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SIMILARITY LAWS FOR LIFTING WINGS OF MINIMUM DRAG AT HYPERSONIC SPEEDS

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SUMMARY

The problem of determining the chordwise and spanwise contours minimizing the drag of a flat-top wing in hypersonic flow is considered under the hypotheses that the pressure distribution is Newtonian and the skin-friction coefficient is constant. It is also assumed that the wing is slender and affine and that certain arbitrarily prescribed values are assigned to -- at most -- the lift, the planform area, the frontal area, the volume, the root chord, the span, and the root thickness. Two similarity laws are determined.

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The Similarity Law for Chordwise Contours permits one to determine the optimum chordwise contour of a wing of arbitrary spanwise contour and chord distribution from the known optimum chordwise contour of a reference wing (a wing of constant trailing edge thickness and constant chord); the aerodynamic and geometric quantities of the latter must be replaced by appropriate proportional quantities of the former, with the proportionality constants depending only on the prescribed spanwise contour and chord distribution.

The Similarity Law for Spanwise Contours permits one to determine the optimum spanwise contour and chord distribution of a wing of arbitrary chordwise contour from the known optimum spanwise contour and chord distribution of a reference wing (a wing with a linear chordwise thickness distribution); the aerodynamic and geometric quantities of the latter must be replaced by appropriate proportional quantities of the former, with the proportionality constants depending only on the prescribed chordwise contour.

1. INTRODUCTION

In a previous report (Ref 1), the basic theory of slender, lifting wings in the hypersonic regime was formulated under the following hypotheses: (a) a plane of symmetry exists between the left-hand and right-hand parts of the wing; (b) no plane of symmetry exists between the upper and lower parts; however, the intersection of these parts is a curve contained in a plane perpendicular to the plane of symmetry, called the reference plane; (c) the wing is slender in both the chordwise and spanwise senses, that is, the squares of both the chordwise and spanwise slopes are small with respect to one; (d) the wing is affine, in the sense that each chordwise section can be generated from the root section by a linear transformation not involving rotation; (e) the free-stream velocity is parallel to the line of intersection of the plane of symmetry and the reference plane; (f) the pressure coefficient is twice the cosine squared of the angle formed by the free-stream velocity and the normal to each surface element; (g) the skin-friction coefficient is constant; and (h) the contribution of the tangential forces to the lift is negligible with respect to the contribution of the normal forces. Direct methods were employed in Ref. 1. Specifically, the class of flat-top wings whose chordwise thickness distribution is a power law and whose spanwise thickness distribution is proportional to some power of the chord distribution was considered, and the thickness ratio, the chordwise power law exponent, and the spanwise power law exponent were determined so as to maximize the lift-to-drag ratio.

It should be noted that the above direct methods supply results which are valid for only particular chordwise and spanwise contours. Therefore, it is important to reformulate the minimal problem by using the indirect methods of the calculus of variations, that is, by eliminating any previous restriction on the class of wings being investigated. Thus, two complementary variational problems arise: (1) to determine the optimum chordwise contour for an arbitrarily prescribed spanwise contour; and (2) to determine the optimum spanwise contour for an arbitrarily prescribed chordwise contour. In each case, the quantity to be minimized is the drag, and constraints may be imposed on aerodynamic quantities (lift) and geometric quantities (planform area, frontal area, volume, root chord, span, and root thickness).

Since the number of possible variational problems is almost without limit, economy of thought leads us to pose the following questions: (1) Is there any similarity law which permits one to determine the optimum chordwise contour of a wing of arbitrary spanwise contour from the known optimum chordwise contour of a reference wing? and (2) Is there any similarity law which permits one to determine the optimum spanwise contour of a wing of arbitrary chordwise contour from the known optimum spanwise contour of a reference wing? It is the purpose of this paper to show that these similarity laws do exist (for the analogous laws concerning lifting bodies, see Ref. 2).

2. AERODYNAMIC QUANTITIES

In order to relate the drag and the lift of a wing to its geometry, we consider a Cartesian coordinate system Oxyz defined as follows (Fig. 1): the origin O is the apex of the wing; the x-axis is the intersection of the plane of symmetry and the reference plane, positive toward the trailing edge; the z-axis is contained in the plane of symmetry, perpendicular to the x-axis, and positive downward; and the y-axis is such that the xyz-system is right-handed.

