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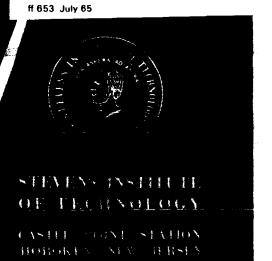
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DEPARTMENT OF PHYSICS

FINAL REPORT

JUNE 1965

by

James L. Anderson

Principles of General Covariance
in its Various Formulations
and
the Related Problems of the Role of Coordinates
in General Relativity

UNCLASSIFIED

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DEPARTMENT OF PHYSICS

STEVENS INSTITUTE OF TECHNOLOGY

CASTLE POINT STATION

HOBOKEN, NEW JERSEY

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FINAL REPORT

NASA Grant No. NsG-326-63

During the grant period we have investigated a number of problems associated with the use of space-time coordinates in physical theory. Part of the results of this investigation are contained in the preprint and reprint at the end of this report. We have given special attention to the use of coordinate conditions in general relativity. The paper "Coordinate Conditions and Canonical Formalisms in Gravitational Theory" reviews the various types of conditions that have been proposed and shows in what sense they are related to each other. It also discusses the construction of a Hamiltonian formalism when coordinate conditions are employed and demonstrates the equivalence of various methods employed in these constructions. In the paper "Maximal Covariance Conditions and Kretchmann's Relativity Group" we examined the question of what constitutes a maximal set of coordinate conditions using the gauge group of electrodynamics as a model. The maximal question bears directly on the Kretchmann definition of the relativity group of a theory. We were able to show that the Kretchmann definition leads, in general, to a trivial group. "Twins, Clocks and Geometry" (reprints submitted November 1964) was concerned with the problem of making space-time measurements and in particular with the problem of constructing a model clock within the framework of a space-time theory. Our main conclusion was that the construction of such a clock is independent of the coordinate system employed in its construction. We also showed that the usual assumption that a model clock reads proper time is, in general, unwarranted since the behavior of such a clock depends on how it is constructed.

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Unfortunately, a number of results having to do with the principle of general covariance have not as yet been written. While the paper "Twins, Clocks and Geometry" alludes to some of these results, the rigorous discussion is only now being completed. The main idea involved is the distinction between the absolute and the dynamical elements of a theory. In special relativity, for example, the metric is an absolute object while in general relativity it is a dynamical object. We have used this distinction to define two different groups of a physical theory, the covariance group and the symmetry group. The former group is the group of transformations that map a solution of a set of equations onto another solution. The symmetry group, on the other hand, is the subgroup of the covariance group that leaves invariant the absolute objects of the theory. Of course, if there are no absolute objects then the two groups are identical. Using this definition we can show that the symmetry group of general relativity is just the Poincare group, regardless of whether the covariance group is the group of all coordinate mappings or just the Poincare group. We have also extended these ideas to the problem of defining internal symmetries of a theory.

Maximal Covariance Conditions and Kretschman's Relativity Group[*][+]

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ABSTRACT

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The general problem of how large a set of covariance conditions one can impose on a theory with a covariance group is discussed. In imposing such conditions one makes the natural requirement that such conditions must be satisfiable by at least one member of every equivalence class of physically distinct solutions. Even so we have shown in the cases of electrodynamics and relatively theory that one can impose a number of conditions sufficient to reduce the covariance group down to the identity. As a consequence we conclude that the Kretschman definition of the Relativity group of a theory is not a useful definition since it always leads to the Relativity group being the identity group.

author

I. INTRODUCTION

Whenever a system of physical laws admits a covariance group it is possible to separate the dynamically possible trajectories into equivalence classes, two trajectories belonging to the same equivalence class if it is possible to transform one into another by an element of the group. In fact, the possibility of such a separation is just what we mean when we say that a system of laws admits a covariance group. When we come to interpret such a system of laws and the dynamically possible trajectories admitted by them it is clear that all of the trajectories belonging to a particular equivalence class must be considered as describing the same physical state of the system. Otherwise the dynamical laws must be considered as being incomplete since they by themselves do not allow a distinction to be made between the various elements of an equivalence class.

As a consequence of the existence of a covariance group, some part of the collection of configuration variables used to describe the trajectories of the system in question is superfluous as far as the characterization of physically distinct states of the system is concerned. The possibility exists therefore in such cases of imposing additional non-covariant restrictions on the configuration variables. These restrictions, being non-covariant, will be satisfied only by a subset of elements in each equivalence class. One can, for instance, eliminate the translational covariance associated with an N-body system by taking the center of mass of the system to be at the origin of coordinates. Another familiar example is the Lorentz gauge condition one

sometimes employs in electrodynamics. In what follows we shall refer to such conditions as covariance conditions.

The need to impose covariance conditions arises most strongly in cases where the covariance group is a gauge group, i.e., a group whose various elements are specified by one or more space-time functions, as is the case for the gauge group of electrodynamics and the group of arbitrary coordinate transformations of space-time theories. Except in special instances one cannot solve the dynamical equations of motion in such cases unless one brings in covariance conditions. They are also needed if one wishes to construct an unambiguous Hamiltonian for such a theory. (1)

In imposing covariance conditions the question arises as to how far one can go, i.e., what are the maximal number of conditions one can impose on a theory. It is to this question that we address ourselves in this paper. From the practical point of view it is desirable to know the answer to this question since it will tell us what part of the collection of configuration variables corresponds to physically observable aspects of the system and consequently how much information is needed in order to specify a state of the system. Furthermore the imposition of covariance conditions may lead directly to the elimination of non-physical degrees of freedom of the system and hence simplify the problem of solving the equations of motion of the system. Of course one would want to effect as great an elimination as possible.

The question of how many covariance conditions can be imposed on a given theory was actually first raised by Kretschmann[2] shortly after the advent of the General Theory of Relativity. Kretschmann criticized the principle of general covariance on the grounds that any theory could be made to conform to it by the device of introducing additional degrees of freedom into the theory. As an alternative he attempted to define what he called the Relativity group of the theory. In order to determine this group one imposes the maximal number of covariance conditions on the configuration variables of the theory, maximal in the sense that any additional conditions would eliminate one or more equivalence classes by being unsatisfiable by any member of these classes. The Relativity group of the theory is then the covariance group of the original dynamical laws plus the supplementary conditions. Of course, if it is always possible to eliminate the covariance group entirely by the judicious choice of covariance conditions, Kretschmann's definition would not be too meaningful.

In this paper we shall discuss those aspects of the problem of the imposition of covariance conditions that bear on the points raised above. Unfortunately one cannot arrive to any general conclusions since the situation varies from group to group and from theory to theory. In specific cases we shall see that it is always possible to reduce the covariance group down to at most a finite parameter Lie group. In these cases such a reduction is sufficient to allow me to construct Dirac brackets for the theory. Any further reduction will in general depend upon the group in question and the configuration variables available for the reduction. However, in important cases of the gauge group

and the full coordinate group we will see that it is possible to restrict
the covariance group down to the identity element even though the theory
possesses weak, proper conservation laws and consequently, by Noether's
theorems, a symmetry group. For these theories Kretschmann's Relativity
group would be empty and hence we conclude that this Relativity group is
not a useful concept in analyzing dynamical laws.

II. GENERAL REQUIREMENTS ON COVARIANCE CONDITIONS

In order that a given covariance condition be compatible with a given
set of dynamical laws it is necessary that it can be satisfied by at least
one member of every equivalence class of possible trajectories. If this
were not the case its imposition would eliminate one or more equivalence
classes and hence change the physical content of the original equations of
motion. Aside from this requirement there are no other hard and fast require-
ments one can make on possible covariance conditions. However, there are a
number of additional requirements one might wish to impose from the stand-
point of convenience.

Since one of the reasons one imposes covariance conditions in the first
place is to enable one to construct a canonical formalism from the corresponding
Lagrangian formalism one would want to require that these conditions can be
expressed in terms of the canonical variables of the theory. If we accept
this requirement then we can no longer use the Lorentz condition of electrody-
namics or the corresponding de Donder conditions of general relativity. The
Lorentz condition involves the time derivative of the scalar potential while

the de Donder conditions involve the time derivatives of the $g_{0\mu}$ components of the metric tensor, neither of which can be expressed in terms of canonical variables. One would also like to require that, once satisfied, a covariance condition will remain satisfied throughout the temporal evolution of the system. We make this requirement in order that the condition can be employed to eliminate non-physical degrees of freedom.

There is an additional requirement one might wish to make, namely that of locality. We impose this condition in keeping with the local nature of most dynamical laws. Suppose that we have carried out a series of local measurements on a system described by a set of equations with a covariance group. In order to check the validity of these equations one would impose covariance conditions in sufficient number to lead to a determination of the configuration variables sufficient to carry out this check. However, if the conditions were non-local a knowledge of the results of the local measurements would in general be insufficient for this purpose.

III. GAUGE INVARIANCE

Before we try to arrive at any general conclusions it will be helpful to examine in some detail one relatively simple group, namely the gauge group of electrodynamics. In spite of its simplicity this group contains many features of the more complicated groups that arise in physics.

