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A PICARD ITERATIVE METHOD FOR ESTIMATING SOLUTIONS

TO CERTAIN DIFFERENTIAL EQUATIONS*

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Introduction

The method for estimating solutions to differential equations discussed here is a numerical implementation of Picard iteration. It is similar to a method given by Axelsson,⁽¹⁾ and can also be regarded as an implicit Runge-Kutta method described by Butcher.⁽²⁾

The justification, given here, of the scheme, establishes no more than the fact that it delivers Runge-Kutta accuracy at the cost of iteration. Experience, however, indicates that in many instances the proposed method is more effective and efficient than standard Runge-Kutta techniques. Examples to this effect and data, for use in implementing the method, are given in the appendix.

Description of the Method

For the purpose of exposition a first order initial value problem will be used. Higher order equations are solved by repeated use of the first order method.

Let $y'(x) = F(x, y)$, $(x_0 < x < x_0 + h)$, and $y(x_0) = a_0$.

If $y'(x)$ can be represented by a convergent Fourier series of Legendre polynomials and if $P_0(u)$, $P_1(u)$, $P_2(u)$... are the Legendre

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polynomials orthogonal over $0 \leq u \leq 1$, then

$$y'(x_0+hu) = \sum_{k=0}^{\infty} (2k+1)P_k(u) \int_0^1 F(x_0+hv, y(x_0+hv))P_k(v)dv$$

and

$$\begin{aligned} y(x_0+hu) = & y(x_0) + h\left\{\frac{1}{2}[P_0(u)-P_1(u)] \int_0^1 F(x_0+hv, y(x_0+hv))dv \right. \\ & \left. + \sum_{k=1}^{\infty} \frac{1}{2}[P_{k-1}(u)-P_{k+1}(u)] \int_0^1 F(x_0+hv, y(x_0+hv))P_k(v)dv \right\} \end{aligned} \quad (1)$$

If $F(x,y)$ satisfies a Lipschitz condition then a convergent Picard iterative process is given by

$$\begin{aligned} y_{\ell+1}(x_0+hu) = & y(x_0) + h\left\{\frac{1}{2}[P_0(u)-P_1(u)] \int_0^1 F(x_0+hv, y_{\ell}(x_0+hv))dv \right. \\ & \left. + \sum_{k=1}^{\infty} \frac{1}{2}[P_{k-1}(u)-P_{k+1}(u)] \int_0^1 F(x_0+hv, y_{\ell}(x_0+hv))P_k(v)dv \right\} \end{aligned} \quad (2)$$

where $y_1(x_0 + hv) = a_0$.

A numerical estimate to the solution of the differential equation is obtained by truncating the above series and approximating the integrals by quadratures. Two quadratures, Gauss and Gauss-Lobatto, have been tried. If Gaussian quadrature is used, the series (2) truncated, and $P_{k-1}(u)-P_{k+1}(u)$ replaced by $-\frac{2(2k+1)}{k(k+1)}(u)(1-u)P'_k(u)$ the result is

$$y_{k+1}(x_0+hu) = y(x_0) + h \left\{ u \sum_{i=1}^m w_i F(x_0+hv_i, y_{\ell}(x_0+hv_i)) \right. \\ \left. - \sum_{k=1}^n \frac{2k+1}{k(k+1)} u(1-u) P'_k(u) \sum_{i=1}^m w_i F(x_0+hv_i, y_{\ell}(x_0+hv_i)) P_k(v_i) \right\}$$

where $w_1 \dots w_m$ and $v_1 \dots v_m$ are the Gaussian weights and the zeros of $P_m(v)$, respectively.

A change in the order of summation yields

$$y_{k+1}(x_0+hu) = y(x_0) + h \sum_{i=1}^m Z_i(u) F(x_0+hv_i, y_{\ell}(x_0+hv_i)) \quad (3)$$

where

$$Z_i(u) = w_i \left[u - \sum_{k=1}^n \frac{2k+1}{k(k+1)} u(1-u) P'_k(u) P_k(v_i) \right] \\ = w_i \int_0^u \sum_{j=1}^n (2j+1) P_j(t) P_j(v_i) dt = w_i \int_0^u K(t, v_i) dt.$$

If $m = n+1$ the indefinite integral $\int_0^x f(u) du$ is given exactly

by $\sum_{i=1}^m Z_i(x) f(v_i)$ when $f(u)$ is a polynomial of degree less than or

equal to n . A least square approximation results if f is a polynomial of degree $n+1$.

Also $Z'_i(v_j) = \delta_{ij}$. Hence $Z'_i(u)$ is the polynomial $l_i(u) =$

$\frac{P_m(u)}{(u-v_i)P'_m(v_i)}$ associated with Lagrangian interpolation. This fact

could have been used to generate the iterative process (3) above.

If (3) converges to a function $y_c(x_0+hu)$ and $\dot{y}$ is used to denote differentiation with respect to (x_0+hu) it is seen that $y_c(x_0+hu)$ is a polynomial such that

$$\dot{y}_c(x_0+hv_i) = F(x_0+hv_i, y_c(x_0+hv_i)), \quad 1 \leq i \leq m.$$

If h is sufficiently small and $F(x,y)$ satisfies a Lipschitz condition the process (3) will converge because if $a_{ij} = Z_j(v_i)$ and A denotes the $m \times m$ matrix $\{a_{ij}\}$ it is seen that $\|A\|$ is bounded.

