NASA CR 71027
Accession No. 53488-64

Copy No. 51 SID 64-29

ORBITAL TRANSFER BY OPTIMUM THRUST DIRECTION AND DURATION

12 February 1964

Prepared by
Stephen A. Jurovics
SPACE SCIENCES LABORATORY

Approved by

E.R. van Driest
Director

NORTH AMERICAN AVIATION, INC.
SPACE and INFORMATION SYSTEMS DIVISION

FOREWORD

The work presented in this report is an extension of the transfer studies made for the Marshall Space Flight Center, and was supported by MSFC under Contract NAS8-5211 (Satellite Rendezvous Study).

ABSTRACT

20128

A three dimensional derivation is presented of the equations and boundary conditions necessary to determine the minimum fuel orbit transfer path by optimizing the thrust direction and duration. The formulation, known as the Mayer problem in the calculus of variations, yields a two point boundary value problem. A Newton-Raphson method was used to attempt convergence of this two point boundary value problem, but it was found to be inadequate. However, with the final orbit unspecified numerous solutions satisfying the Mayer formulation were generated and then compared with the optimum two-impulse transfer between the same two orbits. This comparison is quite revealing; it shows first, that for the restricted class of orbits examined the optimum two-impulse estimate of velocity increment, or fuel required is very good. Second, it demonstrates that although the optimum departure and arrival points obtained from the impulsive and finite thrust solutions may be quite different, the penalty in using the former for design estimates may be quite minor.

author

CONTENTS

	Page
FOREWORD	ii
ABSTRACT	iii
INTRODUCTION	1
EQUATIONS OF MOTION	2
CONTROL EQUATIONS	6
BOUNDARY CONDITIONS	11
CORNER CONDITIONS	16
EULER-LAGRANGE EQUATIONS	16
ITERATIVE METHOD.	19
NUMERICAL RESULTS	20
CONCLUDING REMARKS	22
APPENDIX A - SOLUTION TO EULER-LAGRANGE EQUATIONS DURING COAST	23
REFERENCES	35

INTRODUCTION

In this report we are concerned with the problem of moving a vehicle between two arbitrary orbits in space. The orbits are assumed to have one planet as a common focus which generates a uniform central gravitational field, and the vehicle is assumed to be capable of thrust direction and on-off control. We present a complete derivation, in three dimensions, of the equations and boundary conditions necessary to determine the minimum-fuel orbit transfer path by optimizing the thrust direction and duration, and the departure and arrival points on the initial and final orbits. The Mayer formulation of the calculus of variations is used.

We turn to optimization procedures for finding the transfer path for three reasons: First, the problem of realistic minimum fuel requirements for space maneuvers is one of extreme importance. Second, for the purposes of design studies based on impulsive transfer, it is necessary to know the error made by the assumption of impulses. Third, the optimization technique gives an organized and general way for finding a transfer path; it is a procedure that is of significance no matter what quantity is to be extremized, since it provides a suitable steering program to accomplish the desired mission.

Selection of the optimization technique is primarily decided by what has been reported in the literature, and the experience of the investigator. Either the indirect method-use of Lagrange multipliers-or the direct method-steepest descent-can be used. Reference (2) reports a successful application of the Mayer formulation to the problem of boosting the maximum payload into orbit with a high thrust engine. Reference (3) also uses the same method successfully on the problem of coplanar orbital transfer with very low thrust engines. Both applications utilized the Newton-Raphson method as the principal iterative technique for solving the two-point boundary value problem. These reports were the main factors in this selection and in the initial approach to the two-point boundary value problem used in this study.

I. EQUATIONS OF MOTION*

The kinetic energy per unit mass is:

$$P = 1/2\ (\dot{r}^2 + r^2\dot{\theta}^2 + r^2\cos^2\theta\ \dot{\phi}^2);\ \text{see Fig. 1.}$$

The potential energy per unit mass is:

$$V = \frac{-\mu}{r}\quad (\mu = K\ M_{earth})$$

The Lagrangian, L = P - V:

$$L = 1/2\ (\dot{r}^2 + r^2\dot{\theta}^2 + r^2\cos^2\theta\ \dot{\phi}^2) + \frac{\mu}{r}$$

The three second-order equations of motion are obtained from:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = Q_i,\quad i = 1\text{-}3,$$

where the q_i are r, θ, and φ. The Q_i are the generalized force and moments due to the thrust, T:

$$Q_r = \frac{T}{m}\cos\psi\cos\nu$$

$$Q_\theta = \frac{T}{m}\ r\sin\psi$$

$$Q_\phi = \frac{T}{m}\ r\cos\psi\sin\nu\cos\theta$$

Thus, the three second-order equations of motion are:

$$\ddot{r} - r\dot{\theta}^2 - r\cos^2\theta\ \dot{\phi}^2 + \frac{\mu}{r^2} = \frac{T}{m}\cos\psi\cos\nu \tag{1}$$

$$\frac{d}{dt}(r^2\dot{\theta}) + r^2\cos\theta\sin\theta\ \dot{\phi}^2 = \frac{T}{m}\ r\sin\psi \tag{2}$$

*See also references 4-6.

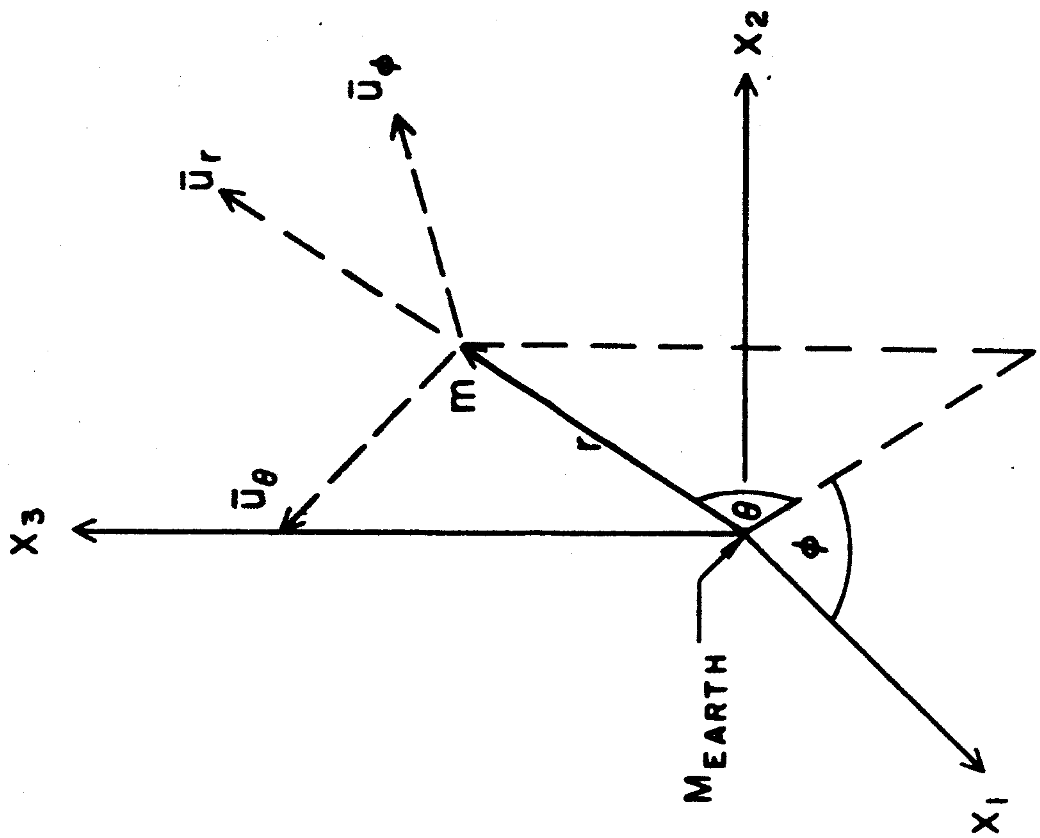


FIG. 1a.

