AERO-ASTRONAUTICS REPORT NO. 14

ONE-DIMENSIONAL APPROACH TO THE MAXIMUM LIFT-TO-DRAG RATIO OF A SLENDER, FLAT-TOP, HYPERSONIC WING

BY

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SUMMARY

An investigation of the lift-to-drag ratio attainable by a slender, flat-top, affine wing at hypersonic speeds is presented under the assumptions that the pressure distribution is Newtonian and the skin-friction coefficient is constant.

It is shown that a value of the thickness ratio exists such that the lift-to-drag ratio is a maximum; this particular value is such that the friction drag is one-third of the total drag. The subsequent optimization of the chordwise and spanwise

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(*) This research was supported by the Langley Research Center of the National Aeronautics and Space Administration under Grant No. NGR-44-006-045.

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contours is reduced to the extremization of products of powers of line integrals related to the lift, the pressure drag, and the skin-friction drag. For the chordwise contour, the variational approach shows that a linear thickness distribution is the best. For the spanwise contour, a thickness distribution proportional to the chord distribution is the best. The lift-to-drag ratio of the variational solution is independent of the chord distribution and depends on the friction coefficient only. For a friction coefficient $C_f = 10^{-3}$, the maximum attainable lift-to-drag ratio is $E = 5.29$.

Author

1. INTRODUCTION

In a previous report (Ref. 1), an investigation of the lift-to-drag ratio attainable by a slender, flat-top, affine wing at hypersonic speeds was presented under the assumptions that the pressure distribution is Newtonian and the skin-friction coefficient is constant. Direct methods were employed and the analysis was confined to the class of wings whose chordwise thickness distribution is a power law and whose spanwise thickness distribution is proportional to some power of the chord distribution. For these special wings, the lift-to-drag ratio depends on three parameters: the thickness ratio, the chordwise power law exponent, and the spanwise power law exponent. Therefore, by means of the ordinary theory of maxima and minima, the combination of parameters maximizing the lift-to-drag ratio can be found.

In this report, the limitations set forth in Ref. 1 are removed and the indirect methods of calculus of variations are employed in order to determine the optimum chordwise and spanwise contours. The hypotheses employed are as follows:

(a) a plane of symmetry exists between the left-hand and right-hand parts of the wing; (b) the upper surface of the wing is flat (reference plane); (c) the wing is slender in both the chordwise and spanwise senses, that is, the squares of both the chordwise and spanwise slopes are small with respect to one; (d) the wing is affine, in the sense that each chordwise section can be generated from the root section by a linear transformation not involving rotation; (e) the free-stream velocity is parallel to the line of intersection of the plane of symmetry and the reference plane; (f) the pressure coefficient is twice the cosine squared of the angle formed by the free-stream velocity and the normal to each surface element; (g) the skin-friction coefficient is constant; and (h) the contribution of the tangential forces to the lift is negligible with respect to the contribution of the normal forces.

2. DRAG AND LIFT

We consider the class of flat-top wings and define a Cartesian coordinate system Oxyz as follows (Fig. 1): the origin O is the apex of the wing; the x-axis is the intersection of the plane of symmetry and the reference plane, positive toward the trailing edge; the z-axis is contained in the plane of symmetry, perpendicular to the x-axis, and positive downward; and the y-axis is such that the xyz-system is right-handed.

We express the geometry of the planform and the thickness distribution on the periphery of the planform in the form

$$\begin{aligned} &\underline{\text{Leading edge}} \qquad && x = m(y) \ , \quad z = o \\ &\underline{\text{Trailing edge}} \qquad && x = n(y) \ , \quad z = t(y) \end{aligned} \tag{1}$$

and write the spanwise chord distribution as

$$c(y) = n(y) - m(y) \tag{2}$$

with

$$0 \le y \le b/2 \tag{3}$$

where b is the wing span. Next, we focus our attention on those wings $z(x, y)$ such that any chordwise contour can be generated from the root contour by means of a linear transformation not involving rotation. The geometry of the lower surface of these affine wings is given by

$$z = t(0)\, A(\xi)\, B(\eta) \tag{4}$$

where the nondimensional chordwise coordinate ξ and the nondimensional spanwise coordinate η are defined as