Now if

$$\dot{y}(x_0 + hu) = F(x_0 + hu, y(x_0 + hu)),$$

then

$$\begin{aligned} \dot{y}(x_0+hu) - \dot{y}_c(x_0+hu) &= F(x_0+hu, y(x_0+hu)) \\ &\quad - F(x_0+hu, y_c(x_0+hu)) \\ &\quad + F(x_0+hu, y_c(x_0+hu)) - \dot{y}_c(x_0+hu). \end{aligned} \tag{4}$$

Let

$$\begin{aligned} F(x_0+hu, y(x_0+hu)) &= F(x_0+hu, y_c(x_0+hu)) \\ &\quad + [y(x_0+hu) - y_c(x_0+hu)]F_2(x_0+hu, \theta(x_0+hu)) \end{aligned}$$

where F_2 denotes partial differentiation with respect to the second, variable.

Substituting in (4) yields

$$y(x_0+hx) - y_c(x_0+hx) = h \int_0^x \exp(g(x,v)) \phi(v) dv$$

where $g(x,v) = h \int_v^x F_2(x_0+ht, \theta(x_0+ht)) dt$ and

$$\phi(v) = F(x_0+hv, y_c(x_0+hv)) - \dot{y}_c(x_0+hv).$$

If $\phi(v)$ has m bounded derivatives then

$$|\phi(v)| = \left| \frac{\phi^{(m)}(\bar{v})}{m!} \right| |P_m(v)| \binom{2m}{m}^{-1} \leq \frac{h^m}{m!} \binom{2m}{m}^{-1} M_1$$

where M_1 is the maximum of $\left| \frac{d^m F(x_0+hv, y_c(x_0+hv))}{d(x_0+hv)^m} \right|$.

Thus

$$|y(x_0+hx) - y_c(x_0+hx)| \leq \frac{h^{m+1}}{m!} \binom{2m}{m}^{-1} M_1 \int_0^x \exp(g(x,v)) dv.$$

for $0 < x < 1$.

A higher order result can be obtained for $x = 1$ if $F(x, y(x))$ is sufficiently well-behaved (has partial derivatives of sufficiently

high order). Let $\phi_1(x_0+hv) = \exp[g(1,v)] = \exp\left(\int_{x_0+hv}^1 F_2(t, \theta(t)) dt\right)$

then

$$\begin{aligned} y(x_0+h) - y_c(x_0+h) &= \int_0^1 \phi_1(x_0+hv) \phi(v) dv \\ &= \int_0^1 \phi_2(x_0+hv) dv \end{aligned}$$

Since $\phi_2(x_0 + h v_i) = 0$, $1 \leq i \leq m$

$$|y(x_0 + h) - y_c(x_0 + h)| \leq \frac{h^{2m}}{(2m-1)!} \left(\frac{2m}{m}\right)^{-2} M_2$$

where M_2 is a maximum of $\left| \frac{d^{2m} \phi_2(x_0 + hv)}{d(x_0 + hv)^{2m}} \right|$.

If Lobatto quadrature is used the series (2) is rearranged in the form

$$y(x_0 + hu) = y(x_0) + h \left\{ \int_0^1 (1-v) F(x_0 + hv, y(x_0 + hv)) dv + \sum_{k=1}^{\infty} \frac{2k+1}{k(k+1)} P_k(u) \int_0^1 v(1-v) P'_k(v) F(x_0 + hv, y(x_0 + hv)) dv \right\}.$$

before truncation.

Truncation and replacement of the integrals by quadrature results in the iterative process

$$y_{\ell+1}(x_0 + hu) = y(x_0) + h \left\{ w_0 F(x_0, y(x_0)) + \sum_{j=1}^{m-1} W_j(u) F(x_0 + hv_j, y_{\ell}(x_0 + hv_j)) \right\}$$

where $W_j(u) = w_j(1-v_j) + w_j \sum_{k=1}^{m-1} \frac{(2k+1)}{k(k+1)} v_j(1-v_j) P'_k(v_j) P_k(u)$,

$w_0 \cdots w_m$ and $v_0 \cdots v_m$ are Lobatto weights and the zeros of $v(1-v) P'_m(v)$ respectively.

This is the previously referenced result of Axelsson. In this case $W'_j(v_i) = \delta_{ij}$ and the iterative process converges to a polynomial $y_c(x_0 + hv)$ such that

$$\dot{y}_c(x_0 + hv_j) = F(x_0 + hv_j, y_c(x_0 + hv_j)) \text{ where } v_j \text{ is a zero of } P'_m(v).$$

However, in this instance $y_c(x_0) \neq y(x_0)$ and order of error terms are computed for $|y(x_0+hu) - y_c(x_0+hu) - y(x_0) + y_c(x_0)|$. For m reasonably large this discrepancy between $y(x_0)$ and $y_c(x_0)$ seems to be an advantage rather than a disadvantage. This may be due to the fact that exact fitting of initial conditions results in more accumulation error.

Conclusion and Results.

The particular methods used here were selected so that classical forms for interpolation error would apply. Computer trials have given satisfactory results when Gaussian quadrature was used to evaluate the integrals involved in the truncation suggested for use with Lobatto quadrature and conversely. This suggests that error analysis should be based on truncation error associated with Fourier series. Efforts in this direction have not, as yet, yielded results more satisfactory than those given.