We refer to the class of flat-top wings (Fig. 2) whose lower surface is described by the relationship

$$z = z(x, y) \tag{1}$$

and assume that the planform geometry and the thickness distribution on the periphery of the planform are expressed as

$$\begin{array}{lll} \underline{\text{Leading edge}} & x = m(y)\,, & z = 0 \\ \underline{\text{Trailing edge}} & x = n(y)\,, & z = t(y) \end{array} \tag{2}$$

Thus, in the light of hypotheses (a) through (c) and (e) through (h) of the introduction, the drag D and the lift L per unit free-stream dynamic pressure q are given by (Ref. 1)

$$D/q = 4\int_0^{b/2}\int_{m(y)}^{n(y)} z_x^3\, dy\, dx + 4C_f \int_0^{b/2} c(y)\, dy$$
$$L/q = 4\int_0^{b/2}\int_{m(y)}^{n(y)} z_x^2\, dy\, dx \qquad (3)$$

where

$$c(y) = n(y) - m(y) \qquad (4)$$

denotes the chord distribution, b the wing span, and C_f the skin-friction coefficient.

2.1. Affine Wing. Next, we employ hypothesis (d) and focus our attention on the class of wings such that any chordwise contour can be generated from the root contour by a linear transformation not involving rotation. The geometry of these affine wings is given by

$$z = t(0)\, A(\xi)\, B(\eta) \qquad (5)$$

where

$$\xi = \frac{x - m(y)}{c(y)} , \qquad \eta = \frac{y}{b/2} \tag{6}$$

denote nondimensional chordwise and spanwise coordinates, $A(\xi)$ is a function describing the chordwise thickness distribution, and $B(\eta) = t(b\eta/2)/t(0)$ is a function describing the spanwise thickness distribution. In the light of Eq. (5), the drag and the lift of an affine wing can be written as (Ref. 1)

$$\begin{aligned} D/q &= bc(0) \left[\tau^3 I_1 J_1 + C_f I_2 J_2 \right] \\ L/q &= bc(0) \tau^2 I_3 J_3 \end{aligned} \tag{7}$$

where $\tau = t(0)/c(0)$ is the thickness ratio of the root airfoil, I_1, I_2, I_3 denote the following integrals depending on the chordwise contour:

$$\begin{aligned} I_1 &= \int_0^1 \dot{A}^3 \, d\xi \\ I_2 &= 1 \\ I_3 &= \int_0^1 \dot{A}^2 \, d\xi \end{aligned} \tag{8}$$

and J_1, J_2, J_3 denote the following integrals depending on the spanwise contour and the chord distribution:

$$J_1 = 2\int_0^1 (B^3/C^2)\, d\eta$$

$$J_2 = 2\int_0^1 C\, d\eta \qquad (9)$$

$$J_3 = 2\int_0^1 (B^2/C)\, d\eta$$

with $C(\eta) = c(b\eta/2)/c(0)$. The functions $A(\xi)$, $B(\eta)$, $C(\eta)$ appearing in Eqs. (8) and (9) satisfy the boundary conditions

$$A(0) = 0\,, \qquad A(1) = 1$$

$$B(0) = 1\,, \qquad C(0) = 1 \qquad (10)$$

3. GEOMETRIC QUANTITIES

For the class of flat-top wings whose lower surface is described by Eq. (1), the planform area S, the frontal area S_f, and the volume V are given by

$$S = 2 \int_0^{b/2} \int_{m(y)}^{n(y)} dy\,dx$$

$$S_f = 2 \int_0^{b/2} t\,dy \tag{11}$$

$$V = 2 \int_0^{b/2} \int_{m(y)}^{n(y)} z\,dy\,dx$$

In particular, if the wing is affine, Eqs. (11) become

$$S = bc(0)\, I_4\, J_4$$

$$S_f = bc(0)\,\tau\, I_5 J_5 \tag{12}$$

$$V = bc^2(0)\,\tau\, I_6 J_6$$

where

$$I_4 = 1$$

$$I_5 = 1 \tag{13}$$

$$I_6 = \int_0^1 A\,d\xi$$

and

$$J_4 = \int_0^1 C\,d\eta$$

$$J_5 = \int_0^1 B\,d\eta \tag{14}$$

$$J_6 = \int_0^1 BC\,d\eta$$

Other possible constraints may have the form

$$\text{Const} = c(0), \quad \text{Const} = b \quad , \quad \text{and/or} \quad \text{Const} = \tau c(0) \tag{15}$$

meaning that the root chord, the span, and/or the root thickness are given.

