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Cylindrical Shock Model of the Plasma Pinch

Report No. 742

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Abstract

A model of the plasma pinch is formulated which represents the imploding current sheet as an impermeable piston that drives a gasdynamic shock wave ahead of it toward the axis of the discharge. This cylindrical piston-shock problem is solved without further reference to electromagnetic effects. First the Lagrangian equations are solved for a parabolic shock trajectory in the r-t plane yielding a first and second approximation for the piston trajectory. To determine the accuracy of the approximation, the same problem is solved for a straight shock in the r-t plane by the method of characteristics in using the Eulerian formulation. It is found that the solutions given by the two methods compare exactly where a solution to the problem using the method of characteristics exists. The results are in qualitative agreement with relevant experimental observations.

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LIST OF SYMBOLS

A	Initial nondimensional velocity of the shock
B	Nondimensional acceleration of the shock
c	Local sound speed of the medium
c_{oo}	Sound speed of undisturbed gas
$H(\psi, t)$	Trajectory of the particle ψ in the r-t plane
L	Initial radius of the piston
M	Mach number of the shock
p	Local pressure
p_1	Pressure immediately after the shock
p_{oo}	Pressure of the undisturbed gas
$P(t)$	Piston trajectory in the r-t plane
R	Gas constant
$R_s(t)$	Shock trajectory in the r-t plane
r	Radial coordinate
t	Time coordinate
t'	Time when the shock crosses a particle ψ
T_o	Nondimensionalizing time
U	Velocity of the shock
u	Local flow velocity
x	Nondimensional position coordinate
X_s	Nondimensional shock trajectory
x_{first}	First approximation to the trajectory of a particle ϕ
γ	Ratio of specific heats
ε	$(\gamma - 1)/(\gamma + 1)$
ρ	Local density
ρ_1	Density immediately after the shock
ρ_{oo}	Density of the undisturbed gas
τ	Nondimensional time
ϕ	Nondimensional stream function identifying a particle
ψ	Stream function: mass between the piston and the particle
ψ_s	Value of the stream function at the shock

I. INTRODUCTION

The object of the research presented here is to formulate a theoretical model of the plasma pinch on the basis of gas dynamics in order to determine what effects observed experimentally might be due only to the gas dynamics of the problem.

The plasma pinch under consideration has been described by Jahn and von Jaskowsky.[1,2,3,4] Briefly, the physical apparatus consists of two plane circular electrodes of four, five, or eight inches diameter, separated by a two inch gap of test gas. Typically the test gas is Argon, although several other gases have been used. The electrodes are connected to two different types of external circuits--simple lumped capacitor arrays, or pulse-forming networks[3,4]--which are discharged through the gap via a gas-triggered switch.[5] It is observed that the discharge across the electrodes begins as a cylindrical sheet at the perimeter of the electrodes and propagates radially inward, driven by the electromagnetic interaction of the discharge current with its own magnetic field. The radial transit time for the current sheet is on the order of a few microseconds. Such pinches have been studied quite extensively by a variety of photographic and internal probe techniques.[1,2,6,7]

In the study which follows, this pinch process is idealized to the problem of a radially driven cylindrical piston, representing the current sheet, generating a gas dynamic strong shock which propagates ahead of the piston into the undisturbed gas in the center of the cylinder.

II. THEORY

A. DEVELOPMENT OF MASTER EQUATION

Consider a cylindrical piston of given height but variable radius. Suppose that suddenly the radius of this piston decreases so rapidly that a strong shock of very high Mach number is driven in front of the wall toward the axis of the cylinder. The question may be posed in two different manners. If we know the path of the piston in the r-t plane, what will the shock path look like? Conversely, if we know the shock path, what piston motion was necessary to drive it? From the formulation of the problem, it is found to be much easier to solve the latter case.

The first attempt to solve this problem was made from the Lagrangian viewpoint of following each particle. The claim is made that if we are considering two particles that are initially (before any shock has touched them) located at different radii in the cylinder, their respective radii will always be such that the particle initially closer to the axis will always be closer to the axis. That is, in the r-t plane, the particle paths will never intersect. This immediately leads to a law for conservation of mass and a method for labeling each particle. Consider Fig. 1 which shows the path of the piston $P(t)$, the path of the shock $R_s(t)$, and the path of any particle $H(\psi, t)$. ψ will be defined such that every particle beginning at the same radius will have the same ψ, but particles beginning at different radii will have different ψ. Once a particle is labeled by a ψ, it will retain this value throughout its history. Physically, the stream function ψ corresponds to the mass between the piston and the particle under question. On the figure, t' refers to the time when the shock passes over the particle ψ. Thus, if the shock trajectory is known, then the stream function may be said to be a function of t'. Conversely, t' may be regarded as a function of ψ.[8,9,10,11] Specifically, we define

$$\psi = \pi \rho_\infty \left[L^2 - R_s(t')^2 \right] \tag{1}$$

L is the initial radius of the chamber. $R_s(t')$ is the radial position of the particle ψ before the shock hits it. ρ_∞ is the density of the undisturbed gas. At any later time, ψ may be found by taking the integral

$$\psi = - \int_{P(t)}^{H(\psi,t)} 2\pi \rho(r,t)\, r\, dr \tag{2}$$

$\rho(r,t)$ is the density at any point on the r-t plane. Since there are only two independent variables, we may consider $r = r(\psi,t)$ or we may say that $\psi = \psi(r,t)$. The Lagrangian point of view makes ψ and t the independent variables.

