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NUMERICAL PROCEDURES FOR ROLL STABILITY STUDIES

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ABSTRACT

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This report illustrates the numerical procedure used by the Aero-Astrodynamics Laboratory in performing roll stability analyses for Saturn type space vehicles. Vehicle control in the roll channel is provided by a position gyro and a rate gyro. The dynamics of the rate gyro are included as a separate equation in the equations of motion. The modes of oscillation in this system are (1) torsion, (2) roll, (3) roll sloshing, (4) rate gyro roll, and (5) swivel engine. Two methods are available for solving the system's eigenvalues: the matrix iteration method and the characteristic equation approach. In both cases the system is solved for the normalized eigenvectors, which provide additional information about the system.

The frequency response of the system, at the control thrust vector point, can be found for a sinusoidal forcing function at various sensor locations.

The equations of motion are derived in the appendix.

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AERO-ASTRODYNAMICS LABORATORY
RESEARCH AND DEVELOPMENT OPERATIONS

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DEFINITION OF SYMBOLS

Symbol	Definition
$A_{(s)}$	Actuator transfer function
a_{or}	Roll position gain factor
a_{1r}	Roll rate gain factor
c	Damping coefficient
D	Dissipation function of total missile
$d_{(x)}$	Distance from gimbal plane to a point on the engine
de	Distance from centerline of vehicle to the swivel engine location
ds	Distance from centerline of vehicle to the slosh mass location
F_s	Swivel thrust
g	Damping factor
I_R	Vehicle moment of inertia in roll about the center of gravity
I_r	Moment of inertia of single engine about missile centerline
I_{CORR}	Correction of the I_R due to the propellant oscillations
$I_{(x)}$	Axial distribution of roll mass moment of inertia
k	Spring constant for mechanical model
K_A	Factor for gain root locus total signal
K_j	Factor for gain root locus individual channel

DEFINITION OF SYMBOLS (Cont'd)

Symbol	Definition
$\overline{K}_1$	Convenience factor for frequency response; equal to one in stability analysis
l_e	Distance from the center of gravity of the swivel engine to the swivel point
M_e	Swivel engine mass
M_s	Sloshing mass of propellant
M_τ	Generalized mass of torsional mode
q_i	Generalized coordinate
Q_i	Generalized force
s	Complex eigenvalue ($\sigma + i\omega$)
Se	First moment of swivel engine about swivel point
T	Kinetic energy of missile
$T_{g(s)}$	Transfer function position gyro filter
$T_{r(s)}$	Transfer function rate gyro filter
V	Potential energy of missile
V_x	Angular velocity at any station x along the missile centerline
β_c	Control deflection
β_e	Engine deflection parallel to missile centerline
δ_A	Artificial phase lag total signal
δ_j	Artificial phase lag individual channel

DEFINITION OF SYMBOLS (Concluded)

Symbol	Definition
δ_{ij}	Kronecker delta (0 if $i \neq j$; 1 if $i = j$)
ζ	Percent critical damping
θ	Rigid body roll deflection
θ_r	Rate gyro roll deflection
θ_μ	Normalized torsional twist
θ_e	Moment of inertia of swivel engine about swivel point
ξ_s	Amplitude of slosh mass of slosh analogy model
σ	Real part of s
τ_μ	Torsional deflection
ω	Natural frequency

SUBSCRIPTS

e	Engine
s	Sloshing
g	Position gyro
r	Rate gyro
τ	Torsion
t	Swivel engine location

TECHNICAL MEMORANDUM X-53368

NUMERICAL PROCEDURES FOR ROLL STABILITY STUDIES

SUMMARY

This report illustrates the numerical procedures used by the Aero-Astrodynamics Laboratory in performing roll stability analyses for Saturn type space vehicles. Vehicle control in the roll channel is provided by a position gyro and a rate gyro. The dynamics of the rate gyro are included as a separate equation in the equations of motion. The modes of oscillation in the system are (1) torsion, (2) roll, (3) roll sloshing, (4) rate gyro roll, and (5) swivel engine. Two methods are available for solving the system's eigenvalues: the matrix iteration method and the characteristic equation approach. In both cases the system is solved for the normalized eigenvectors, which provide additional information about the system.

The frequency response of the system, at the control thrust vector point, can be found for a sinusoidal forcing function at various sensor locations.

The equations of motion are derived in the appendix.

INTRODUCTION

The stability analysis of an elastic space vehicle requires the eigenvalues for the solution of a set of linear, homogeneous differential equations. These differential equations describe the dynamics of the space vehicle and the characteristics of the control system.

The solution is readily obtainable for a simple system, but it is difficult for systems with a more complex control system and large numbers of oscillating modes because of the high order of the characteristic equation and the large magnitude of the coefficients.

To adequately describe the vehicle characteristics, the following modes of oscillation are included in the system:

torsion	(τ)
roll	(θ)
roll sloshing	(ξ_s)
rate gyro roll	(θ_r)
swivel engine	(β_e)

The control command equation, including transfer functions for all filters and the actuator characteristics, is the final equation.

Two numerical methods are presented for solving the system's eigenvalues: the characteristic equation and the matrix iteration approaches. In both cases, the system is solved for the normalized eigenvectors, which provide additional information about the system.

GENERAL APPROACH

Basic Equations

The equations of motion, the control equation, and the equation describing the response characteristics of the rate gyro are homogeneous, linear differential equations. These equations are transformed into a set of homogeneous, linear simultaneous algebraic equations by assuming solutions of time dependency in the form e^{st} by which all differential quotients with respect to time are replaced by the complex operator.

$$s = \sigma + i\omega . \tag{1}$$

With the coordinate (or unknowns) denoted as $X_j (j = 1, 2, 3, \ldots, n)$ and the coefficients of X_j as $d_{ij(s)}$ $(i, j = 1, 2, \ldots, n)$, the set of equations reads

$$\begin{aligned}
d_{11(s)}X_1 + d_{12(s)}X_2 + \ldots\ldots + d_{1n(s)}X_n &= 0 \\
d_{21(s)}X_1 + d_{22(s)}X_2 + \ldots\ldots + d_{2n(s)}X_n &= 0 \\
\ldots \quad \ldots \quad \ldots\ldots \quad \ldots &= 0 \\
d_{n1(s)}X_1 + d_{n2(s)}X_2 + \ldots\ldots + d_{nn(s)}X_n &= 0 \quad .
\end{aligned} \tag{2}$$