$$\xi = \frac{x - m(y)}{c(y)} \quad , \qquad \eta = \frac{y}{b/2} \tag{5}$$

and, hence, vary between the limits 0 and 1. Also, $A(\xi)$ is a function describing the chordwise thickness distribution such that

$$A(0) = 0 \quad , \quad A(1) = 1 \tag{6}$$

and $B(\eta)$ is a function describing the spanwise thickness distribution such that

$$B(0) = 1 \tag{7}$$

With this understanding and in the light of the hypotheses of the introduction, the drag and the lift can be rewritten as

$$\begin{aligned} D/qbc(0) &= \tau^3 I_1 J_1 + C_f J_2 \\ L/qbc(0) &= \tau^2 I_3 J_3 \end{aligned} \tag{8}$$

In Eq. (8), the positive quantities I_1, I_3 are defined by

$$I_1 = \int_0^1 \dot{A}^3 d\xi \quad , \qquad I_3 = \int_0^1 \dot{A}^2 d\xi \tag{9}$$

where $\dot{A} = dA/d\xi$. Also, the positive quantities J_1, J_2, J_3 are defined as

$$J_1 = 2K_1 \; , \quad J_2 = 2K_2 \; , \quad J_3 = 2K_3 \tag{10}$$

where

$$K_1 = \int_0^1 (B^3/C^2) d\eta \;, \quad K_2 = \int_0^1 C d\eta \;, \quad K_3 = \int_0^1 (B^2/C) d\eta \tag{11}$$

Incidentally, the function $C(\eta)$ describes the chord distribution and is such that

$$C(0) = 1 \tag{12}$$

3. LIFT-TO-DRAG RATIO

From the previous formulas, it appears that--if the root chord $c(o)$, the span b, the thickness ratio τ, the chordwise contour $A(\xi)$, the spanwise contour $B(\eta)$, and the chord distribution $C(\eta)$ are given--the drag and the lift can be evaluated from Eqs. (8) through (11). Once these quantities are known, one can determine the aerodynamic efficiency or lift-to-drag ratio

$$E = L/D \tag{13}$$

which, in the light of Eqs. (8), can be written as

$$E = \tau^2 I_3 J_3 / (\tau^3 I_1 J_1 + C_f J_2) \tag{14}$$

and, clearly, is independent of the size of the body.

4. OPTIMUM THICKNESS RATIO

We now assume that the chordwise contour $A(\xi)$, the spanwise contour $B(\eta)$, and the chord distribution $C(\eta)$ are arbitrarily prescribed, and study the effect of the thickness ratio τ on the lift-to-drag ratio (14). Clearly, the lift-to-drag ratio is an extremum when the thickness ratio satisfies the relationship

$$E_\tau = 0 \tag{15}$$

whose explicit form

$$\tau / \sqrt[3]{C_f} = \sqrt[3]{(2/I_1)(J_2/J_1)} = \sqrt[3]{(2/I_1)(K_2/K_1)} \tag{16}$$

means that the friction drag is one-third of the total drag. The associated lift-to-drag ratio is given by

$$E\sqrt[3]{C_f} = \sqrt[3]{(4/27)(I_3^3/I_1^2)(J_3^3/J_1^2 J_2)} = \sqrt[3]{(4/27)(I_3^3/I_1^2)(K_3^3/K_1^2 K_2)} \tag{17}$$

and is a maximum owing to the fact that

$$E_{\tau\tau} = -(2/3C_f)I_3(J_3/J_2) = -(2/3C_f)I_3(K_3/K_2) < 0 \tag{18}$$

5. OPTIMUM CHORDWISE CONTOUR

Next, we consider bodies optimized with respect to the thickness ratio, assume that the spanwise contour $B(\eta)$ and the chord distribution $C(\eta)$ are arbitrarily prescribed, and study the effect of the longitudinal contour $A(\xi)$ on the lift-to-drag ratio (17). Since the lift-to-drag ratio depends on the longitudinal contour through the expression

$$I = I_3^3 / I_1^2 \tag{19}$$

we formulate the following problem: "In the class of functions $A(\xi)$ which satisfy the end conditions (6), find that particular function which extremizes the functional (19), where the integrals I_1, I_3 are defined by Eqs. (9)."

