - \frac{v}{c} \cos \alpha} \tag{2-9}$$

Let us now retransmit the electromagnetic wave from the moving frame S′ to a receiver Re in the stationary frame S as shown in Figure 4. The wave is transmitted from the vehicle at time t' and reaches the receiver Re at time t_2' as seen from the S′ frame.

$$t_2' = t' + \frac{r_2'}{c} \tag{2-10}$$

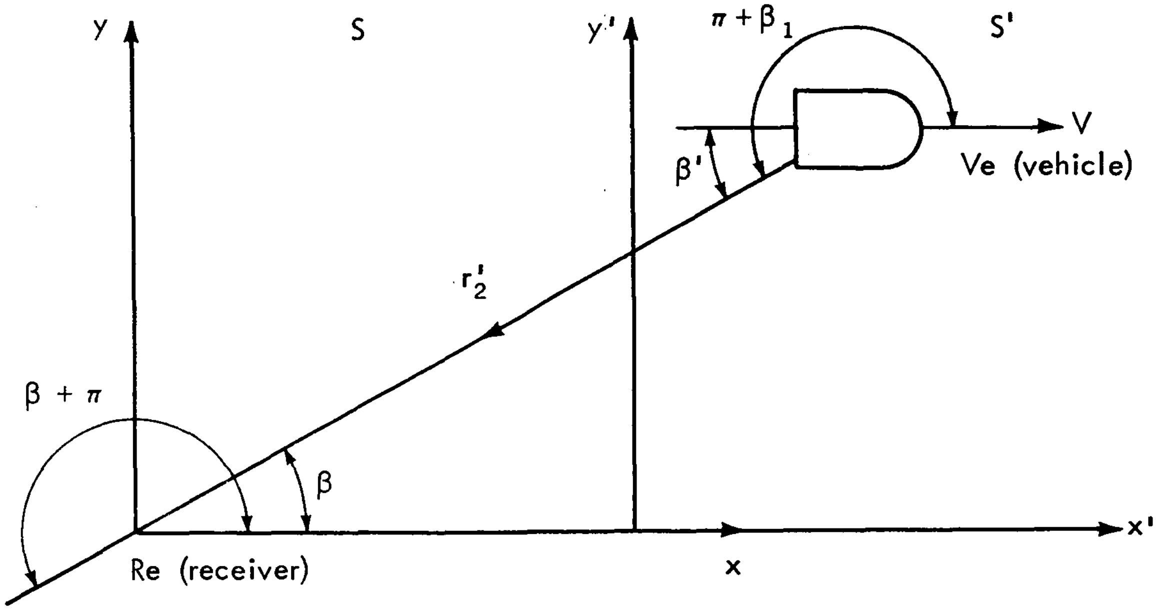

Figure 4–Reference Frames for the Lorentz Transformation Reception by the Ground Receiver Re

The time t_2 of reception in the S frame is again obtained by means of the Lorentz transformation. Observing that the sign of the velocity has changed, the Lorentz transformation reads

$$x' = \frac{x - v t_2}{\sqrt{1 - \frac{v^2}{c^2}}}$$

$$y' = y \tag{2-11}$$

$$z' = z = 0$$

$$t_2' = \frac{t_2 - \frac{v}{c^2} x}{\sqrt{1 - \frac{v^2}{c^2}}}$$

With, see Figure 4,

$$r_2' = x' \cos(\pi + \beta') + y' \sin(\pi + \beta') \tag{2-12}$$

we obtain from Equations (2-10) and (2-11)

$$t' = \frac{1 - \frac{v}{c}\cos\beta'}{\sqrt{1 - \frac{v^2}{c^2}}} \left\{ t_2 - \frac{1}{c} \left(\frac{-\cos\beta' + \frac{v}{c}}{1 - \frac{v}{c}\cos\beta'} x - \frac{\sqrt{1 - \frac{v^2}{c^2}} \sin\beta'}{1 - \frac{v}{c}\cos\beta'} y \right) \right\} \tag{2-13}$$

and the received angular frequency ω_r is thus,

$$\frac{\omega_r}{\omega_v} = \frac{dt'}{dt_2} = \frac{1 - \frac{v}{c}\cos\beta'}{\sqrt{1 - \frac{v^2}{c^2}}} \tag{2-14}$$

and the direction cosine in the S frame is

$$\cos\beta = \frac{\cos\beta' - \frac{v}{c}}{1 - \frac{v}{c}\cos\beta'} \tag{2-15}$$

or

$$\cos\beta' = \frac{\cos\beta + \frac{v}{c}}{1 + \frac{v}{c}\cos\beta} \tag{2-16}$$