To keep a consistent flow of data from other programs needed in equation (2), the following order and definitions of the X_j and equations are used throughout:

Number (i)	Definition	Symbol
1-2	Torsional modes	τ_i
3	Roll	θ
4-5	Roll sloshing	ξ_{si}
6	Rate gyro roll	θ_r
7	Swivel engine	β_e
8	Control	β_c

By using matrix notation, equation (2) can be more conveniently expressed in the form

$$D_{(s)} \{X_j\} = 0, \tag{3}$$

where all the elements of the matrix $D_{(s)}$ except the nth row have the form

$$d_{ij(s)} = s^2 A_{ij} + sB_{ij} + C_{ij}$$
$$(i = 1 \rightarrow n - 1)(j = 1 \rightarrow n) \tag{4}$$

and the elements for the nth row, which describe the control equation and the filter characteristics, as

$$d_{nj(s)} = K_j T_{j(s)}$$
$$(j = 1 \rightarrow n) . \tag{5}$$

The transfer functions $T_{j(s)}$ are of the form

$$T_j = \frac{\sum_{n=0}^{10} a_j s^n}{\sum_{m=0}^{10} b_j s^m} \quad . \tag{6}$$

With the general equation (3) in this form, two basic approaches are available for determining the eignevalues [1]: (1) the matrix iteration and (2) the characteristic equation approach. (The characteristic equation is obtained by expanding the determinant and solving for its roots.) Once the roots (eigenvalues) are determined, it is important to solve for the eigenvector for additional information. This is done by assuming a value for the engine deflection command angle (X_n or β_c) and solving for the resulting eigenvector. Usually the eigenvectors are normalized to X_n or β_c equal to one.

Matrix Iteration for Obtaining Eigenvalue

Equation (3), written as

$$D_{(s)}\{X_j\} = 0, \tag{7}$$

states the transformed equation of the system in matrix form. The problem is to find nontrivial values of s and (X_j) that satisfy equation (3). The eigenvalue problem is nonlinear because the coefficients of $D_{(s)}$ are usually higher degree polynomials in s. Starting with an approximate value of s and (X_j), it is possible to find a set of linear nonhomogeneous equations for the correction terms that have to be added to the approximate value of s and (X_j). By developing equation (3) into a Taylor series at the point s_n and neglecting terms of higher order, the iteration procedure (1) is

$$D_{(s_n)}\{\Delta X_{jn-1} + D'_{(s_n)}\}\{X_{jn}\}\Delta s = -D_{(s_n)}X_{jn} = -Y_{jn}, \tag{8}$$

where

$$D'_{(s_n)} = d\,\frac{D_{(s)}}{ds} .$$

To obtain $d\,\frac{D_{(s)}}{ds}$, each element of $D_{(s)}$ is differentiated with respect to s. The numerical procedure for equation (8) is as follows:

Step I. - Insert first approximation of $s_n = s_n^{(0)}$ - obtained from approximate root programs, sponsor load in choice, or the natural frequencies of the modes - into equation (3) and solve for the approximate eigenvectors (X_j) by setting $X_n^{(0)}$ equal to one.

Step II. - Determine the right-hand side of equation (8) $(-Y_j)$ by using the eigenvectors obtained in Step I.

Step III. - Solve equation (8) for Δs. The ΔX_j are not needed, but could be solved for.

Step IV. - Calculate $s^{(1)} = s^{(0)} + \Delta\, s$.

Step V. - Check s versus a constant $(0.001 + i\, 0.001)$.

a. If either the real or imaginary part is larger than 0.001, begin Step I again with $s^{(1)}$ as $s^{(0)}$ and repeat the above step.

b. If both real and imaginary parts are smaller than 0.001, the program has converged to an eigenvalue. Using the eigenvalue, compute the ratios of the eigenvectors for the engine deflection β_c as described previously.

Step VI. - Print out results.

Experience has shown that normalizing the eigenvectors in Step I by assuming X_n equal to one does not always give the best convergence. The convergence problem can be relieved by selecting the X_j for normalization to correspond with the eigenvalue that you wish to find. This means that provisions must be made in the program for interchanging rows and columns of equation (4) before starting the iteration procedure with Step I.

Characteristic Equation Approach

The other approach for solving equation (4) for its eigenvalues is using the computer to expand the determinant of the coefficients $D_{(s)}$ into a polynomial. In general, two restrictions are imposed on solving the determinant: (1) the determinant must not be larger than 10x10 (in our case, it is smaller) and (2) the elements must be single polynomials in s, not ratios of polynomials. Equation (4) does not usually meet both of the requirements and must be altered. First, the important modes must be determined (not necessary for our system); and second, the determinant must be expanded by minors along the last row, since the last row contains ratios of polynomials.

Equation (4) may be rewritten as

$$D_{(s)} = \sum_{j=1}^{n} (-1)^{(n-j)} \overline{T}_{j(s)} \overline{D}^{j}_{(s)} = 0, \tag{9}$$

where $\overline{D}^{j}_{(s)}$ is the minor of the determinant $D_{(s)}$, taken on the last row. $\overline{T}_{j(s)}$ is a ratio of polynomials and can be wriiten as

$$\overline{T}_{j(s)} = \frac{N_{j(s)}}{P_{j(s)}} . \tag{10}$$

Equation (9) is rationalized and its numerator set equal to zero to solve the characteristic equation. The eigenvalues are the roots of the numerator. $D_{(s)}$ is written in the form

$$D_{(s)} = \sum_{j=1}^{n} \left\{ (-1)^{n-j} \overline{D}^{j}_{(s)} N_{j(s)} \left(\prod_{K \neq j}^{K} P_{K(s)} \right) \right\} = 0 . \tag{11}$$

After the eigenvalues have been determined, equation (8) must be solved for the normalized eigenvectors with an option to bypass.

The matrix iteration has two advantages: (1) only important roots are obtained, and (2) the size of the system is not critically limited. The disadvantages of this method are: (1) more machine time is required, and (2) convergence is poor for high gains with a gain root locus study.

The characteristic equation approach is advantageous because all roots can be found, less machine time is required, and gain root locus studies offer no adverse convergence problems. However, this approach can solve only small systems. Systems with higher than 30th order create problems.