It is well known that the Klein-Gordon equation for a complex field ψ in the absence of electromagnetic fields admits the group of gauge transformations of the first kind

$$\psi \rightarrow \psi' = e^{i\alpha} \psi \tag{1a}$$

$$\psi^* \rightarrow \psi^{*'} = e^{-i\alpha} \psi^* . \tag{1b}$$

where α is an arbitrary constant. The existence of this group leads to the continuity equation for particle number or charge. Nevertheless one can fix the value of α by requiring that

$$\psi^* = \psi \tag{2}$$

at the origin of the coordinate system. This covariance condition satisfies our fundamental requirement on such conditions. Even when ψ happens to be zero at the origin the condition is obviously satisfiable even though α is not determined. However, once we impose the condition (2) on the theory we no longer have the freedom of performing further gauge transformations of the type.[1] If we were to define a Relativity group for such a theory following the procedure of Kretschmann it would be the trivial group consisting of the identity transformation. Nevertheless one still has a conservation law for the theory in spite of the imposition of condition.[2]

One can extend the gauge transformations of the first kind to those of the second kind by allowing α to become an arbitrary space-time function. In order to do so it is necessary to replace ordinary derivatives of the ψ field by "covariant" derivatives defined by[3]

$$\psi_{;\mu} = \psi_{,\mu} + i A_\mu \psi \tag{3}$$

where A_μ is the affinity. By requiring that $\psi_{;\mu}$ transform like ψ under an extended gauge transformation one is led to the transformation law for A_μ, namely

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$$A_\mu \to A_\mu' = A_\mu + \alpha_{,\mu}. \qquad (4)$$

The analogue of the Riemann tensor for this connection is defined in terms of the commutator of two covariant derivatives

$$\psi_{;\mu\nu} - \psi_{;\nu\mu} = - iF_{\mu\nu} \psi \qquad (5)$$

and is given by

$$F_{\mu\nu} = A_{\nu,\mu} - A_{\mu,\nu}. \qquad (6)$$

At this point one could take $F_{\mu\nu} = 0$ as a possible set of equations governing the A_μ. Doing so would obviously not alter the physical content of the original field free Dirac equation. The vanishing of $F_{\mu\nu}$ is the necessary and sufficient condition (in a simply connected domain) that A_μ is the gradient of a scalar. Taking α to be the negative of this scalar would produce an $A_\mu' = 0$ and hence reduce the covariant derivative to an ordinary derivative. However, when the A_μ is not so reduced the covariance group of the Dirac equation plus equation (5) is the group of gauge transformations of the second kind.

The example discussed here is of course just a simplified version of what one does in Special Relativity when one enlarges the covariance group to include arbitrary coordinate transformations by bringing in explicitly the metric tensor $g_{\mu\nu}$. The physical content of the theory is unaltered if one requires that the Riemann-Christoffel tensor $R^\mu_{\;\nu\rho\sigma} = 0$. By making such an extension of the

covariance group it would seem at first sight that the special position
of the inhomogeneous Lorentz group is lost just as the special significance
of the gauge transformations of the first kind is lost in our example
above. It was for just this reason that Kretschmann introduced the notion
of the Relativity group. At least in the case of the gauge group the
Relativity group is not the group of gauge transformations of the first kind
as one might expect.[4]

If one does not require that $F_{\mu\nu} = 0$ but rather that it satisfy the usual
Maxwell equations in the presence of charge it is necessary to impose gauge
conditions on the A_μ in order that one be able to solve these equations. A
number of different conditions have been used for this purpose. Among the most
common of these is the Lorentz condition

$$A^\mu_{,\mu} = 0, \tag{7}$$

We have already pointed out that the Lorentz condition is not useful in a
canonical formalism, containing as it does $A^o_{,o}$. It also suffers from other
defects. By itself it is not sufficient to fix the gauge function α. Any
solution of D'Alembert's equation $\Box^2 \alpha = 0$ can be used to generate a gauge
transformation that is a covariance transformation of Eq. (7). One could
of course work with the Lorentz condition as it stands as one does, for instance,
in most treatments of quantum electrodynamics. But then it is necessary to
check that all final results are invariant with respect to the restricted
gauge group generated by solutions of the D'Alembert equation. However, such

a procedure has led to much confusion in the past, both in quantum electro-dynamics and in the theory of superconductivity.

To overcome the above mentioned difficulty one might try to impose additional conditions on the A_μ that will further restrict the permissible gauge functions. Fock[8] has discussed the question of the uniqueness of solutions of the wave equation. He has shown that, if in addition to satisfying the wave equation, α is uniformly bounded, it, together with its first derivatives, falls off faster than $1/r$ for large spatial distances r and satisfies the outward radiation condition

$$\lim_{r \to \infty} \left\{ \frac{\partial(r\alpha)}{\partial r} + \frac{1}{c} \frac{\partial(r\alpha)}{\partial t} \right\} = 0$$

then α is identically zero. These conditions can be formulated in terms of the A_μ but are non-local in character. But more important they restrict the class of physical solutions of the original Maxwell equations. In particular they exclude all advanced solutions.

A more tractable gauge condition is the radiation gauge condition

$$A^r_{,r} = 0 \tag{8}$$

As in the case of the Lorentz gauge condition the radiation gauge condition does not fix the gauge completely. However it is now possible to find additional conditions that fix the gauge function up to an additive constant without, at the same time, eliminating any equivalence classes of solutions of the original Maxwell equations. A possible condition is

$$n^r A_r = 0 \tag{9}$$

on a boundary surface S. Here $n^r(x)$ is the unit normal on S. One can show

that Eqs. (8) and (9) together with the part of the defining equation (6) for A_r, namely

$$A_{s,r} - A_{r,s} = F_{rs},\tag{10}$$

lead to a unique, up to an additive constant, determination of the vector potential for arbitrary F_{rs} at any time t. Since different F_{rs} correspond to different equivalence classes we see that our gauge conditions (8) and (9) do not rule out any equivalence classes.

To prove our assertion let us suppose we have found a particular solution of Eqs (8) and (10) that does not yet satisfy condition (9). Now perform a gauge transformation that is a covariance transformation of Eq. (8). The gauge function of such a transformation must satisfy

$$\nabla^2 \alpha = 0.\tag{11}$$

We will try to find an α such that $A'_\mu = A_\mu + \alpha_{,\mu}$ also satisfies condition (9). For this to be the case we must have

$$n^r \alpha_{,r} + n^r A_r = 0.\tag{12}$$

This additional restriction on α is compatible with Eq. (11) since we have the chain

$$0 = \int_R \nabla^2 \alpha \, dV = \oint_S \alpha_{,r} \, n^r \, dS = -\oint_S A_r \, n^r \, dS = \int_R A_{,r} \, n^r dS = 0$$

Consequently one can solve the Laplace Eq. (11) with the Neuman type boundary

conditions (12). The solution is

$$\alpha(x) = \int_S G(x,x')n^r(x')A_r(x') \, dS' + \text{const.}$$

where $G(x,x')$ is a Green's function satisfying

$$\nabla'^2 G(x,x') = \delta(x-x')$$

with the boundary condition

$$n^r(x')\partial G(x,x')/\partial x'^r = 1, \quad x'\epsilon S.$$

The conditions (8) and (10) fix the gauge up to an arbitrary function of t. Such a gauge transformation does not effect A_r but does add to the scalar potential A_o a term $\partial\alpha(t)/\partial t$. We can fix α up to an arbitrary constant by requiring, for instance, that $A_o = 0$ at the origin of spatial coordinates. Then since

$$A_{o,r} - A_{r,o} = F_{ro}$$

we have that

$$A_o(x) = \int_o^x A_{o,r}dx^r = \int_o^x (F_{ro} + A_{r,o}) \, dx^r$$

for arbitrary F_{ro}. Finally we can fix the constant by requiring that Eq. (12) is satisfied.

While useful for many prupeses the rad.ation gauge has the defect that,

In order to compute A_μ and its first few derivatives at a point with its help it is necessary to know $F_{\mu\nu}$ over a finite region of space-time. A set of gauge conditions that does not have this defect is the so-called longitudinal gauge condition[6]

$$A_1(x) = 0 \qquad (13)$$

for all $x \epsilon R$. For simplicity we shall assume that the boundary of R is everywhere convex. Then condition (13), together with the defining equation (10) allows one to write

$$A_2 = \int_0^{x^1} F_{12} \, dx^1 + f_2 \, (x^0, x^2, x^3)$$

and

$$A_3 = \int_0^{x^1} F_{13} dx^1 + f_3(x^0, x^2, x^3)$$

where f_2 and f_3 are arbitrary functions of their arguments. Since $F_{\mu\nu}$ satisfies

$$F_{\mu\nu,\rho} + F_{\rho\mu,\nu} + F_{\nu\rho,\mu} = 0$$

it follows that

$$f_{3,2} - f_{2,3} = F_{23}(x^1 = 0) \qquad (14)$$

In order to fix the functions f_2 and f_3 we require, in addition to the condition (13) that

$$A_2 \, (x^1 = 0) = 0 \qquad (15)$$

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so that $f_2 = o$. Then from Eq. (14) it follows that

$$f_3 = \int_0^{x^2} F_{23}(x^1 = 0)dx^2 + g(x^0, x^3)$$

where $g(x^0, x^3)$ is an arbitrary function of its arguments. To determine it we require further that

$$A_3(x^1 = x^2 = 0) = 0 \qquad (16)$$

so that $g = 0$.

The gauge conditions (13), (15), (16) thus lead to a unique determination of A_r corresponding to a given F_{rs}. Again the remaining arbitrariness in the gauge fix by requiring that

$$A_0(x^1 = x^2 = x^3 = 0) = 0 \qquad (17)$$

Then again A_0 will be given by Eq. (12). One sees thus that the longitudinal gauge is quasi-local. One can compute A_μ and its first few derivatives in the neighborhood of a point solely from a knowledge $F_{\mu\nu}$ in this neighborhood.

IV. COORDINATE COVARIANCE

All space-time theories can be made covariant with respect to arbitrary coordinate transformations. These transformations form a group whose elements are specified by the four arbitrary space-time functions that define the transformation. The problem of formulating covariance conditions for this group is

much harder than in the case of the gauge group since, among other things the group is no longer Abelian. Usually they involve restrictions on the metric tensor $g_{\mu\nu}$ and its derivatives. The most used of such conditions are the de Donder conditions.[7] The use of these conditions has been elevated to a physical principle by Fock. For him the resultant coordinates are the analogues of the Cartesian coordinates in Special Relativity. The difficulty with this position of course is that one is able to form scalars other than the Riemann curvature scalar in such a coordinate system and one looses thereby the prime justification for the field equations of General Relativity proposed by Einstein. We have mentioned previously that the de Donder conditions depend upon the time derivatives of $g_{o\mu}$ and hence are not useful in a Hamiltonian formulation of this theory. Furthermore, they suffer from the same defect as does the Lorentz gauge conditions. In order to restrict the possible coordinate transformations compatible with the de Donder conditions one must impose boundary conditions that are satisfied only for a subclass of solutions of the Einstein equations. Fock argues, but does not prove, that after imposition of these additional conditions one is left only with the freedom to perform inhomogeneous Lorentz transformations. This group would then be Kretschmann's Relativity group were it not for the fact that not all solutions are compatible with the boundry conditions.