Equation (2-14) contains the direction cosine of the S′ frame. Substituting $\cos\beta'$ from Equation (2-13) yields

$$\frac{\omega_r}{\omega_v} = \frac{\sqrt{1 - \frac{v^2}{c^2}}}{1 + \frac{v}{c}\cos\beta} \tag{2-17}$$

The two-way Doppler equation is obtained by combining Equations (2-6) and (2-17)

$$\frac{\omega_r}{\omega_t} = \frac{1 - \frac{v}{c}\cos\alpha}{1 + \frac{v}{c}\cos\beta} \tag{2-18}$$

or

$$\frac{\omega_r}{\omega_t} = \frac{1 - \frac{1}{c}\frac{dr_1}{dt}}{1 + \frac{1}{c}\frac{dr_2}{dt}} \tag{2-19}$$

since

$$\frac{dr_2}{dt} \square v \cos\beta \tag{2-20}$$

Equation (2-18) is identical to the "classic" two-way Doppler equation and contains no "relativistic terms." Note that Equation (2-18) contains only the direction cosine of the S frame. This is a logical choice if we want to express the two-way Doppler equation in terms of range rate referenced to the S frame, see Equation (2-19). We could also use the direction cosine in the S′ frame

$$\frac{\omega_r}{\omega_t} = \frac{1 - \frac{v}{c}\cos\beta'}{1 + \frac{v}{c}\cos\alpha'} \tag{2-21}$$

without encountering "relativistic terms." However, if direction cosines are mixed both from the S and S′ frame

$$\frac{\omega_r}{\omega_t} = \frac{\left(1 - \frac{v}{c}\cos\alpha\right)\left(1 - \frac{v}{c}\cos\beta'\right)}{1 - \frac{v^2}{c^2}} \tag{2-22}$$

then the Doppler equation will contain "relativistic terms," but there seems to be no reason for using the Doppler equation in the "mixed" form.

3. VELOCITY ABERRATION

An electromagnetic wave, reflected in a corner reflector on the vehicle, will not be reflected back to the stationary reference frame at the same angle as it was transmitted. Both a non-relativistic and a relativistic derivation of this aberration is given here.

A. Non-Relativistic Derivation

The velocity components of the wave in the stationary reference frame are $c \sin \alpha$ and $c \cos \alpha$ as shown in Figure 5. In the moving reference frame the velocity components are $c \sin \alpha$ and $c \cos \alpha - v$. After reflection the absolute values of the components are $|c \sin \alpha|$ and $|c \cos \alpha - 2v|$. Thus

$$\tan (\alpha + 2\phi) = \frac{c \sin \alpha}{c \cos \alpha - 2v}$$

or

$$\tan 2\phi = \frac{2 \frac{v}{c} \sin \alpha}{1 - \frac{2v}{c} \cos \alpha} \tag{3-1}$$

where 2ϕ is the aberation angle.

B. Relativistic Derivation

The relativistic aberation can be obtained from Equations (2-9) and (2-15) For the corner reflector we have $\alpha' = \beta'$. With $\beta = \alpha + 2\phi$ we thus obtain

$$\frac{\cos \alpha - \frac{v}{c}}{1 - \frac{v}{c} \cos \alpha} = \frac{\cos (\alpha + 2\phi) + \frac{v}{c}}{1 + \frac{v}{c} \cos (\alpha + 2\phi)}$$