Frequency Response

The frequency response of the system to a sinusoidal forcing function at the control thrust vector point is easily obtained. From equation (3) we let σ of the assumed solution s be equal to zero and assume various values for ω within the interested frequency region. For each value of ω, equation (3) is solved for the unknown X_j by assuming X_n (corresponding to β_c equal to one). We set $\overline{K}_1$ equal to zero (see definitions of elements).

Using these frequency-dependent unknowns, a structural transfer function can be obtained between a force applied at the swivel point and any sensor location.

1. Between the thrust vector and a position gyro

$$\frac{\theta_{i(x_g)}}{F} = \frac{\theta_{(\omega)}}{F} + \sum_{\mu} \frac{\tau_{\mu(\omega)}}{F} \, \theta_{\mu(x_g)} \tag{12}$$

(x_g is the position gyro location).

2. Between the thrust vector and a rate gyro,

$$\frac{\dot{\theta}_{r(x_r)}}{F} = i\omega \left(\frac{\theta}{F} + \sum_{\mu} \frac{\tau_{\mu}}{F} \, \theta_{\mu(x_r)} \right) . \tag{13}$$

Other frequency responses can be obtained from equation (13) by solving for the ratios of certain unknowns.

BASIC PROCEDURES FOR ANALYZING

The analysis to be done is either a conventional gain root locus or a phase root locus with a constant gain. In conventional graphical techniques, the phase root locus is much more difficult to construct than the gain root locus; however, for the numerical procedures presented in the report, these problems are eliminated. A combination phase and gain program provides more information than a conventional root locus diagram [2].

For a phase root locus equation (5) is written as

$$K_n \beta_c = -A_{(s)} \sum_{j=1}^{n-1} K_j^* T_{j(s)} X_j \; . \tag{14}$$

Because this is the control equation written in general form with all the transfer functions of the various control loops, a phase root locus can now be computed for the total system or for each control loop separately by writing K_j^*'s in the form

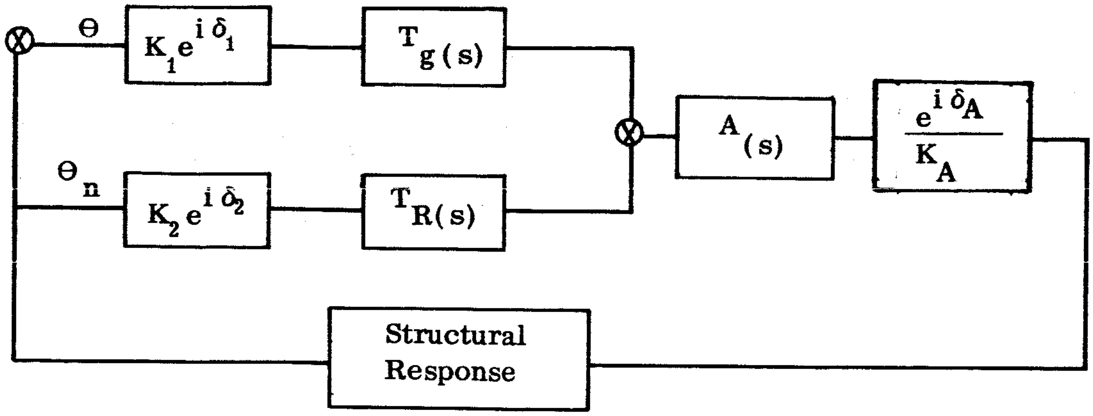
$$K_j^* = K_j e^{i\delta_j} \qquad \text{and} \qquad K_n = \frac{e^{i\delta_A}}{K_A} \tag{15}$$

Equation (13) then becomes

$$e^{i\delta_A} \beta_c = -\left(A_{(s)} \sum_{j=1}^{n-1} T_{j(s)} K_j e^{i\delta_j} X_j \right) K_A \; . \tag{16}$$

Now K_j contains the open loop gains of each loop as a parameter and is kept constant as the phase δ_j is varied and the roots of equation (3) are computed for each value of δ_j.

The following block diagram depicts the system and procedure.



The matrix iteration procedure works much better for the phase root locus, since the complex elements introduced into the control equation do not cause any convergence problems. This is not always true in the root solving procedure for polynomials.

EQUATIONS OF ELEMENTS

The coefficients of $s^2 A_{ij}$ are as follows:

Torsional Modes ($i = 1 \rightarrow 2$)

$j = 1 \rightarrow 2$

$A_{ij} = M_{\tau i} \delta_{ij}$

$j = 3 \rightarrow 6$

$A_{ij} = 0$

$j = 7$

$A_{i7} = \text{deSe}\,\theta_{\mu(x_t)}$

$j \quad = 8$

$A_{i8} = 0$

Roll ($i = 3$)

$j \quad = 1 \rightarrow 2$

$A_{3j} = 0$

$j \quad = 3$

$$A_{33} = +(I_R - I_{CORR}) + \sum_{n=1}^{2} M_{sn} ds_n^{\,2}$$

$j \quad = 4 \rightarrow 5$

$$A_{3j} = \sum_{n=1}^{2} M_{sn} ds_n \qquad \begin{array}{ll} n = 1 & \text{when } j = 4 \\ n = 2 & \text{when } j = 5 \end{array}$$

$j \quad = 6$

$A_{36} = 0$

$j \quad = 7$

$A_{37} = deSe$

$j \quad = 8$

$A_{38} = 0$

Sloshing Roll ($i = 4 \rightarrow 5$)

$j \quad = 1 \rightarrow 2$

$A_{ij} = 0$

$j \quad = 3$

$$A_{i3} = ds_i$$

$$j = 4 \rightarrow 5$$

$$A_{ij} = 0 \quad \text{when} \quad i \neq j$$

$$A_{ij} = 1 \quad \text{when} \quad i = j$$

$$j = 6 \rightarrow 8$$

$$A_{ij} = 0$$

Rate Gyro Roll $(i = 6)$

$$j = 1 \rightarrow 5$$

$$A_{6j} = 0$$

$$j = 6$$

$$A_{66} = \frac{1}{\omega_r^2}$$

$$j = 7 \rightarrow 8$$

$$A_{6j} = 0$$

Swivel Engine $(i = 7)$

$$j = 1 \rightarrow 2$$

$$A_{7j} = \frac{deSe\,\theta_{j(x_t)}}{\theta_e \omega_e^2}$$

$$j = 3$$

$$A_{73} = \frac{deSe}{\theta_e \omega_e^2}$$

$j = 4 \rightarrow 6$

$A_{7j} = 0$

$j = 7$

$A_{77} = \frac{1}{\omega_e^2}$

$j = 8$

$A_{78} = 0$

The coefficients of $s\, B_{ij}$ are as follows:

