NASA CR 71091

N-PORT RECTANGULAR-SHAPED DISTRIBUTED RC NETWORKS

By

Y. Fu and J.S. Fu

February 14, 1966

Southern Methodist University
Dallas, Texas
75222

This work is supported by NASA
Institutional Grant No. NGR 44-007-006

ABSTRACT

The three-layer structure consisting of a perfect dielectric sandwiched in between a resistive film and a pure conductor has been considered as an n-port distributed RC network. Heizer has pointed out that under certain conditions this structure can generate rational 2-port parameters. This work, using a different but simpler derivation, has established a necessary and sufficient condition for that in a generalized n-port structure.

I. INTRODUCTION

Being a model for the molecular electronic system, distributed RC networks have drawn considerable attention during the last decade. Following others [1]-[7], Heizer [8]-[10] first pointed out that distributed systems can also be made to have rational short-circuit admittance parameters by choosing appropriate geometry of a two-port rectangular three-layer structure. Later, Barker [11] and Woo and Hove [12] applied his result to practical circuit realizations. Heizer introduced a condition for giving the rational functions but did not mention the uniqueness.

This work, using a different but simpler derivation, furnishes a necessary and sufficient condition for generating rational functions of that structure. Furthermore, it is generalized to n-port structures. While the formulation may seem deceptively simple, it does present a unified and comprehensive analysis of the various network properties which are indespensible to the synthesis problem. In the limiting case, the resulting solution can also provide an insight into the transcendental network functions.

II. THE BASIC DIFFERENTIAL EQUATION

Consider the network representation of an elementary length, Δx, of a layer structure with both normalized dimensions and coordinates (Fig. 1). $C_a(x)$ and $C_b(x)$ are the capacitance per unit length

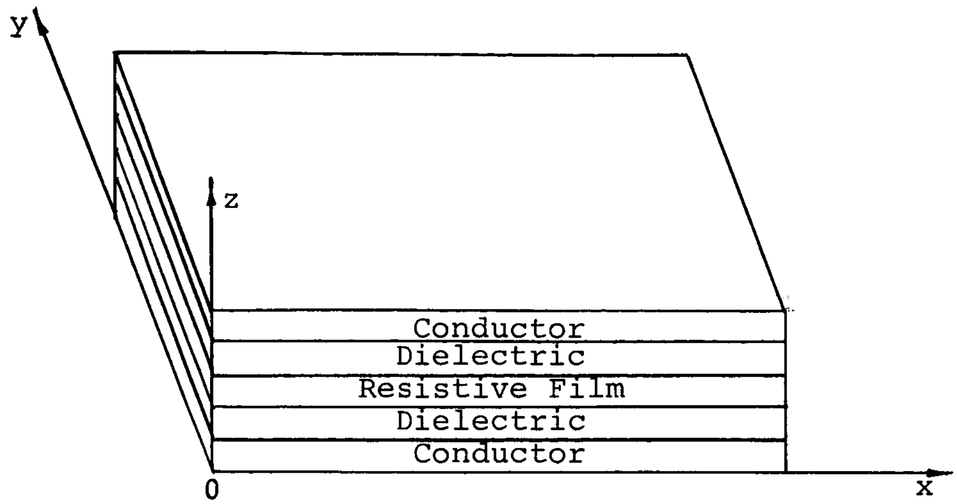

Fig.1a- Distributed RC Layer Structure

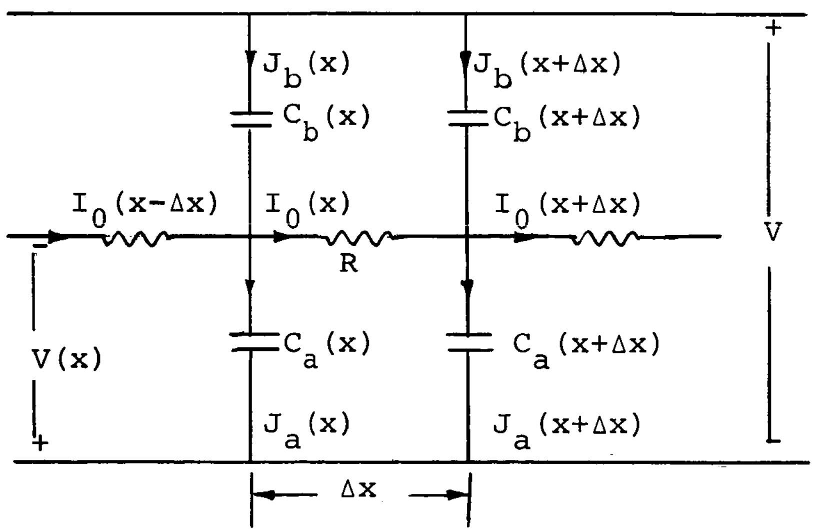

Fig.1b- Schematic Representation of Fig.1a in an Elementary Length Δx in the x-direction between the resistive film and the lower and the upper conductors respectively. The resistive film has a resistance R per unit length in the x-direction, but no resistance in the y-direction.** $J_a(x)$ and $J_b(x)$ are currents per unit length, directed in the z-direction. $I_0(x)$ is the current in the resistive film in the x-direction. $v(x)$ is the voltage across the capacitance $C_a(x)$. Further, we assume that the wavelength in the dielectric is much longer than the dimensions of the structure. Then