The functional (19) is a product of powers of integrals whose end points are fixed and is governed by the theory set forth in Ref. 2. In this reference, it is shown that the previous problem is equivalent to that of extremizing the integral

$$\tilde{I} = \int_0^1 F(\dot{A}, \lambda)\, d\xi \tag{20}$$

where the fundamental function F is defined as

$$F = \dot{A}^2 - \lambda \dot{A}^3 \tag{21}$$

and the undetermined, constant Lagrange multiplier is given by

$$\lambda = 2\, I_3 / 3\, I_1 \tag{22}$$

Since the fundamental function does not contain the variable A explicitly, the extremal solution is described by the Euler equation (see, for instance, Chapter 1 of Ref. 3)

$$d\, F_{\dot{A}} / d\xi = 0 \tag{23}$$

which admits the first integral

$$F_{\dot{A}} = \text{Const} \tag{24}$$

After this equation is explicitly written as

$$2\, \dot{A} - 3\, \lambda\, \dot{A}^2 = \text{Const} \tag{25}$$

we see that the slope $\dot{A}$ is constant. Therefore, the solution of the Euler equation (25) has the form

$$A = C_1 \xi + C_2 \tag{26}$$

where, because of the end conditions (6), the integration constants take the values

$$C_1 = 1 \ , \quad C_2 = 0 \tag{27}$$

In conclusion, the optimum contour is described by

$$A = \xi \tag{28}$$

meaning that <u>a linear thickness distribution is the best in the chordwise sense</u>.

For this variational solution, the integrals (9) are given by

$$I_1 = I_3 = 1 \tag{29}$$

and the Lagrange multiplier (22) becomes

$$\lambda = 2/3 \tag{30}$$

Finally, the optimum values of the thickness ratio (16) and the lift-to-drag ratio (17) become

$$\tau/\sqrt[3]{C_f} = \sqrt[3]{2K_2/K_1} \quad , \quad E\sqrt[3]{C_f} = \sqrt[3]{(4/27)\,(K_3^3/K_1^2K_2)} \tag{31}$$

Incidentally, the solution obtained maximizes the lift-to-drag ratio, owing to the fact that

$$F_{\dot{A}\dot{A}} = 2 - 6\lambda\dot{A} = -2 < 0 \tag{32}$$

6. OPTIMUM SPANWISE CONTOUR

Finally, we consider configurations optimized with respect to the thickness ratio τ and the longitudinal contour $A(\xi)$, assume that the chord distribution $C(\eta)$ is arbitrarily given, and study the effect of the spanwise contour $B(\eta)$ on the lift-to-drag ratio (31-2). Since the lift-to-drag ratio depends on the spanwise contour through the expression

$$K = K_3^3/K_1^2K_2 \tag{33}$$

we formulate the following problem: "In the class of functions $B(\eta)$ which satisfy the initial condition (7), find that particular function which extremizes the functional (33), where the integrals K_1, K_2, K_3 are defined by Eqs. (11)."

For each given chord distribution $C(\eta)$, the functional (33) is a product of powers of integrals and is governed by the theory set forth in Ref. 2. Therefore, the previous problem is equivalent to that of extremizing the integral

$$\widetilde{K} = \int_0^1 F(\eta, B, \lambda_1, \lambda_2)\, d\eta \tag{34}$$

where the fundamental function is defined as

$$F = B^2/C - \lambda_1 B^3/C^2 - \lambda_2 C \tag{35}$$

and the undetermined, constant Lagrange multipliers are given by

$$\lambda_1 = 2K_3/3K_1 \quad , \quad \lambda_2 = K_3/3K_2 \tag{36}$$

Since the fundamental function does not contain the derivative $\dot{B}$ explicitly, the extremal solution is described by the Euler equation (see, for instance, Chapter 1 of Ref. 3)

$$F_B = 0 \tag{37}$$

which, in explicit form, is given by

$$2(B/C) - 3\lambda_1 (B/C)^2 = 0 \tag{38}$$

and admits the solution

$$B/C = \text{Const} \tag{39}$$

Because of the conditions (7) and (12), the value of the constant is one. Hence, the optimum spanwise thickness distribution is represented by

$$B = C \tag{40}$$

that is, it is proportional to the prescribed chord distribution. For this variational solution, the integrals (11) are given by

$$K_1 = K_2 = K_3 = \int_0^1 C d\eta \tag{41}$$

and the Lagrange multipliers (36) become

$$\lambda_1 = 2/3 \quad , \quad \lambda_2 = 1/3 \tag{42}$$

Finally, the optimum values of the thickness ratio and the lift-to-drag ratio (31) become

$$\tau / \sqrt[3]{C_f} = \sqrt[3]{2} \quad , \quad E\sqrt[3]{C_f} = 2/3\sqrt[3]{2} \tag{43}$$