Torsional Modes $(i = 1 \rightarrow 2)$

$j = 1 \rightarrow 2$

$B_{ij} = 2\zeta_{\tau i}\omega_{\tau i}M_{\tau i}\delta_{ij}$

$j = 3 \rightarrow 8$

$B_{ij} = 0$

Roll $(i = 3)$

$j = 1 \rightarrow 8$

$B_{3j} = 0$

Sloshing Roll $(i = 4 \rightarrow 5)$

$j = 1 \rightarrow 3$

$B_{ij} = 0$

$j = 4 \rightarrow 5$

$B_{ij} = 2\zeta_{si}\omega_{si}\delta_{ij}$

$j = 6 \rightarrow 8$

$B_{ij} = 0$

Rate Gyro Roll $(i = 6)$

$j = 1 \rightarrow 2$

$B_{6j} = -\theta_{j(x\phi r)}$

$j = 3$

$B_{63} = -1$

$j = 4 \rightarrow 5$

$B_{6j} = 0$

$j = 6$

$$B_{66} = \frac{2\zeta_r}{\omega_r}$$

$j = 7 \rightarrow 8$

$B_{6j} = 0$

Swivel Engine $(i = 7)$

$j = 1 \rightarrow 6$

$B_{7j} = 0$

$j = 7$

$$B_{77} = \frac{2\zeta_e}{\omega_e}$$

$j = 8$

$$B_{78} = -\frac{2\zeta_e}{\omega_e}\overline{K}_1$$

The constant coefficients of C_{ij} are as follows:

Torsional Modes ($i = 1 \rightarrow 2$)

$j = 1 \rightarrow 2$

$$C_{ij} = \omega_{\tau i}^2 M_{\tau i} \delta_{ij}$$

$j = 3 \rightarrow 6$

$$C_{ij} = 0$$

$j = 7$

$$C_{i7} = \theta_{\mu(x_t)} deF_s$$

$j = 8$

$$C_{i8} = 0$$

Roll ($i = 3$)

$j = 1 \rightarrow 6$

$$C_{3j} = 0$$

$j = 7$

$$C_{37} = deF_s$$

$j = 8$

$$C_{38} = 0$$

Sloshing Roll ($i = 4 \rightarrow 5$)

$j = 1 \rightarrow 3$

$C_{ij} = 0$

$j = 4 \rightarrow 5$

$C_{ij} = \omega_{si}^{2} \delta_{ij}$

$j = 6 \rightarrow 8$

$C_{ij} = 0$

Rate Gyro Roll ($i = 6$)

$j = 1 \rightarrow 5$

$C_{6j} = 0$

$j = 6$

$C_{66} = 1$

$j = 7 \rightarrow 8$

$C_{6j} = 0$

Swivel Engine ($i = 7$)

$j = 1 \rightarrow 6$

$C_{7j} = 0$

$j = 7$

$C_{77} = 1$

$j = 8$

$C_{78} = -\bar{K}_1$

Control Equation (i = 8)

$$j = 1 \rightarrow 2$$

$$C_{8j} = -a_{or}\theta_{j(x\phi r)}A_{(s)}T_{g(s)}K_1e^{i\delta_1}$$

$$j = 3$$

$$C_{83} = -a_{or}A_{(s)}T_{g(s)}K_1e^{i\delta_1}$$

$$j = 4 \rightarrow 5$$

$$C_{8j} = 0$$

$$j = 6$$

$$C_{86} = -a_{1r}A_{(s)}T_{r(s)}K_2e^{i\delta_2}$$

$$j = 7$$

$$C_{87} = 0$$

$$j = 8$$

$$C_{88} = \frac{e^{i\delta_A}}{K_A}$$

APPENDIX

Roll Equations

The equations of motion for the roll stability program are derived using Lagrange's equation. The torsion, roll, roll sloshing, and swivel engine equations were derived, while the rate gyro equation was obtained from Reference 3. The longitudinal acceleration was eliminated from the equations of motion because of its negligible influence on the final results.

1. Lagrange's Equation

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} + \frac{\partial D}{\partial \dot{q}_i} = Q_i \; , \tag{1}$$

where $L = T - V$.

T is the kinetic energy, and V is the potential energy of the system, while D is the dissipation function, and Q_i's are the generalized forces.

The generalized coordinates, q_i, for the derivation are:

$q_1 = \tau_\mu$	Torsional twist
$q_2 = \theta$	Rigid body roll angle
$q_3 = \xi_s$	Roll sloshing displacement
$q_4 = \theta_r$	Rate gyro roll
$q_5 = \beta_e$	Swivel engine

2. Kinetic Energy

a. Airframe

The kinetic energy of the airframe depends on the rotation of each element, which is the sum of the rigid body roll angle and the torsional twist. The angular velocity about the missile centerline is

$$V_{A(x)} = \dot{\theta} + \sum_{\mu} \dot{\tau}_{\mu} \theta_{\mu(x_t)} . \tag{2}$$

The kinetic energy is given by

$$T_A = \frac{1}{2} \int_A I_{(x)} V^2_{A(x)} \, dx , \tag{3}$$

where $I_{(x)}$ is the axial distribution of the roll mass moment of inertia.

$$T_A = \frac{1}{2} \int_A I_{(x)} \left[\dot{\theta}^2 + 2\dot{\theta} \sum_{\mu} \dot{\tau}_{\mu} \theta_{\mu(x_t)} \right] dx$$

$$+ \frac{1}{2} \int_A I_{(x)} \left[\sum_{\mu} \dot{\tau}_{\mu} \theta_{\mu(x_t)} \right]^2 dx$$

$$T_A = \frac{1}{2} \dot{\theta}^2 \int_A I_{(x)} \, dx + \dot{\theta} \sum_{\mu} \dot{\tau}_{\mu} \int_A I_{(x)} \theta_{\mu(x_t)} \, dx$$

$$+ \frac{1}{2} \sum_{i,j} \dot{\tau}_i \dot{\tau}_j \int_A I_{(x)} \theta_{i(x_t)} \theta_{j(x_t)} \, dx .$$

But $$\int_A I_{(x)} \, dx = I_R , \tag{4}$$

where I_R is the total moment of inertia of the airframe about the centerline and

$$\int_A I_{(x)} \theta_{\mu(x_t)} \, dx = 0 \quad \text{(end torque)}. \tag{5}$$

Also,

$$\int_A I_{(x)} \theta_{i(x_t)} \theta_{j(x_t)} dx = \delta_{ij} \begin{Bmatrix} i \neq j = 0 \\ \\ i = j = M_{\tau i} \end{Bmatrix} \tag{6}$$

where δ_{ij} is the kronecker delta.