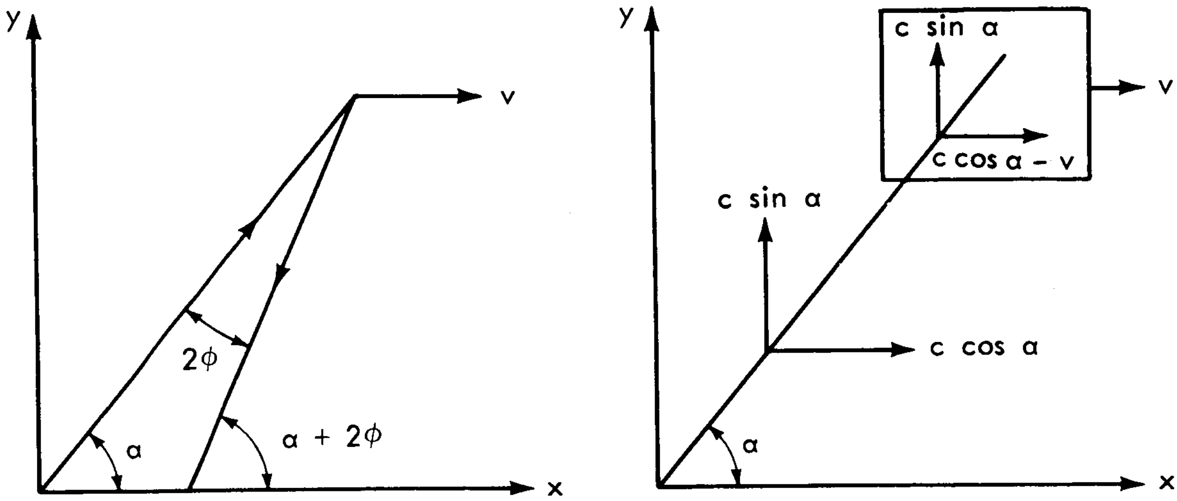

Figure 5—Non-relativistic Velocity Aberration

Solving for 2ϕ yields

$$\tan 2\phi = \frac{2\frac{v}{c}\sin\alpha - \frac{v^2}{c^2}\sin 2\alpha}{1 - 2\frac{v}{c}\cos\alpha + \frac{v^2}{c^2}\cos 2\alpha} \tag{3-2}$$

or

$$\tan\phi = \frac{\frac{v}{c}\sin\alpha}{1 - \frac{v}{c}\cos\alpha} \tag{3-2a}$$

The difference in non-relativistic and relativistic aberration is best seen if we expand Equations (3-1) and (3-2) into series.

Non-Relativistic:

$$\tan 2\phi = \frac{2v}{c}\sin\alpha\left(1 + \frac{v}{c}\cos\alpha + \frac{4v^2}{c^2}\cos^2\alpha + \cdots\right) \tag{3-3}$$

Relativistic:

$$\tan 2\phi = \frac{2v}{c} \sin \alpha \left(1 + \frac{v}{c} \cos \alpha + \frac{v^2}{c^2} + \cdots \right) \tag{3-4}$$

The difference between the two derivations is in the second order term and the relative difference is therefore of the magnitude $(v/c)^2$.

If we instead require that the angles of transmission and reception at the ground are identical, i.e., $\alpha = \beta$, then the angle of transmission β' from the vehicle has to offset from the angle of reception α' at the vehicle. With the offset angle 2ψ

$$2\psi + \alpha' = \beta' \tag{3-5}$$

we obtain from Equations (2-9) and (2-15)

$$\tan 2\psi = -\frac{2\frac{v}{c}\sin \alpha' + \frac{v^2}{c^2}\sin 2\alpha'}{1 + 2\frac{v}{c}\cos \alpha' + \frac{v^2}{c^2}\cos 2\alpha'} \tag{3-6}$$

or

$$\tan \psi = -\frac{\frac{v}{c}\sin \alpha'}{1 + \frac{v}{c}\cos \alpha'} \tag{3-7}$$

4. INTEGRATION OF DOPPLER FREQUENCY

In the previous Chapters we derived the Doppler frequency as a function of the range rate of a moving vehicle. In performing range and range rate measurements we are faced with a different problem: If we measure the Doppler frequency what can we say about the range and range rate?

Frequency cannot be measured instantaneously. The most commonly used methods for measuring the Doppler frequency is to count the number of Doppler cycles during a fixed time period or to measure the time required for the reception of a fixed number of Doppler cycles. Both methods imply an integration of Doppler frequency and the integrated Doppler frequency will be discussed in this Chapter.