In applying iterative schemes of the nature discussed here it is desirable to have a set of reasonably good starting values. For the first step x_0 to x_0+hu the starting values used were $y(x_1) = y(x_0)$. For subsequent steps starting values were obtained by extrapolating the polynomial $y_c(x)$ to the right. In most cases this worked rather well, but it is possible that other methods of prediction would serve better.

The Lobatto quadrature scheme was used to obtain the results reported here. The equations

$$y(x_0 + hv_j) = y(x_0) + w_0 F(x_0) + h \sum_{i=1}^{m-1} W_i(v_j) F(x_0 + hv_j, y(x_0 + hv_j))$$

were solved by Gauss Siedel iteration. For the purpose of comparison the differential equations used were solved by the iterative technique, based on a 9 point Lobatto quadrature, with a step size h , and a given convergence criterion. The same differential equations were then solved using a fifth order Runge-Kutta method with several step sizes. Of the several Runge-Kutta solutions the one judged to be most nearly equivalent (in accuracy and efficiency) to the iterative solution was selected for comparison.

The programs were written in Fortran and run in double precision on the IBM 7094. Run times that are given were obtained through a call clock command and do not include program compile time. The true answer as well as the estimated answer was computed and compared for each point used in the computational routines.

Error values are given at selected points. The errors given are the computed value of the known solution minus the estimated solution. The step size h , the run time t , in seconds, and the convergence criterion ϵ are given.

The iterative process was discontinued when at the rightmost point v_j

$$|F(x_0 + hv_j, y_{l+1}(x_0 + hv_j)) - F(x_0 + hv_j, y_l(x_0 + hv_j))|$$

became less than ϵ .

In each instance reported a nine point Lobatto quadrature was

used with the iterative process.

The numbers $Z_i(x_j)$ and $W_i(x_j)$ are given for 5, 9, and 13 point Gauss and Gauss-Lobatto quadrature. They were computed in double precision on the IBM 7094 and can serve as input data for computer programs designed for using these iterative methods for solving differential equations.

REFERENCES

1. Axelsson, O. BIT 4 1964, 69-86.
2. Butcher, J. C. Math Comp. 18 (1964) 50-64, 233-244.

APPENDIX I

The following tables give the errors in estimates to the solution of sample differential equations when obtained by a fifth order Runge-Kutta method (R K Error) as compared to the errors obtained when an iterative process (I Error) was used.

Table 1

Differential equation

$$y' = \frac{1}{x^3} [y^3 + 3xy^2 + 4x^2y + x^3] \quad \frac{1}{e} < x \leq \frac{1}{e} + 6.5$$

$$y = \frac{x}{\sqrt{4-2 \log_e x}} - x.$$

Runge-Kutta

Step size $h = .005$

Run time $t = 20.00$ sec.

Iterative Method

Step size $h = 0.05$

Run time $t = 15.13$ sec.

Convergence constant $\epsilon = 1.0 \times 10^{-11}$

Table 2

Differential equation

$$y' = -50y + y \sin x + e^{-8x}(42 - \sin x), 0 < x \leq 1$$

$$y = e^{-8x}$$

Runge-Kutta:

Step size $h = 0.001$ Run time $t = 31.93$ sec.

Iterative method:

Step size, $h = 0.05$ Run time $t = 35.22$ sec.Convergence constant $\epsilon = 1.0 \times 10^{-13}$

Table 3

$$y'' = 9y - 20 \sin x, 0 < x \leq 3.0$$

$$y = e^{-3x} + 2 \sin x$$

Runge-Kutta:

Step size $h = 0.01$ Run time $t = 7.77$ sec.

Iterative method:

Step size $h = 0.25$ Run time $t = 7.63$ sec.Convergence constant $\epsilon = 1.0 \times 10^{-11}$

TABLE 1

<u>X</u>	<u>R K Error</u>	<u>I Error</u>
1/e + 0.1	-0.202865D-12	-0.174860D-14
0.2	-0.356825	-0.280331
0.3	-0.503402	-0.352495
0.4	-0.655614	-0.413558
0.5	-0.819205	-0.610622
0.6	-0.101588D-11	-0.785482
0.7	-0.119160	-0.960342
0.8	-0.140382	-0.112687D-13
0.9	-0.163546	-0.128785
1.0	-0.188765	-0.146549
1.1	-0.216204	-0.164868
1.2	-0.245992	-0.184297
1.3	-0.278271	-0.204281
1.4	-0.313260	-0.227040
1.5	-0.351146	-0.251465
1.6	-0.932108	-0.277555
1.7	-0.436362	-0.307531
1.8	-0.484184	-0.336397
1.9	-0.535821	-0.369704
2.0	-0.591576	-0.401900
2.1	-0.651761	-0.439648
2.2	-0.716826	-0.482391
2.3	-0.787120	-0.532351
2.4	-0.863048	-0.587307
2.5	-0.945066	-0.647815
2.6	-0.102912D-10	-0.713318
2.7	-0.112962	-0.779376
2.8	-0.123338	-0.856537
2.9	-0.134575	-0.939248
3.0	-0.146752	-0.102695D-12
3.1	-0.159964	-0.111854
3.2	-0.174312	-0.122790