$$v(x) - I_0(x)R\Delta x - v(x+\Delta x) = 0 \qquad (1)$$

and

$$sC_a(x)\Delta x v(x) + I_0(x) - sC_b(x)\Delta x\{V - v(x)\} - I_0(x-\Delta x) = 0 \quad . \qquad (2)$$

Eliminate $I_0(x)$ from (1) and (2) to give

$$-\frac{v(x+\Delta x)-2v(x)+v(x-\Delta x)}{(\Delta x)^2} + s\{C_a(x)+C_b(x)\}Rv(x) - sC_b(x)RV = 0. \qquad (3)$$

If the second derivative of $v(x)$ exists, the difference equation becomes the following second order linear differential equation as $\Delta x \to 0$,

---

**This assumption can be realized approximately by making many conducting stripes parallel to the y-axis on the resistive film. However, by using uniform resistive film instead, Barker [11], Woo and Hove [12] obtained experimental results which agree fairly close to the theoretical calculations under this assumption.

$$\frac{d^2v(x)}{dx^2} - sR\{C_a(x)+C_b(x)\}v(x) = - sC_b(x)RV \quad . \tag{4}$$

(4), an analogy of the telegraph equation of the transmission lines, is considered as the differential equation for the structure shown in Fig. 2.

III. RECTANGULAR LAYER STRUCTURES

In Fig. 3, a distributed circuit is described. The resistive film has R ohms per unit length in the x-direction, and no resistance in the y-direction. The total capacitance due to the dielectric between the resistive film and the conductors in the x-direction per unit length is C farads. The conductor is cut into (n+1) pieces (Fig. 3); the voltage at conductor 0 is the reference. The capacitance between the film and the conductor k is $C_k(x)$, $k = 0, 1, \ldots, n$. Thus this structure is actually an n-port network (Fig. 4). The short-circuit parameters can be obtained as follows:

(a)

$$y_{ii} = \left.\frac{I_i}{V_i}\right|_{V_k = 0} \qquad k = 1, 2, \ldots, n; \; k \neq i \tag{5}$$