Another set of coordinate conditions that have been used were first introduced by Kretschmann[2] and later used by Komar[8] for the purpose of constructing coordinate invariant quantities. To set these conditions one

first constructs four scalars from the curvature tensor. (There are just four such scalars that may be non-zero when the Ricci tensor vanishes.) Let these scalars be K^α. One then uses coordinates such that

$$x^\alpha = K^\alpha (x). \qquad (18)$$

These conditions have the great advantage that they are truely local conditions and furthermore can be easily incorporated into a canonical scheme. However, whenever a particular solution $g_{\mu\nu}$ of the field equations possesses an intrinsic symmetry, e.g., the Schwartzschild solution, one or more of the scalars K^α vanish. Consequently the conditions (19) do not fullfill Kretschmann's requirement, as he himself pointed out.

A set of coordinate conditions that does meet all of our requirements is those leading to a Riemann normal coordinate system. In a given space Riemann coordinates are obtained by first constructing all of the geodesics passing through some point P_o. Each such geodesic can be characterized by the unit vector tangent to it at P_o and hence by the components δ^μ of this vector along four mutually orthogonal directions at P_o. Since, in a finite region around P_o there is only one geodesic that passes through a given point P of this region and P_o, we can assign to the point P the unique coordinates

$$y^\mu = \delta^\mu S \qquad (19)$$

where S is the geodesic distance between P_o and P and δ^μ are the components of the unit tangent vector to the geodesic passing through

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P_0 and P. The y^μ are then the Riemannian coordinates of the point

P. In this coordinate system it is well known that all components of the

affine connection $\Gamma^\mu_{\rho\sigma}$ vanish at the origin (the point P_0). Furthermore, it

can be shown that, in this coordinate system, at any point

$$\Gamma^\mu_{\rho\sigma} y^\rho y^\sigma = 0. \tag{20}$$

Conversely, if eq. (20) is satisfied at all points and the components of the

affine connection vanish at the origin then we have a Riemannian coordinate

system.

Let us consider these conditions (20) on a space-like hypersurface where

y^0 is the time-like coordinate corresponding to the time-like direction at P_0.

They can be written in the form

$$\Gamma^0_{rs} y^r y^s = 0 \tag{21}$$

and

$$\overset{3}{\Gamma}{}^u_{rs} y^r y^s = 0 \tag{22}$$

where $\overset{3}{\Gamma}{}^u_{rs}$ are the Cristoffel symbols constructed from the three-dimensional

metric g_{rs} and its inverse e^{rs} induced on the hypersurface by the four-dimensional

metric $g_{\mu\nu}$. Now Dirac[14] has shown that the momentum densities p^{rs} canonically

conjugate to g_{rs} are related to Γ^0_{rs} by the relation

$$\Gamma^0_{ab} = (-g^{00})^{1/2} K^{-1} (g_{ra}g_{sb} - \tfrac{1}{2}g_{rs}g_{ab})p^{rs} \tag{23}$$

where K^2 is the determinant of g_{rs}. Thus condition (21) can be rewritten as

$$(g_{ra}g_{sb} - \tfrac{1}{2}g_{rs}g_{ab})\, p^{ab}\, y^r y^s = 0 \tag{24}$$

and is thus seen to depend only on the canonical variables g_{rs} and p^{rs} and the coordinates.

Condition (24) is linear in p^{rs} and so can be solved for one of its components in terms of the other thus leading to the elimination of this variable and its conjugate from consideration as dynamical field variables. Condition (22) is still too complicated for such pruposes. However we notice that, because of condition (22) the y^r form a Riemannian coordinate system on the hypersurface $y^0 = 0$. We now make use of a result due to Vermeil[10] that the metric on this hypersurface must be of the form

$$g_{rs} = \epsilon_{rau}\epsilon_{sbv}h^{ab}y^u y^v + \overset{o}{g}_{rs} \tag{25}$$

where ϵ_{rau} is the Levi-Civita tensor density of rank three and $\overset{o}{g}_{rs}$ is the three-dimensional metric at the origin and equal to δ_{rs}. If we multiply this equation by y^s and sum we have that

$$g_{rs}y^s = y^r \tag{26}$$

as an alternate form of the coordinate conditions that lead to Riemannian

coordinates on the hypersurface. The conditions (26) are now linear in g_{rs} and can thus be solved for three of its components in terms of the other three. Thus we have succeeded in finding a set of coordinate conditions applicable to the hypersurface $y^o = 0$ that satisfy all of our requirements since, aside from picking the point P_o and the construction of a local orthogonal coordinate system there, Riemannian coordinates are uniquely determined in a finite region around P_o.

Our construction is still not complete however since our conditions (24) and (26) only apply on the hypersurface $y^o = 0$. We can fill in coordinates in the rest of space-time by first drawing the geodesic that passes through P_o and is tangent to the time-like axis there. This line will be our y^o axis throughout the finite region of space-time considered. Times measured along it will just be equal to the proper-time figured from P_o as the starting point. Taking now some point on this axis we use it for the origin of a new Riemannian coordinate system. Following the procedure employed at the origin we construct a space-like hypersurface through this point and normal to the tangent to the y^o axis at this point. We then require that conditions (24) and (26) be satisfied on this new surface. Since it is uniquely defined these conditions will lead to a unique coordinate system on this surface. We see that in this manner we can construct a unique coordinate system that satisfies all of our basic requirements by requiring that (24) and (26) be satisfied everywhere in our space-time domain.

There remains now only the question of the characterization of the point P_o and the four mutually orthogonal directions at P_o used to define the components

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of the tangent vectors to the geodesics passing through this point. If the metric itself has symmetries it will in general be impossible to characterize these quantities uniquely. For our purposes however it is only necessary to find a method that does not exclude these symmetric solutions even though it does not lead to a unique characterization in these cases. This was just the procedure we followed in the gauge case when we used the condition (2). One possibility is to pick P_0 as the point where the absolute value of the curvature scalar is an absolute minimum. The orthogonal directions at P_0 could then be fixed by taking gradients of the scalars K^{α} at this point. The condition that P_0 is the point where $|R|$ is a minimum does not rule out the case when $|R|$ has no minimum, it just doesn't determine P_0 in this case. However, once we impose this condition (and in fact, even before we impose it) the group of transformations leading from one coordinate system satisfying conditions (24) and (26) to another no longer involve uniform translations. Likewise, by fixing the orthogonal directions at P_0 we no longer have the possibility of performing homogeneous Lorentz transformations. In any case the coordinate conditions (24) and (26) do not lead to an interesting Relativity group for General Relativity and certainly do not lead to the inhomogeneous Lorentz group.

If one were to restrict oneself to Special Relativity one could not use the metric alone to restrict the covariance group beyond the inhomogeneous Lorentz group since this is just the symmetry group of the metric in this case. However one can make use of other dynamical objects to effect a further reduction. (We assume there are such objects available since the case of an empty flat space

is not very interesting.) Thus, if we had a scalar field ϕ we could proceed to use it in place of the curvature scalar R but following the same procedure as outlined in the above paragraph. Consequently, even in Special Relativity it is seen that the maximal set of coordinate conditions is sufficient to reduce the covariance group to the identity.

V. CONCLUSIONS

We have seen, in the two important cases of electrodynamics and relativity theory, how it is possible to impose a number of covariance conditions on the theory sufficient to reduce the respective covariance groups from gauge groups to at most Lie groups on a finite dimensional parameter space. Then, depending on the variables available for the purpose one can effect a further reduction of the covariance group down to the identity transformation. In effecting this additional reduction one must be sure that no physical state of the system is eliminated thereby. In electrodynamics one can effect this final reduction provided one has a source field for the electrodynamic field. In gravitational theory the situation is less clear because of the greater complexity of the covariance group involved. However it appears certain that one can always effect a reduction to a group which is at most a subgroup of the Poincare group.

Aside from their pertinance to the general problem of the reduction of the number of dynamical fields in theories with gauge groups these conclusions bear on Kretschmann's definition of the Relativity group of a theory. At least as it stands the definition does not lead to the desired group of the theory, e.g. the Poincare group for space-time theories, but rather leads to the trivial group of the identity transformation. While we have been unable to prove such a conclusion for an arbitrary convariance group it seems to be the case.

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1) J. L. Anderson, Rev. Mod. Phys. **36**, 929 (1964).

2) E. Kretschmann, Ann. Physik **53**, 575 (1917).

3) Throughout this paper we will employ a notation in which Greek suffixes take one the values 0,1,2,3 while lower case Roman suffixes take on the values 1,2,3. The usual summation convention holds for these indicies and, unless otherwise stated, they will be raised and lowered with the help of the Minkowski metric $\eta_{\mu\nu}$ of signature -2. We also use the comma notation to denote ordinary derivative, i.e., $\psi_{,\mu} \equiv \partial\psi/\partial x^{\mu}$.

4) The significance of the gauge group of the first kind is that it is a symmetry group A_{μ} in the sense that for such transformations $A_{\mu}' = A_{\mu}$. As such it leads to the conservation of charge.

5) V. Fock, The Theory of Space, Time and Gravitation, Pergamon Press, London, 1959.

6) First discussed by us in Les Theories Relativistes de la Gravitation, Centre National de la Recherche Scientific, Paris, 1962. pg. 378.

7) T. de Donder, La Gravifique Einsteinne, Gauthier-Villars, Paris, 1921.

8) A. Komar, Phys. Rev. **111**, 1182 (1958).

9) P. A. M. Dirac, Phys. Rev. **114**, 924 (1959).

10) H. Vermeil, Math. Ann., **79**, 289 (1918).

11) The discussion given here is only valid for complete Lagrangian theories, i.e., theories all of whose equations of motion can be derived from a variational principle. It would therefore not apply to the example given in Sec. III of the complex scalar field when $F_{\mu\nu} = 0$ or to the case of special relativity where $R^{\rho}_{\sigma\mu\nu} = 0$ since neither of these equations can be obtained from a variational principle.

