

ANALYSIS OF RECTANGULAR RC DISTRIBUTED
CIRCUITS WITH SHAPED ELECTRODES

NASA CR 71095

by

K. W. Heizer

February 9, 1966

Southern Methodist University
Dallas, Texas

This work was supported by NASA
Institutional Grant No. NGR 44-007-006

FACILITY FORM 602

N66-19536	
(ACCESSION NUMBER)	(THRU)
23	1
(PAGES)	(CODE)
CR 71095	10
(NASA CR OR TMX OR AD NUMBER)	(CATEGORY)

GPO PRICE \$ _____
CFSTI PRICE(S) \$ _____
Hard copy (HC) 1.00
Microfiche (MF) .50

SUMMARY

19536

An analysis is given for a distributed RC network which consists of a resistive layer separated by an insulator from three separate electrodes. A conducting tab is placed completely across two opposing ends of the resistive layer. These tabs form two terminals of a five-terminal network. The three electrodes form the other terminals. The short circuit admittance parameters for this network are derived and consideration is given to special interconnections. The short circuit transfer admittances between an electrode and either end of the resistive layer may easily be made rational. An interconnection provides the possibility of a rational short circuit transfer admittance which does not possess a zero at the origin. The method can be extended to similar multielectrode networks.

Author

ANALYSIS OF RECTANGULAR RC DISTRIBUTED CIRCUITS WITH SHAPED ELECTRODES

Introduction. Circuit theory for linear, lumped, passive, bilateral, finite networks has been well developed. The distributed circuit, while offering different properties of which some are advantageous, has not been fully explored. Some of the recent methods of synthesis with distributed circuits have been concerned with the interconnection of a number of uniform structures [1], [2]. Another method makes use of a non-uniform resistance and capacitance of a single structure [3]. A basic building block which lies between these approaches uses a uniform resistance and a partitioned capacitance [4], [5]. A logical extension of the latter circuit is the subject of this paper. It is necessary that the properties of the structure be established before the synthesis problem can be solved.

The type of distributed RC network discussed in this paper is illustrated schematically in Fig. 1. It is composed of a rectangular uniform resistive layer of width W and length L . A conducting tab is placed along each end of the resistive layer. Terminals 1 and 2 connect to these tabs. At any point along the resistive layer, the capacity per unit area to each of the three conducting plates is defined as $C^{(0)}(x,y)$, $C^{(3)}(x,y)$ and $C^{(4)}(x,y)$ as shown. The total capacity per unit area is constant. That is

$$C^{(0)}(x,y) + C^{(3)}(x,y) + C^{(4)}(x,y) = C. \quad (1)$$

A simple construction that accomplishes the capacity requirement is a three layer structure as shown in Fig. 2. Two of the three capacitances are zero at any point; the third is equal to C . Although this is a very useful structure, it is but a special case for the analysis given.

In the analysis the interelectrode capacitances are assumed to be zero. Any non-zero value can be considered to be a lumped element connected external to the idealized distributed circuit.

Analysis. The potential at any point on the resistive layer with respect to terminal zero is defined as V and is a function of x and y . The relations to be solved with the proper boundary conditions are

$$\nabla \cdot \bar{J}_s = -sCV + J_1 \quad (2)$$

and

$$\nabla V = -R\bar{J}_s \quad (3)$$

Where

\bar{J}_s is a surface current vector in amperes per unit length.

J_1 is an input scalar current density in amperes per unit area.

R is the layer resistivity in ohms per square.

C is the total capacity in farads per unit area.

The term J_1 in (1) allows for potentials on the other electrodes and is a function of position.

$$J_1 = sC^{(3)}V_3 + sC^{(4)}V_4 \quad (4)$$

The surface current density may be eliminated between (2) and (3) to give (5).

$$\nabla \cdot \nabla V = RCsV - J_1R \quad (5)$$

The coordinates are shown in Fig. 2. In the first case, let all terminals excluding number 1 be connected together. Voltages and currents are defined in Fig. 3. This connection imposes the following boundary conditions:

$$V = V_1 \text{ at } x = 0.$$

$$V = 0 \text{ at } x = L.$$

$$J_y = 0 \text{ at } y = 0 \text{ and at } y = W.$$

$$J_1 \text{ is zero everywhere.}$$

The solution to (5) in this case is the classical solution to the distributed RC circuit with shorted terminals.

$$V = \frac{V_1 \sinh [\gamma(L-x)]}{\sinh [\gamma L]} \quad (6)$$

and

$$J_x = \frac{\gamma \cosh [\gamma(L-x)]}{R \sinh [\gamma L]} \quad (7)$$

when $\gamma = \sqrt{RCs}$.