Let

$$C_a(x) = C - C_i(x), \qquad C_b(x) = C_i(x)$$

$$\gamma^2 = sRC, \qquad C_i(x) = Cf_i(x), \qquad V = V_i$$

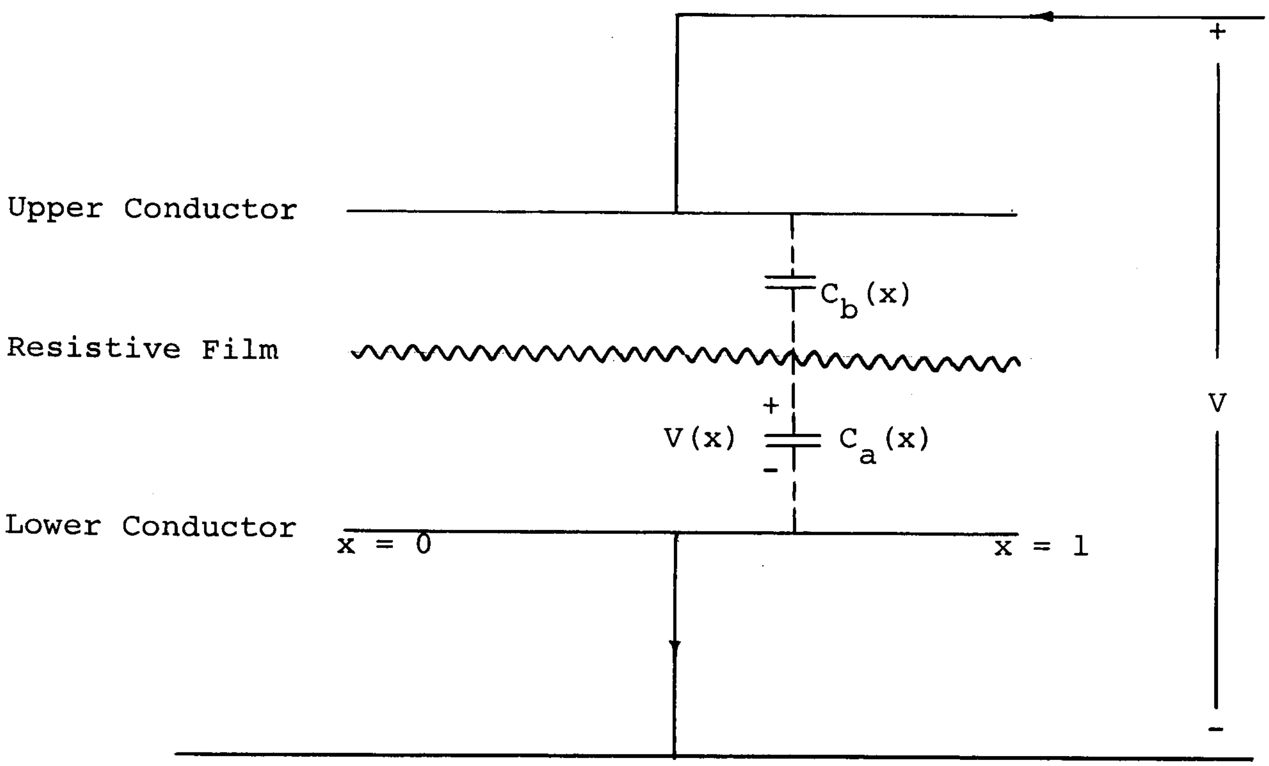

Fig.2- Schematic Representation of Layer Structure of Fig.1a

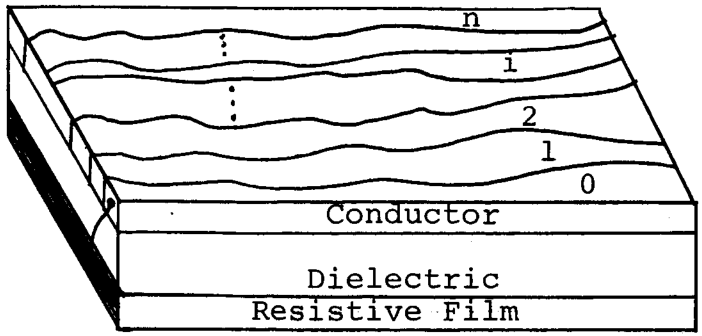

Fig. 3- N-Port Distributed RC Structure

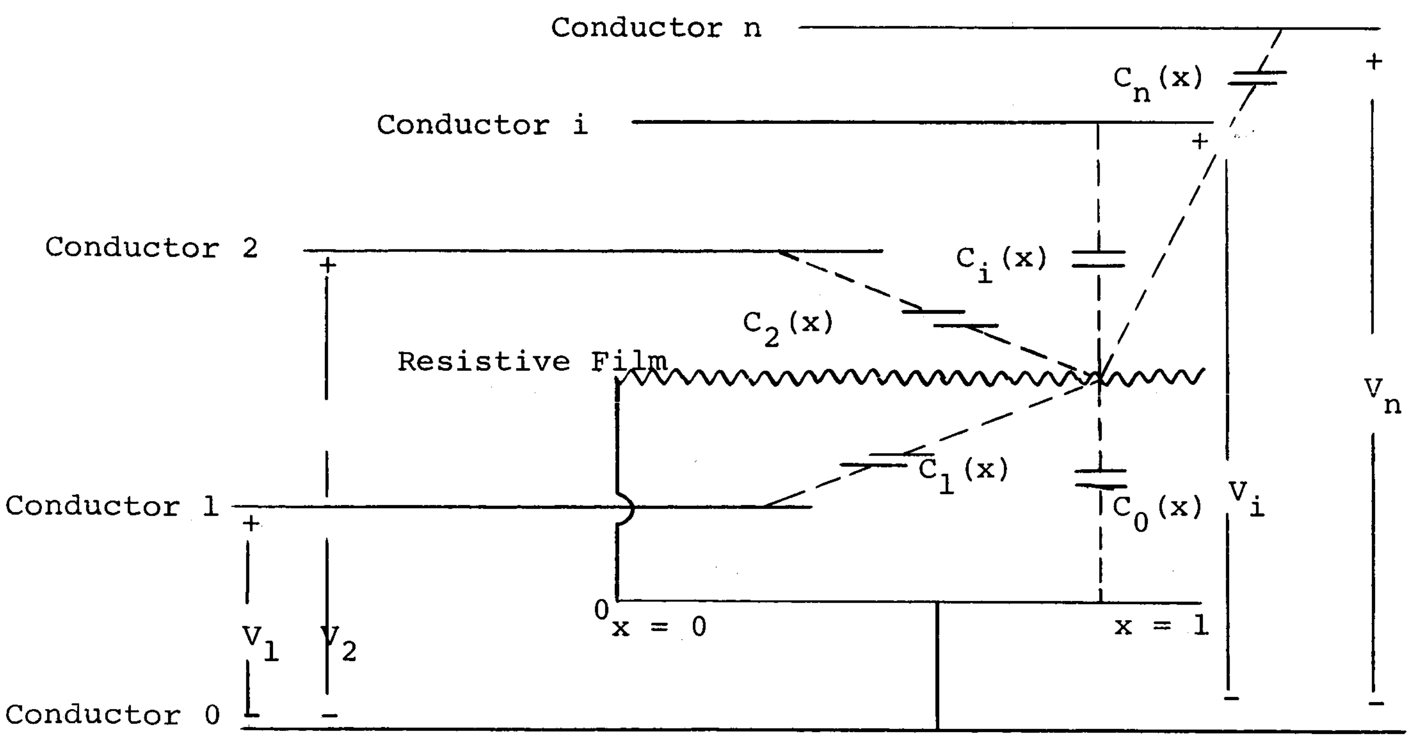

Fig.4- Schematic Representation of N-Port Distributed RC Structure of Fig.3

be substituted into (4):

$$\frac{d^2v(x)}{dx^2} - \gamma^2 v(x) = -\gamma^2 f_i(x)V_i \quad . \tag{6}$$

The general solution of (6) is

$$v(x) = Ae^{\gamma x} + Be^{-\gamma x} + V_i g_i(\gamma,x) \quad , \tag{7}$$

where $g_i(\gamma,x) = (1 - \frac{D^2}{\gamma^2})^{-1} f_i(x)$ is not an algebraic equation, its right side expresses the operation of $(1 - \frac{D^2}{\gamma^2})^{-1}$ on the function $f_i(x)$ with $D \equiv \frac{d}{dx}$. A and B can be determined from the two boundary conditions:

$$v(x) = 0 \qquad \text{at } x = 0 \ , \tag{8a}$$

and

$$\frac{dv(x)}{dx} = 0 \qquad \text{at } x = 1 \ . \tag{8b}$$

therefore

$$A = \frac{-V_i}{\gamma(e^{\gamma}+e^{-\gamma})}\{\gamma e^{-\gamma} g_i(\gamma,0) + g_i'(\gamma,1)\} \tag{9a}$$