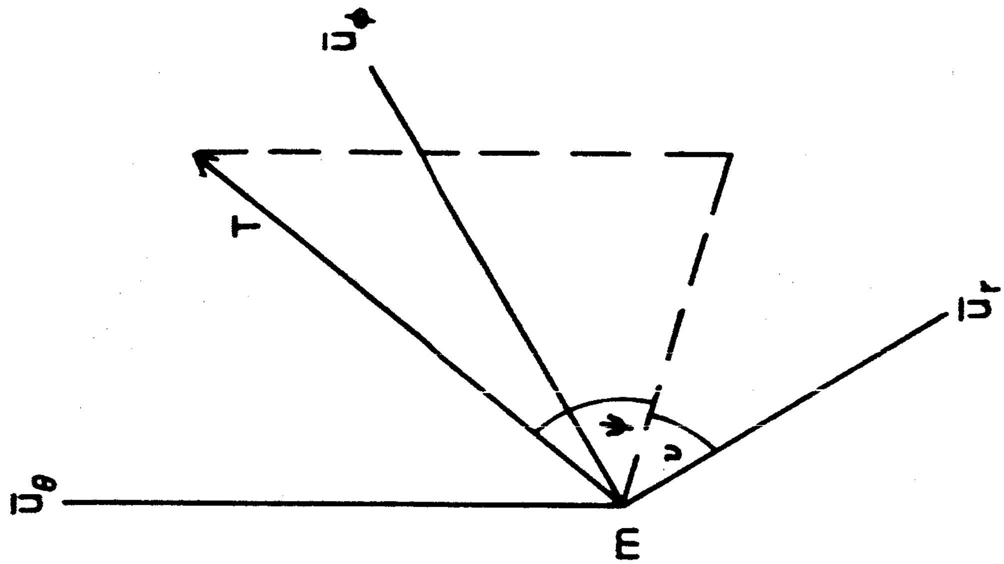


FIG. 1b.

COORDINATE SYSTEMS

$$\frac{d}{dt}(r^2 \cos^2 \theta \dot{\phi}) = \frac{T}{m} r \cos \psi \sin \nu \cos \theta \tag{3}$$

We want the thrust, T, to be either on or off. Hence, we define $T = c\beta$, where c = an effective exhaust velocity, and β = mass flow rate.

$$\text{Check dimensions:} \quad \left|T\right| = F = \frac{ML}{T^2}; \quad \left|c\beta\right| = \frac{L}{T} \cdot \frac{M}{T}$$

Expanding (2) and (3), and noting that we cannot have $\theta = \pm \frac{\pi}{2}$, we get the following seven first-order equations of motion, where new variables ρ, x, y are defined as indicated:

$$w_1 \equiv \dot{r} - \rho = 0 \tag{4}$$

$$w_2 \equiv \dot{\theta} - x = 0 \tag{5}$$

$$w_3 \equiv \dot{\phi} - y = 0 \tag{6}$$

$$w_4 \equiv \dot{\rho} - r x^2 - r \cos^2\theta\, y^2 + \frac{\mu}{r^2} - \frac{c\beta}{m} \cos \psi \cos \nu = 0 \tag{7}$$

$$w_5 \equiv \dot{x} + \frac{2 \rho x}{r} + \cos \theta \sin \theta\, y^2 - \frac{c\beta}{mr} \sin \psi = 0 \tag{8}$$

$$w_6 \equiv \dot{y} - 2 \tan \theta \cdot x y + \frac{2 \rho y}{r} - \frac{c\beta \cos \psi \sin \nu}{mr \cos \theta} = 0 \tag{9}$$

$$w_7 \equiv \dot{m} + \beta = 0 \tag{10}$$

The optimum path (for min. fuel expenditure) that is to be found must satisfy the equations of motion, and this is represented by constraints, $w_i = o$, $i = 1\text{-}7$.

There is one further constraint to be added: We require the thrust to be on or off--no throttling. This is expressed by:

$$w_8 \equiv \beta(\beta - \beta_{max.}) = 0$$

Hence, problem variables are:

Dependent			Independent
Dynamic and kinematic		Control	
r	ρ	ψ	t
θ	x	ν	
ϕ	y	β	
m			

Denoting all dependent variables by z, the constraints can be expressed as:

$$w_i = \dot{z}_i - f_i(z_j) = 0 \quad i = 1 - 8, \; j = 1 - 10$$

II. DERIVATION OF OPTIMIZATION PROBLEM*

A. Since the quantity that we want to minimize only enters in the boundary conditions (we use the Mayer formulation of the calculus of variations), let us first obtain the Euler-Lagrange equations associated with the control variables ψ, ν, β.

$$F = \lambda_i(t)\, w_i(\dot{z}_i, z_j)$$

Require:

$$\frac{d}{dt}\frac{\partial F}{\partial \dot{z}_i} - \frac{\partial F}{\partial z_i} = 0 \qquad z_i = \nu, \psi, \beta$$

$$\frac{\partial F}{\partial \dot{z}_i} = 0 \quad \therefore \quad \frac{\partial \lambda_j w_j}{\partial z_i} = \lambda_j \frac{\partial w_j}{\partial z_i} = 0$$

1.) $z = \nu$

$$\lambda_4\left(\frac{c\beta}{m}\cos\psi\sin\nu\right) + \lambda_6\left(\frac{-c\beta\cos\psi\cos\nu}{mr\cos\theta}\right) = 0$$

$$\frac{c\beta}{m}\left[\lambda_4\cos\psi\sin\nu - \lambda_6\frac{\cos\psi\cos\nu}{r\cos\theta}\right] = 0$$

If $\beta = 0$, then $T = 0$ and ψ and ν have no meaning, and we simply compute the $\lambda_i(t)$ by a closed-form solution which is given in Appendix A. For $\beta \neq 0$, and c and $m \neq 0$ for all t:

$$\cos\psi\left[\lambda_4\sin\nu - \frac{\lambda_6\cos\nu}{r\cos\theta}\right] = 0$$

$$\therefore \text{Either } \psi = \pm\frac{\pi}{2}, \text{ or } \tan\nu = \frac{\lambda_6}{\lambda_4 r\cos\theta} \tag{11}$$

$$\therefore \sin\nu = \frac{\lambda_6}{\pm D_\nu}, \quad \cos\nu = \frac{\lambda_4 r\cos\theta}{\pm D_\nu} \qquad (12), (13)$$

*See also references (7)-(9).

where

$$D_\nu = \sqrt{\lambda_6^{\,2} + \lambda_4^{\,2}\, r^2 \cos^2\theta}$$

Note: From equations (7) and (9), that if $\psi = \pm\pi/2$, the ν terms drop out, as expected on physical grounds.

2.) $z = \psi$

$$\lambda_4 \left[\frac{c\beta}{m} \sin\psi \cos\nu\right] + \lambda_5 \left[\frac{-c\beta}{mr} \cos\psi\right] + \lambda_6 \left[\frac{c\beta \sin\psi \sin\nu}{mr \cos\theta}\right] = 0$$

If $\beta = 0$, then the argument is the same as above. For $\beta \neq 0$:

$$\sin\psi \left[r\lambda_4 \cos\nu + \frac{\lambda_6 \sin\nu}{\cos\theta}\right] - \lambda_5 \cos\psi = 0$$

Insert (12) and (13) for $\sin\nu$ and $\cos\nu$, and collect terms:

$$\sin\psi \left[\frac{\pm D_\nu}{\cos\theta}\right] - \lambda_5 \cos\psi = 0$$

$$\frac{\sin\psi}{\cos\psi} = \tan\psi = \frac{\lambda_5 \cos\theta}{\pm D_\nu} \tag{14}$$

3.) $z = \beta$

$$\lambda_4 \left[\frac{-c}{m} \cos\psi \cos\nu\right] + \lambda_5 \left[\frac{-c}{mr} \sin\psi\right] + \lambda_6 \left[\frac{-c\cos\psi \sin\nu}{mr\cos\theta}\right]$$

$$+ \lambda_7 (1) + \lambda_8 \left[(\beta - \beta_{max}) + \beta\right] = 0 \tag{15}$$

This equation yields λ_8, but it is of no significance in this problem.

B. To reconcile the sign ambiguities in 1.) and 2.), above, and to determine when $\beta = 0$, $\beta = \beta_{max.}$, we turn to the Weierstrass necessary condition.

This condition states that for a minimum, $E \geq 0$:

$$E = F(Z_i^*, \dot{Z}_i^*) - F(Z_i, \dot{Z}_i) - \sum_i (\dot{Z}_i^* - \dot{Z}_i) \frac{\partial F}{\partial \dot{Z}_i}$$

Z_i^* differs from Z_i by a <u>finite</u>, but admissible amount.