When (7) is evaluated at the ends, the currents I_{11} and I_{21} are readily determined.

$$I_{11} = \frac{WV_1 \gamma \coth [\gamma L]}{R} \quad (8)$$

$$-I_{21} = \frac{WV_1 \gamma}{R \sinh [\gamma L]} \quad (9)$$

Hence,

$$y_{11} = \frac{W \gamma \coth [\gamma L]}{R} \quad (10)$$

and

$$-y_{21} = \frac{W \gamma}{R \sinh [\gamma L]} \quad (11)$$

The current I_{31} can be determined by the relation

$$I_{31} = -s \int_0^W \int_0^L C^{(3)}(x,y) V dx dy \quad (12)$$

In some applications (12) is easily applied directly. In the general case it is convenient to express $C^{(3)}(x,y)$ by a two-dimensional Fourier series. Let

$$C^{(3)}(x,y) = C \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} C_{nm}^{(3)} \sin\left(\frac{n\pi x}{L}\right) \cos\left(\frac{m\pi y}{W}\right) \quad (13)$$

This apparently restricted form is quite general since the area specified is only one-fourth of the fundamental area. The substitution of (13) into (12) with the indicated integration performed yields

$$I_{31} = -\frac{WV_1\pi}{RL} \sum_{n=1}^{\infty} \frac{nsC_{no}^{(3)}}{s + \lambda_{no}} \quad (14)$$

and

$$-Y_{31} = \frac{W\pi}{RL} \sum_{n=1}^{\infty} \frac{nsC_{no}^{(3)}}{s + \lambda_{no}} \quad (15)$$

where

$$\lambda_{no} = \frac{(n\pi)^2}{L^2 RC} \quad .$$

By similar operations

$$-y_{41} = \frac{W\pi}{RL} \sum_{n=1}^{\infty} \frac{nsC_{no}^{(4)}}{s + \lambda_{no}} . \quad (16)$$

The other short circuit parameters may be determined from the interconnections shown in Fig. 4. For these connections, the boundary conditions are:

$$\begin{aligned} V &= 0 \text{ at } x = 0 \text{ and at } x = L \\ \bar{J}_y &= 0 \text{ at } y = 0 \text{ and at } y = W \\ J_1 &= V_3 s C^{(3)}(x, y) . \end{aligned}$$

In this case the solution of (5) is

$$V = \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} V_{nm} \sin\left(\frac{n\pi x}{L}\right) \cos\left(\frac{m\pi y}{W}\right) \quad (17)$$

$$\text{where } V_{nm} = \frac{V_3 s C_{nm}^{(3)}}{s + \lambda_{nm}} \quad (18)$$

and

$$\lambda_{nm} = \frac{\left(\frac{n\pi}{L}\right)^2 + \left(\frac{m\pi}{W}\right)^2}{RC} . \quad (19)$$

The surface current density may be obtained by taking the gradient of V as indicated in (3). The component in the x direction is

$$J_x = -\frac{1}{R} \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} V_{nm} \left(\frac{n\pi}{L}\right) \cos\left(\frac{n\pi x}{L}\right) \cos\left(\frac{m\pi y}{W}\right) . \quad (20)$$

The currents I_{13} and I_{23} may be determined from (20) by putting $x = 0$, and $x = W$ respectively and integrating from 0 to W . The corresponding short circuit admittance parameters are then determined to be

$$-Y_{13} = \frac{W\pi}{RL} \sum_{n=1}^{\infty} \frac{nsC_{no}^{(3)}}{s + \lambda_{no}} \quad (21)$$

and

$$Y_{23} = \frac{W\pi}{RL} \sum_{n=1}^{\infty} \frac{nsC_{no}^{(3)} \cos(n\pi)}{s + \lambda_{no}} \quad (22)$$

The current I_{43} is determined by the relation

$$I_{43} = -s \int_0^W \int_0^L C^{(4)}(x,y) V \, dx dy \quad (23)$$

where V is defined by (17). With a change in superscript, $C^{(4)}(x,y)$ can be expanded in the same manner as $C^{(3)}(x,y)$ in (13). The evaluation of (23) reveals

$$-Y_{43} = \frac{CLW}{4} \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} \frac{C_{nm}^{(3)} C_{nm}^{(4)} s^2}{s + \lambda_{nm}} \quad (24)$$