$$B = \frac{-V_i}{\gamma(e^{\gamma}+e^{-\gamma})}\{\gamma e^{\gamma} g_i(\gamma,0) - g_i'(\gamma,1)\} \tag{9b}$$

and

$$v(x) = V_i\{g_i(\gamma,x) - \frac{\cosh\gamma(x-1)}{\cosh\gamma} g_i(\gamma,0) - \frac{\sinh\gamma x}{\gamma\cosh\gamma} g_i'(\gamma,1)\} \quad . \tag{10}$$

The current I_i from the conductor i to the resistive film is

$$I_i = \int_0^1 \{V_i - v(x)\} \, sC_i(x)dx \quad . \tag{11}$$

Substitute (10) into (11) and (5); we have the short-circuit driving point admittance

$$y_{ii} = sC\int_0^1 \{1 - g_i(\gamma,x) + \frac{\cosh\gamma(x-1)}{\cosh\gamma} g_i(\gamma,0) + \frac{\sinh\gamma x}{\cosh\gamma} g_i'(\gamma,1)\} f_i(x)dx \quad . \tag{12}$$

(b)

$$y_{ij} = \left.\frac{I_j}{V_i}\right|_{V_k = 0} \qquad k = 1, 2, \ldots, n; \; k \neq i \tag{13}$$

The current I_j, $j \neq i$, from the conductor j to the resistive film is

$$I_j = -\int_0^1 v(x) sC_j(x)dx = -sC\int_0^1 v(x) f_j(x)dx \tag{14}$$

Substitute (10) into (13) and (14), we have the short-circuit transfer admittance

$$y_{ij} = -sC\int_0^1 \{g_i(\gamma,x) - \frac{\cosh\gamma(x-1)}{\cosh\gamma} g_i(\gamma,0) - \frac{\sinh\gamma x}{\gamma\cosh\gamma} g_i'(\gamma,1)\} f_j(x)dx \quad . \tag{15}$$