Therefore,

$$T_A = \frac{1}{2} I_R \dot{\theta}^2 + \frac{1}{2} \sum_{\mu} M_{\tau\mu} \dot{\tau}^2_{\mu} \ . \tag{7}$$

b. Liquid propellant

The outer clustered tanks of a Saturn type space vehicle are combined into two tanks for the spring-mass model (Fig. 1). Springs and dashpots are used to attach the oscillating masses to the tank wall.

The angular velocity at any station x is

$$V_{s(x)} = ds_n \dot{\theta} + \dot{\xi}_s \tag{8}$$

where ds_n is the distance from the centerline of the vehicle to the sloshing mass $(n = 1,2)$.

Hence, the kinetic energy is given by

$$T_{(s)} = \frac{1}{2} \sum_n M_{sn} V_{s(x)}^2 = \frac{1}{2} \sum_n M_{sn} (ds_n \dot{\theta} + \dot{\xi}_s)^2 \, dx$$

$$T_{(s)} = \frac{1}{2} \sum_n M_{sn} ds_n^2 \dot{\theta}^2 + \sum_n M_{sn} ds_n \dot{\theta} \dot{\xi}_s + \frac{1}{2} \sum M_{sn} \dot{\xi}_s^2 \ . \tag{9}$$

c. Swivel engine

The angular velocity of the gimbal rotation is given by (Fig. 2):

$$V_{\theta e(x)} = -\left(\dot{\theta} + \sum_{\mu} \dot{\tau}_{\mu} \theta_{\mu(x_t)}\right) . \tag{10}$$

Therefore, the linear velocity at the gimbal station is

$$V_{e(x)} = - de\left(\dot{\theta} + \sum_{\mu} \dot{\tau}_{\mu} \theta_{\mu(x_t)}\right) , \tag{11}$$

where de is the distance from the missile centerline to the swivel engine. The velocity of a point on the engine relative to the gimbal is given by $d_{(x)}\dot{\beta}_e$. Consequently, the velocity of a point on the engine is given by

$$V_{e(x)} = d_{(x)}\dot{\beta}_e + de\left(\dot{\theta} + \sum_{\mu} \dot{\tau}_{\mu} \theta_{\mu(x_t)}\right) , \tag{12}$$

and the kinetic energy of the system is

$$T_e = \frac{1}{2} \int_e M_{e(x)} \left[d_{(x)}\dot{\beta}_e + de\left(\dot{\theta} + \sum_{\mu} \dot{\tau}_{\mu} \theta_{\mu(x_t)}\right)\right]^2 dx . \tag{13}$$

But $$\int_e M_{e(x)} d^2_{(x)} dx = \theta_e \tag{14}$$

when θ_e is the moment of inertia of the engine about the gimbal plane; and

$$de^2 \int_e M_{e(x)} dx = I_r = 0, \tag{15}$$

where I_r is the moment of inertia of the engine about the missile centerline.

$$\int_e M_{e(x)} d_{(x)} \, dx = Se \; , \tag{16}$$

where Se is the static moment of inertia of the engine about the gimbal.

Hence, the kinetic energy of the engine is

$$T_e = \frac{1}{2} \theta_e \dot{\beta}_e^2 + deSe\dot{\theta}\dot{\beta}_e + deSe\dot{\beta}_e + \sum_{\mu} \dot{\tau}_{\mu} \theta_{(x_t)} \; . \tag{17}$$

3. Potential Energy

a. Airframe

The torsional vibration comprises the total potential energy of the airframe.

$$V_A = \frac{1}{2} \sum_{\mu} M_{\tau\mu} \omega_{\tau\mu}^2 \tau_{\mu}^2 \; , \tag{18}$$

where $\omega_{\tau\mu}$ is the torsional natural frequency and $M_{\tau\mu}$ is the generalized torsional mass.

b. Liquid propellant

The potential energy usually arises from three sources, but in roll only the strain energy contained in the spring of the mechanical analogy is used.

$$V_s = \frac{1}{2} k_n \xi_s^2 = \frac{1}{2} \sum_n M_{sn} \omega_{sn}^2 \xi_s^2 \; , \tag{19}$$

where k_n is the spring constant of the slosh mass.

c. Swivel engine

The swivel engine is coupled with the roll control system, and the dynamics of the swivel engine can be described by the simple spring-mass-damper system. The actual potential energy is determined by an effective spring constant and damping coefficient of the gimbaling system.

$$V_e = \frac{1}{2} k_e (\beta_e - \beta_c)^2 = \frac{1}{2} \theta_e \omega_e^2 (\beta_e - \beta_c)^2 , \tag{20}$$

where k_e is the spring constant and ω_e is the natural frequency of the swivel engine.

4. Dissipation Function

a. Airframe

The dissipation of the empty airframe arises from its structural damping, which is a function of the torsional twist and not of its frequency.

$$D_A = \frac{1}{2} \sum_{\mu} M_{\tau\mu} \omega_{\tau\mu} g_{\tau\mu} \dot{\tau}_{\mu}^2 \quad , \tag{21}$$

where $g_{\tau\mu}$ is the torsional damping of the structure.

b. Liquid propellant

The dissipation of the liquid propellant is represented by an equivalent dashpot associated with the mechanical model.

$$D_s = \frac{1}{2} c_s \dot{\xi}_s^2 = \frac{1}{2} \sum_{n} M_{sn} \omega_{sn} g_{sn} \dot{\xi}_s^2 , \tag{22}$$

where g_{sn} is the propellant damping.

c. Swivel engine

The damping of the swivel engine is of minor importance and can be neglected, but the viscous damping of the actuator is proportional to the difference in the velocities of the command signal and the engine deflection;

$$D_e = \frac{1}{2} c_e (\dot{\beta}_e - \dot{\beta}_c)^2 = \frac{1}{2} g_e \theta_e \omega_e (\dot{\beta}_e - \dot{\beta}_c)^2 \tag{23}$$

where g_e is the swivel engine damping.