The potential between the resistive layer and terminal 3 is $V - V_3$. Hence, the driving current is

$$I_{33} = s \int_0^W \int_0^L C^{(3)}(x,y)(V_3-V) dx dy \quad (25)$$

from which one obtains

$$y_{33} = sCWL \left[\frac{1}{\pi} \sum_{n=1}^{\infty} \frac{C_{no}^{(3)}(1-(-1)^n)}{n} - \frac{1}{4} \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} \frac{(C_{nm}^{(3)})^2 s}{s + \lambda_{nm}} \right]. \quad (26)$$

In a similar manner

$$y_{44} = sCWL \left[\frac{1}{\pi} \sum_{n=1}^{\infty} \frac{C_{no}^{(4)}(1-(-1)^n)}{n} - \frac{1}{4} \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} \frac{(C_{nm}^{(4)})^2 s}{s + \lambda_{nm}} \right]. \quad (27)$$

The same procedure yields

$$y_{24} = \frac{n\pi W}{2L} \sum_{n=1}^{\infty} \frac{sC_{no}^{(4)} \cos(n\pi)}{s + \lambda_{no}} \quad (28)$$

and

$$y_{22} = \frac{W\gamma \coth[\gamma L]}{R}. \quad (29)$$

The network is bilateral, hence all short circuit admittance parameters are determined.

The five-terminal network which has been analyzed may now be interconnected in a variety of ways. For example, if terminals 1 and 2 are connected to the reference terminal, and if further $C^{(3)}(x,y)$ and $C^{(4)}(x,y)$ are expressible in a finite trigonometric

series, a three-terminal network is obtained which has rational short circuit admittance parameters. The physical construction with $C^{(3)}(x,y)$ and $C^{(4)}(x,y)$ expressible with a finite trigonometric series is difficult to produce. The structure illustrated in Fig. 2 is easy to construct and possesses interesting features.

Another example of interconnection is the case where terminals "1" and "3" are connected together to form terminal "a", terminals "2" and "4" are connected together to form terminal "b", and terminal "0" is used as a common terminal to form a two-port. The corresponding short circuit parameters are expressible in terms of the parameters of the five-terminal network.

$$y_{aa} = y_{11} + y_{13} + y_{31} + y_{33} \quad (30)$$

$$y_{ab} = y_{14} + y_{12} + y_{34} + y_{32} \quad (31)$$

$$y_{bb} = y_{22} + y_{24} + y_{42} + y_{44} \quad (32)$$

A special case of this structure is also of interest. Let $C^{(3)}(x,y) = 0$. The short circuit transfer admittance becomes $y_{14} + y_{12}$. The series form of y_{14} as given in (16) converges rapidly if the slope of the amplitude function is positive and the overall loss is not too great. In other cases it may converge very slowly. For these cases, it is computationally shorter to use (12) directly. Assuming the form given in Fig. 2 where electrode number 4 covers the area bounded below by the x axis, and above by the expression $f(x)$, it is seen from (6) and (12) that

$$-y_{41} = \frac{sC}{\text{Sinh} [\gamma L]} \int_0^L f(x) \text{Sinh} [\gamma(L-x)] dx \quad (33)$$

This integral can usually be evaluated in a straightforward manner. However, it may be observed that it is a convolution integral and may be evaluated as

$$g(L) = \int_0^L f(x) \sinh [\gamma(L-x)] dx = \mathcal{L}^{-1} \left[F(P) \frac{\gamma}{p^2 - \gamma^2} \right] \Big|_x = L \quad (34)$$

where $F(P)$ is the Laplace transform of $f(x)$ and P is the Laplace transform variable. It should also be noted that linear operations on $f(x)$ result in the same linear operations on $g(L)$. With this in mind, a table can quickly be constructed for y_{41} and $f(x)$ for a number of cases.

Table I is such an abbreviated table which gives a few interesting results.