The only variables which admit of such strong variation are ν, ψ, and β, where, for example:

$$\psi = \psi \text{ or } \psi + \pi; \quad \nu = \nu \text{ or } \nu + \pi; \quad \beta = 0 \text{ or } \beta_{max}.$$

Now, the third term in E is identically zero since there are no constraints involving $\dot{\psi}$, $\dot{\nu}$, $\dot{\beta}$.

$$\therefore \; E = \lambda_i(t)\, w_i(Z_j^*, \dot{Z}_i^*) - \lambda_i(t)\, w_i(Z_j, \dot{Z}_i) \geq 0 \quad j, i = 1\text{-}8$$

$$E = \lambda_i(t)\left[\dot{Z}_i^* - f_i(Z_j^*)\right] - \lambda_i(t)\left[\dot{Z}_i - f_i(Z_j)\right] \geq 0$$

$$E = \lambda_i(t)\, f_i(Z_j) - \lambda_i(t)\, f_i(Z_j^*) \geq 0$$

or

$$\lambda_i(t)\, f_i(Z_j) \geq \lambda_i(t)\, f_i(Z_j^*) \tag{16}$$

Applying (16) we get:

$$\lambda_4\left[\frac{c\beta}{m}\cos\psi\cos\nu\right] + \lambda_5\left[\frac{c\beta}{mr}\sin\psi\right] + \lambda_6\left[\frac{c\beta\cos\psi\sin\nu}{mr\cos\theta}\right]$$

$$+ \lambda_7(-\beta) + \lambda_8\left(-\beta\left[\beta - \beta_{max}\right]\right) \geq \lambda_4\left[\frac{c\beta^*}{m}\cos\psi^*\cos\nu^*\right]$$

$$+ \lambda_5\left[\frac{c\beta^*}{mr}\sin\psi^*\right] + \lambda_6\left[\frac{c\beta^*\cos\psi^*\sin\nu^*}{mr\cos\theta}\right] + \lambda_7(-\beta^*)$$

$$+ \lambda_8\left[-\beta^*(\beta^* - \beta_{max})\right]$$

Note, first, that the λ_8 term $\equiv 0$.

Now, factoring out a β and β^* yields, in the notation of ref. (8):

$$\beta k - \beta^* k^* \geqslant 0$$

where

$$k = \frac{c}{m}\left(\lambda_4 \cos\psi \cos\nu + \frac{\lambda_5}{r} \sin\psi + \frac{\lambda_6 \cos\psi \sin\nu}{r\cos\theta}\right) - \lambda_7$$

For $k = k^*$; $\beta \neq \beta^*$; $k(\beta - \beta^*) \geqslant 0$

If $k > 0$, then $\beta > \beta^* \Rightarrow \beta = \beta_{max}$ (17a)

If $k < 0$, then $\beta < \beta^* \Rightarrow \beta = 0$ (17b)

Thus, we have the engine on-off criteria.

For $\beta = \beta^*$, $k \neq k^*$;

$$\lambda_4 \cos\psi \cos\nu + \frac{\lambda_5}{r} \sin\psi + \frac{\lambda_6 \cos\psi \sin\nu}{r\cos\theta} \geqslant \lambda_4 \cos\psi^* \cos\nu^* + \frac{\lambda_5}{r}\sin\psi^* + \frac{\lambda_6 \cos\psi^*}{r\cos\theta}\sin\nu^* \tag{18}$$

a.) $\psi = \psi^*$; $\nu \neq \nu^* \Longrightarrow \nu = \nu$ or $\nu + \pi\ (= \nu^*)$

Hence, (18) becomes

$$\lambda_4 \cos\psi \cos\nu + \frac{\lambda_6 \cos\psi \sin\nu}{r\cos\theta} \geqslant 0$$

Using (12) and (13):

$$\cos\psi \frac{\lambda_4^2 r^2 \cos^2\theta + \lambda_6^2}{\pm D_\nu\, r\cos\theta} \geq 0$$

or,

$$\cos\psi\left[\frac{\pm D_\nu}{r\cos\theta}\right] \geq 0$$

Since $r > 0$, and $\frac{-\pi}{2} < \theta < \frac{\pi}{2}$, the above yields

$$\left.\begin{array}{l} + D_\nu,\ \cos\psi > 0 \\ - D_\nu,\ \cos\psi < 0 \end{array}\right] \tag{19}$$

Physically, we will most probably be confined to

$$\frac{-\pi}{2} < \psi < \frac{\pi}{2} \Rightarrow + D_\nu .$$

b.) $\nu = \nu^*;\ \psi \neq \psi^* \Rightarrow \psi = \psi$ or $\psi + \pi\ (= \psi^*)$

From (14) and (19):

$$\tan\psi = \frac{\lambda_5\cos\theta}{\pm D_\nu}$$

$$\sin\psi = \frac{\lambda_5\cos\theta}{\pm D_\psi},\quad \cos\psi = \frac{\pm D_\nu}{\pm D_\psi} \tag{20}$$

where

$$D_\psi = \sqrt{D_\nu^2 + \lambda_5^2\cos^2\theta}$$

From (18) again:

$$\lambda_4\cos\psi\cos\nu + \frac{\lambda_5\sin\psi}{r} + \frac{\lambda_6\cos\psi\sin\nu}{r\cos\theta} \geq 0$$

Substituting (12), (13), and (20) and clearing yields:

$$\frac{\pm D_\psi}{r \cos \theta} \geq 0$$

Again, since $r \cos \theta > 0$, this requires $+ D_\psi$ (21)

C. There is a first integral, since the Lagrangian, F, does not involve time explicitly.

$$\frac{\partial F}{\partial \dot{z}_k} \dot{z}_k = C; \quad \frac{\partial \lambda_i(t) w_i (\dot{z}_i, z_j)}{\partial \dot{z}_k} \dot{z}_k = C$$

$$\lambda_i(t) \frac{\partial (\dot{z}_i - f_i (z_j))}{\partial \dot{z}_k} \dot{z}_k = C$$

Hence,

$$\lambda_1 \dot{r} + \lambda_2 \dot{\theta} + \lambda_3 \dot{\phi} + \lambda_4 \dot{\rho} + \lambda_5 \dot{x} + \lambda_6 \dot{y} + \lambda_7 \dot{m} = C \tag{22}$$

D. Boundary Conditions

The boundary conditions to be applied come from two sources: Those implied by the physics of the problem, and the remainder from the transversality condition

$$\left[d G + (F - \frac{\partial F}{\partial \dot{z}_k} \dot{z}_k) dt + \frac{\partial F}{\partial \dot{z}_k} dz_k \right]_0^T = 0, \tag{23}$$

where G is the function to be minimized.

1.) To clarify the derivation of the boundary conditions, let us first consider that the two orbits are coplanar. We reiterate the problem: Find the minimum fuel path to transfer between two coplanar orbits by optimizing the thrust direction (ν) and duration ("Bang-bang" control). The departure and arrival points on the initial and final orbits are not specified, but the total time of transfer is specified. The geometry is shown in Figure 2.

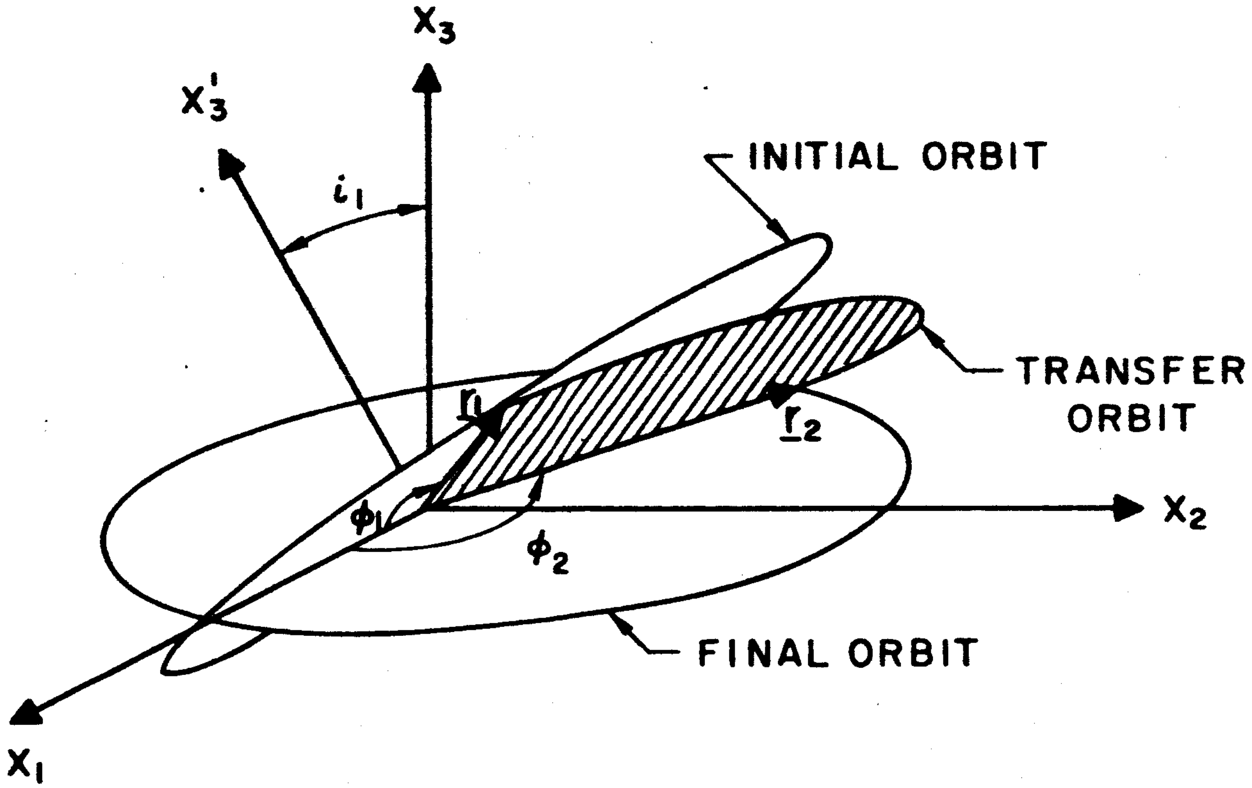


FIG. 2. TRANSFER GEOMETRY

Thus, we have a system of 10 first-order differential equations for the variables:

$$r, \phi, \rho, y, m, \lambda_1, \lambda_3, \lambda_4, \lambda_6, \lambda_7$$

This system thus requires 10 boundary conditions. The seven specified by the physics of the problem are: (i = initial, f = final).