Let us integrate the two-way angular Doppler frequency at the receiver during a time interval T_2. We obtain

$$\int_{t_2}^{t_2+T_2} \omega_d \, dt_2 = \int_{t_2}^{t_2+T_2} (\omega_r - \omega_t) \, dt_2$$

$$= \int_{t_2}^{t_2+T_2} \left(\omega_t \frac{dt_1}{dt_2} - \omega_t \right) dt_2 = \omega_t \left[t_1 - t_2 \right]_{t_2}^{t_2+T_2} \tag{4-1}$$

by using Equations (1-9) and (1-12). But from Equation (1-18)

$$t_2 = t_1 + \tau_1 - \tau_2$$

and thus

$$\int_{t_2}^{t_2+T_2} \omega_d \, dt_2 = - \omega_t \left[\tau_1 + \tau_2 \right]_{t_2}^{t_2+T_2} \tag{4-2}$$

In Figure 6 (1) and (2) denote the vehicle positions corresponding to the beginning and the end of the integration interval T_2. With the notations in Figure 6, Equation (4-2) may also be written

$$\int_{t_2}^{t_2+T_2} \omega_d \, dt_2 = - \omega_t \left\{ (\tau_{12} + \tau_{22}) - (\tau_{11} + \tau_{21}) \right\} \tag{4-3}$$

Equation (4-3) shows clearly that the integrated Doppler frequency shift is a measurement of the difference in propagation delay when the wave travels the path Tr → (1) → Re and the path Tr → (2) → Re. It is seen from Figure 6 that the vehicle is in position (1) at vehicle time t