TABLE I

Transfer Admittance as a Function
of Electrode Shape

$f(x)$	$-Y_{41}$
1. $\delta(x-x_0)$	$\frac{sC \sinh [\gamma(L-x_0)]}{\sinh [\gamma L]}$
2. $U(x-x_0)$	$\frac{sC [\cosh [\gamma(L-x_0)] - 1]}{\gamma \sinh [\gamma L]}$
3. k	$\frac{k s C [\cosh [\gamma L] - 1]}{\gamma \sinh [\gamma L]}$
4. x	$\frac{1}{R} \left[1 - \frac{\gamma L}{\sinh [\gamma L]} \right]$
5. $(x-a)U(x-a)$	$\frac{s C}{\sinh [\gamma L]} \left[\frac{-(L-a)}{\gamma} + \frac{\sinh [\gamma(L-a)]}{\gamma^2} \right]$
6. x^2	$\frac{s C}{\sinh [\gamma L]} \left[\frac{-2 - \gamma^2 L^2 + 2 \cosh [\gamma L]}{\gamma^3} \right]$
7. $\sin ax$	$\frac{s C}{\sinh [\gamma L]} \left[\frac{a \sinh [\gamma L] - \gamma \sin [aL]}{a^2 + RCs} \right]$

$f(x)$

$-y_{41}$

$$8. \quad \sin \frac{m\pi}{L} x \quad \frac{\frac{m\pi}{RL} s}{s + \frac{(m\pi)^2}{L^2 RC}}$$

$$9. \quad U(x-a) \sin [b(x-a)] \quad \frac{s C}{\sinh [\gamma L]} \left[\frac{b \sinh [\gamma(L-a)] - \gamma \sin [b(L-a)]}{RCs + b^2} \right]$$

$$10. \quad \cos ax \quad \frac{s C \gamma}{\sinh [\gamma L]} \left[\frac{\cosh [\gamma L] - \cos [aL]}{RCs + a^2} \right]$$

It is interesting to note that if $f(x)$ is Wx/L , then

$$-y_{21} - y_{41} = \frac{1}{R} . \quad (35)$$

If one further adds to $f(x)$ a function of the form

$$f_2(x) = \sum_{m=1}^N a_m \sin \frac{m\pi x}{L} , \quad (36)$$

then the short circuit transfer function is a rational function. Contrary to previous results, this rational function does not possess a zero at the origin. The position of the poles in this case are restricted to values of $-\lambda_{n0}$ as given in (15). The total function of $f(x)$ is, of course, restricted to the area of the rectangular structure.

The analysis here has dealt with a three-electrode system. However, the method is applicable to a similar structure with many elements.

Conclusion: A method of analysis has been developed for the rectangular distributed circuit described. This circuit is a two-dimensional system and therefore has the possibilities of a doubly infinite number of poles. When the capacitance of one of the electrodes to the resistive sheet can be expressed in a doubly finite trigonometric series, then the driving point and transfer admittances of this electrode are rational functions. A practical structure produces a short circuit transfer admittance which is rational and which does not necessarily have a zero at the origin. A relation was established for the possible location of the poles.

Acknowledgement

The author wishes to express his appreciation to Dr. Yumin Fu, Dr. Julia Fu and Dr. F.W. Tatum for suggestions on the manuscript.

REFERENCES

- [1] R.W. Wyndrum, Jr., "The Realization of Monomorphic Thin Film Distributed RC Networks", IEEE International Convention Record (USA), Vol. 13, Pt. 10, pp. 90-95; March, 1965.
- [2] R.P. O'Shea, "Synthesis Using Distributed RC Networks", IEEE International Convention Record (USA), Vol. 13, pt. 7, pp. 18-29; March, 1965.
- [3] R.A. Rohrer, J.A. Resh, and R.A. Hoyt, "Distributed Network Synthesis for a Class of Integrated Circuits", IEEE International Convention Record (USA), Vol. 13, pt. 7, pp. 100-112; March, 1965.
- [4] K.W. Heizer, "Distributed RC Networks with Rational Transfer Functions", Coordinated Science Lab., University of Illinois, Urbana, Report No. R-153; September, 1962.
- [5] D.G. Barker, "Synthesis of Active Filters Employing Thin Film Distributed Parameter Networks", IEEE International Convention Record (USA), Vol. 13, pt. 7, pp. 119-126; March, 1965.
- [6] K.W. Heizer, "Rational Parameters with Distributed Networks", IEEE Transactions on Circuit Theory, Vol. 10, No. 4, pp. 531-532; December, 1963.