Incidentally, the solution obtained maximizes the lift-to-drag ratio, owing to the fact that

$$F_{BB} = 2/C - 6\lambda_1 B/C^2 = -2/C < 0 \tag{44}$$

7. DISCUSSION AND CONCLUSIONS

In the previous sections, the optimization of the lift-to-drag ratio of a slender flat-top, affine wing at hypersonic speeds is presented under the assumptions that the pressure distribution is Newtonian and the skin-friction coefficient is constant.

It is shown that a value of the thickness ratio exists which maximizes the lift-to-drag ratio; this particular value is such that the friction drag is one-third of the total drag. The subsequent optimization of the chordwise and spanwise contours is reduced to the extremization of products of power of line integrals related to the lift, the pressure drag, and the skin-friction drag. For the chordwise contour, the variational approach shows that a linear thickness distribution is the best. For the spanwise contour, the variational approach indicates the thickness distribution is to be proportional to the chord distribution. The maximum lift-to-drag ratio is independent of the chord distribution and depends on the friction coefficient only. For $C_f = 10^{-3}$, the lift-to-drag ratio of the variational solution is 5.29.

In closing, the following comments are pertinent:

(a) The main drawback of the slender wings considered here is the severe heat transfer occurring at the lines of intersection between the surfaces composing the vehicle. Consequently, the present sharp-edge configurations must be replaced by faired configurations in which the transition from one surface to another occurs with a finite curvature. If this is done, lift-to-drag ratios smaller than those predicted here are to be expected.

(b) To design a practical hypersonic vehicle, the present idealized configurations are to be modified by additional elements, such as control surfaces. Hence, a further reduction in the lift-to-drag ratio is to be expected.

(c) The cumulative detrimental effect of the considerations (a) and (b) can be offset to some degree by inclining the upper surface at a negative angle with respect to the flow, that is, by taking advantage of the added lift produced by the flow expansion. If this is done, it is probable that a lift-to-drag ratio in the neighborhood of 4 to 5 can be achieved in practice. This value is sufficiently high to encourage further studies of hypersonic cruise vehicles, suborbital vehicles, and vehicles for maneuverable reentry from outer space.

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LIST OF CAPTIONS

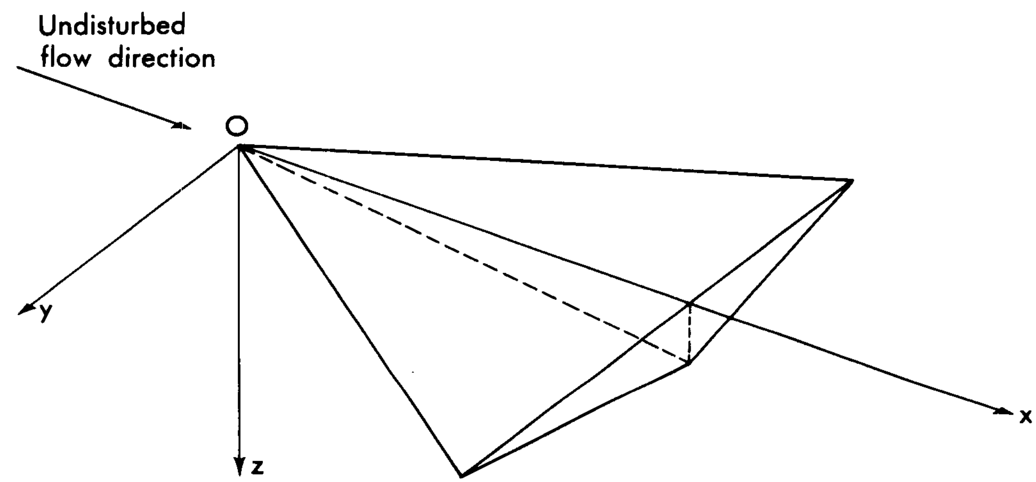

Fig. 1 Coordinate system.