$$t = t_2 - \tau_{21} \tag{4-4}$$

and in position (2) at vehicle time t + T

$$t + T = t_2 + T_2 - \tau_{22} \tag{4-5}$$

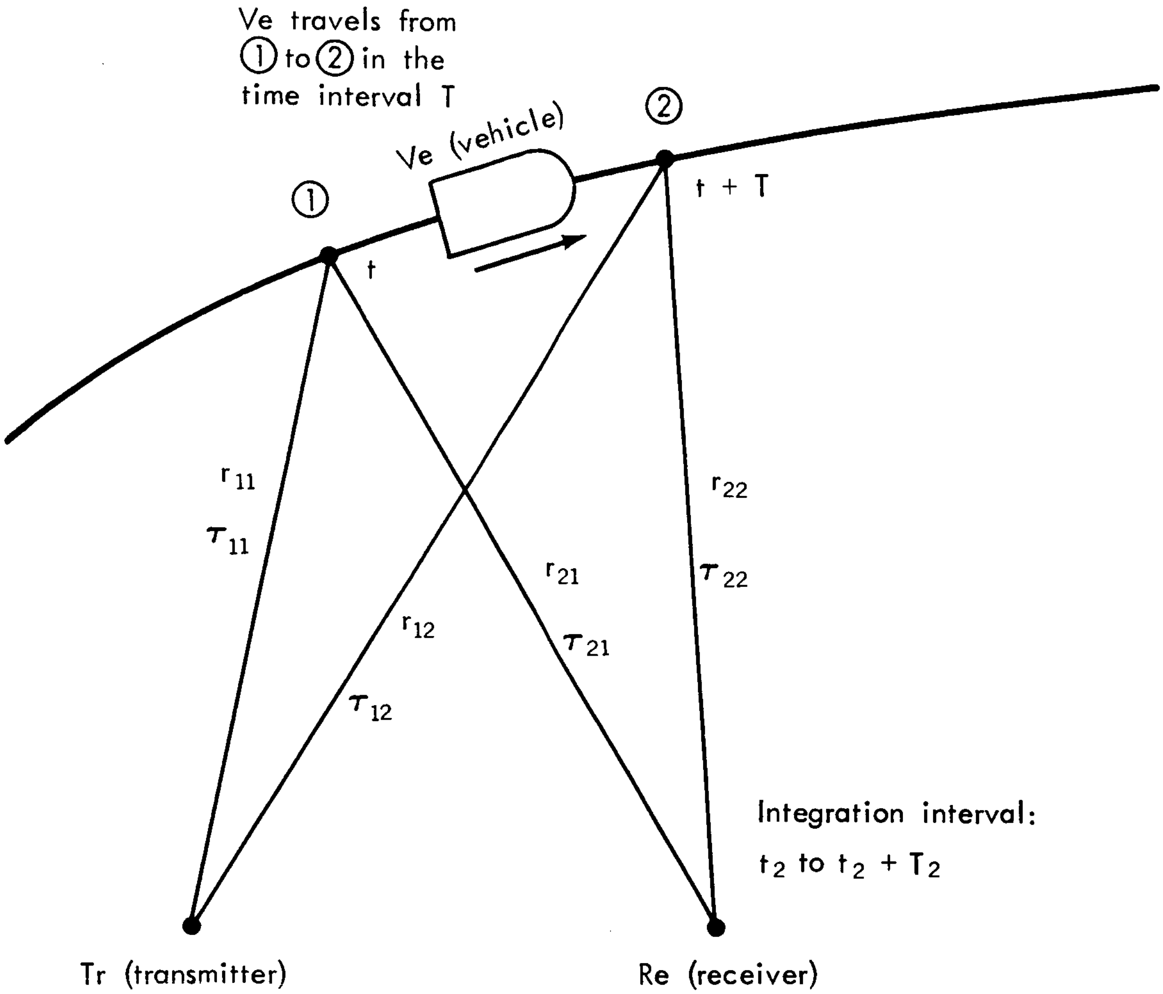

Figure 6–Integration of Doppler Frequency

The above equations are general to the extent that no assumptions have been made about the index of refraction of the propagation media.

If we want to interpret the measured difference in propagation delay as a difference in range, then assumptions have to be made about the index of refraction. Let us assume that the propagation media is vacuum. Then

$$\tau_{11} = \frac{r_{11}}{c}, \quad \tau_{12} = \frac{r_{12}}{c} \text{ etc.}$$

and

$$\int_{t_2}^{t_2+T_2} \omega_d \, dt_2 = -\frac{\omega_t}{c} \{(r_{12} + r_{22}) - (r_{11} + r_{21})\} \tag{4-6}$$

where

$$\left.\begin{array}{l} r_{11} = r_1 \\ \\ r_{21} = r_2 \end{array}\right\} \text{ at } \quad t = t_2 - \frac{r_{21}}{c}$$

$$\left.\begin{array}{l} r_{12} = r_1 \\ \\ r_{22} = r_2 \end{array}\right\} \text{ at } \quad t + T = t_2 + T_2 - \frac{r_{22}}{c}$$

With the following use full notations

$$r_{12} - r_{11} = \Delta r_1$$

$$r_{22} - r_{21} = \Delta r_2$$

$$\int_{t_2}^{t_2+T_2} \omega_d \, dt_2 = \Delta\phi = \text{integrated Doppler frequency}$$

Equation (4-6) can be written

$$\Delta r_1 + \Delta r_2 = -\frac{c\Delta\phi}{\omega_t} \tag{4-7}$$

In other words, the integrated Doppler frequency $\Delta\phi$ is proportional to a change in range.

The time T required for the vehicle to travel from 1 to 2 is according to Equations (4-4) and (4-5)

$$T = T_2 - \frac{\Delta r_2}{c} \tag{4-8}$$

If we define the average range rate as

$$\dot{r}_{a1} = \frac{\Delta r_1}{T} \tag{4-9}$$

$$\dot{r}_{a2} = \frac{\Delta r_2}{T}$$

then Equation (4-7) yields

$$\dot{r}_{a1} + \dot{r}_{a2} = - \frac{c\Delta\phi}{\omega_t T} \tag{4-10}$$

At the ground receiver Re only the time interval T_2 is known and not the time interval T. From Equations (4-8) and (4-9) we obtain

$$T = \frac{T_2}{1 + \frac{\dot{r}_{a2}}{c}}$$

and thus

$$\dot{r}_{a1} + \dot{r}_{a2} \left(1 + \frac{\Delta\phi}{\omega_t T_2}\right) = - \frac{c\Delta\phi}{\omega_t T_2} \tag{4-11}$$