5. External Forces

The only external force considered in this derivation is the swivel thrust. The drag, aerodynamic forces, and forces caused by the flowing propellants are neglected.

$$Q_{\tau\mu} = deF_s \theta_{\mu(x_t)} \sin(-\beta_e) = -deF_s \theta_{\mu(x_t)} \sin \beta_e,$$

and we may write

$$\sin \beta_e \cong \beta_e$$

$$Q_{\tau\mu} = -deF_s \theta_{\mu(x_t)} \beta_e \tag{24}$$

$$Q_\theta = deF_s \sin(-\beta_e) = -deF_s \beta_e \tag{25}$$

$$Q_{\xi s} = Q_{\theta n} = Q_{\beta_e} = 0\,. \tag{26}$$

6. Equations of Motion

a. The kinetic potential, L, is given by

$$\begin{aligned} L = {} & \frac{1}{2} I_R \dot{\theta}^2 + \frac{1}{2} \sum_\mu M_{\tau\mu} \dot{\tau}_\mu^2 + \frac{1}{2} \sum_n M_{sn} ds_n^2 \dot{\theta}^2 + \sum_n M_{sn} ds_n \dot{\theta} \dot{\xi}_s \\ & + \frac{1}{2} \sum_n M_{sn} \dot{\xi}_s^2 + \frac{1}{2} \theta_e \dot{\beta}_e^2 + deSe\dot{\theta}\dot{\beta}_e + deSe\dot{\beta}_e \sum_\mu \dot{\tau}_\mu \theta_{\mu(x_t)} \\ & - \frac{1}{2} \sum_\mu M_{\tau\mu} \omega_{\tau\mu}^2 \tau_\mu^2 - \frac{1}{2} \sum_n M_{sn} \omega_{sn}^2 \xi_s^2 - \frac{1}{2} \theta_e \omega_e^2 \beta_e^2 \\ & + \theta_e \omega_e^2 \beta_e \beta_c - \frac{1}{2} \theta_e \omega_e^2 \beta_c^2\,. \end{aligned} \tag{27}$$

b. The dissipation function, D, is given by

$$D = \frac{1}{2}\sum_{\mu} M_{\tau\mu}\omega_{\tau\mu}g_{\tau\mu}\dot{\tau}^2_{\mu} + \frac{1}{2}\sum_{n} M_{sn}\omega_{sn}g_{sn}\dot{\xi}^2_{s}$$

$$+ \frac{1}{2} g_e\theta_e\omega_e\dot{\beta}^2_e - g_e\theta_e\omega_e\dot{\beta}_e\dot{\beta}_c + \frac{1}{2} g_e\theta_e\omega_e\dot{\beta}^2_c \ . \tag{28}$$

c. The equations of motion are written with the aid of Lagrange's equation:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} + \frac{\partial D}{\partial \dot{q}_i} = Q_i \ .$$

(1) Torsion equation (τ_μ)

$$\frac{\partial L}{\partial \dot{\tau}_\mu} = \sum_{\mu} M_{\tau\mu}\dot{\tau}_\mu + deSe\dot{\beta}_e \sum_{\mu}\theta_{\mu(x_t)}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\tau}_\mu}\right) = \sum_{\mu} M_{\tau\mu}\ddot{\tau}_\mu + deSe\theta_{\mu(x_t)}\ddot{\beta}_e$$

$$\frac{\partial L}{\partial \tau_\mu} = -\sum_{\mu} M_{\tau\mu}\omega^2_{\tau\mu}\tau_\mu$$

$$\frac{\partial D}{\partial \dot{\tau}_\mu} = \sum_{\mu} M_{\tau\mu}\omega_{\tau\mu}g_{\tau\mu}\dot{\tau}_\mu$$

$$Q_{\tau\mu} = -\, deF_s\theta_{\mu(x_t)}\beta_e$$

Hence, the torsional equation is

$$\sum_{\mu} M_{\tau\mu}\ddot{\tau}_{\mu} + \sum_{\mu} M_{\tau\mu}\omega_{\tau\mu}g_{\tau\mu}\dot{\tau}_{\mu} + \sum_{\mu} M_{\tau\mu}\omega^2_{\tau\mu}\tau_{\mu} + \text{deSe}\theta_{\mu(x_t)}\ddot{\beta}_e$$

$$+ \text{deF}_s\theta_{\mu(x_t)}\beta_e = 0 \; . \tag{29}$$

(2) Roll equation (θ)

$$\frac{\partial L}{\partial \dot{\theta}} = I_R\dot{\theta} + \sum_n M_{sn} ds_n^2 \dot{\theta} + \sum_n M_{sn} ds_n \dot{\xi}_s + \text{deSe}\dot{\beta}_e$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = I_R\ddot{\theta} + \sum_n M_{sn} ds_n^2 \ddot{\theta} + \sum_n M_{sn} ds_n \ddot{\xi}_s + \text{deSe}\ddot{\beta}_e$$

$$\frac{\partial L}{\partial \theta} = 0$$

$$\frac{\partial D}{\partial \dot{\theta}} = 0$$

$$Q_\theta = -\,\text{de}\;F_s\beta_e$$

Hence, the roll equation is

$$I_R\ddot{\theta} + \sum_n M_{sn} ds_n^2 \ddot{\theta} + \sum_n M_{sn} ds_n \ddot{\xi}_s + \text{deF}_s\ddot{\beta}_e + \text{deF}_s\beta_e = 0 \; . \tag{30}$$

(3) Sloshing roll equation (ξ_s)

$$\frac{\partial L}{\partial \dot{\xi}_s} = \sum_n M_{sn} ds_n \dot{\theta} + \sum_n M_{sn} \dot{\xi}_s$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\xi}_s}\right) = \sum_n M_{sn} ds_n \ddot{\theta} + \sum_n M_{sn} \ddot{\xi}_s$$

$$\frac{\partial L}{\partial \xi_s} = -\sum_n M_{sn} \omega^2_{sn} \xi_s$$

$$\frac{\partial D}{\partial \dot{\xi}_s} = \sum_n M_{sn} \omega_{sn} g_{sn} \dot{\xi}_s$$

$$Q_{\xi s} = 0$$

Hence, the sloshing roll equation is

$$\sum_n M_{sn} ds_n \ddot{\theta} + \sum_n M_{sn} \ddot{\xi}_s + \sum_n M_{sn} \omega_{sn} g_{sn} \dot{\xi}_s + \sum_n M_{sn} \omega^2_{sn} \xi_s = 0 .$$

Dividing through by the sloshing mass,

$$ds_n \ddot{\theta} + \ddot{\xi}_s + 2\omega_{sn} \zeta_{sn} \dot{\xi}_s + \omega^2_{sn} \xi_s = 0. \qquad (31)$$

(4) Rate gyro roll equation (θ_r)

The rate gyro equation is obtained from Reference 3, but can easily be converted to roll.