Coordinate Conditions and Canonical Formalisms in Gravitational Theory*

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I. INTRODUCTION

An expression for the Hamiltonian of general relativity has been given by Dirac[1] that is especially simple in form and so allows one to proceed further with the study of the canonical formalism of this theory than was possible using the older, more complicated expressions for this quantity.[2(a),(b)] This Hamiltonian is given by

$$H = \int \{(-g^{00})^{-\frac{1}{2}}\mathcal{3C}_L + g_{0r}\mathcal{3C}^r\}\, d^3x, \qquad (1.1)$$

where $\mathcal{3C}_L$ is the Hamiltonian constraint of the theory and $\mathcal{3C}^r$ are the longitudinal constraints. They have the form

$$\mathcal{3C}_L = K^{-1}(g_{ra}g_{sb} - \tfrac{1}{2}g_{rs}g_{ab})\, p^{rs}p^{ab} + K(^3R) \qquad (1.2)$$

and

$$\mathcal{3C}^r = p_{|s}^{rs}, \qquad (1.3)$$

where p^{rs} is the momenta conjugate to the spatial part of the metric g_{rs}, K^2 is the determinant of g_{rs}, and ^3R is the scalar curvature formed from g_{rs}.[3]

As it stands, the Hamiltonian (1.1) is incomplete because of the appearance of $g_{0\mu}$ in it. In the formulation given by Dirac that leads to this Hamiltonian, the momenta $p^{0\mu}$ conjugate to $g_{0\mu}$ vanish weakly. This has as a consequence that the degrees of freedom associated with $g_{0\mu}$ disappear from the Hamiltonian formalism. It therefore becomes necessary to assign values to the $g_{0\mu}$ by some means in order to make the Hamiltonian definite. The method of assignment has varied from author to author and has resulted in seeming disparate expressions for the Hamiltonian. We discuss here the various methods used to fix the $g_{0\mu}$, the Hamiltonians that result therefrom, and the relations between these various methods.

II. COORDINATE CONDITIONS

In all the methods that we discuss here the values of the $g_{0\mu}$ are fixed by the introduction of coordinate conditions into the theory, albeit not necessarily in the most obvious manner. One might imagine that the simplest thing to do would be to require that one always work in a coordinate system in which the $g_{0\mu}$ are specific space–time functions.[4] Thus one might introduce a Gaussian normal coordinate system in which $g_{0\mu} = -\delta_{0\mu}$. Such a coordinate system can be shown to always exist in every finite region of space–time that is topologically Euclidean.

There are, however, a number of reasons why one does not assign values to the $g_{0\mu}$ directly. For one thing it does not fix the coordinate system uniquely. In general it leaves arbitrary one surface $x^0 = \text{const}$ as well as the coordinates in that surface. In order to remove this arbitrariness it is necessary to impose additional conditions on the metric and its derivatives on a $t^0 = \text{const}$ surface. Since the values of the $g_{0\mu}$ are already fixed this means that these additional conditions must involve the g_{rs}, the p^{rs}, and their spatial derivatives. Thus one might require, as does Dirac,[5] that one of the $t^0 = \text{const}$ surfaces should have a maximum three-dimensional "area." This condition can be expressed in terms of the g_{rs} and p^{rs} by requiring that

$$p \equiv g_{rs}p^{rs} \approx 0. \qquad (2.1)$$

To fix the coordinates within this surface we could require in addition that the harmonic conditions in three dimensions,

$$(Ke^{rs})_{,s} \approx 0 \qquad (2.2)$$

be satisfied. Here e^{rs} is the inverse of the matrix formed from g_{rs}:

$$e^{ra}g^\alpha{}_s = \delta_s{}^r. \qquad (2.3)$$

While we are free to impose these conditions on any one $t^0 = \text{const}$ surface, they will not, in general, remain satisfied on the other $t^0 = \text{const}$ surfaces since their time derivatives, formed by taking their Poisson brackets with the Hamiltonian (1.1), will not vanish weakly, either as a consequence of the coordinate conditions (2.2), (2.3), or the secondary constraint equations

$$\mathcal{3C}_L \approx 0 \qquad (2.4)$$

and

$$\mathcal{3C}^r \approx 0. \qquad (2.5)$$

It would thus be extremely awkward to work with such conditions in studying the dynamical evolution of a metric field.

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[1] P. A. M. Dirac, Proc. Roy. Soc. (London) A246, 333 (1958)

[2] (a) P. G. Bergmann, R. Penfield, R. Schiller, and H. Zatzkis, Phys. Rev. 80, 81 (1950). (b) F. A. E. Pirani and A. Schild, Phys. Rev. 79, 986 (1950); Phys. Rev. 87, 452 (1952).

[3] We follow here the notation used by Dirac in his papers. Latin indices run from 1 to 3 and represent the spatial part of an index set. Indices are to be raised or lowered with the help of g_{rs}. The notation $|s$ indicates the three-dimensional covariant derivative formed using g_{rs}. Finally, we use a metric with signature $+2$.

[4] See, for example, P. G. Bergmann and A. I. Janis, Phys. Rev. 111, 1191 (1958).

[5] P. A. M. Dirac, Phys. Rev. 114, 924 (1959).

The difficulty that the secondary coordinate conditions (2.3), (2.4) do not remain satisfied in the course of time if we first assign values to the $g_{0\mu}$ is directly related to a second difficulty of this way of doing things and is associated with the existence of the secondary constraints (2.4) and (2.5). They arise as a consequence of requiring that $p^{0\mu}$ vanish weakly throughout the temporal evolution of the metric field. Since they depend only on the g_{rs} and p^{rs} they represent restrictions on the values that one can assign to these quantities initially. Furthermore, since $\mathfrak{IC}_L$ and $\mathfrak{IC}^r$ are generators of infinitesimal coordinate transformations,[6] all physical observables, which are *a fortiori* invariants, must have vanishing Poisson brackets with the $\mathfrak{IC}_L$ and $\mathfrak{IC}^r$. The construction of initial conditions and observables is therefore greatly complicated by the existence of the secondary constraints. Furthermore, the existence of constraints such as (2.4) and (2.5) implies that not all of the apparent degrees of freedom of the field are real. As in the electrodynamic analog where only the transverse components of the field represent dynamical degrees of freedom of the field, only a subset of the g_{rs}, p^{rs} represent true dynamical degrees of freedom of the metric field.

In attempting to overcome these difficulties associated with the existence of the secondary constraints, many authors have made use of coordinate conditions. It is clear, for instance, that once a coordinate system is fixed, any functional of the g_{rs} and p^{rs} constructed in that coordinate system is an observable. However for this purpose of constructing observables the coordinate system must be fixed up to a point where any additional arbitrariness resides in at most a finite dimensional Lie group of transformations. Thus it would not do simply to assign values to the $g_{0\mu}$; we must impose additional, secondary conditions such as (2.1) and (2.2) on the g_{rs} and p^{rs}. If these conditions would remain satisfied throughout the course of time then we could alleviate to some extent the other above mentioned difficulties. In particular we could then use the conditions on the g_{rs} and p^{rs} together with the constraint equations (2.4) and (2.5) to eliminate the nonphysical degrees of freedom from the field equations for the metric as is done in electrodynamics. There the Coulomb gauge condition $\nabla \cdot \mathbf{A} = 0$ together with the secondary constraint $\nabla \cdot \mathbf{P} = -\rho$ is used to eliminate the longitudinal components of the vector potential $\mathbf{A}^L$ and the conjugate momenta $\mathbf{P}^L$ from the theory. If we could effect this elimination of the nonphysical degrees of freedom of the metric field through the use of coordinate conditions, then the remaining degrees of freedom would constitute the physically observable parts of the metric and at the same time could be assigned arbitrary (to the extent that the reality conditions on the metric are not violated) initial values.

If we are going to introduce coordinate condition into the theory we see that we would like them to involve restrictions on the g_{rs} and p^{rs} that remain satisfied in the course of time. In almost all of the work that involves coordinate conditions this is accomplished by determining values for the $g_{0\mu}$ that keep the coordinate conditions satisfied once they are satisfied initially. In other words, one first decides upon the conditions that involve the g_{rs} and p^{rs} and then finds those values for the $g_{0\mu}$ that maintain these conditions in the course of time. Thus in electrodynamics, the Coulomb gauge condition $\nabla \cdot \mathbf{A} = 0$ will remain satisfied in the course of time provided we require that the scalar potential ϕ satisfies $-\nabla^2 \phi = \rho$. Had we started out by setting $\phi = 0$, we could still have found a gauge in which $\nabla \cdot \mathbf{A} = 0$ at some one time but, because of the field equation $-\nabla \cdot (d\mathbf{A}/dt + \nabla\phi) = \rho$, it would not remain zero in the future. It should be noted that, while neither the condition $\nabla \cdot \mathbf{A} = 0$ nor the condition $-\nabla^2 \phi = \rho$ is sufficient by itself to fix the gauge, the two together, along with the boundary conditions that $\phi = 0$ and $\mathbf{n} \cdot \mathbf{A} = 0$ on the boundary of the spatial region under investigation (n is the outward directed normal to this boundary) are sufficient to fix the gauge up to an arbitrary constant, which is the best we can hope to do.

In general relativity, various authors have used different types of conditions and have woven them into the cannonical formalism in different ways, leading to what appears at first sight to be unrelated expressions for the resulting Hamiltonian for the theory. The coordinate conditions fall into two main classes, those which involve the coordinates themselves such as those by Arnowitt, Deser, and Misner[7] and Bergmann and Komar[8] and those which do not and which are used by Dirac[9] and the author.[10]

III. MODIFIED CANONICAL FORMALISMS

The use made of coordinate conditions varies from author to author and depends upon the type of coordinate conditions employed. We give a brief survey of these different approaches in this section and then show their equivalence in the next section.

Dirac was the first to set up a formalism[11] that would allow one to make explicit use of coordinate conditions in the canonical formalism of general relativity. His method is general enough to be applicable to all types of coordinate conditions. Furthermore, as we shall show, the other approaches are all variants on this general method. We therefore begin our survey with a description of this method.