$$p_i \text{ (or } h_i), \; e_i \text{ (or } E_i), \; \omega_i, \; m_i$$

$$p_f \text{ (or } h_f), \; e_f \text{ (or } E_f), \; \omega_f \tag{24}$$

p, e, ω are semi-latus rectum, eccentricity, and argument of perigee, respectively. h and E are angular momentum and total energy.

We derive the three remaining conditions from Equation (23) and thus we are obliged finally to select the quantity to be optimized. Since we wish to compare our results with minimum impulsive orbital transfer, let us consider minimizing the characteristic velocity,

$$G = c \ln \frac{m_i}{m_f}$$

Equation (23) becomes, utilizing (22);

$$\left[\frac{-c}{m_f} + \lambda_7\right] dm_f + \left[\frac{c}{m_i} - \lambda_7\right] dm_i + \Big[- C\, dt + \lambda_1\, dr$$

$$+ \lambda_3\, d\phi + \lambda_4\, d\rho + \lambda_6\, dy\Big]_0^T = 0 \tag{25}$$

Since m_i is specified, $dm_i = 0$. Also, $dt\big]_0^T = 0$, which implies C = unknown. Thus,

$$\lambda_7 = \frac{c}{m} \text{ at } t = T \tag{26}$$

This is our eighth boundary condition. The remaining two come from

$$\left[\lambda_1\, dr + \lambda_3\, d\phi + \lambda_4\, d\rho + \lambda_6\, dy\right]_0^T = 0, \tag{27}$$

where we use orbit equations to relate the differentials in terms of the given parameters p, e, and w. To do this we note:

$$r = \frac{p}{1 + e\cos(\phi - \omega)} \equiv f(\phi) \tag{28}$$

$$dr = f'(\phi)\, d\phi$$

$$E/m = 1/2(\dot{r}^2 + r^2\dot{\phi}^2) - \frac{\mu}{r} \tag{29}$$

$$d(E/m) = \dot{r}d\dot{r} + \dot{\phi}^2 r dr + r^2\dot{\phi}d\dot{\phi} + \frac{\mu}{r^2}dr = 0$$

$$h/m = r^2\dot{\phi} \tag{30}$$

$$d(h/m) = 2r\dot{\phi}dr + r^2 d\dot{\phi} = 0$$

Expressing all the differentials in terms of $d\phi$, the two boundary conditions then are

$$f'(\phi)\left[\lambda_1 - \frac{\lambda_4}{\dot{r}}\left(- r\dot{\phi}^2 + \frac{\mu}{r^2}\right) - \frac{2\lambda_6\dot{\phi}}{r}\right] + \lambda_3 = 0, \text{ at } t = 0, T.$$

These two equations can be put in a more revealing form. Substituting $\dot{\rho}$ and $\dot{y}$ from the equations of motion, we find

$$\dot{r}\lambda_1 + \dot{\phi}\lambda_3 + \dot{\rho}\lambda_4 + \dot{y}\lambda_6 = \frac{c\beta}{m}\left[\frac{\lambda_6 \sin\nu}{r} + \lambda_4\cos\nu\right] \text{ at } t = 0, T$$

Utilizing Equation 17 from p. 9, with $\psi \equiv 0$ and $\lambda_5 \equiv 0$, we see that the right side of the above equation is

$$\beta k + \beta\lambda_7, \text{ or}$$

$$\dot{r}\lambda_1 + \dot{\phi}\lambda_3 + \dot{\rho}\lambda_4 + \dot{y}\lambda_6 + \dot{m}\lambda_7 = \beta k \text{ at } t = 0, T \tag{31}$$

This thus identifies the constant, C (Equation (22)) as equal to βk at the end points.

Further, if $C \neq 0$ at $t = 0$, (31) implies that $k(0) = k(T)$.

2.) We can now proceed to derive, rather succintly, the boundary conditions for the three dimensional case. The problem requires fourteen boundary conditions since there are fourteen first order differential conditions for the variables:

$$r, \theta, \phi, \rho, x, y, m, \lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6, \lambda_7.$$

The physics of the problem now yields eleven conditions while the transversality (Equation (23)) yields three, exactly as in the planar case. The additional four physical constraints are that the vehicle's position and velocity are to be in the specified initial and final planes.

We list the fourteen conditions in terms of their origin:

(a) From the final point ($t = T$), there are five: By choosing the final plane to have zero inclination the two additional constraints at the final point are simply $\theta(T) = 0$ and $\dot{\theta}(T) = 0$. The other three are Equations (28), (29) and (30) applied to the final point.

(b) From the initial point ($t = 0$), there are six: One of the six is the specification of initial mass, while five are orbit equations. The initial orbital plane is taken to have an inclination i and to have its ascending node on the x_1 axis as in Figure 2. The departure point angle called ϕ_1 in Figure 2 is replaced so that ϕ represents the angle in the x_1, x_2 plane as in Figure 1. The five orbital equations may be taken as: Equations (28), (29), (30),

$$\sin \phi = \tan \theta \cot i, \tag{33}$$

and

$$y r^2 \cos^2 \theta = \frac{h}{m} \cos i. \tag{34}$$

(c) From the transversality condition, there are three:

$$\lambda_7 = \frac{c}{m} \text{ at } t = T \tag{35}$$

is obtained exactly as before. The remaining two equations are:

$$\left[\lambda_1\, dr + \lambda_2\, d\theta + \lambda_3\, d\phi + \lambda_4\, d\rho + \lambda_5\, dx + \lambda_6\, dy \right]_{t = 0} = 0 \tag{36}$$

and

$$\left[\lambda_1\, dr + \lambda_3\, d\phi + \lambda_4\, d\rho + \lambda_6\, dy \right]^{t = T} = 0 \tag{37}$$

In addition it should be pointed out that just as in the planar case Equation (36) and Equation (37) are equivalent to

$$\beta(0)\, k\,(0) = C \tag{38}$$

$$\beta(T)\, k\,(T) = C \tag{39}$$

Finally, for use in computation it must be indicated that equation (36) along with the total differentials of the five orbit equations (28, 29, 30, 33, and 34) constitute a set of six homogenous equations, the determinant of whose coefficients is the required relationship. This is the generalization of Equation (31) for the initial point. For the final point the generalization is the same as in the planar problem.

E. Corner Conditions

The points at which the thrust goes on or off give rise to discontinuities in the $\dot{z}_k$. The mathematical criterion needed to join different positions of the extremal arc is supplied by the Erdmann-Weierstrass corner condition:

$$\left(\frac{\partial F}{\partial \dot{z}_k}\right)_- = \left(\frac{\partial F}{\partial \dot{z}_k}\right)_+$$

or

$$\left(\lambda_k\right)_- = \left(\lambda_k\right)_+ , \quad k = 1\text{-}7 \tag{40}$$

$$\left[- F + \frac{\partial F}{\partial \dot{z}_k}\, \dot{z}_k\right]_- = \left[- F + \frac{\partial F}{\partial \dot{z}_k}\, \dot{z}_k\right]_+$$

or,

$$C_- = C_+ \tag{41}$$

We observe that any of the seven conditions which comprise (40) would not apply if the value of the physical variable were specified at the discontinuity. Similarly, (41) would not apply if the time of the discontinuity were specified.