<u>X</u>	<u>R K Error</u>	<u>I Error</u>
3.3	-0.189910	-0.133726
3.4	-0.206829	-0.146216
3.5	-0.225404	-0.159150
3.6	-0.245602	-0.173111
3.7	-0.267706	-0.187960
3.8	-0.291904	-0.204530
3.9	-0.318488	-0.223931
4.0	-0.347707	-0.243347
4.1	-0.379920	-0.265633
4.2	-0.415536	-0.289809
4.3	-0.454994	-0.316482
4.4	-0.498867	-0.346639
4.5	-0.547771	-0.379501
4.6	-0.602495	-0.417998
4.7	-0.663981	-0.461797
4.8	-0.733312	-0.510480
4.9	-0.811897	-0.563882
5.0	-0.901365	-0.627164
5.1	-0.100384D-09	-0.698330
5.2	-0.112194	-0.779820
5.3	-0.125900	-0.875299
5.4	-0.141933	-0.992761
5.5	-0.160842	-0.112532D-11
5.6	-0.183375	-0.128763
5.7	-0.210539	-0.147837
5.8	-0.243708	-0.171374
5.9	-0.284853	-0.200550
6.0	-0.336834	-0.238786
6.1	-0.403955	-0.288924
6.2	-0.492985	-0.359445
6.3	-0.615131	-0.464961
6.4	-0.790129	-0.655564
6.5	-0.105585D-08	-0.111120D-10

TABLE 2

<u>X</u>	<u>R K Error</u>	<u>I Error</u>
0.05	-0.430929D-10	-0.260902D-14
0.10	-0.323417	-0.205391
0.15	-0.219075	-0.133226
0.20	-0.146663	-0.929811D-15
0.25	-0.980460D-11	-0.596744
0.30	-0.655379	-0.402455
0.35	-0.438074	-0.270616
0.40	-0.292833	-0.121430
0.45	-0.197339	-0.780625D-16
0.50	-0.130868	-0.503069
0.55	-0.875007D-12	-0.390312
0.60	-0.585052	-0.197758D-15
0.65	-0.391213	-0.149186
0.70	-0.261617	-0.100397
0.75	-0.174968	-0.678710D-16
0.80	-0.117029	-0.448859
0.85	-0.782897D-13	-0.219117D-15
0.90	-0.523726	0.167509
0.95	-0.350413	0.113841
1.00	-0.234483	0.764362D-16

TABLE 3

<u>X</u>	<u>R K Error</u>	<u>I Error</u>
0.25	-0.248001D-10	-0.657918D-12
0.50	-0.193078D-11	-0.577315
0.75	0.541772D-10	-0.325961
1.00	0.162107D-09	-0.247579D-13
1.25	0.377101	0.376143D-12
1.50	0.820256	0.104871D-11
1.75	0.174841D-08	0.232425
2.00	0.370493	0.496280
2.25	0.783976	0.105229D-10
2.50	0.165871D-07	0.222832
2.75	0.361689	0.471761
3.00	0.742893	0.998713

Quadrature points and weights for the 5, 9, and 13 point Gauss integration formulas

X(1) =	0.4691007703066802D-01	W(1) =	0.1184634425280945D-00
X(2) =	0.2307653449471585D-00	W(2) =	0.2393143352496830D-00
X(3) =	0.5000000000000000D 00	W(3) =	0.2844444444444444D-00
X(4) =	0.7692346550528415D 00	W(4) =	0.2393143352496830D-00
X(5) =	0.9530899229693319D 00	W(5) =	0.1184634425280945D-00

X(1) =	0.1591988024618701D-01	W(1) =	0.4063719413078700D-01
X(2) =	0.8198444633668200D-01	W(2) =	0.9032408034742850D-01
X(3) =	0.1933142836497050D-00	W(3) =	0.1303053482014675D-00
X(4) =	0.3378732882980955D-00	W(4) =	0.1561735385200015D-00
X(5) =	0.5000000000000000D 00	W(5) =	0.1651196775006300D-00
X(6) =	0.6621267117019044D 00	W(6) =	0.1561735385200015D-00
X(7) =	0.8066857163502950D 00	W(7) =	0.1303053482014675D-00
X(8) =	0.9180155536633180D 00	W(8) =	0.9032408034742850D-01
X(9) =	0.9840801197538129D 00	W(9) =	0.4063719413078700D-01

X(1) =	0.7908472640705988D-02	W(1) =	0.2024200239265800D-01
X(2) =	0.4120080038851098D-01	W(2) =	0.4606074991886400D-01
X(3) =	0.9921095463334501D-01	W(3) =	0.6943675510989350D-01
X(4) =	0.1788253302798300D-00	W(4) =	0.8907299038097299D-01
X(5) =	0.2757536244817765D-00	W(5) =	0.1039080237684445D-00
X(6) =	0.3847708420224325D-00	W(6) =	0.1131415901314485D-00
X(7) =	0.5000000000000000D 00	W(7) =	0.1162757766154370D-00
X(8) =	0.6152291579775676D 00	W(8) =	0.1131415901314485D-00
X(9) =	0.7242463755182235D 00	W(9) =	0.1039080237684445D-00
X(10) =	0.8211746697201699D 00	W(10) =	0.8907299038097299D-01
X(11) =	0.9007890453666549D 00	W(11) =	0.6943675510989350D-01
X(12) =	0.9587991996114890D 00	W(12) =	0.4606074991886400D-01
X(13) =	0.9920915273592940D 00	W(13) =	0.2024200239265800D-01

Constants associated with Gauss quadrature for the 5, 9, and 13 point formulas respectively