4. SIMILARITY LAW FOR OPTIMUM CHORDWISE CONTOURS

Now, suppose that the spanwise contour $B(\eta)$ and the chord distribution $C(\eta)$ are arbitrarily given, meaning that the quantities J_1, ..., J_6 are known a priori. Assume that the skin-friction coefficient is given and that certain arbitrarily prescribed values are assigned to--at most--the lift, the planform area, the frontal area, the volume, the root chord, the span, and the root thickness. Observing that the right-hand sides of Eqs. (7), (12), and (15) depend on the parameters $c(0)$, b, τ and the chordwise contour $A(\xi)$, we formulate the following variational problem: "In the class of parameters $c(0)$, b, τ and functions $A(\xi)$ which satisfy the conditions (7-2), (10-1), (12), and (15), find that particular set which minimizes the total drag (7-1)."

Rather than solving this problem as stated, we introduce the following modified aerodynamic and geometric quantities:

$$\tilde{D} = (2/J_1)\, D\,, \qquad \tilde{C}_f = (J_2/J_1)\, C_f$$

$$\tilde{L} = (2/J_3)\, L\,, \qquad \tilde{S} = (1/J_4)\, S \tag{16}$$

$$\tilde{S}_f = (1/J_5)\, S_f\,, \qquad \tilde{V} = (1/J_6)\, V$$

and rewrite Eqs. (7) and (12) in the form

$$\tilde{D}/2q = bc(0)\left[\tau^3 I_1 + \tilde{C}_f I_2\right]$$

$$\tilde{L}/2q = bc(0)\,\tau^2 I_3$$

$$\tilde{S} = bc(0)\, I_4 \tag{17}$$

$$\tilde{S}_f = bc(0)\,\tau I_5$$

$$\tilde{V} = bc^2(0)\,\tau I_6$$

We observe that, since the spanwise contour and the chord distribution are given, the modified skin-friction coefficient $\tilde{C}_f$, lift $\tilde{L}$, planform area $\tilde{S}$, frontal area $\tilde{S}_f$, and volume $\tilde{V}$ are known a priori. Furthermore, the modified drag $\tilde{D}$ is proportional to the actual drag. This being the case, the previous minimal problem is equivalent to that of finding, in the class of parameters $c(0)$, b, τ and functions $A(\xi)$

which satisfy the conditions (10-1), (15), and (17-2) through (17-5), that particular set which minimizes the functional (17-1).

We note that, for a flat-top wing of constant trailing edge thickness and constant chord, that is, for

$$B = C = 1 , \qquad 0 \le \eta \le 1 \tag{18}$$

the expressions (9) and (14) become

$$\begin{aligned} J_1 &= 2 , & J_2 &= 2 , & J_3 &= 2 \\ J_4 &= 1 , & J_5 &= 1 , & J_6 &= 1 \end{aligned} \tag{19}$$

with the implication that

$$\begin{aligned} D/2q &= bc(0) \left[\tau^3 I_1 + C_f I_2 \right] \\ L/2q &= bc(0) \tau^2 I_3 \\ S &= bc(0) I_4 \\ S_f &= bc(0) \tau I_5 \\ V &= bc^2(0) \tau I_6 \end{aligned} \tag{20}$$

Since Eqs. (17) and (20) are formally identical, the following Similarity Law for Chordwise Contours can be stated: "The parameters c(0), b, τ and the function $A(\xi)$ which optimize the chordwise contour of a wing of arbitrary spanwise contour and chord distribution are identical with those optimizing the chordwise contour of a wing of constant trailing edge thickness and constant chord providing the aerodynamic and geometric quantities of the latter are replaced by the modified aerodynamic and geometric quantities (16) of the former."