F. Euler-Lagrange Equations

Here we write down the differential equations for the Lagrange multipliers, which come from the Euler necessary condition in the calculus of variations:

$$\frac{d}{dt}\left[\frac{\partial F}{\partial \dot z_k}\right] - \frac{\partial F}{\partial z_k} = 0; \quad z_k = r,\ \theta,\ \phi,\ \rho,\ x,\ y,\ m \tag{42}$$

$$F = \lambda_j w_j = \lambda_j(t)\left[\dot z_j - f_j(z_\ell)\right]$$

$$\frac{d}{dt}\left(\lambda_j(t)\,\delta_{jk}\right) = -\lambda_j \frac{\partial f_j}{\partial z_k}(z_\ell)$$

$$\dot\lambda_k = -\lambda_j \frac{\partial f_j}{\partial z_k}(z_\ell) \tag{43}$$

Using equations (4) - (10), equation (43) yields:

$$\dot\lambda_1 = -\lambda_4\left[x^2 + y^2\cos^2\theta + \frac{2\mu}{r^3}\right] - \frac{\lambda_5}{r^2}\left[2\rho x - \frac{c\beta\sin\psi}{m}\right]$$

$$-\frac{\lambda_6}{r^2}\left[2\rho y - \frac{c\beta\cos\psi\sin\nu}{m\cos\theta}\right] \tag{44}$$

$$\dot\lambda_2 = -\lambda_4(-2ry^2\cos\theta\sin\theta) + \lambda_5 y^2\cos 2\theta$$

$$-\lambda_6\left[2xy\sec^2\theta + \frac{c\beta}{mr}\cos\psi\sin\nu\tan\theta\sec\theta\right] \tag{45}$$

$$\dot\lambda_3 = 0 \tag{46}$$

$$\dot\lambda_4 = -\lambda_1 + \frac{2\lambda_5 x}{r} + \frac{2\lambda_6 y}{r} \tag{47}$$

$$\dot\lambda_5 = -\lambda_2 - 2rx\lambda_4 + \frac{2\lambda_5\rho}{r} - 2\lambda_6 y\tan\theta \tag{48}$$

$$\dot{\lambda}_6 = -\lambda_3 - 2ry\,\lambda_4 \cos^2\theta + 2\,\lambda_5\, y \cos\theta \sin\theta - 2\,\lambda_6 \left[x \tan\theta - \frac{\rho}{r}\right] \quad (49)$$

$$\dot{\lambda}_7 = \frac{\beta}{m}\left[\frac{c}{m}\,\lambda_4 \cos\psi \cos\nu + \frac{c\,\lambda_5 \sin\psi}{mr} + \frac{c\,\lambda_6 \cos\psi \sin\nu}{mr\cos\theta}\right] \quad (50)$$

or,

$$\dot{\lambda}_7 = \frac{\beta}{m}\left[k + \lambda_7\right],$$

from Section II-B-1.

III. ITERATIVE METHOD

The equations (4) - (10) and (44) - (50), plus the control equations for the switching function, k, and the steering angles, ψ and ν, are a set of differential and algebraic equations whose boundary values at $t = 0$ and $t = T$ must meet the specified conditions at those two points. We are thus faced with the well-known two-point boundary value problem. The Newton-Raphson method, and a "Matrix Modification" technique were selected as the first iterative techniques to attempt convergence of the two-point boundary value problem. Both these methods are fully explained in reference (2), and only a brief description of the convergence characteristics of this method on this problem will be given here.

The iterative techniques have so far been only applied to the coplanar case because it was felt that until a fast and reliable method was available for that problem it was rather hopeless to tackle the three-dimensional case. Reference (3) reported success with this technique for low-thrust engines, but in this case when the thrust-to-weight ratio (T/W) is between one and ten, it does not seem to be able to handle the problem. One comment about a T/W of ten is in order; the iterative procedure begins by first obtaining the optimum two-impulse transfer. We then have the optimum departure and arrival points, velocity increment necessary, time for the transfer, and initial and final thrust direction. Hence, if we assume an engine with a $T/W = 10$, we have almost an impulsive vehicle, and if the final time is set equal to the impulsive time for transfer plus the time necessary to burn fuel yielding a velocity increment equal to or slightly greater than the impulsive solution, we can expect that the finite-thrust solution will be very close to the impulsive solution in all respects. Once this one has been obtained, we can then proceed to decrease the T/W to 8, 6, 4, etc., obtaining solutions for all these, until we are down to precisely the engine in which we are interested.

Now, the Newton-Raphson method applied to the coplanar problem has the behavior of converging on the transversality condition first, equation (31), and then keeping that satisfied, move very slowly towards meeting the orbit conditions, p, e, and ω. The conclusion, so far, is that the method is inadequate for this complex and sensitive problem. However, several modifications of the method, and its use, are being studied, and it may yet prove capable. If not, other iterative methods for handling the two-point boundary value problem are being studied, and will be tried if the Newton-Raphson proves conclusively unsatisfactory.

NUMERICAL RESULTS

In the introduction to this paper, three reasons for turning to optimization procedures for the solution of the minimum fuel orbital transfer problem were given. This section gives an indication of the answer to the second statement; i.e., the comparison with two-impulse orbital transfer. The answer is not conclusive since the switching function time history was restricted to one coast period, and the second burn period was terminated as soon as

$$k\,(t) = k\,(0)\ ;\ \text{see equation (31)}.$$

Thus, a rather restricted class of initial and final orbits was considered; all orbit pairs intersected, and in most cases the intersection was quite shallow.

The following table presents some of the results gathered from this restricted comparison. The first column is the thrust-to-weight ratio at the initial orbit; for example, a vehicle of 1000 slugs mass, with fuel-flow rate, β, of 1 slug/sec., has a specific impulse of 300 sec. if the $(T/W)_i = .7118$, at a distance of 6058 miles from the center of the earth. In the second column, the percentage difference in velocity increment is given; $V_F = c \ln \frac{m_i}{m_f}$, and V_I is equal to the total velocity increment from the two-impulse minimization. Total $\Delta\phi$, in the third column was computed as follows:

$$\text{Total } \Delta\phi = \left|\phi_{i,F} - \phi_{i,I}\right| + \left|\phi_{f,F} - \phi_{f,I}\right|$$

Thus it represents the total deviation in the departure and arrival points between this finite thrust solution—subscript F—and the impulsive solution—subscript I. The last column gives an approximation to the penalty in velocity increment, or fuel, if the departure and arrival point of the impulsive solution is used instead of the points specified by the finite thrust solution. This estimate was obtained in the following way: Reference (1) presents contour maps of minimum transfer velocity on a ϕ_i, ϕ_f plot. By differencing the value at $(\phi_{i,I}, \phi_{f,I})$ with the value at $(\phi_{i,F}, \phi_{f,F})$, and dividing by V_I, we obtain an estimate of the penalty in velocity, or fuel, that would be incurred. We emphasize that this is an approximation; but in view of the results in the second column, it is probably a reasonable one.

Finite Thrust Versus Two Impulse Comparison

$(T/W)_i$	$\frac{(V_F - V_I)}{V_I} 10^2$	Total $\Delta\phi$, deg.	$\frac{\Delta V}{V_I} 10^2$ Penalty
10	.086	20.0	.135
10	.136	26.3	.410
8	.203	21.1	.352
8	.236	27.9	.401
6	.143	18.7	.365
6	.501	29.8	.685
4	.278	34.4	.874
4	.354	32.2	.247
2	.224	24.8	.611
2	.293	28.7	.631
.7118	.095	72.8	1.89
.7118	.194	13.0	.407

We observe from the first and second columns, that if orbit transfers with realistic vehicles are restricted to be completed in one orbit, then the time constraint—obtained from the impulsive solution—placed upon these finite thrust solutions is also realistic, and, ipso facto, the fuel requirement for the transfer obtained from the two-impulse solution is a very good estimate of that which would actually be needed. This is, of course, with the assumption that the finite thrust transfer vehicle departs and arrives at the proper point, for we see that the discrepancies in ϕ_i and ϕ_f can be quite sizable. However, from the fourth column, we note that the penalty in fuel, or velocity, for using the optimum ϕ_i, ϕ_f from the impulsive solution rather than those specified by the finite thrust solution may be quite minor; however, this was a rather restricted comparison, and a good deal more numerical results are necessary before any even tentative generalizations in this direction are possible.

CONCLUDING REMARKS

The Mayer formulation of the calculus of variations has been used to derive, in three dimensions, the equations and boundary conditions necessary to determine the minimum fuel orbit transfer path by optimizing the thrust direction and duration, and the departure and arrival points on the initial and final orbits. The closed-form solution to the Euler-Lagrange equations, which apply along the coast arc has also been derived, rather explicitly, and has been verified by some of the numerical integrations indicated in the preceding section.

The numerical results section is considerably leaner than desired. One conclusion, therefore, is that the multivariable Newton-Raphson iteration technique is inadequate for this complex and sensitive problem. This is a useful, albeit frustrating result. A more gratifying result is the favorable comparison of two-impulse and finite thrust orbit transfer solutions. Restrictive as it is, it should be of interest to design personnel, for it is the first proven indication, to this writer's knowledge, of the real utility of the impulsive solution and how much a design based on it differs from the optimum.