The rate gyro is included as a separate equation because it is nonideal. Hence, the rate gyro roll equation is

$$\frac{\ddot{\theta}_r}{\omega_r^2} + \frac{2\zeta_r}{\omega_r}\dot{\theta}_r + \theta_r = \left(\dot{\theta} + \sum_\mu \dot{\tau}_\mu \theta_{\mu(x\phi r)}\right) \tag{32}$$

$$-\sum_\mu \dot{\tau}_\mu \theta_{\mu(x\phi r)} - \dot{\theta} + \frac{\ddot{\theta}_r}{\omega_r^2} + \frac{2\zeta_r}{\omega_r}\dot{\theta}_r + \theta_r = 0 \ . \tag{32a}$$

(5) Swivel engine equation (β_e)

$$\frac{\partial L}{\partial \dot{\beta}_e} = \theta_e \dot{\beta}_e + deSe\dot{\theta} + deSe \sum_\mu \dot{\tau}_\mu \theta_{\mu(x_t)}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\beta}_e}\right) = \theta_e \ddot{\beta}_e + deSe\ddot{\theta} + deSe \sum_\mu \ddot{\tau}_\mu \theta_{\mu(x_t)}$$

$$\frac{\partial L}{\partial \beta_e} = \theta_e \omega_e^2 \beta_c - \theta_e \omega_e^2 \beta_e$$

$$\frac{\partial D}{\partial \dot{\beta}_e} = g_e \theta_e \omega_e \dot{\beta}_e - g_e \theta_e \omega_e \dot{\beta}_c$$

$$Q_{\beta_e} = 0$$

Hence, the swivel engine equation is

$$\text{deSe}\sum_{\mu}\ddot{\tau}_{\mu}\theta_{\mu(x_t)} + \text{deSe}\ddot{\theta} + \theta_e\ddot{\beta}_e + 2\zeta_e\omega_e\theta_e\dot{\beta}_e + \theta_e\omega_e^2\beta_e$$

$$- 2\zeta_e\omega_e\theta_e\dot{\beta}_c - \theta_e\omega_e^2\beta_c = 0 \; .$$

Dividing by $\theta_e\omega_e^2$:

$$\frac{\text{deSe}\sum_{\mu}\ddot{\tau}_{\mu}\theta_{\mu(x_t)}}{\theta_e\omega_e^2} + \frac{\text{deSe}\ddot{\theta}}{\theta_e\omega_e^2} + \frac{\ddot{\beta}_e}{\omega_e^2} + \frac{2\zeta_e}{\omega_e}\dot{\beta}_e + \beta_e - \frac{2\zeta_e}{\omega_e}\beta_c - \beta_c = 0 \; . \qquad (33)$$

(6) Control Equation

$$\beta_c = a_{or}\left(\theta + \sum_{\mu}\tau_{\mu}\theta_{\mu(x\phi r)}\right) + a_{1r}\theta_r \qquad (34)$$

$$-a_{or}\sum_{\mu}\tau_{\mu}\theta_{\mu(x\phi r)} - a_{or}\theta - a_{1r}\theta_r + \beta_c = 0 \qquad (35)$$

With filters and actuator

$$-a_{or}\sum_{\mu}\tau_{\mu}\theta_{\mu(x\phi r)}A_{(s)}T_{g(s)}K_{1e}^{i\delta_1} - a_{or}T_{g(s)}A_{(s)}\theta K_1e^{i\delta_1}$$

$$-a_{1r}T_{r(s)}A_{(s)}\theta_r K_2e^{i\delta_2} + \frac{e^{i\delta_A}}{K_A}\beta_c = 0 \; . \qquad (35)$$