	$Z_1(x_i)$	$Z_2(x_i)$	$Z_3(x_i)$
$x(1) =$	0.5193904511595521D-01	0.1083143016873644D-02	-0.1546726163707234D-01
$x(2) =$	0.1342261416863164D-00	0.1024518421758591D-00	-0.2331735754219876D-02
$x(3) =$	0.1137762880042245D-00	0.2600046516806412D-00	0.1422222222222222D-00
$x(4) =$	0.1151573009105928D-00	0.2462013800279819D-00	0.2867761801986642D-00
$x(5) =$	0.1241680057083205D-00	0.2242546215345454D-00	0.2999117060815167D-00
$x(1) =$	0.1952276506997044D-01	-0.4797308121617143D-02	0.7258648848675012D-03
$x(2) =$	0.4497585504801112D-01	0.4186056636877524D-01	-0.5627508176694419D-02
$x(3) =$	0.3812031695894941D-01	0.1010843514298415D-00	0.6013392172781163D-01
$x(4) =$	0.4215886942842090D-01	0.8457657301086466D-01	0.1446978850484127D-00
$x(5) =$	0.3994037286983455D-01	0.9294337227271687D-01	0.1242571104107755D-00
$x(6) =$	0.4063398525461718D-01	0.9015029390824517D-01	0.1313355191092380D-00
$x(7) =$	0.4114299850520293D-01	0.8878291286050957D-01	0.1325532716816294D-00
$x(8) =$	0.3990714059694486D-01	0.9263838204870172D-01	0.1265617981057803D-00
$x(9) =$	0.4125524773846499D-01	0.8834648399454648D-01	0.1335748771198865D-00
$x(1) =$	0.9930620765386001D-02	-0.31286715733329502D-02	0.1595581533971721D-02
$x(2) =$	0.2216415922660012D-01	0.2213668265818997D-01	-0.4392909517101449D-02
$x(3) =$	0.1914739892433289D-01	0.5102594721954042D-01	0.3295221522933987D-01
$x(4) =$	0.2100236885868644D-01	0.4311266596464565D-01	0.7716821173728725D-01
$x(5) =$	0.1969756245740481D-01	0.4806335008237965D-01	0.6498629817168876D-01
$x(6) =$	0.2060754999701716D-01	0.4474042818430168D-01	0.7222236635140502D-01
$x(7) =$	0.2003793958110275D-01	0.4680042908513713D-01	0.6787869345449576D-01
$x(8) =$	0.2030187597830581D-01	0.4582652909136529D-01	0.6998835979915766D-01
$x(9) =$	0.2030219662272786D-01	0.4587867985394867D-01	0.6969323430524976D-01
$x(10) =$	0.2009433354638438D-01	0.4654503674219237D-01	0.6859866702017138D-01
$x(11) =$	0.2043710693970312D-01	0.4541173769430608D-01	0.7059456237779322D-01
$x(12) =$	0.2004502313519012D-01	0.4671930313158011D-01	0.6824969034192294D-01
$x(13) =$	0.2038882234858161D-01	0.4556892620608015D-01	0.7032682670673343D-01

	$Z_4(x_i)$	$Z_5(x_i)$	$Z_6(x_i)$
$x(1) =$	0.1505971371513746D-01	-0.5704563180226033D-02	0.3933373546286507D-02
$x(2) =$	-0.6887044778298955D-02	0.3306141617501556D-02	-0.4281204370335962D-02
$x(3) =$	-0.2069031643095820D-01	0.4687154523869931D-02	0.1867259116563845D-02
$x(4) =$	0.1368624930738237D-00	-0.1576269915822185D-01	0.3494299746569697D-02
$x(5) =$	0.2382311922328093D-00	0.6652439741213924D-01	-0.1382690673524536D-01
$x(1) =$	0.2022116442937438D-02	-0.3576945453047080D-02	0.8189289777560006D-01
$x(2) =$	-0.4497977205883807D-03	0.3347233209259896D-02	0.1633635269222500D-00
$x(3) =$	-0.7189988402248413D-02	0.5109831364488537D-03	0.1566233362405918D-00
$x(4) =$	0.7428064074439646D-01	-0.1048180413815822D-01	0.1541514220770598D-00
$x(5) =$	0.1700004452552532D-00	0.8255983875031500D-01	
$x(6) =$	0.1526792387734259D-00	0.1756014816387883D-00	
$x(7) =$	0.1543062794034407D-00	0.1646086943641820D-00	
$x(8) =$	0.1604547428903392D-00	0.1617724442913703D-00	
$x(9) =$	0.1522401649737106D-00	0.1686966229536772D-00	
$x(1) =$	-0.4905291973990891D-03	-0.3731498215961110D-03	0.1008030221537516D-02
$x(2) =$	0.1627542416286917D-02	-0.2439087816796075D-04	-0.1046720593719212D-02
$x(3) =$	-0.5298844134214556D-02	0.1622966700530239D-02	0.1852290361616503D-03
$x(4) =$	0.4221087967686996D-01	-0.6291636536663532D-02	0.2026804428315042D-02
$x(5) =$	0.9873067204257991D-01	0.4974567644736508D-01	-0.7554499239960935D-02
$x(6) =$	0.8391747654501761D-01	0.1143340010714254D-00	0.5522576032594931D-01
$x(7) =$	0.9187307472407894D-01	0.9896227176122852D-01	0.1232140251350764D-00
$x(8) =$	0.8796255863961845D-01	0.1060167074891409D-00	0.1090926209714729D-00
$x(9) =$	0.8889877237353785D-01	0.1037074129304848D-00	0.1142180024208337D-00
$x(10) =$	0.9015399777307768D-01	0.102820753565627D-00	0.1138650075870671D-00
$x(11) =$	0.8748853597923650D-01	0.1057145081404699D-00	0.1114260785260884D-00
$x(12) =$	0.9072827248815059D-01	0.1019543423672432D-00	0.1151338710639914D-00
$x(13) =$	0.8782309601962829D-01	0.1054015305062236D-00	0.1115828566513467D-00