5. SIMILARITY LAW FOR OPTIMUM SPANWISE CONTOURS

Now, suppose the chordwise contour $A(\xi)$ is arbitrarily given, meaning that the quantities $I_1, \ldots, I_6$ are known a priori. Assume that the skin-friction coefficient is given and that certain arbitrarily prescribed values are assigned to--at most--the lift, the planform area, the frontal area, the volume, the root chord, the span, and the root thickness. Observing that the right-hand sides of Eqs. (7), (12), and (15) depend on the parameters $c(0)$, b, τ, the spanwise contour $B(\eta)$, and the chord distribution $C(\eta)$, we formulate the following variational problem: "In the class of parameters $c(0)$, b, τ and functions $B(\eta)$, $C(\eta)$ which satisfy the conditions (7-2), (10-2), (12), and (15), find that particular set which minimizes the total drag (7-1)."

Rather than solving this problem as stated, we introduce the following modified aerodynamic and geometric quantities:

$$\widetilde{D} = (1/I_1)\, D\,, \qquad \widetilde{C}_f = (I_2/I_1)\, C_f$$

$$\widetilde{L} = (1/I_3)\, L\,, \qquad \widetilde{S} = (1/I_4)\, S \tag{21}$$

$$\widetilde{S}_f = (1/I_5)\, S_f\,, \qquad \widetilde{V} = (1/2I_6)\, V$$

and rewrite Eqs. (7) and (12) in the form

$$\begin{aligned} \widetilde{D}/q &= bc(0)\left[\tau^3 J_1 + \widetilde{C}_f J_2\right] \\ \widetilde{L}/q &= bc(0)\, \tau^2 J_3 \\ \widetilde{S} &= bc(0)\, J_4 \\ \widetilde{S}_f &= bc(0)\, \tau J_5 \\ 2\widetilde{V} &= bc^2(0)\, \tau J_6 \end{aligned} \tag{22}$$

We observe that, since the chordwise contour is given, the modified skin-friction coefficient $\widetilde{C}_f$, lift $\widetilde{L}$, planform area $\widetilde{S}$, frontal area $\widetilde{S}_f$, and volume $\widetilde{V}$ are known a priori. Furthermore, the modified drag $\widetilde{D}$ is proportional to the actual drag. This being the case, the previous minimal problem is equivalent to that of finding, in the class of parameters c(0), b, τ and functions $B(\eta)$, $C(\eta)$ which satisfy the conditions (10-2), (15), and (22-2) through (22-5), that particular set which minimizes the functional (22-1).

We note that, for a flat-top wing whose chordwise thickness distribution is linear, that is, for

$$A = \xi \quad , \quad 0 \le \xi \le 1 \tag{23}$$

the expressions (8) and (13) become

$$\begin{aligned} I_1 &= 1 , \quad I_2 = 1 , \quad I_3 = 1 \\ I_4 &= 1 , \quad I_5 = 1 , \quad I_6 = 1/2 \end{aligned} \tag{24}$$

with the implication that

$$\begin{aligned} D/q &= bc(0) \left[\tau^3 J_1 + C_f J_2 \right] \\ L/q &= bc(0)\, \tau^2 J_3 \\ S &= bc(0)\, J_4 \\ S_f &= bc(0)\, \tau J_5 \\ 2V &= bc^2(0)\, \tau J_6 \end{aligned} \tag{25}$$

Since Eqs. (22) and (25) are formally identical, the following Similarity Law for Spanwise Contours can be stated: "The parameters c(0), b, τ and the functions

$B(\eta)$, $C(\eta)$ which optimize the spanwise contour and the chord distribution of a wing of arbitrary chordwise contour are identical with those optimizing the spanwise contour and the chord distribution of a wing of linear chordwise thickness distribution, providing the aerodynamic and geometric quantities of the latter are replaced by the modified aerodynamic and geometric quantities (21) of the former."

6. DISCUSSION AND CONCLUSIONS

In the previous sections, the minimum drag problem is considered for the class of flat-top, slender, affine wings flying at hypersonic speeds under the assumptions that the pressure distribution is Newtonian and the skin-friction coefficient is constant. It is also assumed that certain arbitrarily prescribed values are assigned to -- at most -- the lift, the planform area, the frontal area, the volume, the root chord, the span, and the root thickness. Under these hypotheses, two similarity laws are obtained.