Suppose $f_i(x)$ is continuous in the interval (0,1); it can be expressed by a polynomial as follows:

$$f_i(x) = \sum_\mu a_{i\mu} x^\mu \quad , \tag{16}$$

All short-circuit parameters can then be obtained from (12) and (15). In this case, they are not rational functions because $g_i(\gamma,0)$ and $g_i'(\gamma,1)$ cannot vanish simultaneously for all values of γ, and therefore the arbitrary constants A and B of (9a) and (9b) cannot be zero simultaneously. However, the cuts $f_i(x)$ satisfy the Dirichlet conditions, they can be expressed in Fourier series:

$$f_i(x) = \frac{\alpha_0}{2} + \sum_\mu \alpha_\mu \cos\mu\theta x + \sum_\mu \beta_\mu \sin\mu\theta x \tag{17}$$

then

$$g_i(\gamma,x) = \frac{\alpha_0}{2} + \sum_\mu \frac{\mu\theta\alpha_\mu \sin\mu\theta x}{1 + \frac{\mu^2\theta^2}{\gamma^2}} + \sum_\mu \frac{\mu\theta\beta_\mu \cos\mu\theta x}{1 + \frac{\mu^2\theta^2}{\gamma^2}} \quad , \tag{18}$$

and

$$g_i'(\gamma,x) = -\sum_\mu \frac{\mu\alpha_\mu \theta \sin\mu\theta}{1 + \frac{\mu^2\theta^2}{\gamma^2}} + \sum_\mu \frac{\mu\theta\beta_\mu \cos\mu\theta}{1 + \frac{\mu^2\theta^2}{\gamma^2}} \quad . \tag{19}$$

In order to have both A and B be zero, we need $g_i\,(\gamma,o) = 0$ and $g_i'(\gamma,1) = o$ simultaneously. That is

$$g_i(\gamma,0) = \frac{\alpha_0}{2} + \sum_\mu \frac{\alpha_\mu}{1 + \frac{\mu^2\theta^2}{\gamma^2}} = 0 \tag{20}$$

and

$$g_i'(\gamma,1) = -\sum_{\mu} \frac{\mu\theta\alpha_{\mu}\sin\mu\theta}{1+\frac{\mu^2\theta^2}{\gamma^2}} + \sum_{\mu} \frac{\mu\theta\beta_{\mu}\cos\mu\theta}{1+\frac{\mu^2\theta^2}{\gamma^2}} = 0 \quad . \tag{21}$$

It follows from (20) that

$$\alpha_{\mu} = 0 \; ; \qquad \mu = 0,\ 1,\ 2,\ \ldots$$

and from (21), we have

$$\mu\theta = (2m+1)\frac{\pi}{2} \; ; \qquad m = 0,\ 1,\ 2,\ \ldots$$

Therefore, we have established that the rectangular structure of Fig. 3 can give rise to rational short-circuit admittance parameters if and only if the distribution of the capacitances $C_i(x) = Cf_i(x)$ is of the following form:

$$f_i(x) = \sum_{m=0}^{M} \beta_{im}\sin(2m+1)\frac{\pi x}{2} \quad , \tag{22}$$

where M is a finite integer. This expression shows that $f_i(x)$ is a periodic function with period 4, or the interval (0,1) is actually a quarter of a period, and that this function is antisymmetrical with respect to the origin at $x = 0$, and symmetrical with respect to the axis of $x = 1$. For instance, if $f_i(0) \neq 0$, an infinite number of terms would be necessary, or M must be infinite.

If $f_i(x)$ is expressed by (22), we have

$$g_i(\gamma,x) = \sum_{m=0}^{M} \frac{\beta_{im}\sin(2m+1)\frac{\pi x}{2}}{1 + \frac{(2m+1)^2\pi^2}{4\gamma^2}} \tag{23}$$

Substitute (22) and (23) into (12), we have

$$y_{ii} = \frac{2sC}{\pi} \sum_{m=0}^{M} \frac{\beta_{im}}{2m+1} - \frac{s^2C}{2} \sum_{m=0}^{M} \frac{\beta_{im}^2}{s + \lambda_m} \tag{24}$$

where $\lambda_m = \frac{(2m+1)^2\pi^2}{4RC}$. Similarly, we calculate y_{ij} from (15) to give

$$y_{ij} = - \frac{s^2C}{2} \sum_{m=0}^{M} \frac{\beta_i\beta_j}{s + \lambda_m} . \tag{25}$$

Both results will be reduced to Heizer's form [10] if n-port is reduced to 2-port, but the sign of his result in his equation (10) is obviously wrong.

It is interesting to notice that the Fourier series of the waveform f(x), shown in Fig. 5, is

$$f(x) = \sum_{m=1}^{\infty} \frac{4}{\pi(2m+1)}\sin\frac{(2m+1)\pi x}{2} . \tag{26}$$