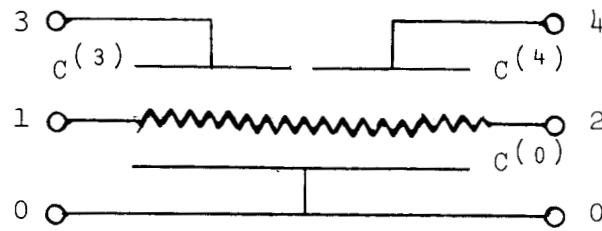


Fig. 1
Circuit Representation

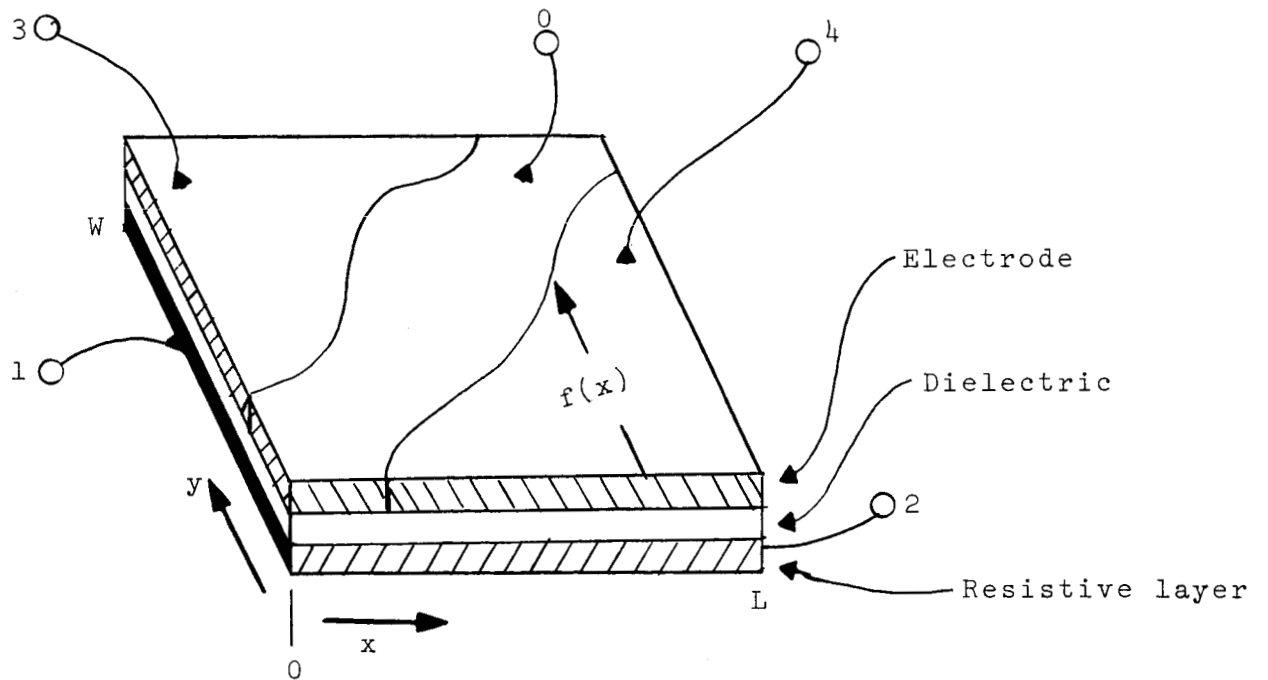


Fig. 2
Possible Layer Structure

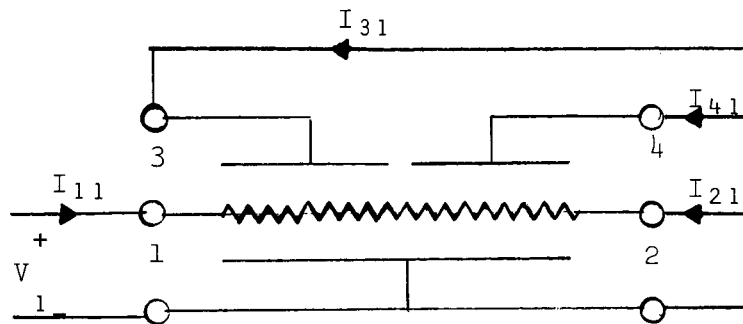


Fig. 3
First Connection

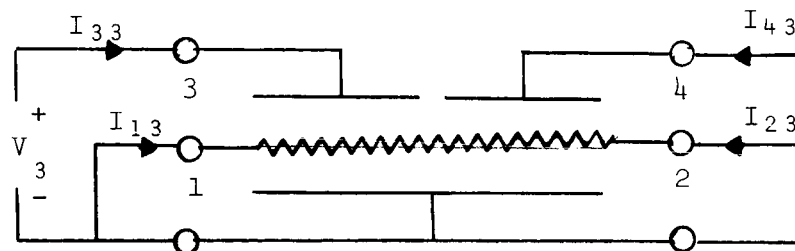


Fig. 4
Second Connection