$z_7(x_i)$ $z_8(x_i)$ $z_9(x_i)$

$x(1) = -0.3269528918417616D-02$
 $x(2) = 0.3743550095685014D-02$
 $x(3) = -0.2247923480163348D-02$
 $x(4) = -0.1030170907767974D-02$
 $x(5) = 0.6048237790686522D-02$
 $x(6) = -0.1439253684693946D-01$
 $x(7) = 0.7017142647365413D-01$
 $x(8) = 0.1359328563781599D-00$
 $x(9) = 0.1295794833166042D-00$
 $-0.1977596352881684D-02$
 $-0.2314301701271591D-02$
 $0.1541167486920333D-02$
 $0.1737864391817276D-03$
 $-0.2619291925284954D-02$
 $0.5747507336561018D-02$
 $-0.1076027108241271D-01$
 $0.4846351397865458D-01$
 $0.9512138846904347D-01$
 $-0.6180535576785867D-03$
 $0.7300535838423765D-03$
 $-0.5058043244164042D-03$
 $0.32089226168441256D-05$
 $0.6968213109533487D-03$
 $-0.1521675247633463D-02$
 $0.2516877221836513D-02$
 $-0.4338660867223821D-02$
 $0.2111442911081716D-01$

$x(1) = -0.1403462541525888D-02$
 $x(2) = 0.1704827912482847D-02$
 $x(3) = -0.1213540376726331D-02$
 $x(4) = -0.1882795163713604D-03$
 $x(5) = 0.2920751334819008D-02$
 $x(6) = -0.8950200175866587D-02$
 $x(7) = 0.5813788830771849D-01$
 $x(8) = 0.1252259767913835D-00$
 $x(9) = 0.1133550252804524D-00$
 $x(10) = 0.1164640561315507D-00$
 $x(11) = 0.1174893169922765D-00$
 $x(12) = 0.1145709487032313D-00$
 $x(13) = 0.1176792391569990D-00$
 $0.1558733480261906D-02$
 $-0.1992280933390860D-02$
 $0.1715511604342134D-02$
 $-0.7234174547625405D-03$
 $-0.1076412289969134D-02$
 $0.4048969160279629D-02$
 $-0.1007243500377368D-01$
 $0.5791582980536463D-01$
 $0.1206960893720130D-00$
 $0.1111147857027928D-00$
 $0.1129563610961925D-00$
 $0.1141883107254122D-00$
 $0.1121335599098900D-00$
 $-0.1493506737535639D-02$
 $0.1953681401908337D-02$
 $-0.1806484370570707D-02$
 $0.1087270411482209D-02$
 $0.2006108381117409D-03$
 $-0.2108683720484488D-02$
 $0.4945752007179485D-02$
 $-0.1042597730307978D-01$
 $0.5416234732071715D-01$
 $0.1101996603055120D-00$
 $0.1022850570664783D-00$
 $0.1039324146458969D-00$
 $0.1042811735899208D-00$

$z_{10}(x_i)$ $z_{11}(x_i)$ $z_{12}(x_i)$

$x(1) =$	0.1249894360329498D-02	-0.8900715980852730D-03	0.4918237128495405D-03
$x(2) =$	-0.1655282106718598D-02	0.1187064769842383D-02	-0.6585532133771114D-03
$x(3) =$	0.1584454399185946D-02	-0.1157807267078672D-02	0.6490122244763297D-03
$x(4) =$	-0.1081007391551662D-02	0.8380880886469057D-03	-0.4842868240122152D-03
$x(5) =$	0.1742180080693148D-03	-0.2564791928296877D-03	0.1820700649131806D-03
$x(6) =$	0.1110431738971008D-02	-0.5516046918130503D-03	0.2342208280202921D-03
$x(7) =$	-0.2800084340182285D-02	0.1558061657772427D-02	-0.7396791678095740D-03
$x(8) =$	0.5155513833448983D-02	-0.2785611243255571D-02	0.1320321736799672D-02
$x(9) =$	-0.9657681659642808D-02	0.4450456938577630D-02	-0.2002600165371035D-02
$x(10) =$	0.4686211070319212D-01	-0.7731456626511050D-02	0.2948083954855250D-02
$x(11) =$	0.9437183451606401D-01	0.3648453987828800D-01	-0.4965197299604835D-02
$x(12) =$	0.8744544796697074D-01	0.7382966462701336D-01	0.2392406726021961D-01
$x(13) =$	0.8956351957740057D-01	0.6784117357508063D-01	0.4918942149340451D-01

$$Z_{13}(x_i)$$

$x(1) = -0.1468199649862095D-03$
 $x(2) = 0.1969792464338922D-03$
 $x(3) = -0.1951045559783830D-03$
 $x(4) = 0.1476688361511866D-03$
 $x(5) = -0.6019424085263412D-04$
 $x(6) = -0.5987359369799953D-04$
 $x(7) = 0.2040627988693346D-03$
 $x(8) = -0.3655476115985035D-03$
 $x(9) = 0.5444399231101236D-03$
 $x(10) = -0.7603664750684390D-03$
 $x(11) = 0.1094603458614414D-02$
 $x(12) = -0.1922156845614520D-02$
 $x(13) = 0.1031138161868777D-01$