For the case of the coinciding ground transmitter and receiver antenna, i.e. $\Delta r_1 = \Delta r_2$, Equation (4-11) reduces to

$$\dot{r}_a = - \frac{\frac{c\Delta\phi}{\omega_t T_2}}{2 + \frac{\Delta\phi}{\omega_t T_2}} \tag{4-12}$$

Equation (4-11) is the integrated two-way Doppler equation. Observe the formal similarity between the integrated Equation (4-11) and the non-integrated Equation (1-13b). If we define the average angular Doppler frequency ω_{da}

$$\omega_{da} = \frac{1}{T_2} \int_{t_2}^{t_2+T_2} \omega_d \, dt_2 = \frac{\Delta\phi}{T_2}$$

we see that Equation (4-11) contains only averaged quantities or finite differences. Equation (1-13b) on the other hand contains only differentials and has therefore an "instantaneous" character. Although the Equations (4-11) and (1-13b) are formally similar, they are only identical in the limit when $T_2 \to 0$. The difference between the average range rate $\dot{r}_a$ and the instantaneous range rate $\dot{r}$ may be considerable, see reference (3).

In order to associate a time with $\dot{r}_a$ the middle of the integration interval $t + \frac{1}{2} T$, expressed in vehicle time, is chosen. This choice is somewhat arbitrary but has the advantage of minimizing the difference between $\dot{r}_a$ and $\dot{r}$, see reference (3). With the aid of Equations (4-4) and (4-5) this time can be expressed in ground receiver time

$$t + \frac{1}{2} T = t_2 + \frac{T_2}{2} - \frac{1}{2} (\tau_{21} + \tau_{22}) \tag{4-13}$$

or

$$t + \frac{1}{2} T = t_2 + \frac{T_2}{2} - \frac{1}{2c} (r_{21} + r_{22}) \tag{4-13a}$$

where wave propagation through vacuum is assumed in the last equation.

5. LIMITATIONS OF RANGE MEASUREMENTS AND DEPENDENCE BETWEEN RANGE AND RANGE RATE MEASUREMENTS

Range measurements consist of the measurement of the two-way propagation delay of an electromagnetic wave. The same antenna is used both for transmission and reception. In order to compare range and range rate measurements we therefore consider the two-way Doppler equations for the special case of coinciding transmitting and receiving antennas. The Equation (4-3) for the integrated Doppler frequency reduces for this case to

$$\int_{t_2}^{t_2+T_2} \omega_d \, dt_2 = -2\omega_t (\tau_2 - \tau_1) \tag{5-1}$$

with notations as shown in Figure 7. From this equation we can draw a number of interesting conclusions:

1. The integrated two-way Doppler frequency is proportional to the difference between the two-way propagation delays at the beginning and at the end of the integration interval. A range measurement consists of the measurement of the two-way propagation delay. The integrated two-way Doppler frequency is thus proportional to the difference of two range measurements, namely the measurements of r_2 and r_1 as shown in Figure 7.