[6] J. L. Anderson and P. G. Bergmann, Phys. Rev. **83**, 1018 (1951)

[7] R. Arnowitt, S. Deser, and C. W. Misner, Phys. Rev. **116**, 1322 (1959); Phys. Rev. **117**, 1595 (1960).

[8] P. G. Bergmann and A. Komar, Phys. Rev. Letters **4**, 432 (1960).

[9] P. A. M. Dirac, Phys. Rev. **114**, 924 (1959).

[10] J. L. Anderson, *Les Theories Relativistes de la Gravitation* (Centre National de la Recherche Scientifique, Paris, 1962), p. 372.

[11] P. A. M. Dirac, Can. J. Math. **2**, 129 (1950).

A. Dirac's Method

In order to make use of the constraint equations (2.4) and (2.5) to eliminate a corresponding number of degrees of freedom from the theory they must somehow be converted into strong equations. (In Dirac's terminology, an equation is considered weak if its satisfaction violates the basic Poisson bracket relations of the theory while a strong equation is consistent with these relations). One cannot use them directly for this purpose by, for example, solving them for some of the canonical variables in terms of the others and making use of these solution to eliminate the variables solved for from the Hamiltonian. By doing so one would change the equations of motion of the theory since, as they stand, the constraint equations are weak equations and so can be used to eliminate variables from the theory only after all Poisson brackets have been computed.

In order to convert the constraint equations into strong equations Dirac introduced a new type of bracket expression into the theory which we shall refer to as a Dirac bracket (Db) in contradistinction to a Poisson bracket (Pb). The ability to introduce these new bracket expressions into the theory depends upon the concomitant introduction of coordinate conditions into the theory. Because the coordinate conditions must single out a particular coordinate system, given the values of the metrical quantities, they cannot be an invariant and consequently they must have non-vanishing Pb's with the constraints $\math3C_L$ and $\math3C^r$. They have the effect of converting the constraint equations from what Dirac calls first-class constraints (the Pb's of a first-class constraint with all the other constraints and the Hamiltonian must vanish weakly) into second class constraints (some of the Pb's with other constraints do not vanish even weakly) if the coordinate conditions themselves are considered to be constraints.

To define Db expressions we first compute the Pb's of all the second-class constraints of the theory with each other. Let us call all of the second-class constraint equations of the theory χ equations and distinguish between the different χ's by an index that can and will take on both discrete and continuous values. We next construct the inverse $c_{ss'}$ to the matrix $[\chi_s, \chi_{s'}]$ of the Pb's of the χ's with each other. It satisfies

$$c_{ss'}[\chi_{s'}, \chi_{s''}] = \delta_{ss''}, \tag{3.1}$$

where $\delta_{ss''}$ is a generalized delta, i.e., a Dirac delta function when s and s'' vary continuously. The matrix $c_{ss'}$ was shown by Dirac to always exist provided that no linear combination of the χ's is first class. The Db of any two functionals A and B of the canonical variables is then defined by the equation

$$[A, B]_D = [A, \chi_s]c_{ss'}[\chi_{s'}, B] + [A, B] \tag{3.2}$$

where the bracket expressions appearing on the right side of this equation are ordinary Pb's. Dirac has shown

that the Db's satisfy all of the usual relations, e.g., the Jacobi identity, satisfied by the Pb's. Furthermore, and this is the main reason for introducing them, the Db of a χ with anything is automatically zero as can be seen immediately from the above definition. As a consequence one can now consider the x's as strong equations and can be used before a Db is calculated.

A particular simplification in the construction of Db's occurs when one or more of the χ's are just the canonical variables p themselves so that the corresponding equations of constraint look like

$$p_\iota \approx 0, \tag{3.3}$$

where p_ι can be either a momentum or coordinate variable. Then, since Eqs. (3.3) are supposed to be second-class equations, the canonical conjugates q_ι must appear in enough of the other χ equations to allow us to solve for these variables in terms of the remaining variables so that we have

$$q_\iota \approx f_\iota(\bar{q}, \bar{p}), \tag{3.4}$$

where $\bar{q}$ and $\bar{p}$ represent the remaining canonical variables. In this case the Db reduces to an ordinary Pb where Eqs. (3.3) and (3.4) have been used as strong equations to eliminate the p_ι and q_ι from the quantities whose Db one wishes to compute and where differentiations are only with respect to the $\bar{q}$'s and $\bar{p}$'s. Here one accomplishes a reduction in the number of degrees of freedom of the system directly. In the more general case one retains all of the original canonical variables but the use of the Db effectively takes account of the fact that constraint equations together with the coordinate conditions effectively serve to reduce the number of degrees of freedom of the system.

In order that one can replace the Pb's by Db's in the theory it is necessary that the equations of motion are not altered and this is the case provided that the Pb's of all the χ's with the Hamiltonian of the theory vanish at least weakly. If this is so then

$$[g, H]_D = [g, H] - [g, \chi_s]c_{ss'}[\chi_{s'}, H] = [g, H]$$

$$= \dot{g}$$

showing that one can indeed use the Db's in constructing the equation of motion for any dynamical variable g.

In applying the above procedure to the case of general relativity Dirac introduced the coordinate condition (2.1) into the theory as an additional constraint on the canonical variables. While the Pb of p with the $\math3C^r$ vanishes its Pb with $\math3C_L$ does not. Thus $\math3C_L$ is converted into a second-class constraint. Dirac then introduced a new set of canonical variables that include p, its conjugate $\ln \kappa$ where $\kappa = K^{\frac{1}{2}}$, $\bar{p}^{rs} = \kappa^2(p^{rs} - \frac{1}{3}pe^{rs})$ and $\tilde{g}_{rs} = g_{rs}/\kappa^2$. He then indicated that one would proceed by solving the constraint equations $\math3C_L = 0$ for $\ln \kappa$ in terms of $\bar{p}^{rs}$ and $\tilde{g}_{rs}$. However, at this point a

difficulty occurs that is not common to other theories with constraints. The Hamiltonian of general relativity (1.1) is a homogeneous, linear functional of the constraints so that one cannot simply substitute the expression obtained for $\ln \kappa$ by solving the constraint equations directly into the Hamiltonian since this would lead to a vanishing modified Hamiltonian. To overcome this difficulty Dirac rewrote the Hamiltonian (1.1) in the form

$$H = H_{\text{main}} + \int [\{(-g^{00})^{-\frac{1}{2}} - 1\} \mathfrak{IC}_L + g_{0r} \mathfrak{IC}^r] \, d^3x, \quad (3.5)$$

where

$$H_{\text{main}} = \int \{\mathfrak{IC}_L - (K^{-1}(K^2 e^{rs}), r), s\} \, d^3x.$$

This means that the Hamiltonian of Eq. (3.5) differs from that of Eq. (1.1) by a surface term. Adding or subtracting such a surface term from the Hamiltonian does not change the equations of motion computed directly from this Hamiltonian. However, Arnowitt, Deser, and Misner[12] have argued that the effective Hamiltonian obtained after the elimination of degrees of freedom from the theory by direct substitution might depend upon which complete divergence is subtracted from the original Hamiltonian H.

If we could be sure that we could treat the equations $p \approx 0$ and $\mathfrak{IC}_L \approx 0$ as strong equations then, of course, it would be immaterial which divergence is subtracted from H. However the following argument shows that it is not completely obvious that we can treat these equations as strong equations. In order for this to be so two conditions must be met. First, both p and $\mathfrak{IC}_L$ must have weakly vanishing Pb's with H (either H will do for this condition to be met) and second, the Pb matrix $[p, \mathfrak{IC}_L']$ must possess an inverse. But we see that these two conditions are in fact incompatible with each other. Thus, to satisfy the first condition we must have

$$[p, H] = \int \{(-g^{00})^{-\frac{1}{2}}[p, \mathfrak{IC}_L'] \, d^3x = 0 \quad (3.6)$$

which we could imagine being satisfied by an appropriate choice of g^{00}. However, if $[p, \mathfrak{IC}_L']$ were to possess an inverse we could conclude that g^{00} would, in fact, have to be zero. It would not do to have g^{00} zero as this would imply, among other things, that the determinant of $g_{\mu\nu}$ is singular. We must have, therefore, that the matrix $[p, \mathfrak{IC}_L']$ is singular. In fact, whenever one imposes coordinate conditions that do not explicitly depend upon the coordinates one must make sure that the Pb matrix formed using these conditions and the constraints is singular or else we would be forced to set $g^{00} = 0$ in order that the coordinate conditions remain satisfied in the course of time. The quantity $(-g^{00})^{-\frac{1}{2}}$

[12] R. Arnowitt, S. Deser, and C. W. Misner, J. Math. Phys. 1, 434 (1960).

is then determined by requiring that it be a null vector of the Pb matrix.

One can check that in the present case the Pb matrix does in fact possess null vectors since, when we work out the Pb in Eq. (3.6) we obtain an equation for $(-g^{00})^{-\frac{1}{2}}$ which in general possesses nonzero solutions.

If it is true that the Pb matrix is singular, then we cannot construct its inverse in the usual fashion and so cannot construct the corresponding Db's. Note that in Dirac's proof of existence of an inverse it was necessary that no linear combination of the χ's used in constructing the Pb matrix was first class. But in our case, the linear combination $\int (-g^{00})^{-\frac{1}{2}} \mathfrak{IC}_L \, d^3x$ is still first class so that the condition for the existence of an inverse is violated. One could try to work only with those combinations of constraints and coordinate conditions that are second class. To do this it would be necessary to introduce a complete set of functions $g(x, x')$ such that $\int g(x, x') (g'^{00})^{-\frac{1}{2}} \, d^3x' = 0$ and work with the reduced set of constraints $\int g(x, x') \mathfrak{IC}_L' \, d^3x'$ instead of the $\mathfrak{IC}_L$ in constructing the Pb matrix with the p's. While this matrix would be nonsingular and so possess an inverse one would still have to show that one could determine $\ln \kappa$ in terms of $\tilde{g}_{rs}$ and $\tilde{p}^{rs}$ from the reduced set of constraints, in order to justify the direct elimination, of p and $\ln \kappa$ from the theory. Rather than do this we shall show directly that the equations of motion obtained by direct elimination are equivalent to those obtained before the redundant variables are eliminated.