Quadrature points and weights for the 5, 9, and 13 point Gauss - Lobatto integration formulas

X(1)	=	0.		W(1)	=	0.4999999999999999D-01
X(2)	=	0.1726731646460115D--00		W(2)	=	0.2722222222222222D-00
X(3)	=	0.5000000000000000D 00		W(3)	=	0.3555555555555555D-00
X(4)	=	0.8273268353539885D 00		W(4)	=	0.2722222222222222D-00
X(5)	=	1.0000000000000000D 00		W(5)	=	0.4999999999999999D-01

X(1)	=	0.	W(1)	=	0.138888888888888888D+01
X(2)	=	0.5012100229426986D+01	W(2)	=	0.8274768078040274D+01
X(3)	=	0.1614068602446311D+00	W(3)	=	0.1372693562500809D+00
X(4)	=	0.3184412680869109D+00	W(4)	=	0.17321422554865231D+00
X(5)	=	0.5000000000000000D+00	W(5)	=	0.1857596371882086D+00
X(6)	=	0.6815587319130890D+00	W(6)	=	0.17321422554865231D+00
X(7)	=	0.8385931397553689D+00	W(7)	=	0.1372693562500809D+00
X(8)	=	0.9498789977057301D+00	W(8)	=	0.8274768078040274D+01
X(9)	=	1.0000000000000000D+00	W(9)	=	0.138888888888888888D+01

X(1)	#	0.	W(1)	=	0.6410256410256410D+02
X(2)	=	0.2334507667891805D+01	W(2)	=	0.3890084337340946D+01
X(3)	=	0.7682621767406389D+01	W(3)	=	0.6749096334480414D+01
X(4)	=	0.1569057654591213D+00	W(4)	=	0.9182343260177502D+01
X(5)	=	0.2585450894543319D+00	W(5)	=	0.1103838967830550D+00
X(6)	=	0.3753565349468800D+00	W(6)	=	0.1220078951533382D+00
X(7)	=	0.5000000000000000D+00	W(7)	=	0.1259654246667233D+00
X(8)	=	0.6246434650531200D+00	W(8)	=	0.1220078951533382D+00
X(9)	=	0.7414549105456680D+00	W(9)	=	0.1103838967830550D+00
X(10)	=	0.8430942345408786D+00	W(10)	=	0.9182343260177502D+01
X(11)	=	0.9231737823259361D+00	W(11)	=	0.6749096334480414D+01
X(12)	=	0.9766549233210818D+00	W(12)	=	0.3890084337340946D+01
X(13)	=	1.0000000000000000D+00	W(13)	=	0.6410256410256410D+02

Constants associated with Gauss - Lobatto quadrature for the 5, 9, and 13 point formulas respectively

	$W_1(x_i)$	$W_2(x_i)$	$W_3(x_i)$
X(1)=	0.1524774528788104D-00	-0.4044011245821461D-01	0.1063582422541544D-01
X(2)=	0.2888636342763056D-00	0.1777777777777778D-00	-0.1664141205408352D-01
X(3)=	0.2615863979968066D-00	0.3959956680137700D-00	0.1197447693434116D-00
X(4)=	0.2923674247642566D-00	0.2888888888888888D-00	0.3687436863468540D-00
X(1)=	0.4762215980278096D-01	-0.2046876619410781D-01	0.1602232535614350D-01
X(2)=	0.8658404112901086D-01	0.7333736062164268D-01	-0.1970099596570858D-01
X(3)=	0.8115507322904750D-01	0.1464508516034278D-00	0.8912877679760983D-01
X(4)=	0.8369955136739388D-01	0.1327247851468220D-00	0.1873230701258434D-00
X(5)=	0.8203382619747552D-01	0.1404240930909589D-00	0.1652136519161402D-00
X(6)=	0.8339513797201167D-01	0.1345140321606501D-00	0.1796450883685010D-00
X(7)=	0.8201390811386579D-01	0.1403341653028478D-00	0.1663281426026987D-00
X(8)=	0.8444682958457459D-01	0.1302217352692886D-00	0.1888333161687236D-00
X(1)=	0.2250590196440397D-01	-0.1056219508653100D-01	0.9630383818578735D-02
X(2)=	0.4060957291353151D-01	0.3645813412320934D-01	-0.1135851653983146D-01
X(3)=	0.3824120745200046D-01	0.7162840061216503D-01	0.4811103831717491D-01
X(4)=	0.3925552286702953D-01	0.6564994013786940D-01	0.9839923700183464D-01
X(5)=	0.3867158443833413D-01	0.6859007170291548D-01	0.8862500305997267D-01
X(6)=	0.3906898856442434D-01	0.6671750788320492D-01	0.9388288378423614D-01
X(7)=	0.3876468098695678D-01	0.6810238717563348D-01	0.9027059966516704D-01
X(8)=	0.3902111225698279D-01	0.6695892367940613D-01	0.9313754480500402D-01
X(9)=	0.3878507982073752D-01	0.6799821459132483D-01	0.9059189434479433D-01
X(10)=	0.3902402180350362D-01	0.6695444186132200D-01	0.9311232027511560D-01
X(11)=	0.3874757881543321D-01	0.6815623565001856D-01	0.9023479849702715D-01
X(12)=	0.3926949190984054D-01	0.6589298839537678D-01	0.9563017390854064D-01