As we know from Fig. 3 that

$$f(x) = \sum_{i=0}^{n} f_i(x) \qquad \text{for all } x$$

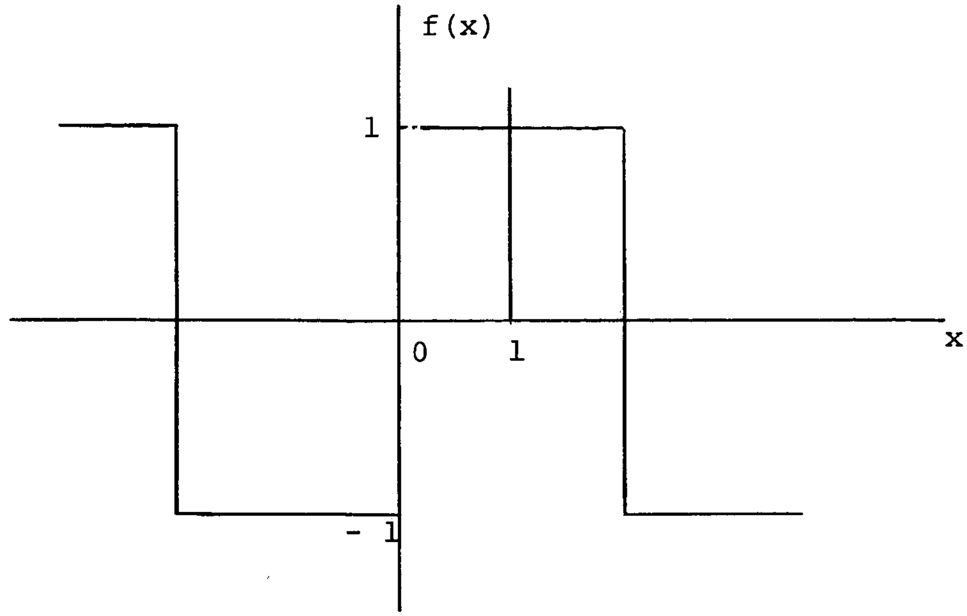

Fig.5- A Rectangular Wave Antisymmetric About x = 0
and Symmetric About x = 1.

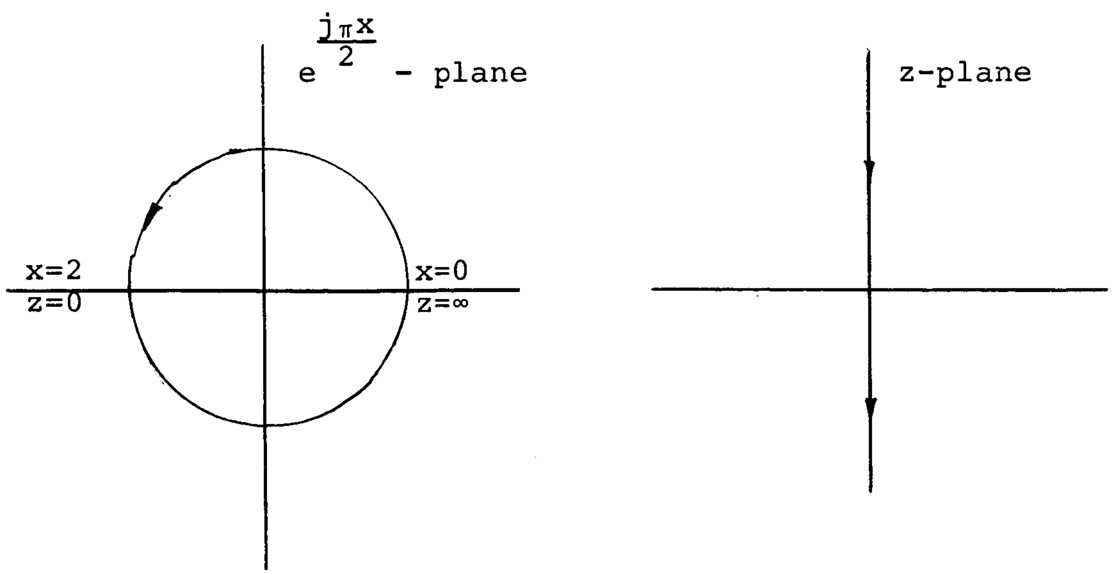

Fig.6- Bilinear Transformation

a relationship among all the Fourier coefficients of the same components for the different $f_i(x)$ is thus obtained as follows:

$$\sum_{i=0}^{n} \beta_{im} = \frac{4}{\pi(2m+1)} \quad . \tag{27}$$

Consequently, we know that, among all the expansions of $f_i(x)$; $i = 0, 1, 2, \ldots, n$, in the form of (22) at least one of them must contain an infinite number of terms. Or in the notation and coordinates we adopted, all y_{ij}; $i \neq 0$ and $j \neq 0$, will be rational if we choose all the $f_i(x)$; $i = 1, 2, \ldots, n$; $i \neq 0$, possessing a finite number of terms, since $f_0(x)$ is not involved in the calculations.