To demonstrate the equivalence between the two sets of equations of motion let $H*$ be the Hamiltonian obtained from H given by Eq. (3.5) by setting $p = 0$ and substituting in for $\ln \kappa$ the expression obtained by solving for this quantity in the constraint equation $\mathfrak{IC}_L = 0$ in terms of $\tilde{g}_{rs}$ and $\tilde{p}^{rs}$. Then the time derivative of any functional $F(\tilde{g}_{rs}, \tilde{p}^{rs})$ is given, when we use $H*$ as our Hamiltonian, by

$$\dot{F} = [F, H*]. \quad (3.7)$$

In computing $[F, H*]$ we must keep in mind that the $\tilde{g}_{rs}$ and $\tilde{p}^{rs}$ appear in $H*$ both as they appear in H directly, and through the factors $\ln \kappa$ that appear in H. Thus we have that

$$\dot{F} = [F, H_{\text{main}}]* - \int [F, \ln \kappa *][p, H_{\text{main}}]* \, d^3x$$

$$+ \left[F, \int g_{0r} \mathfrak{IC}^r(\tilde{g}_{rs}, \tilde{p}^{rs}) \, d^3x \right]* \quad (3.8)$$

where we have made use of the chain rule for Poisson brackets and where a $*$ after a quantity means that it is to be evaluated by setting $p = 0$ and $\ln \kappa$ equal to its value in terms of $\tilde{g}_{rs}$ and $\tilde{p}^{rs}$. It is this expression for $\dot{F}$ that we must compare with the one obtained using H directly and then starring all quantities, that is, with

the equation

$$\dot{F}=[F,H]*. \qquad (3.9)$$

We have that

$$[F,H]*=[F,H_{\text{main}}]*=\left[F,\int N_0\mathfrak{IC}_L\,d^3x\right]*$$

$$+\left[F,\int g_{0r}\mathfrak{IC}^r\,d^3x\right]*, \qquad (3.10)$$

where we have introduced the abbreviation N_0 for $\{(-g^{00})^{-\frac{1}{2}}-1\}$.

We see that the first and last terms on the right-hand side of Eq. (3.10) agree with the first and last terms on the right-hand side of Eq. (3.11) so that all we must do is to show that the middle terms are equivalent. For this purpose we expand $\mathfrak{IC}_L$ in a functional Taylor series about $\ln \kappa=\ln \kappa*$. Since $\mathfrak{IC}_L*=0$ we have

$$\mathfrak{IC}_L=-\int[p',\mathfrak{IC}_L]*(\ln \kappa'-\ln \kappa'*)\,d^3x'+\cdots. \qquad (3.11)$$

Then we have

$$\left[F,\int N_0\mathfrak{IC}_L\,d^3x\right]*=\int N_0[F,\mathfrak{IC}_L]*\,d^3x$$

$$=\iint N_0[p',\mathfrak{IC}_L]*[F,\ln \kappa'*]\,d^3x\,d^3x'. \qquad (3.12)$$

Now we note that we can rewrite Eq. (3.6) in terms of N_0 as

$$[p',H_{\text{main}}]*+\int N_0[p',\mathfrak{IC}_L]*\,d^3x=0. \qquad (3.13)$$

Substituting into the right-hand side of Eq. (3.12) we have

$$\left[F,\int N_0\mathfrak{IC}_L\,d^3x\right]=-\int[F,\ln \kappa'*][p',H_{\text{main}}]\,d^3x' \qquad (3.14)$$

which proves the desired result that $[F,H]*=[F,H*]$.

To complete the elimination of redundant degrees of freedom from the theory Dirac requires, in addition to the condition $p=0$, the conditions

$$(\kappa^2 e^{rs})_{,s}=0. \qquad (3.15)$$

He uses these conditions rather than the harmonic conditions (2.2) since $\kappa^2 e^{rs}$ has a vanishing Pb with p while Ke^{rs} does not. The use of the conditions (2.2) would require a change in the Pb relations between nongravitational variables when one goes over to Db's while the use of the conditions (3.15) would not.

Rather than use the conditions (3.15) to eliminate additional degrees of freedom directly as before, Dirac chooses to introduce the Db's in this case and obtains

an approximate solution for the inverse of the Pb matrix.

In discussing Dirac's procedure for eliminating redundant degrees of freedom from the theory we have worked with the variables κ, p, $\tilde{g}_{rs}$ and $\tilde{p}^{rs}$. They have the advantage that the $\mathfrak{IC}^r$ can be expressed directly in terms of the $\tilde{g}_{rs}$ and $\tilde{p}^{rs}$ since they do not depend upon κ. This also allows one to eliminate the degrees of freedom associated with $\mathfrak{IC}_L$ before dealing with those associated with $\mathfrak{IC}^r$ since p has a vanishing Pb with the $\mathfrak{IC}^r$. In the more general case one would introduce a twelve independent functionals of the original canonical variables g_{rs} and p^{rs} grouped into three sets of four each according to the scheme

$$\{g_{rs},p^{rs}\}\Rightarrow\{g_A,p^A;g_\iota;p^\iota\} \qquad A=1,2;\iota=1,\cdots,4. \qquad (3.16)$$

The four functionals g_ι, p^ι are solved for in the constraint equations (2.4) and (2.5) and are referred to as the "solved for" variables. The four functionals g_ι are used to impose the coordinate conditions and are called the coordinate variables. Finally the remaining four functionals g_A, p^A that describe the dynamical evolution of the system are termed the dynamical variables. In general we would want g_ι and p^ι to be conjugates of each other and have vanishing Pb's with the g_A and p^A. Then if we require that

$$g_\iota\approx0 \qquad (3.17)$$

as coordinate conditions, we again must have the Pb matrix formed with the g_ι, $\mathfrak{IC}_L$, and the $\mathfrak{IC}^r$ be singular in order to avoid requiring that $(-g^{00})^{-\frac{1}{2}}$ be zero in order to insure that the conditions (3.17) remain satisfied in the course of time. The proof that one can treat these equations together with the constraint equations as strong equations is then only slightly more complicated than in the previous case.

B. Anderson's Method[10]

The method devised by the author to eliminate redundant degrees from the theory differs in approach from that used by Dirac although not in the final results. To simplify the discussion we work with the Dirac variables $\ln \kappa$, p, $\tilde{g}_{rs}$ and $\tilde{p}^{rs}$ although again this imposes no serious restrictions on the procedure and it can be carried through with only slightly more trouble in the more general case of the scheme indicated in (3.16). Again we solve the equation $\mathfrak{IC}_L=0$ for $\ln \kappa$ in terms of g_{rs} and $\tilde{p}^{rs}$. We now introduce in place of $\ln \kappa$ a new variable λ through the equation

$$\ln \kappa=\ln \kappa*+\lambda, \qquad (3.17)$$

where a $*$ after a quantity has the same meaning as before. Setting $\lambda=0$ is equivalent to treating the constraint equation $\mathfrak{IC}_L=0$ as a strong equation. We now determine $(-g^{00})^{-\frac{1}{2}}$ in H given by Eq. (3.5) by

the requirement that λ appears in H only to higher powers than the first. Any factors of λ higher than the first do not contribute to the equations of motion when we set $\lambda = 0$ after the Pb's have been computed. Thus the effective Hamiltonian for the theory is just $H*$ again. To show that this effective Hamiltonian gives the same equations of motion as the original one we must first determine $(-g^{00})^{-\frac{1}{2}}$ to use in the original Hamiltonian.

When we substitute into H for $\ln \kappa$ using Eq. (3.17) and expand about $\lambda = 0$ we obtain

$$H = H*_{\text{main}} + \int [p, H_{\text{main}}] * \lambda \, d^3x + \cdots$$

$$+ \iint N_0'[p, \mathcal{K}_L'] * \lambda \, d^3x \, d^3x' + \cdots, \quad (3.18)$$

where the dots indicate the terms containing higher powers of λ than the first. We see that the terms linear in λ can be made to vanish by choosing N_0 such that it satisfies the equation

$$[p, H_{\text{main}}] * = \int N_0'[p, \mathcal{K}_L'] * d^3x' = 0.$$

We see that this is just the same condition (3.13) used to fix N_0 so that $[p, H]$ is again zero. In the present case, since $\ln \kappa$ no longer appears in the Hamiltonian except through powers of λ higher than the first we can conclude that $[p, H]$ is again zero. Thus the two methods are entirely equivalent and we do not need to give a separate proof of the consistency of our method.

So far we have used coordinate conditions that do not depend explicitly on the coordinates, e.g., we have set $p = 0$ or more generally would set the coordinate variables of (3.16), the g_{i}, equal to zero. In order to insure the consistency of the approach it was necessary that the Pb matrix of the p^i with $\mathcal{K}_L$ and the $\mathcal{K}^r$ have at least one null vector. We now examine a number of procedures that make use of coordinate conditions that depend explicitly on the coordinates. Thus, Arnowitt, Deser, and Misner (referred to in what follows as ADM) take as coordinate conditions.

$$g_i \approx x^i. \quad (3.19)$$

Bergmann and Komar (referred to hereafter as BK), although they do not explicitly introduce new variables as in Eq. (3.16), work in a coordinate system in which the coordinates at a point of the space-time manifold are numerically equal to the values of the four scalars one can form from the $g_{\mu\nu}$ and their first and second derivatives that do not vanish when $R_{\mu\nu} = 0$.[13,14] In either case the requirement that the coordinate conditions remain satisfied throughout the course of time

[13] E. Kretschmann, Ann. Physik **53**, 575 (1917).
[14] A. Komar, Phys. Rev. **111**, 1182 (1958).

is now that

$$\delta_i^4 = [g_i, H] \quad (3.20)$$

or

$$\delta_i^4 = \int \{ (-g^{00})^{-\frac{1}{2}}[g_i', \mathcal{K}_L] + g_{0r}[g_i', \mathcal{K}^r] \} \, d^3x. \quad (3.21)$$

Now the Pb matrix of the g_i with the constraints should possess an inverse in order to be able to determine $(-g^{00})^{-\frac{1}{2}}$ and g_{0r} from Eq. (3.21). We now discuss how these types of coordinate conditions are used by BK and ADM.