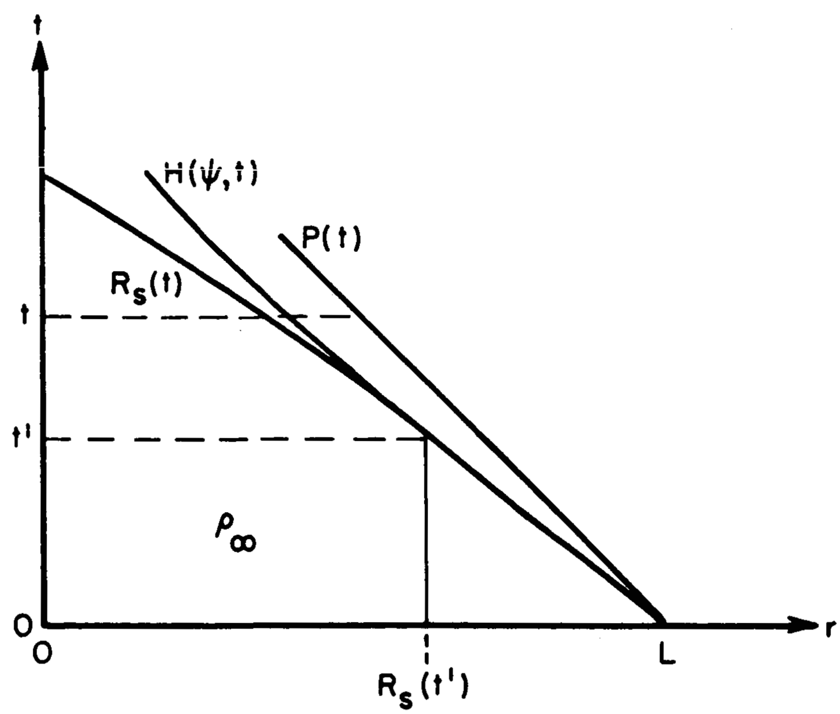


Figure 1

The momentum equation is seen to be

$$\rho \frac{\partial^2 r}{\partial t^2} = - \frac{\partial p}{\partial r} \tag{3}$$

where p is the pressure acting on the particle to accelerate it.

The process is assumed to be entirely isentropic except for a jump in entropy as the shock crosses the particle's path. Therefore, the ratio p/ρ^γ is a constant for each particle as it travels from the shock toward the axis. This constant is given by the conditions immediately after the shock. The pressure and density of a particle immediately after the shock has crossed it will be denoted by the subscript 1. γ is the ratio of specific heats, assumed constant.

$$\frac{p}{\rho^\gamma} = \frac{p_1}{\rho_1^\gamma} = \frac{p_1\ (\psi)}{\rho_1^\gamma\ (\psi)}$$

Note that p/ρ^γ is a function only of the streamline. This function is known if the shock trajectory is known. That is, p_1 and ρ_1 can be found by using the strong shock relations in conjunction with the perfect gas law. Specifically

$$\rho_1 = \frac{\gamma + 1}{\gamma - 1} \rho_\infty = \frac{\rho_\infty}{\varepsilon} \tag{4}$$

where $\varepsilon \equiv \dfrac{\gamma - 1}{\gamma + 1}$

and $p_1 = p_\infty \dfrac{2\gamma}{\gamma+1} M^2 = p_\infty \gamma (1-\varepsilon) \dfrac{U^2}{c_\infty^2} = p_\infty \gamma (1-\varepsilon) \dfrac{\dot{R}_s^2(t')}{\gamma \dfrac{p_\infty}{\rho_\infty}}$

or

$$p_1 = \rho_\infty (1 - \epsilon) \dot{R}_s^2(t') \tag{5}$$

Thus the isentropic condition is

$$\frac{p}{\rho^\gamma} = \rho_\infty^{1-\gamma} \epsilon^\gamma (1 - \epsilon) \dot{R}_s^2(t') \tag{6}$$

In these relations

p_∞	is the pressure of the undisturbed gas
M	is the Mach number of the shock
U	is the velocity of the shock
c_∞	is the speed of sound in the undisturbed gas
$\dot{R}_s(t')$	is the velocity of the shock as it crosses the particle $\psi = \psi(t')$. That is, $\dot{R}_s^{\,2}$ is the square of the derivative of the shock trajectory with respect to time evaluated at the time t'.