It is hoped, and rather optimistically felt, that one of the iteration techniques currently under study for solving the two-point boundary value problem will be effective in this endeavor. With this accomplished, an unrestricted variety of problems with an equally unrestricted genus of propulsion systems will be able to be expediently solved. The two-impulse solution is obviously not universally a good estimate for design, or even applicable. When low-thrust ion or nuclear propulsion systems are being considered, and interplanetary transfers are being studied, it will be distinctly advantageous, if not imperative, that the capability begun herein be a reality.

APPENDIX A

SOLUTION TO EULER-LAGRANGE EQUATIONS DURING COAST

With the thrust off ($\beta = 0$), the equations of motion are

$$\ddot{r} = r\dot{\phi}^2 - \frac{\mu}{r^2} \tag{A1}$$

$$r\ddot{\phi} = -2\,\dot{r}\dot{\phi} \tag{A2}$$

for coplanar orbits. The solution to these involves four arbitrary constants; p_c, e_c, ω_c - the elements of the coast orbit - and ϕ_c, the angle at which the coast is begun.

The Euler-Lagrange equations are:

$$\dot{\lambda}_4 = -\lambda_1 + \frac{2\lambda_6\dot{\phi}}{r} \tag{A3}$$

$$\dot{\lambda}_6 = -\lambda_3 - 2\,\lambda_4\, r\dot{\phi} + \frac{2\lambda_6\dot{r}}{r} \tag{A4}$$

$$\lambda_1 = \frac{1}{\dot{r}}\left[C - \lambda_3\dot{\phi} - \lambda_4\ddot{r} - \lambda_6\ddot{\phi}\right] \tag{A5}$$

$$\dot{\lambda}_7 = 0;\ \lambda_7 = \lambda_7 \text{ at beginning of coast} \tag{A6}$$

First, change the independent variable from t to ϕ:

$$\lambda'_4\,\dot{\phi} = -\lambda_1 + \frac{2\lambda_6\dot{\phi}}{r} \tag{A7}$$

$$\lambda'_6\,\dot{\phi} = -\lambda_3 - 2r\dot{\phi}\lambda_4 + \frac{2\dot{r}}{r}\lambda_6 \tag{A8}$$

Putting (A5) in (A7), and collecting terms, yields

$$\lambda'_4 \dot{\phi} \dot{r} - \lambda_4 \ddot{r} - \lambda_3 \dot{\phi} + C - \lambda_6 \left(\frac{2\dot{r}\dot{\phi}}{r} + \ddot{\phi} \right) = 0 \qquad \text{(A9)}$$

The solution to (A1) and (A2) is given by

$$r^2 \dot{\phi} = h = \sqrt{\mu p_c}$$

$$r = \frac{p_c}{1 + e_c \cos(\phi - \omega_c)}$$

We find $\dot{r}$ by

$$\dot{r} = \frac{r^2}{p_c} \dot{\phi}\, e_c \sin(\phi - \omega_c) = \frac{h\, e_c}{p_c} \sin(\phi - \omega_c)$$

From (A2), $\frac{2\dot{r}\dot{\phi}}{r} + \ddot{\phi} = 0$; thus (A9) becomes

$$\lambda'_4 - \frac{\lambda_4 \ddot{r}}{\dot{\phi}\dot{r}} = \frac{\lambda_3}{\dot{r}} - \frac{C}{\dot{\phi}\dot{r}} \qquad \text{(A10)}$$

Defining true anomaly as $\theta \equiv \phi - \omega_c$, and using θ as the independent variable, we get upon substituting the equations of motion solution:

$$\frac{d\lambda_4}{d\theta} - \lambda_4 \cot\theta = \frac{\lambda_3 p}{h\, e \sin\theta} - \frac{C p^3}{h^2 e \sin\theta \,[1 + e\cos\theta]^2} \qquad \text{(A11)*}$$

where the subscript c is now omitted.

Substituting the orbit solution in equation (A8) we get

$$\frac{d\lambda_6}{d\theta} - \frac{\lambda_6\, 2 e \sin\theta}{1 + e\cos\theta} = \frac{-\lambda_3 p^2}{h\,[1 + e\cos\theta]^2} - \frac{2 p \lambda_4(\theta)}{1 + e\cos\theta} \qquad \text{(A12)}$$

---

*We note the singularity in this equation at $\theta = 0, \pi$, and that the limit approaches $\pm\infty$ on opposite sides of the singularity; the handling of this is discussed below.

We obtain the solution to (A11) first. The homogeneous equation is

$$\int \frac{d\lambda_4}{\lambda_4} = \int \cot\theta \, d\theta$$

$$\lambda_4 = K_1 \sin\theta \qquad \text{(A13)}$$

Applying variation of constants, we insert (A13) into (A11), letting $K_1 = K_1(\theta)$.

$$K_1'(\theta) = \frac{C_1}{\sin^2\theta} - \frac{C_2}{\sin^2\theta\,[1 + e\cos\theta]^2}$$

where $C_1 \equiv \dfrac{\lambda_3 p}{he}$, $C_2 \equiv \dfrac{Cp^3}{h^2 e}$

$$K_1(\theta) = -C_1 \cot\theta + C_3 - C_2 \int \frac{d\theta}{\sin^2\theta\,[1 + e\cos\theta]^2} \qquad \text{(A14)}$$

Letting $u = [1 + e\cos\theta]^{-2}$, $dv = \csc^2\theta\, d\theta$, we get

$$\int u\,dv = -\cot\theta\,[1 + e\cos\theta]^{-2} + 2e\int \frac{\cos\theta\, d\theta}{[1 + e\cos\theta]^3}$$

Using ref. (10), we find

$$2e\int \frac{\cos\theta\, d\theta}{[1 + e\cos\theta]^3} = \frac{e}{(1 - e^2)}\left[\frac{\sin\theta}{(1 + e\cos\theta)^2} + \int \frac{[-2e + \cos\theta]\, d\theta}{[1 + e\cos\theta]^2}\right]$$

Multiple use of #317 and #309 in ref. (10) yields

$$\int \frac{d\theta}{(1 + e\cos\theta)^2} = \frac{1}{(1 - e^2)}\left[\frac{-e\sin\theta}{1 + e\cos\theta} + \frac{2}{\sqrt{1 - e^2}}\tan^{-1}\frac{\sqrt{1 - e^2}\,\tan\frac{1}{2}\theta}{1 + e}\right]$$

where $-\pi < \theta < \pi$ and $0 \leq e < 1$ - elliptical transfer orbits only.

Again, using #315 and #309 we obtain

$$\int \frac{\cos\theta \, d\theta}{(1 + e\cos\theta)^2}$$

Collecting terms we get:

$$\int \frac{d\theta}{\sin^2\theta\,[1 + e\cos\theta]^2} = -\cot\theta\,[1 + e\cos\theta]^{-2}$$

$$+ \frac{e}{(1 - e^2)}\left[\frac{\sin\theta}{(1 + e\cos\theta)^2} \quad - \frac{2e}{(1 - e^2)}\left[\frac{-e\sin\theta}{1 + e\cos\theta}\right.\right.$$

$$\left. + \frac{2}{\sqrt{1 - e^2}}\tan^{-1}\frac{\sqrt{1 - e^2}\tan 1/2\,\theta}{1 + e}\right] + \frac{1}{1 - e^2}\left(\frac{\sin\theta}{1 + e\cos\theta}\right.$$

$$\left.\left. - \frac{2e}{\sqrt{1 - e^2}}\tan^{-1}\frac{\sqrt{1 - e^2}\tan 1/2\,\theta}{1 + e}\right)\right] + C_4 \equiv L + C_4$$

$$K_1(\theta) = -C_1\cot\theta + C_3 - C_2\left(-\cot\theta\,[1 + e\cos\theta]^{-2}\right.$$

$$+ \frac{e}{1 - e^2}\left[\frac{\sin\theta}{(1 + e\cos\theta)^2} + \frac{\sin\theta}{(1 - e^2)(1 + e\cos\theta)}[2e^2 + 1]\right.$$

$$\left.\left. - \frac{6e}{(1 - e^2)^{3/2}}\tan^{-1}(ARG)\right] + C_4\right)$$

Defining the constant $C_3 - C_2 C_4 \equiv \overline{K}_1$, we have:

$$\lambda_4(\theta) = - C_1 \cos\theta + \bar{K}_1 \sin\theta - C_2 \sin\theta \left(- \cot\theta\,[1 + e\cos\theta]^{-2} \right.$$

$$+ \frac{e}{1 - e^2} \left[\frac{\sin\theta}{(1 + e\cos\theta)^2} + \frac{\sin\theta\,(2e^2 + 1)}{(1 - e^2)(1 + e\cos\theta)} \right.$$

$$\left.\left. - \frac{6e}{(1 - e^2)^{3/2}} \tan^{-1}(\text{ARG}) \right] \right) \qquad \text{(A15)}$$

where

$$\text{ARG} \equiv \frac{\sqrt{1 - e^2}\,\tan\theta/2}{1 + e},$$

and $\bar{K}_1$ is determined such that $\lambda_4(\phi_c - \omega_c) \equiv \lambda_4(\theta_c)$ is satisfied.