C. The Method of Bergmann and Komar[8]

In general the Pb matrix depends upon δ functions and their derivatives so that Eqs. (3.21) become a set of coupled differential equations for the $(-g^{00})^{-\frac{1}{2}}$ and g_{0r}. Similarly, to find the inverse to the Pb matrix one must in general solve a set of coupled differential equations. However, BK observed that in the special case where the g_i are all scalars the Pb matrix involves only δ functions and not their derivatives. The reason for this lies in the fact that the $\mathcal{K}_L$ and $\mathcal{K}^r$ are the generators of coordinate transformations and the transformation law for scalars does not involve the derivatives of the descriptors (the arbitrary functions) of the transformation. Thus is the generator of the transformation is given by

$$C = \int \omega_4 \mathcal{K}_L + \omega_r \mathcal{K}^r \} \, d^3x, \quad (3.22)$$

where the ω_4 and ω_r are the descriptors of the transformation, we must have, if g_i is a scalar, that

$$-\delta g_i = g_{i,4}\omega_4 + g_{i,r}\omega_r. \quad (3.23)$$

But since $\delta g_i = [g_i, C]$ we have that

$$\delta g_i = \int \{ \omega_4'[g_i, \mathcal{K}_L'] + \omega_r'[g_i, \mathcal{K}'^r] \} \, d^3x' \quad (3.24)$$

and since the ω's are arbitrary functions of x we can conclude that the Pb matrix does not involve derivatives of the δ function. Consequently, solving Eqs. (3.21) or finding the inverse of the Pb matrix involves only algebraic operations. In this regard, however, one must be sure that the g_i do not depend upon $(g^{00})^{-\frac{1}{2}}$ or g_{0r}; otherwise what we have said would not be true and in fact, if this were the case, Eqs. (3.21) would become nonlinear in these quantities. Fortunately such is not the case as BK were able to show; the four nonvanishing scalars formed from g the $g_{\mu\nu}$ and their first and second derivatives depend only upon the g_{rs}, p^{rs} and their spatial derivatives.

Once having set up their coordinate system BK proceed to construct a set of invariants. Since the constraints are the generators of infinitesimal coordinate transformations the condition that a functional F of the g_{rs} and p^{rs} be an invariant is that the

Pb's between it and all of the constraints must vanish. Without the use of coordinate conditions the construction of such quantities is extremely difficult and only a few such quantities are known at present. However, once one introduces coordinate conditions into the theory one can construct invariants relatively easily. In fact, corresponding to each functional F there is another one $\bar{F}$ that has vanishing Pb's with the constraints.

For the construction of $\bar{F}$ and in what follows it is convenient to introduce the notation

$$\{\mathfrak{K}_\iota\} = \{\mathfrak{K}^r, \mathfrak{K}_L\} \qquad (3.25)$$

and

$$\{N_\iota\} = \{g_{0r}, (-g^{00})^{-\frac{1}{2}}\}, \qquad (3.26)$$

where again ι (not a tensor index) runs from 1 to 4. Then, given an F we add to it a linear combination of the constraints and coordinate conditions according to

$$\bar{F} = F + \int \alpha_\iota (g_\iota - x^\iota) + \beta_\iota \mathfrak{K}_\iota\} \, d^3x, \qquad (3.27)$$

where the coefficients α_ι and β_ι are determined by the condition that the Pb's between $\bar{F}$ and $(g_\iota - x^\iota)$ and $\mathfrak{K}_\iota$ vanish. If the Pb matrix has in inverse one can determine the coefficients in Eq. (3.27) directly. Thus if $G_{\iota\kappa}(x, x')$ is the reciprocal of the Pb matrix so that[16]

$$\int [g_\iota, \mathfrak{K}_\lambda''] G_{\lambda\kappa}(x'', x') \, d^3x'' = \delta_{\iota\kappa} \, \delta(x - x'), \qquad (3.28)$$

we can determine the α_ι by the requirement that $[\bar{F}, \mathfrak{K}_\iota] = 0$. If we multiply this requirement by $G_{\iota\kappa}$, sum over the indices, and integrate, we obtain

$$\alpha_\iota = -\int [F, \mathfrak{K}_\kappa'] G_{\kappa\iota}(x', x) \, d^3x'. \qquad (3.29)$$

The β's can be obtained similarly. If the g_ι are scalars $G_{\iota\kappa}$ can be found immediately and hence so can the α's and β's.

As long as we work only with the barred quantities we see that the constraint equations and coordinate conditions can be treated as strong equations since they all have vanishing Pb's with any barred quantity. In this regard then the BK procedure is equivalent so that of introducing Db's into the theory. In fact one can show immediately that

$$[\bar{F}, \bar{K}] = [F, K]_D. \qquad (3.30)$$

If we compute the total time derivative of $\bar{F}$ we have, as usual, that

$$d\bar{F}/dt = \partial\bar{F}/\partial t + [\bar{F}, H]. \qquad (3.31)$$

Since H is a homogeneous linear functional of the $\mathfrak{K}_\iota$

the second term on the right-hand side of Eq. (3.31) vanishes [this same result would hold of course if we had used H given by Eq. (3.5) instead of that given by (1.1) since we are not using the constraint or coordinate equations as strong equations here]. Furthermore we see from Eq. (3.27) that $\partial\bar{F}/\partial t = -\int \alpha_4 \, d^3x$ so we have finally

$$\frac{d}{dt} \bar{F} = -\int \alpha_4 \, d^3x \qquad (3.32)$$

or

$$\frac{d}{dt} \bar{F} = \iint [F, \mathfrak{K}_\iota'] G_{\iota 4}(x', x) \, d^3x \, d^3x'. \qquad (3.33)$$

We notice however that N_ι, as determined from Eq. (3.21) by multiplying this equation by $G_{\iota\lambda}$, is given by

$$N_\iota' = \int G_{\iota 4}(x', x) \, d^3x. \qquad (3.34)$$

If we make use of this result in Eq. (3.33) we see that the total time derivative of $\bar{F}$ is just given by

$$\frac{d}{dt} \bar{F} = \left[F, \int N_\iota \mathfrak{K}_\iota \, d^3x \right] = [F, H] \qquad (3.35)$$

so that

$$d\bar{F}/dt = dF/dt. \qquad (3.36)$$

This result is, of course, not surprising since $\bar{F}$ is weakly equal to F and Eq. (3.36) only holds as a weak equation.

D. The Method of Arnowitt, Deser, and Misner[7]

The Hamiltonian employed by ADM is obtained in the first instance from the Lagrangian formulation of general relativity by making use of methods developed by Schwinger.[16] By adding suitable surface terms to the action of general relativity, $I = \int (-gR)^{\frac{1}{2}} \, d^4x$, they show that it can be brought into the form

$$I = \int \{p^{rs}\partial_4 g_{rs} - N_\iota \mathfrak{K}_\iota\} \, d^4x, \qquad (3.37)$$

where the quantities appearing herein are defined as before. The p^{rs} appearing here as the canonical conjugates to g_{rs} are defined by

$$p^{rs} = -K(\Gamma^{rs} - e^{rs}g_{ab}\Gamma^{ab}) \qquad (3.38)$$

where $\Gamma_{rs} = -N_4\Gamma^4{}_{rs}$ is the second fundamental form of the surface $t = x^4 = $ const. In analogy with the parameterized action for a particle system they look upon the N_ι as Lagrange multipliers associated with the coordinate covariance of the action I. Varying these quantities gives the constraint equations (2.4) and (2.5) while a variation of g_{rs} and p^{rs} gives the remainder of the Einstein field equations. One sees that the Hamil-

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[16] We have neglected here the fact that the Pb's $[g_\iota, g_\kappa']$ are not necessarily zero so that, strictly speaking, $G_{\iota\kappa}(x, x')$ is not reciprocal of the total Pb matrix. However only the reciprocal of $[g_\iota, H_\kappa']$ is needed to determine α_ι.

[16] J. Schwinger, Phys. Rev. **82**, 914 (1951); **91**, 713 (1953).

tonian leading to these latter equations, $\int N_i \mathfrak{IC}_i \, d^3x$, is identical to the one obtained by Dirac. We point out here that the constraint equations are just the Gauss–Codazi equations that are the conditions that a surface $t=$ const with first and second fundamental forms g_{rs} and Γ^{rs}, respectively, can be imbedded in a four-dimensional space described by the equations $R_{\mu\nu}=0$.

The point of departure of the ADM method from the others described above lies in the coordinate conditions employed by them and the uses to which they are put in obtaining an effective Hamiltonian. They express their coordinate conditions in terms of the linear orthogonal decompositions of g_{rs} and p^{rs}. If f_{rs} is any symmetric matrix then one can decompose it using a generalized Helmholtz theorem according to

$$f_{rs}=f_{rs}{}^{TT}+f_{rs}{}^{T}+(f_{r,s}+f_{s,r}) \qquad (3.39)$$

where $f_{rs}{}^{TT}$ is the traceless, transverse part of f_{rs} and satisfies the conditions $f_{rs,s}{}^{TT}=0$ and $f_{rr}{}^{TT}=0$; $f_{rs}{}^{T}$ is transverse and satisfies thereby the condition $f_{rs,s}{}^{T}=0$ and is determined by its trace $f^{T}=f_{rr}{}^{T}$. When one inserts these decompositions for g_{rs} and p^{rs} into the action (3.37) one obtains

$$I=\int \{p^{rsTT}\partial_4 g_{rs}{}^{TT}-(-g^{T}{}_{,rr})\partial_4\{-(2\nabla^2)^{-1}p^{T}\}$$
$$+2(-p^{r}{}_{,ss}-p^{s}{}_{,rs})\partial_4 g_r-N_i\mathfrak{IC}_i\} \, d^4x, \qquad (3.40)$$

where $1/\nabla^2$ is the inverse of the Cartesian Laplacian with "suitable" boundary conditions. The form of this action suggested to ADM that they take $g_{rs}{}^{TT}$ and p^{rsTT} to be the dynamical variables of the theory, g^T and p^r as the solved for variables and p^T and g_r as the coordinate variables. Because g^T and p^r alone appear in the linear parts of the constraint equations one hopes that they can be solved for by an iterative procedure. The coordinate conditions imposed by ADM are

$$+(2\nabla^2)^{-1}p^{T}\approx t \qquad (3.41a)$$

and

$$g_r\approx x^r. \qquad (3.41b)$$

They also use the conditions

$$-(2\nabla^2)^{-1}(p^{T}+\nabla^2 p^{L})\approx t \qquad (3.42a)$$

and

$$g_r-(4\nabla^2)^{-1}g^{T}{}_{,r}\approx x^r. \qquad (3.42b)$$

These latter conditions have the property that for $g_{rs}{}^{TT}=0$, the spatial metric reduces to isotropic form, i.e., $g_{rs}=(1+\tfrac{1}{2}g^{T})\delta_{rs}$.

