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TWO-DIMENSIONAL APPROACH TO THE MAXIMUM LIFT-TO-DRAG RATIO OF A SLENDER, FLAT-TOP, HYPERSONIC WING

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SUMMARY

19661

An investigation of the lift-to-drag ratio attainable by a slender, flat-top, wing at hypersonic speeds is presented under the assumptions that the planform shape is given, the pressure distribution is Newtonian, and the skin-friction coefficient is constant. The methods of the calculus of variations in two independent variables are employed. It is shown that the optimum wing has a constant chordwise slope so that the chordwise thickness distribution is a

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linear function of the distance from the leading edge and the trailing edge thickness distribution is proportional to the chord distribution. Also, the friction drag is one-third of the total drag. The lift-to-drag ratio and the thickness ratio of the variational solution are independent of the planform shape and depend on the friction coefficient only. For a friction coefficient $C_f = 10^{-3}$, the maximum attainable lift-to-drag ratio is $E = 5.29$ and the corresponding thickness ratio is $\tau = 0.126$.

1. INTRODUCTION

In previous reports (Refs. 1 and 2), the lift-to-drag ratios attainable by slender, flat-top wings at hypersonic speeds were determined under the assumptions that the pressure distribution is Newtonian and the skin-friction coefficient is constant. The analysis was confined to the class of affine wings, that is, wings such that each chordwise section can be generated from the root section by a linear transformation not involving rotation. Because of these hypotheses, each of the surface integrals associated with the lift, the pressure drag, and the skin-friction drag degenerates into the products of two line integrals depending on the chordwise and spanwise contours, respectively. These chordwise and spanwise contours were determined so as to maximize the lift-to-drag ratio by either direct methods (Ref. 1) or the indirect methods of the calculus of variation in one independent variable (Ref. 2).

In this report, the limitations set forth in Refs. 1 and 2 are removed and the indirect methods of the calculus of variations in two independent variables (see, for instance, Ref. 3) are employed in order to determine the optimum configuration of a wing not necessarily affine. The hypotheses employed are as follows: (a) a plane of symmetry exists between the left-hand and right-hand parts of the wing; (b) the upper surface of the wing is flat (reference plane); (c) the wing is slender in both the chordwise and spanwise senses, that is, the squares of both the chordwise and spanwise slopes are small with respect to one; (d) the free-stream velocity is parallel to the line of intersection of the plane of symmetry and the reference plane; (e) the pressure coefficient is twice the cosine squared of the angle formed by the free-stream velocity and the normal to each surface element; (f) the skin-friction coefficient is constant; and (g) the contribution of the tangential forces to the lift is negligible with respect to the contribution of the normal forces.

2. DRAG AND LIFT

We consider the class of flat-top wings and define a Cartesian coordinate system Oxyz as follows (Fig. 1): the origin O is the apex of the wing; the x-axis is the intersection of the plane of symmetry and the reference plane, positive toward the trailing edge; the z-axis is contained in the plane of symmetry, perpendicular to the x-axis, and positive downward; and the y-axis is such that the xyz-system is right-handed.

We express the geometry of the planform and the thickness distribution on the periphery of the planform in the form

$$\begin{aligned} &\underline{\text{Leading edge}} \qquad x = m(y) \ , \ z = 0 \\ &\underline{\text{Trailing edge}} \qquad x = n(y) \ , \ z = t(y) \end{aligned} \tag{1}$$

and write the chord distribution as

$$c(y) = m(y) - n(y) \tag{2}$$

with

$$0 \leq y \leq b/2 \tag{3}$$

where b is the wing span.

In the light of hypotheses (a) through (g) of the introduction, the drag D and the lift L per unit free-stream dynamic pressure q can be written as (Ref. 1)

$$D/q = 4 \int_0^{b/2} \int_{m(y)}^{n(y)} (p^3 + C_f)\, dxdy$$

$$L/q = 4 \int_0^{b/2} \int_{m(y)}^{n(y)} p^2\, dxdy \tag{4}$$

where p denotes the derivative $\partial z/\partial x$. As a consequence, the lift-to-drag ratio

$$E = L/D \tag{5}$$

becomes

$$E = \frac{\int_0^{b/2} \int_{m(y)}^{n(y)} p^2\, dxdy}{\int_0^{b/2} \int_{m(y)}^{n(y)} (p^3 + C_f)\, dxdy} \tag{6}$$

3. VARIATIONAL PROBLEM

We now assume that the planform shape is given, that is, the functions m(y) and n(y) are arbitrarily prescribed. We also assume that the trailing edge thickness distribution t(y) is free. Then, we formulate the following problem: "In the class of functions z(x, y) which satisfy the boundary condition (1-1), find that particular function such that the lift-to-drag ratio (6) is a maximum."

The functional (6) is the ratio of two surface integrals and its extremization is governed by the theory set forth in Ref. 4. Therefore, the previous problem is equivalent to that of extremizing the integral

$$I = \int_0^{b/2} \int_{m(y)}^{n(y)} F(p, E)\, dxdy \tag{7}$$

where the fundamental function is defined as

$$F = p^2 - E\,(p^3 + C_f) \tag{8}$$

and the undetermined constant E is defined by Eq. (6).