$W_4(x_i)$ $W_5(x_i)$ $W_6(x_i)$

X(1) = -0.1149868205703775D-01
X(2) = 0.1162053155494231D-01
X(3) = -0.1774204374450480D-01
X(4) = 0.9287981859410430D-01
X(5) = 0.20350168093226954D-00
X(6) = 0.1741391056332445D-00
X(7) = 0.1972583192452336D-00
X(8) = 0.1603628117913832D-00

0.6886112883838056D-02
-0.6430832881957887D-02
0.8000603570398158D-02
-0.1410881463931891D-01
0.8408547868892599D-01
0.1929152514522473D-00
0.1571919301303900D-00
0.2066436764161251D-00

-0.3064809052772344D-02
0.2755324089417813D-02
-0.3154736840887400D-02
0.4544571103258888D-02
-0.9181495353354941D-02
0.6393199562843250D-01
0.1577381224441807D-00
0.1006532756378319D-00

X(1) = -0.8760675121219126D-02
X(2) = 0.8459654734337807D-02
X(3) = -0.1201338381768474D-01
X(4) = 0.5673973627964141D-01
X(5) = 0.1191279632932073D-00
X(6) = 0.1058292095123678D-00
X(7) = 0.1135131888948562D-00
X(8) = 0.1078640266974486D-00
X(9) = 0.1126779869057236D-00
X(10) = 0.1080240039316659D-00
X(11) = 0.1132648897501973D-00
X(12) = 0.1035064560705869D-00

0.7545257559339001D-02
-0.6708455749633218D-02
0.7705995585255798D-02
-0.1213211873405804D-01
0.6180294414753740D-01
0.1324718421698889D-00
0.1162122485010693D-00
0.1262539484904354D-00
0.1183304519841301D-00
0.1256861782339282D-00
0.1175846430308481D-00
0.1325051477927818D-00

-0.6042738377735769D-02
0.5149508957944579D-02
-0.5401152711501094D-02
0.6851040163539482D-02
-0.1161521413802020D-01
0.62982712333336168D-01
0.1375806388047412D-00
0.1191143845010360D-00
0.1313665773744765D-00
0.1208159157130564D-00
0.1320081630470009D-00
0.1117574104587091D-00

$w_9(x_i)$ $w_8(x_i)$ $w_7(x_i)$

$x(1) = 0.7337726665385972D-03$
 $x(2) = -0.6474571916055687D-03$
 $x(3) = 0.7138545829301656D-03$
 $x(4) = -0.9518705869914940D-03$
 $x(5) = 0.1592607551357494D-02$
 $x(6) = -0.3836360348606863D-02$
 $x(7) = 0.3512552097762362D-01$
 $x(8) = 0.1149494662431836D-00$

$x(1) = 0.4423252120331013D-02$
 $x(2) = -0.3678283082958297D-02$
 $x(3) = 0.3677443172014303D-02$
 $x(4) = -0.4246053334830466D-02$
 $x(5) = 0.5795646652129312D-02$
 $x(6) = -0.1046394701654116D-01$
 $x(7) = 0.6020495100593010D-01$
 $x(8) = 0.1341400138891274D-00$
 $x(9) = 0.1143018995719641D-00$
 $x(10) = 0.1287163508974140D-00$
 $x(11) = 0.1144626375913912D-00$
 $x(12) = 0.1394767271857962D-00$

$-0.2880992966853821D-02$
 $0.2359892852916177D-02$
 $-0.2294090124729455D-02$
 $0.2519870083844098D-02$
 $-0.3129292111641543D-02$
 $0.4554687270604282D-02$
 $-0.8744066510194582D-02$
 $0.5364416050198016D-01$
 $0.1223972805980247D-00$
 $0.1019242420534665D-00$
 $0.1191445719077605D-00$
 $0.9066079084878427D-01$

$0.1588634104255761D-02$
 $-0.1288887673611363D-02$
 $0.1231538258024383D-02$
 $-0.1314112202235829D-02$
 $0.1552832936795423D-02$
 $-0.2059451182757377D-02$
 $0.3198429542178954D-02$
 $-0.6575804399420828D-02$
 $0.4371239428628549D-01$
 $0.1031819491382807D-00$
 $0.8219304878024176D-01$
 $0.1122780133418735D-00$

$w_{11}(x_i)$
 $w_{10}(x_i)$

$x(1) =$	-0.6652723041991583D-03	0.1532645585121915D-03
$x(2) =$	0.5365191579291990D-03	-0.1231784304059803D-03
$x(3) =$	-0.5072512462759685D-03	0.1157635529606618D-03
$x(4) =$	0.5320396643430366D-03	-0.1202688835442142D-03
$x(5) =$	-0.6114238304451365D-03	0.1361623864756474D-03
$x(6) =$	0.7734554614339171D-03	-0.1681451909772956D-03
$x(7) =$	-0.1099108358235690D-02	0.2292589349960756D-03
$x(8) =$	0.1841023206897263D-02	-0.3546794933814033D-03
$x(9) =$	-0.4137437268676304D-02	0.6596359213559183D-03
$x(10) =$	0.3103282922429366D-01	-0.1708729540280823D-02
$x(11) =$	0.7805315843119087D-01	0.1639494140824754D-01
$x(12) =$	0.4828907338524434D-01	0.5432347029612771D-01