If one uses the coordinate conditions and constraint equations as strong equations, the action (3.40) reduces to the canonical form

$$I=\int \{p^{rsTT}\partial_4 g_{rs}{}^{TT}-\mathfrak{IC}(g_{rs}{}^{TT},\, p^{rsTT})\} \, d^4x, \qquad (3.43)$$

where $\mathfrak{IC}=\nabla^2 g^{T}$ is the solution of the constraint equation $\mathfrak{IC}_L=0$. The form of this action suggests that one

can use $H=\int \mathfrak{IC} \, d^3x$ as the Hamiltonian for obtaining the equations of motion for $g_{rs}{}^{TT}$ and p^{rsTT}. In a separate paper,[12] ADM show that the equations of motion obtained using this Hamiltonian are identical to those obtained using the Hamiltonian given in Eq. (1.1) in the coordinate system fixed by Eqs. (3.41).

One can arrive at the same expression for the effective Hamiltonian as did ADM by requiring that the coordinate conditions of the type in Eqs. (3.19) remain satisfied throughout the course of time. This will be the case provided the Pb matrix possesses an inverse and then the N_i are given by Eqs. (3.34). Let us now expand the $\mathfrak{IC}_i$ about $p^i=p^i*$ where a star after a quantity has the same meaning as before. We have that

$$H=\int N_i\left\{\mathfrak{IC}_i*+\int [g_\kappa',\mathfrak{IC}_i]*(p'^\kappa-p'^\kappa*) \, d^3x'+\cdots\right\} d^3x, \qquad (3.44)$$

where again the dots represent terms containing higher powers of $(p^\kappa-p^\kappa*)$. Since these terms will not contribute to the equations of motion we can again ignore them. Also, as before, $\mathfrak{IC}_i*$ vanishes. Therefore, when we substitute the expression for N_i from Eq. (3.34) into the above expression for the Hamiltonian we obtain

$$H=\int (p^4-p^4*) \, d^3x. \qquad (3.45)$$

If we are only interested in the equations of motion of the dynamical variables g_A and p^A we can drop the first term in the integrand. Thus the effective Hamiltonian becomes

$$H=-\int p^4* \, d^3x \qquad (3.46)$$

which is just the result obtained by ADM. Since we have not used the coordinate conditions or constraint equations as strong equations in deriving this result we see that immediately that the equations of motion for the dynamical variables obtained using this Hamiltonian will be the same as those obtained using the original Hamiltonian (1.1).

IV. RELATION BETWEEN THE METHODS

At first sight the effective Hamiltonian obtained by Dirac and by the author, $H=H*_{main}$, seems to bear little relation to that obtained by ADM and indirectly by BK. The main reason for this apparent difference lies in the different types of coordinate conditions employed in the two cases. To explore this difference further we see that if we were to require that $g_i=0$ we could not alternately require that $g_i=x^i$. In the former case we saw that Pb matrix must be singular in order that N_4 not be zero while in the latter case it

should possess an inverse in order that we can determine N_i as in Eq. (3.34).

While we cannot use the same g_i in the two types of coordinate conditions we can use closely related ones. Thus, if we required that $g_i = x^i$, we saw that the N_i could be obtained from Eq. (3.21) which we rewrite as

$$\delta_i{}^4 = \int N_\kappa'[g_i, \mathcal{K}_\kappa'] \, d^3x'. \tag{4.1}$$

Suppose now that we introduce a new set of g_i coordinate variables $\bar{g}_i$ that are obtained from the g_i by differential operations performed on these latter quantities. Thus, for instance, we might take $\bar{g}_i = \nabla^2 g_i$, etc., where ∇^2 is now the generalized Lapacian operator for the metric g_{rs}. Then it follows immediately from Eq. (4.1) that

$$0 = \int N_\kappa'[\bar{g}_i, \mathcal{K}_\kappa'] \, d^3x. \tag{4.2}$$

Thus the same N_i that satisfy Eqs. (4.1) can satisfy Eqs. (4.2). But these latter equations are just the conditions that the coordinate conditions $\bar{g}_i = 0$ remain satisfied in time. However, if $g_i = x^i$ and $\bar{g}_i = \nabla^2 g_i$ we have immediately that $\bar{g}_i = 0$ so that the conditions that the equations $\bar{g}_i = 0$ and $g_i = x^i$ remain satisfied in time are seen to be directly related to each other as above. Furthermore we see that the Pb matrix $[\bar{g}_i, \mathcal{K}_\kappa']$ will not possess an inverse and that the N_i will be null vectors of this matrix. The difference in the structure of the two Pb matrices is due, in the final analysis, to the fact that the conditions $\bar{g}_i = 0$ contain less information than do the conditions $g_i = x^i$. Thus, if the g_i are scalars, the latter type of conditions determine the coordinate system uniquely, since presumably the values of the g_i are known at each point of the space-time manifold. However, while the related conditions $\bar{g}_i = x^i$ imply the conditions $\bar{g}_i = 0$ the converse is not true without additional information, e.g., boundary conditions. This is reflected in the fact that, if the Pb matrix $[\bar{g}_i, \mathcal{K}_\kappa']$ is singular it means that some linear combination of the $\mathcal{K}$ still vanishes only weakly and can therefore be used as the generator of a subgroup of the original symmetry group generated by the full set of $\mathcal{K}_i$.

To see how effective Hamiltonians that arise from the use of the two different types of coordinate conditions are related to each other we shall consider the special case where we use the Dirac coordinate conditions (2.1) and (3.15) and the variables p, κ, $\bar{g}_{rs}$ and $\bar{p}^{rs}$. In terms of these quantities the Hamiltonian constraint is

$$\mathcal{K}_L \equiv \mathcal{K}_{\text{main}} + 4(c^{rs}\kappa_{,r})_{,s} \approx 0. \tag{4.3}$$

If we were to follow the Dirac procedure we would solve this equation for κ, set its conjugate $2\kappa^{-1}p$ equal to zero which is equivalent to setting p equal to zero

since κ cannot vanish, and taking as the Hamiltonian $\int \mathcal{K} *_{\text{main}} d^3x$. Alternately we could solve the above equation for $4 (e^{rs}\kappa_{,r})_{,s}$. The conjugate to this quantity is $\frac{1}{2}\bar{\nabla}^{-2}(\kappa^{-1}p)$ where $\bar{\nabla}^{-2}$ is the inverse to the Laplacian belonging to the metric $\bar{g}_{rs}$. Since it is not unique we suppose that it is fixed by appropriate boundary conditions. If now the Pb matrix of $\frac{1}{2}\bar{\nabla}^{-2}(\kappa^{-1}p)$ with $\mathcal{K}_L$ is not singular we can require, as a coordinate condition, that

$$-(2\bar{\nabla}^2)^{-1}(\kappa^{-1}p) \approx t. \tag{4.4}$$

We see that this condition implies that $p = 0$ but not conversely.

Now according to ADM the Hamiltonian in the coordinate system in which Eq. (4.4) is valid is just $\{\int 4(e^{rs}\kappa_{,r})_{,s} d^3x\} *$ where as before the star means that the starred quantity is to be expressed in terms of the solution of the constraint equation. But from Eq. (4.3) we see that this quantity is just $\{\int \mathcal{K}_{\text{main}} d^3x\} *$ which is just equal to $H *_{\text{main}}$, the Hamiltonian that follows from the Dirac procedure. A similar discussion can be carried out for other types of coordinate conditions. In general one finds that the Hamiltonian obtained by the ADM procedure is equal to that following Dirac whenever the coordinate conditions used in the two cases are related as in the above case.

V. QUANTIZATION OF GENERAL RELATIVITY

Since all of the canonical schemes were devised by their authors with the idea of using them in the construction of a quantized version of general relativity we would like to conclude this survey with a brief comparison of the several attempts in this direction. Roughly speaking, the aim of Dirac, the author, and ADM was to eliminate redundant degrees of freedom from the theory by the introduction of coordinate conditions. Although they differed in detail it was necessary that they be able to solve the constraint equations for certain functions of the canonical variables, the quantities we have called g_i. To date none of the authors have discovered a set of g_i that would allow one to solve the constraint equations in closed form. Consequently these authors must resort to some sort of iteration scheme to solve the constraint equations for the g_i based upon a weak-field approximation. But it seems to the author that the use of a weak-field iteration procedure destroys just that aspect of general relativity that might be responsible for features which are quantitatively different from those found in conventional quantum field theories, namely its intrinsic nonlinearity.

The approach of BK is essentially different from that of the other authors in that they do not use the coordinate conditions to eliminate degrees of freedom from the theory but merely to construct an observable $\bar{F}$ from every F by means of Eq. (3.27). For this purpose it is not necessary to solve any constraint

equations. In the quantized version of this formulation of the theory one would look for an operator representation for the $\bar{F}$'s which would reproduce the classical commutator algebra between the various $\bar{F}$'s obtained from their Pb's. Since now one has many more observables than degrees of freedom the observables are not all independent of one another and so one has certain consistency conditions to satisfy that are not present when one eliminates degrees of freedom from the theory directly. It is not clear at present whether or not one can satisfy these consistency requirements. If they can be satisfied then the BK procedure would have the advantage over the other schemes of quantization that it does not require a weak-field approximation procedure to obtain its results.