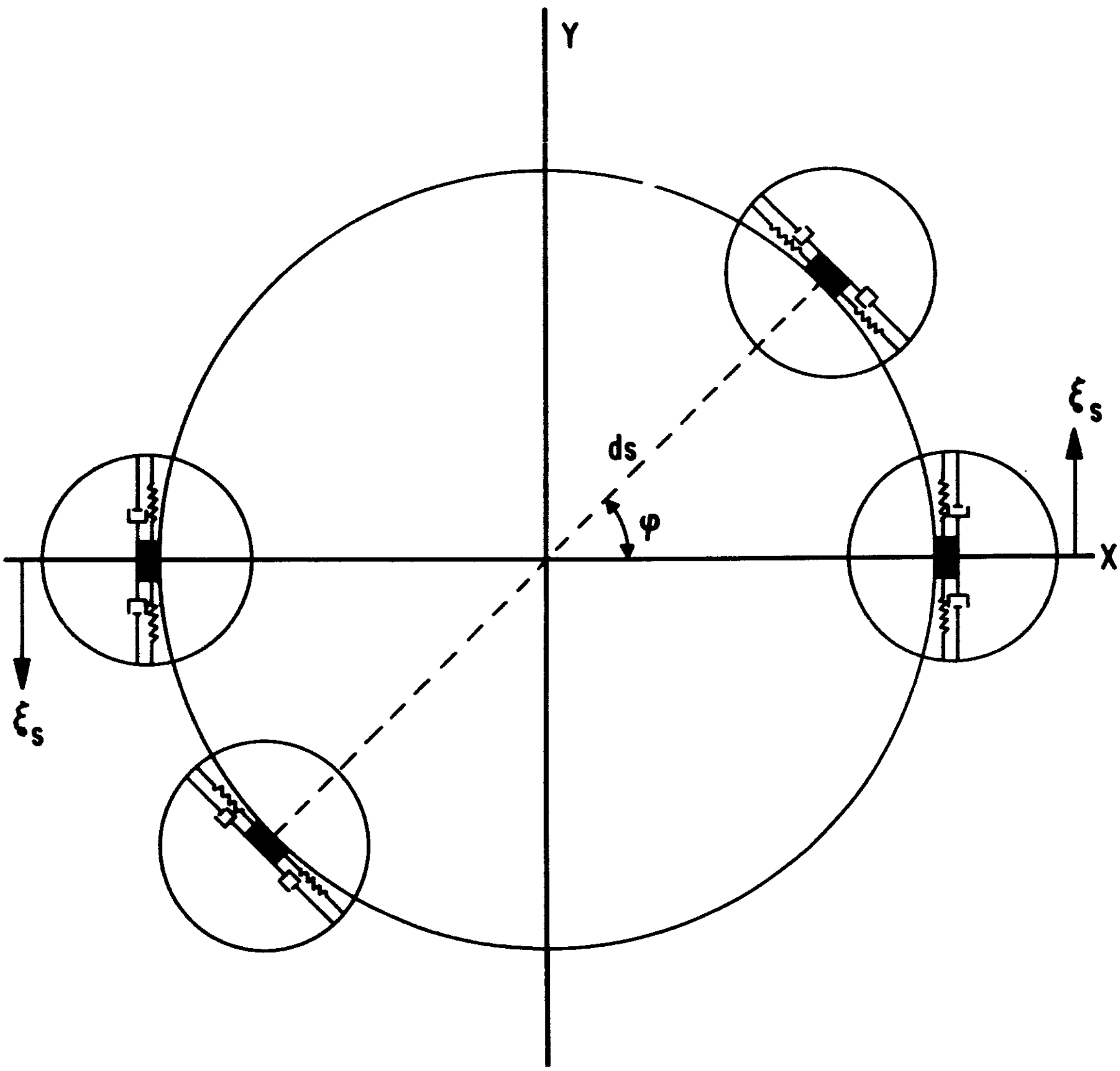


FIGURE 1. SLOSH MECHANICAL MODEL

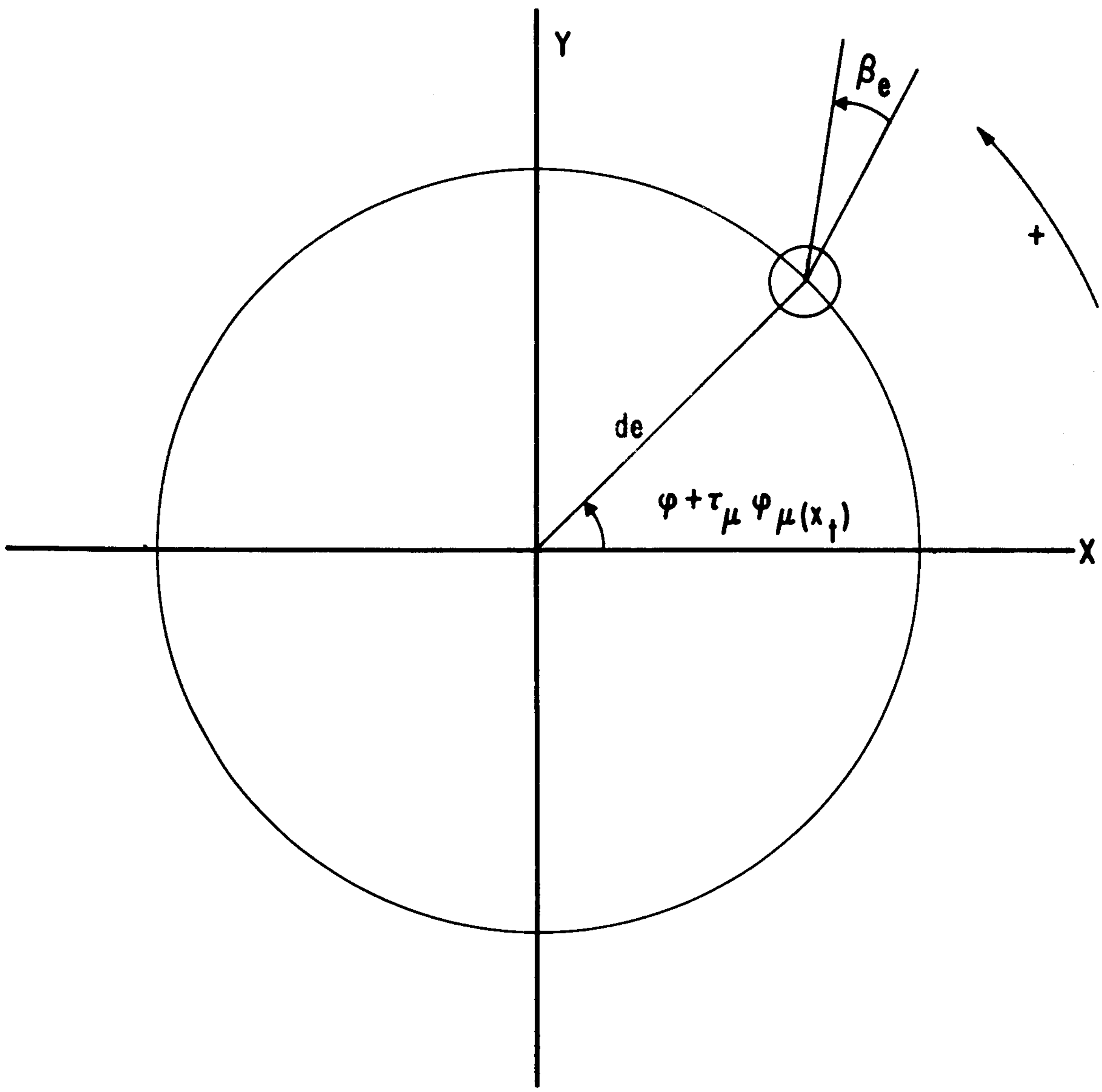


FIGURE 2. SWIVEL ENGINE MODEL

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NASA TM X-53368 APPROVAL

NUMERICAL PROCEDURES FOR ROLL STABILITY STUDIES

By Philip J. Hays

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This report has also been reviewed and approved for technical accuracy.

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