Turning now to equation (A12), we have for the homogeneous solution:

$$\lambda_6 = K_2 (1 + e\cos\theta)^{-2}$$

Using the form (A14) for $K_1(\theta)$ in the equation for $\lambda_4(\theta)$, substituting the homogeneous solution for $\lambda_6(\theta)$, above, into (A12) and considering that $K_2 = K_2(\theta)$, yields the differential equation for $K_2(\theta)$:

$$K_2'(\theta)\,[1 + e\cos\theta]^{-2} = - C_1'\,[1 + e\cos\theta]^{-2} - 2p\,[1 + e\cos\theta]^{-1}$$

$$\left[- C_1\cos\theta + C_3\sin\theta - C_2\sin\theta \int \frac{d\theta}{\sin^2\theta\,[1 + e\cos\theta]^2} \right]; \quad C_1' \equiv \frac{\lambda_3 p^2}{h}$$

Thus,

$$K_2'(\theta) = - C_1' - 2p\,[1 + e\cos\theta] \left[- C_1\cos\theta + C_3\sin\theta \right.$$

$$\left. - C_2\sin\theta \int \frac{d\theta}{\sin^2\theta\,[1 + e\cos\theta]^2} \right]$$

Now:

$$\int \left[- C_1' - 2\,p\,[1 + e\cos\theta] \left(- C_1 \cos\theta + C_3 \sin\theta \right) \right] d\,\theta$$

$$= 2\,p\,\left(C_1 \sin\theta + C_3 \cos\theta + \frac{e\,C_1}{2} \sin\theta \cos\theta - \frac{e\,C_3}{2} \sin^2\theta\right) + C_5 \qquad \text{(A16)}$$

Finally, we need:

$$2\,p\,C_2 \int \sin\theta\,[1 + e\cos\theta] \left(\int \frac{d\,\theta}{\sin^2\theta\,[1 + e\cos\theta]^2} \right) d\,\theta \qquad \text{(A17)}$$

Let:

$$u = \int \frac{d\,\theta}{\sin^2\theta\,[1 + e\cos\theta]^2}\,; \qquad dv = \sin\theta\,[1 + e\cos\theta]\,d\,\theta$$

$$v = - \cos\theta + \frac{e}{2}\sin^2\theta$$

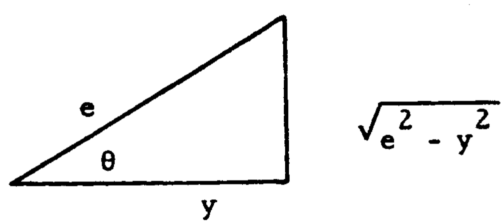
$$\int u\,dv = uv - \int \frac{[- \cos\theta + e/2 \sin^2\theta]}{\sin^2\theta\,[1 + e\cos\theta]^2}\,d\theta$$

$$\int u\,dv = uv + \int \frac{\cos\theta\,d\,\theta}{\sin^2\theta\,[1 + e\cos\theta]^2} - \frac{e}{2} \int \frac{d\,\theta}{[1 + e\cos\theta]^2}$$

For the first integral let

$$y = e\cos\theta$$

$$dy = - e\sin\theta\,d\,\theta$$



$$\sin\theta = \frac{\sqrt{e^2 - y^2}}{e}$$

$$\int \frac{\cos\theta \, d\theta}{\sin^2\theta\,[1 + e\cos\theta]^2} = -e \int \frac{y\, dy}{[1 + y]^2 (e^2 - y^2)^{3/2}}$$

Now, let

$$z = 1 + y;$$

$$y^2 = z^2 - 2z + 1$$

Then:

$$\int \frac{\cos\theta \, d\theta}{\sin^2\theta\,[1 + e\cos\theta]^2} = -e \int \frac{(z - 1)\, dz}{z^2 Z^{3/2}}$$

where

$$Z = -z^2 + 2z + e^2 - 1,$$

$$z = 1 + e\cos\theta$$

Using reference (10), #190 and #197:

$$\int \frac{\cos\theta \, d\theta}{\sin^2\theta\,[1 + e\cos\theta]^2} = -e \int \frac{dz}{z Z^{3/2}} + e \int \frac{dz}{z^2 Z^{3/2}}$$

$$= \frac{e}{(e^2 - 1)} \left[\frac{-1}{Z^{1/2}} \left(1 + \frac{1}{z} + \frac{3}{e^2 - 1}\right) - \int \frac{dz}{z Z^{1/2}} \left(1 + \frac{3}{(e^2 - 1)}\right) + 3 \int \frac{dz}{Z^{3/2}} \left(1 + \frac{1}{e^2 - 1}\right) \right] + C_6 \qquad \text{(A18)}$$

where

$$\int \frac{dz}{z Z^{1/2}} = \frac{1}{\sqrt{1 - e^2}} \sin^{-1} \left(\frac{z + (e^2 - 1)}{z e}\right)$$

and

$$\int \frac{dz}{Z^{3/2}} = \frac{z - 1}{e^2 Z^{1/2}}$$

Thus, from previous results in $\lambda_4(\theta)$ and collecting the above, we find:

$$K_2(\theta) = (A16) + 2\,p\,C_2 \left(uv + (A18) - \frac{e}{2} \quad \frac{1}{(1-e^2)} \left[\frac{-\,e\sin\theta}{1 + e\cos\theta} \right.\right.$$

$$\left.\left. + \frac{2}{\sqrt{1-e^2}} \tan^{-1} \frac{\sqrt{1-e^2}\tan\theta/2}{1+e} + C_7 \right] \right) \tag{A19}$$

where u and v are defined below equation (A17).

Collecting terms, we get:

$$\lambda_6(\theta) = [1 + e\cos\theta]^{-2} \left[\overline{K}_2 + 2p(C_1 \sin\theta + C_3\cos\theta + \frac{e\,C_1}{2}\sin\theta\cos\theta \right.$$

$$+ \frac{e\,C_3}{2}\sin^2\theta) + 2\,p\,C_2 \left([L + C_4] \; [-\cos\theta + \frac{e}{2}\sin^2\theta] \right.$$

$$+ \frac{e}{(e^2-1)} \left[-\frac{1}{Z^{1/2}} \left(1 + \frac{1}{z} + \frac{3}{e^2-1} \right) - \int \frac{dz}{z\,Z^{1/2}} \left(1 + \frac{3}{e^2-1} \right) \right.$$

$$\left. + 3 \int \frac{dz}{Z^{3/2}} \left(1 + \frac{1}{e^2-1} \right) \right] + \frac{-\,e}{2\,(1-e^2)} \left(- \frac{e\sin\theta}{1 + e\cos\theta} \right.$$

$$\left.\left.\left. + \frac{2}{\sqrt{1-e^2}} \tan^{-1}(ARG) \right) \right) \right] \tag{A20}$$

We note that the constants C_3 and C_4 appear explicitly in (A20). To eliminate this, we consider all terms containing them, namely:

$$2\,p \left(C_3 \cos\theta - \frac{e\,C_3}{2}\sin^2\theta \right) + 2\,p\,C_2\,C_4\,(-\cos\theta + \frac{e}{2}\sin^2\theta)$$

$$= C_3\,(2\,p\cos\theta - p\,e\sin^2\theta) - C_2\,C_4\,(2\,p\cos\theta - p\,e\sin^2\theta)$$

$$= \overline{K}_1\,p\,(2\cos\theta - e\sin^2\theta),$$

where $\overline{K}_1$ is the constant we determine from the initial conditions on $\lambda_4(\theta)$.