The Similarity Law for Chordwise Contours permits one to determine the optimum chordwise contour of a wing of arbitrary spanwise contour and chord distribution from the known optimum chordwise contour of a reference wing (a wing of constant trailing edge thickness and constant chord). Conversely, the Similarity Law for Spanwise Contours permits one to determine the optimum spanwise contour and chord distribution of a wing of arbitrary chordwise contour from the known optimum spanwise contour and chord distribution of a reference wing (a wing with a linear chordwise thickness distribution). Finally, the simultaneous use of these similarity laws leads to the idea that--by combining the results valid for flat-top wings of constant trailing edge thickness and constant chord with those valid for wings of linear chordwise thickness distribution--truly three-dimensional, slender, affine, lifting wings of minimum drag can be determined.

REFERENCES

1. MIELE, A., Lift-to-Drag Ratios of Slender Wings at Hypersonic Speeds, Rice University, Aero-Astronautics Report No. 13, 1966.

2. MIELE, A., Similarity Laws for Lifting Bodies of Minimum Drag at Hypersonic Speeds, Rice University, Aero-Astronautics Report No. 11, 1965.

LIST OF CAPTIONS

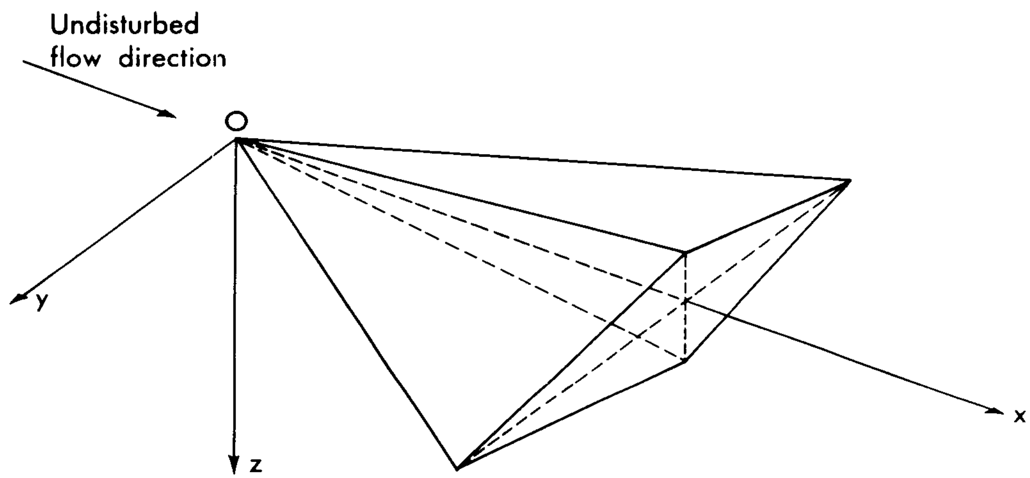

Fig. 1 Coordinate system.

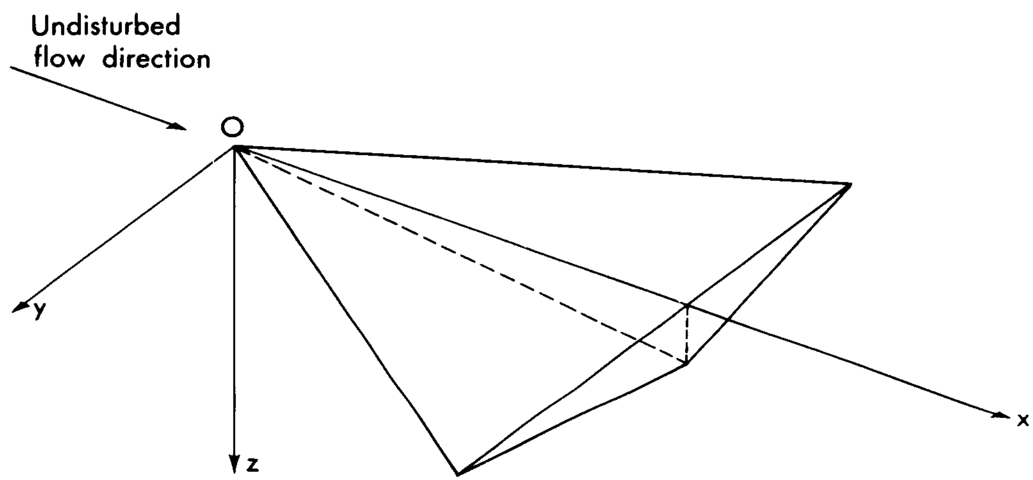

Fig. 2 Flat-top wing.