IV. OTHER RELATIONS AMONG THE FOURIER COEFFICIENTS

Physically it is necessary that $f_i(x) \geqq 0$ for all $0 \geqq x \geqq 1$. Two necessary conditions can be easily obtained by inspection:

$$\sum_{m=0}^{M} (2m+1)\beta_{im} \geqq 0 \tag{28}$$

$$\sum_{m=0}^{M} \frac{\beta_{im}}{2m+1} \geqq 0 \tag{29}$$

It is easy to see that (28) follows from the fact that when x is very small, we have

$$\sin\frac{(2m+1)\pi x}{2} \cong \frac{(2m+1)\pi x}{2} \quad ,$$

and that (29) follows from the fact that

$$\int_0^1 f_i(x)dx \geqq 0 \quad .$$

The necessary and sufficient conditions will be derived in the next paragraph.

The function $f_i(x)$, a Fourier series having only odd sine terms, must be antisymmetrical with respect to the origin and symmetrical about the axis of x = 1. Furthermore, the capacitance must be positive, that is, $f_i(x) \geqq 0$ for $0 \leqq x \leqq 2$. Or equivalently we require that the imaginary part of the new function $f_i^*(x)$ be positive, where

$$f_i^*(x) = \sum_{m=0}^{M} \beta_{im} e^{\frac{j(2m+1)\pi x}{2}} \tag{30}$$

and $j \equiv \sqrt{-1}$. Consider the transformation shown in Fig. 6

$$e^{\frac{j\pi x}{2}} = \frac{z - 1}{z + 1} \quad ,$$

or

$$z = j\cot\frac{\pi x}{4} \quad .$$

Define $F_i(u) = f_i^*(x)$, where u is the imaginary part of z, and assume the $f_i(x)$ has M+1 terms, we have

$$F_i(u) = \sum_{m=0} \beta_{im}\left(\frac{ju - 1}{ju + 1}\right)^{2m+1} \tag{31}$$

Simplify (31) to give

$$F_i(u) = \frac{-1}{(u^2+1)^{2M+1}} \sum_{m=0}^{M} \beta_{im}(u^2+1)^{2M-2m}(1-ju)^{4m+2}$$

and

$$\text{Im } F_i(u) = \frac{1}{(u^2+1)^{2M+1}} \sum_{m=0}^{M} \beta_{im}(u^2+1)^{2M-2m}\left\{\sum_{r=1,3,..}^{4m+1} \binom{4m+2}{r}(-1)^{\frac{r-1}{2}}\right\} \tag{32}$$

To require that Im $F_i(u) \geqq 0$, for $0 \leqq u \leqq \infty$, it is the same as the polynominal

$$\sum_{m=0}^{M} \beta_{im}(\mu+1)^{2M-2m}\left\{\sum_{r=1,3,..}^{4m+1} \binom{4m+2}{r}(-\mu)^{\frac{r-1}{2}}\right\} \geqq 0 \tag{33}$$

for $0 \leqq \mu \leqq \infty$, where $\mu = u^2$. The positiveness of (33) can be tested by the well-known Sturm's theorem.

V. THE POLES AND ZEROS OF 2-PORT SHORT-CIRCUIT PARAMETERS

The properties of the 2-port short-circuit parameters can be determined from (24) and (25) for M finite or infinite. In the latter case, the finite series is essentially the Mittag-Leffler expansion of a transcendental function. Rewrite (24) and (25), for the 2-port, and let $\lambda = \frac{\pi^2}{4RC}$ and we have

$$y_{22} = \frac{2sC}{\pi} \sum_{m=0}^{M} \frac{\beta_{2m}}{2m+1} - \frac{s^2C}{2} \sum_{m=0}^{M} \frac{\beta_{2m}^2}{s+(2m+1)^2\lambda} \tag{34}$$

and

$$-y_{12} = \frac{s^2C}{2} \sum_{m=0}^{M} \frac{\beta_{1m}\beta_{2m}}{s+(2m+1)^2\lambda} \tag{35}$$

Like the lumped RC element networks, y_{12} may have complex zeros, but the poles can only be at $s = -(2m+1)^2\lambda$ on the real axis. The poles and zeros of y_{22} are interlaced on the negative real axis. The open circuit transfer function [13] $G_{12} = -y_{12}/y_{22}$ must have a zero at the origin.

In special case, if $f_1(x) = f_2(x)$, or when the conductor is cut symmetrically, the $\beta_{1m} = \beta_{2m}$ for all m, and then

$$y_{22} = \frac{2sC}{\pi} \sum_{m=0}^{M} \frac{\beta_{2m}}{2m+1} + y_{12} \tag{36}$$

In this case, the difference between y_{12} and y_{22} behaves like a pure capacitance. The open-circuit transfer function G_{12} has all its transmission zeros restricted to the negative real axis starting with one at the origin.