$$\therefore \lambda_6(\theta) = [1 + e\cos\theta]^{-2}\Bigg[\bar{K}_2 + 2pC_1 \sin\theta\left(1 + \frac{e\cos\theta}{2}\right)$$

$$+ \bar{K}_1 p(2\cos\theta - e\sin^2\theta) + 2pC_2\Bigg(L\left(-\cos\theta + \frac{e}{2}\sin^2\theta\right)$$

$$+ \frac{e}{e^2 - 1}\left[-\frac{1}{Z^{1/2}}\left(1 + \frac{1}{z} + \frac{3}{e^2 - 1}\right) - \int \frac{dz}{z Z^{1/2}}\left(1 + \frac{3}{e^2 - 1}\right)\right.$$

$$\left. + 3\int \frac{dz}{Z^{3/2}}\left(1 + \frac{1}{e^2 - 1}\right)\right] + \frac{-e}{2(1 - e^2)}\Bigg(\frac{-e\sin\theta}{1 + e\cos\theta}$$

$$+ \frac{2}{\sqrt{1 - e^2}}\tan^{-1}\left(\frac{\sqrt{1 - e^2}\tan\theta/2}{1 + e}\right)\Bigg)\Bigg)\Bigg] \qquad \text{(A21)}$$

We note that equation (A11) has a singularity at $\theta = 0$ or π ($\phi = \omega_c$ or $\phi = \omega_c + \pi$). If it is necessary to evaluate λ_4 across either of these points, we have, from the first integral (A5), a solution.

$$\lim_{\substack{\phi \to \omega_c \\ \phi \to \omega_c + \pi}} \left[\begin{array}{l} r = \dfrac{p}{1 \pm e} \\ \dot{r} = 0 \\ \dot{\phi} = \dfrac{h}{p^2}(1 \pm e)^2 \\ \ddot{\phi} = 0 \\ \ddot{r} = \dfrac{\pm e\mu(1 \pm e)^2}{p^2} \end{array}\right.$$

where the upper sign is used for $\phi \to \omega_c$ ($\theta = 0$), and the lower for $\phi \to \omega_c + \pi$ ($\theta = \pi$).

We thus find, from (A5):

$$\lim_{\substack{\theta \to 0 \\ \theta \to \pi}} \qquad \lambda_4 = \frac{C - \lambda_3 \ddot{\phi}}{\ddot{r}}$$

$$\lim_{\theta \to 0} \lambda_4 = \frac{Cp^2}{e\mu(1+e)^2} - \frac{\lambda_3 h}{e\mu} \tag{A22}$$

We can derive (A22) in a different, and more fruitful manner. Rewrite equation (A11) as:

$$\frac{d\lambda_4}{d\theta} = \frac{1}{\sin\theta}\left[\lambda_4 \cos\theta + \frac{\lambda_3 P}{he} - \frac{Cp^3}{h^2 e\,[1 + e\cos\theta]^2}\right]$$

Since we require continuity of the multipliers, the bracketed quantity must approach zero just as $\sin\theta$ does as $\theta \to 0$. Solving, then, for λ_4 at $\theta = 0$, gives:

$$\lambda_4 = \frac{Cp^2}{e\mu(1+e)^2} - \frac{\lambda_3 h}{e\mu}$$

Thus, we know that

$$\lim_{\substack{\theta \to 0 \\ \theta \to \pi}} \qquad \frac{d\lambda_4}{d\theta} \to \frac{0}{0}$$

We can thus use L'Hospital's Rule and derive two approximate differential equations for $\lambda_4(\theta)$. In the neighborhood of $\theta = 0$,

$$\frac{d\lambda_4}{d\theta} = -\lambda_4\theta - \frac{2\,Cp^2\theta}{\mu(1+e)^3}$$

In the neighborhood of $\theta = \pi$,

$$\frac{d\lambda_4}{d\theta} = -\lambda_4(\theta - \pi) - \frac{2\,Cp^2(\theta - \pi)}{\mu(1-e)^3}$$

Solving these two equations, we obtain:

$$\lambda_4(\theta)_{\theta \sim 0} = -\frac{2\,Cp^2}{\mu(1+e)^3} + \overline{K}_3 \exp\left(-\frac{\theta^2}{2}\right)$$

$$\lambda_4(\theta)_{\theta \sim \pi} = \frac{-2\,Cp^2(\theta - \pi)}{\mu(1-e)^3} + \overline{K}_4 \exp\left(-\theta\left[\frac{\theta}{2} - \pi\right]\right)$$

We can similarly approximate (A12), and obtain

$$\lambda_6(\theta)_{\theta \to 0} = \exp\left(\frac{\theta^2 e}{1+e}\right)\left[\frac{\sqrt{\pi}}{2}\operatorname{erf}(\theta)\left(\sqrt{\frac{1+e}{e}}\left[-\frac{\lambda_3 p^2}{h(1+e)^2} + \frac{4Cp^3}{\mu(1+e)^4}\right]\right.\right.$$

$$\left.\left. - \frac{2p\bar{K}_3}{1+e}\sqrt{\frac{2(1+e)}{1+3e}}\right) + \bar{K}_5\right]$$

where erf (θ) is the error function, or probability integral:

$$\operatorname{erf}(\theta) = \frac{2}{\sqrt{\pi}}\int_0^{\theta} e^{-u^2}\,du$$

$$\lambda_6(\theta)_{\theta \sim \pi} = \exp\left(\frac{2e\theta}{1-e}\left[\pi - \frac{\theta}{2}\right]\right)\left[-\frac{\lambda_3 p^2 \theta}{h(1-e)^2}\exp\left(\frac{A_1\pi^2}{2}\right)\right.$$

$$\left[1 + \frac{A_1\pi}{2}(\theta - \pi)\right] + \frac{Cp^3}{e\mu(1-e)^3}\exp\left(A_1\theta\left[\pi - \frac{\theta}{2}\right]\right) -$$

$$\left.\frac{2p\bar{K}_4\theta}{1-e}\exp\left(\frac{A_2\pi^2}{2}\right)\left[1 + \frac{A_2\pi}{2}(\theta - \pi)\right] + \bar{K}_6\right]$$

where

$$A_1 = \frac{-2e}{1-e}$$

$$A_2 = \frac{1-3e}{1-e}$$

Since we do not have the switching function, k, as an explicit function of θ, some iterative method is needed to find the first θ at which k crosses from negative to positive values. Simply using two points and a slope to find a parabola for extrapolation works quite well. Writing k as:

$$k = \frac{c}{m}\left(\lambda_4 \cdot \frac{\lambda_4 r}{D_\nu} + \frac{\lambda_6}{r} \cdot \frac{\lambda_6}{D_\nu}\right) - \lambda_7$$

where

$$D_\nu = +\sqrt{\lambda_6^2 + \lambda_4^2 r^2}$$

and

$$\frac{d\,\lambda_7}{d\,\theta} = 0$$

we find

$$\frac{d\,k}{d\,\theta} = \frac{c}{m\,D_\nu}\left[\,r\,\lambda_4\,\frac{d\,\lambda_4}{d\,\theta} + \frac{\lambda_6}{r}\,\frac{d\,\lambda_6}{d\,\theta} - \frac{\lambda_6^2\,e\,\sin\theta}{p}\right]$$

REFERENCES

1. McCue, G., "Optimum Two-Impulse Orbital Transfer and Rendezvous Between Inclined Elliptical Orbits, " AIAA Journal, Vol. 1, pp. 1865-1872 (1963).

2. Jurovics, S., "Boost Optimization Program, " North American Aviation, Inc., SID 62-1015, September 27, 1962; revised April 19, 1963.

3. McKay, J., et al; "Optimum Low Acceleration Trajectories for Earth-Mars Transfer, " Proc. of the IAS Symposium on Vehicle Systems Optimization, New York, November 1961.

4. Moulton, F. R., An Introduction to Celestial Mechanics, Macmillan, New York, 1914.

5. Goldstein, H., Classical Mechanics, Addison-Wesley, Reading, Mass., 1959.

6. Burns, R., and Singleton, L., "Lunar Escape, " George C. Marshall Space Flight Center, MTP-P&VE-F-63-3, July 16, 1963.

7. Breakwell, J., "The Optimization of Trajectories, " J. Soc. Indust. Appl. Math., Vol. 7, No. 2, June, 1959, pp. 215-247.

8. Leitman, G., ed, Optimization Techniques, Academic Press, New York, 1962

9. Richman, J., "Two-Point Boundary Value Problem of the Calculus of Variation for Optimum Orbits, " George C. Marshall Space Flight Center, MTP-AERO-63-12; February 6, 1963, pp. 29-56.

10. Pierce, B. O., A Short Table of Integrals, Fourth Edition, Ginn and Company, New York, 1956.

ACKNOWLEDGMENTS

The author expresses his gratitude to Mr. H. W. Bell and Dr. D. F. Bender for several consoling consultations, and for assistance in problem formulation. He is also indebted to Mr. G. A. McCue for assistance in the initial phases of program construction, and for supplying the two-impulse solution data. Lastly, the author expresses his appreciation to Mrs. S. Wall for programming, and incorporating into the original program the pertinent equations in Appendix A, and for assistance in obtaining the numerical results presented.