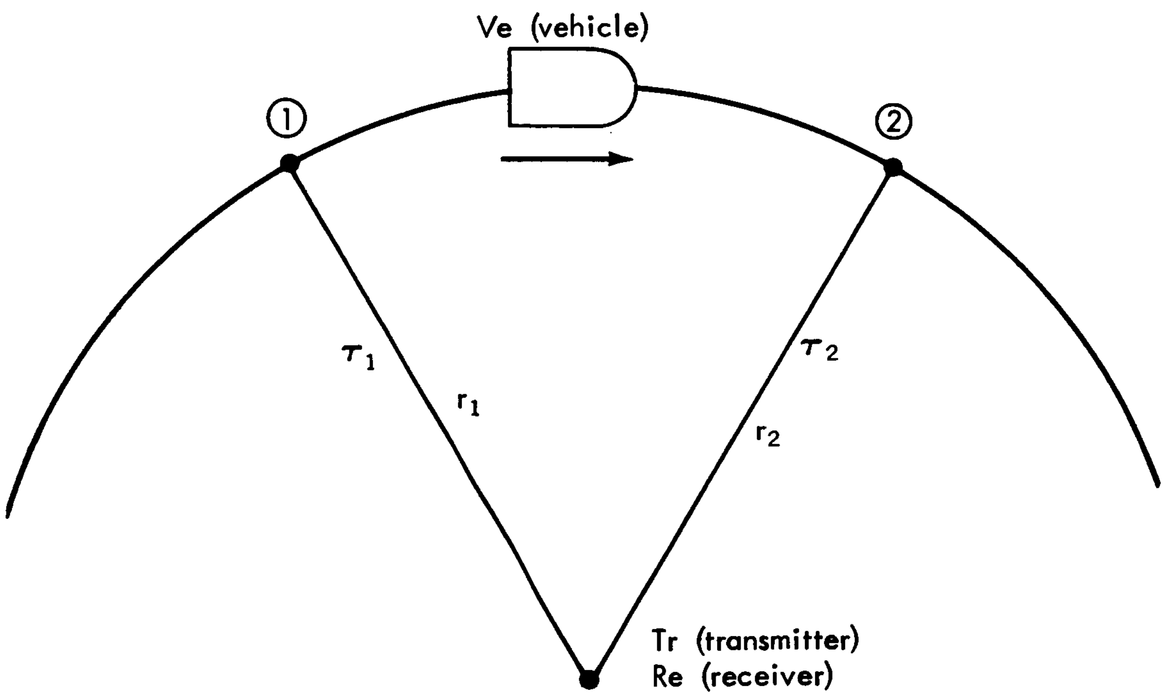

Figure 7–Comparison of Integrated Doppler Frequency and Range Measurement

2. Only the propagation properties along r_1 and r_2 are of importance and these propagation properties are the same for the integrated Doppler frequency and for range measurements of r_1 and r_2. Note that the propagation properties of the propagation medium between r_1 and r_2 are of no importance.

3. If we write the range in the form

$$r = r_0 + \int_{t_0}^{t} \dot{r}\, dt$$

we can consider r_0 as an integration constant. The integrated Doppler frequency, i.e., a range rate measurement, will not give us any information about r_0. A series of range measurements therefore contains more information than the corresponding range rate measurements.

A very important consequence of the above conclusions is: Range and range rate measurements are not independent measurements as far as propagation properties are concerned. Both are measurements of time delay and a range rate measurement is identical to the difference between two range measurements. Measuring errors introduced by the ground equipment are, on the other hand, to a certain extent independent. An example is the oscillator drift, which will effect the integrated Doppler frequency or range rate during the hole integration interval but a range measurement only during the propagation time T. For a complete analysis of these errors the particular measuring scheme employed has to be known and no general statements can therefore be made.

The above conclusion 3 states that a series of range measurements contains more information than the corresponding range rate measurements. If this is so, why are range rate measurements used? The answer is that range rate measurements can be performed much more precisely than range measurements. In order to resolve the ambiguities in range measurements, the electromagnetic wave (carrier) has to be modulated (side tones, pseudo random codes). The bandwidth allocated for this modulation determines the capability to resolve the fine ambiguities. The range accuracy is limited by the fine ambiguity resolution and is thus bandwidth limited. Most of the information is therefore obtained from the integrated range-rate or Doppler frequency and the range measurements are only used for determination of the integration constant r_0. As long as the integration process is not interrupted, only one range measurement is required for the determination of r_0. However, several range measurements may be used in order to reduce the error in r_0 by statistical methods.

Another interesting question of great practical importance is: What is the contribution of range measurements to orbit determination if both range and range rate measurements are available? From the above, one might suspect that the contribution is rather limited and this suspicion is verified by recent error analysis studies and actual experience. In reference (4) it has been shown by means of error analysis that range rate and angle measurements are adequate for orbit determination of the translunar, lunar and transearth phases of the Apollo missions. From reference (5) (table 1, page 7), we learn that only range rate and angular measurements were used for the Ranger VII orbit determination. Except for near earth phases of missions, the contribution of range measurements to orbit determination is therefore rather limited.

A rather high price has to be payed in bandwidth and hardware for obtaining range measurements. It should therefore be seriously considered if range measurements are really required, especially for deep space and galactic missions where bandwidth and weight are at an extra premium.

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