VI. A MODIFIED CONNECTION

Different short-circuit transfer admittances can be obtained if a modification of the connection of the n-port structure of Fig. 3 is made. Instead of being cut into (n+1) pieces, the conductor plate is cut into n pieces and are named 0, 2, 3, ..., and n. Disconnect the conductor 0 and the resistive film and use the resistive film at $x = 0$ as port 1. The short-circuit transfer admittance $y_{ij}(i,j = 2, 3, \ldots, n;\ i \neq j)$ and the driving-point admittance $y_{ii}(i = 2, 3, \ldots, n)$ are unchanged but $y_{1i} = y_{i1}$ $(i = 1, 2, \ldots, n)$ are changed. They can be obtained as follows:

When $V_k = 0$ $(k = 1, 2, \ldots, n;\ k \neq i)$, the current I_1 is from (1)

$$I_1 = I_0(0) = -\frac{1}{R}\frac{dv}{dx}\bigg|_{x=0} .$$

By (10) and (23), we have

$$y_{1i} = -\frac{\pi s}{2R}\sum_{m=0}^{M}\frac{(2m+1)\beta_{im}}{s+\lambda_m} . \tag{37}$$

y_{11} can be obtained by applying a current source I_1 to the resistive film at $x = 0$ and short all other ports. From the well-known equation

$$\frac{d^2v(x)}{dx^2} - \gamma^2 v(x) = 0 , \tag{38}$$

with the boundary conditions

$$-\frac{1}{R}\frac{dv(x)}{dx}\bigg|_{x=0} = I_1 \quad \text{and} \quad \frac{dv(x)}{dx}\bigg|_{x=1} = 0 \quad ,$$

we may find $V_1 = v(0)$. By the definition of y_{11}, we obtain

$$y_{11} = \left(\frac{sC}{R}\right)^{\frac{1}{2}} \tanh\sqrt{RCs} \quad . \tag{39}$$

The interesting thing is that by this connection y_{1j} has a single zero at $s = 0$ while the other connection has a double zero there. Another difference is that y_{11} can no longer be rational. The reason is as follows: For the present connection, any non-trivial solution of (38) can be nothing but transcendental while in the previous connection, discussed in section III, the differential equation (6) is non-homogeneous and accordingly we can adjust the particular integral such that the transcendental complementary function is cancelled out. Heizer [9] had this result for y_{11} but did not give the reason why it cannot be rational. Since y_{12} has only one zero at the origin, therefore, in the two-port case, the open-circuit transfer function may have no transmission zero at the origin.

Acknowledgment

The authors wish to thank Professor K.W. Heizer for many illuminating discussions.

REFERENCES

1. Happ, W.W. and Castro, P.S., "Distributed Parameter Circuits and Microsystems Electronics," Proc. NEC, Vol. 16, p. 448-460, 1960.

2. Happ, W.W., "Synthesis of Distributed-Parameter RC Networks," Proc. IRE, Vol. 50, p. 483-484, April, 1962.

3. Wyndrum, Jr., R.W., "The Exact Realization of Distributed RC Networks," New York University, College of Engineering, Technical Report 400-76, 1963.

4. O'Shea, R.P., "Synthesis Using Distributed RC Networks," 1965 IEEE International Convention Record, Part 7, p. 18-29.

5. Mulligan, Jr., J.H., "The Roles of Network Theory in Solid-State Electronics," IRE Transactions on Circuit Theory, Vol. CT-9, p. 323-331, September, 1963.

6. Su, K.L., "Analysis of the Trigonometric RC Line and Some Applications," IEEE Transactions on Circuit Theory, Vol. CT-11, p. 158-160, March, 1964.

7. Ghausi, M.S. and Herskowitz, G.J., "The Transient Response of Tapered Distributed RC Networks," IEEE Transactions on Circuit Theory, Vol. CT-10, p. 443-445, September, 1963.

8. Heizer, K.W., "Distributed RC Networks with Rational Transfer Functions," Coordinated Science Laboratory, University of Illinois, Urbana, Illinois, Report No. R-153, September, 1962.

9. Heizer, K.W., "Distributed RC Networks with Rational Transfer Functions," IRE Transactions on Circuit Theory, Vol. CT-9, p. 356-362, December, 1962.

10. Heizer, K.W., "Rational Parameters with Distributed Networks," IEEE Transactions on Circuit Theory, Vol. CT-10, p. 531-532, December, 1963.

11. Barker, D.G., "Synthesis of Active Filters Employing Thin Film Distributed Parameter Networks," 1965 IEEE International Convention Record, Part 7, p. 119-126.

12. Woo, B.B. and Hove, R.G., "Synthesis of Rational Transfer Functions with Thin Film Distributed-Parameter RC Active Networks," 1965 National Electronics Conference.

13. Van Valkenburg, M.E., "Introduction to Modern Network Synthesis," John Wiley and Sons, Inc., 1960.