Since the fundamental function depends on p only, standard methods of the calculus of variations in two independent variables show that the Euler equation (see, for instance, Chapter 3 of Ref. 3)

$$\partial F_p/\partial x = 0 \tag{9}$$

admits the following first integral:

$$F_p = A(y) \tag{10}$$

where A(y) is any function of the spanwise coordinate. Since the trailing edge thickness distribution is free, the solution of the Euler equation (9) must satisfy the following natural boundary condition:

<u>Trailing edge</u> $$F_p = 0 \tag{11}$$

with the consequence that

$$A(y) = 0 \tag{12}$$

In the light of Eq. (12), the first integral (10) can be rewritten as

$$F_p = 0 \tag{13}$$

and its explicit form

$$p\,(2-3Ep) = 0 \tag{14}$$

is solved by

$$p = 0 \quad \text{or} \quad p = 2/3E \tag{15}$$

The next step consists of applying the Legendre condition

$$F_{pp} \leq 0 \tag{16}$$

and observing that its explicit form is

$$p \geq 1/3\ E \tag{17}$$

This condition is violated by the solution (15-1) and met by the solution (15-2).

From the previous discussion, we see that the extremal surface must satisfy the differential equation

$$p = 2/3E \tag{18}$$

and, therefore, has a constant chordwise slope. Because of this property, the evaluation of the integrals appearing in Eq. (6) is immediate and leads to the relation

$$E = \frac{(2/3E)^2}{(2/3E)^3 + C_f} \tag{19}$$

which in turn yields the following maximum lift-to-drag ratio:

$$E = 2/3 \sqrt[3]{2C_f} \tag{20}$$

A mathematical consequence of Eqs. (18) and (20) is the relationship

$$p = \sqrt[3]{2C_f} \tag{21}$$

which, upon integration, leads to

$$z = \sqrt[3]{2C_f}\ x + B(y) \tag{22}$$

where B(y) is any function of the spanwise coordinate. After the function B(y) is determined from the boundary condition (1-1) as follows:

$$B(y) = -\sqrt[3]{2C_f}\ m(y) \tag{23}$$

the optimum geometry can be rewritten as

$$z = \sqrt[3]{2C_f}\ [x - m(y)] \tag{24}$$

Therefore, the chordwise thickness distribution is a linear function of the distance from the leading edge. Furthermore, the trailing edge thickness distribution is given by

$$t(y) = \sqrt[3]{2C_f}\ c(y) \tag{25}$$

and is proportional to the chord distribution; hence, the thickness ratio $\tau = t(y)/c(y)$ is given by

$$\tau = \sqrt[3]{2C_f} \tag{26}$$

Finally, if S denotes the wing surface and the coefficients of drag and lift are defined as

$$C_D = D/qS \quad , \quad C_L = L/qS \tag{27}$$

the following relationships are readily obtained:

$$C_D = 6C_f \quad , \quad C_L = 2(2C_f)^{2/3} \tag{28}$$

Since the friction drag coefficient is $2C_f$, one concludes that the friction drag of the extremal surface is one-third of the total drag.

4. DISCUSSION AND CONCLUSIONS

In the previous sections, the optimization of the lift-to-drag ratio of a slender, flat-top wing at hypersonic speeds is presented under the assumptions that the pressure distribution is Newtonian and the skin-friction coefficient is constant. It is shown that the optimum wing has a constant chordwise slope so that the chordwise thickness distribution is a linear function of the distance from the leading edge and the trailing edge thickness distribution is proportional to the chord distribution. Also, the friction drag is one-third of the total drag. The lift-to-drag ratio and the thickness ratio of the variational solution are independent of the planform shape and depend on the friction coefficient only. For a friction coefficient $C_f = 10^{-3}$, the maximum attainable lift-to-drag ratio is $E = 5.29$ and the corresponding thickness ratio is $\tau = 0.126$.

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LIST OF CAPTIONS

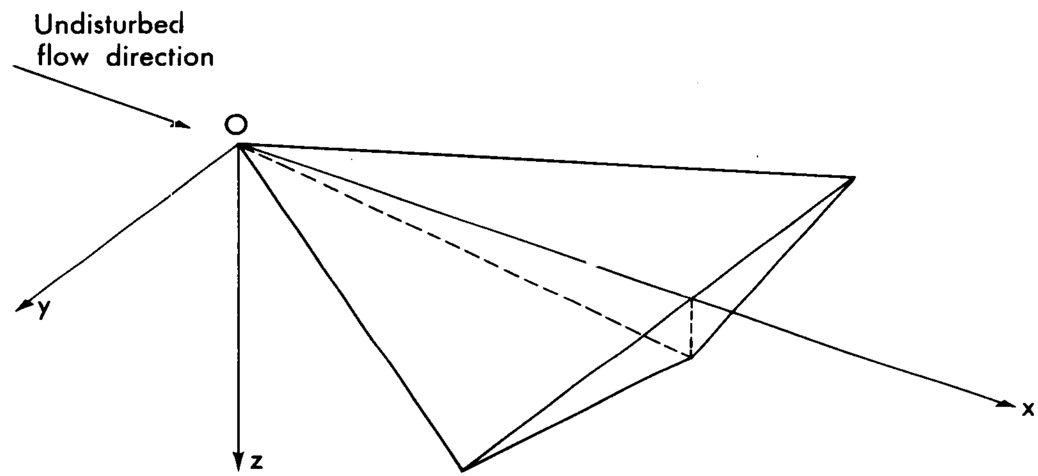

Fig. 1   Coordinate system.