These conservation relations will now be combined into one relationship.* First, by differentiation, equation (2) becomes

$$d\Psi = - 2\pi\rho\, r\, dr = -\pi\rho\, d(r^2) \tag{7}$$

The minus sign is a result of the stream function increasing as r is decreasing. After dividing through by ρ , this relation may now be integrated from the shock to an arbitrary stream line

ψ to yield

$$R_s^2(t) - H^2(\psi,t) = \int_{\psi_s}^{\psi} \frac{d\psi}{\pi\rho} = \frac{1}{\pi}\int_{\psi_s}^{\psi} \frac{1}{p^{\frac{1}{\gamma}}} \left(\frac{p}{\rho^{\gamma}}\right)^{\frac{1}{\gamma}} d\psi$$

where ψ_s is the value of ψ at the shock at any time t. That is, $\psi_s = \psi_s(t)$. Note that this integration is done at the time t and not at the time t'. Substituting in the isentropic relation, we find

$$H^2(\psi,t) = R_s^2(t) + \frac{1}{\pi}\int_{\psi}^{\psi_s}\left[\rho_\infty^{1-\gamma}\ \varepsilon^{\gamma}(1-\varepsilon)\dot{R}_s^2(t')\right]^{\frac{1}{\gamma}} \frac{d\psi}{p^{\frac{1}{\gamma}}} \tag{8}$$

If $R_s = R_s(t)$ is given, then we may find a function $t - t(R_s)$. For example, if $R_s = L - At - Bt^2$, then

$$t(R_s) = t = \frac{-A + \sqrt{A^2 - 4B(R_s - L)}}{2B} \tag{9}$$

From the definition of ψ we note that

$$\psi_s(t) = \pi\rho_\infty L^2 - \pi\rho_\infty R_s^{\ 2}(t)$$

where ψ_s is the value of ψ at the shock at any time t. Hence

$$R_s(t) = \sqrt{L^2 - \frac{\psi_s}{\pi\rho_\infty}}$$

Therefore

$$t(R_s) = t\left(\sqrt{L^2 - \frac{\psi_s}{\pi\rho_\infty}}\right)$$

giving

$$\dot{R}_s(t') = t\left(\sqrt{L^2 - \frac{\psi}{\pi\rho_\infty}}\right)$$

which is a function only of ψ. Recall that t' is defined as the time when the particle ψ was crossed by the shock. In the example given above, equation (9) becomes

$$t' = t\left(\sqrt{L^2 - \frac{\psi}{\pi\rho_\infty}}\right) = \frac{-A + \sqrt{A^2 - 4B\left[\sqrt{L^2 - \frac{\psi}{\pi\rho_\infty}} - L\right]}}{2B}$$

yielding

$$\dot{R}_s(t') = -A - 2Bt' = -\sqrt{A^2 - 4B\left(\sqrt{L^2 - \frac{\psi}{\pi\rho_\infty}} - L\right)} \qquad (10)$$

This relation will be used later. Let us now determine an expression for p. From the momentum equation, we find that

$$\frac{\partial p}{\partial r}\,dr = -\rho\frac{\partial^2 r}{\partial t^2}\,dr = \rho\frac{\partial^2 r}{\partial t^2}\,\frac{d\psi}{2\pi\rho r}$$

having made use of equation (7). Upon integration from the shock to any particle ψ, this becomes

$$p(\psi,t) = p_1(t) + \frac{1}{2\pi}\int_{\psi_s(t)}^{\psi}\frac{\partial^2 r(\psi,t)}{\partial t^2}\,\frac{d\psi}{r(\psi,t)}$$

Note that this integration is done at the time t, any arbitrary time after the shock has crossed the particle ψ. Substituting equation (5) into this expression, the pressure is seen to be

$$P(\psi,t) = \rho_\infty (1-\epsilon)\,\dot{R}_s^2(t) + \frac{1}{2\pi}\int_{\psi_s(t)}^{\psi} \frac{\partial^2 r(\psi,t)}{\partial t^2}\,\frac{d\psi}{r(\psi,t)} \tag{11}$$

Putting this information back into equation (8), we find the master equation

$$H^2(\psi,t) = R_s^2(t) - \frac{\epsilon}{\pi\rho_\infty}\int_{\psi_s(t)}^{\psi}\left\{\frac{\rho_\infty(1-\epsilon)\,\dot{R}_s^2\left(t' = t\left(\sqrt{L^2 - \frac{\psi}{\pi\rho_\infty}}\right)\right)}{\rho_\infty(1-\epsilon)\,\dot{R}_s^2(t) + \frac{1}{2\pi}\int_{\psi_s(t)}^{\psi}\frac{\partial^2 r(\psi,t)}{\partial t^2}\,\frac{d\psi}{r(\psi,t)}}\right\}^{\frac{1}{\gamma}} d\psi \tag{12}$$

Let us nondimensionalize the quantities appearing in the equation as follows

$$x = \frac{r}{L}\ ; \quad X_s = \frac{R_s}{L}\ ; \quad \emptyset = \frac{\psi}{\pi L^2}\ ; \quad \tau = \frac{t}{T_o}$$

Note that $0 \leq x$, X_s, and $\emptyset \leq 1$. T_o is chosen so that τ is in microseconds, corresponding to the observation that all discharges have a pinch time of a few microseconds. The factor nondimensionalizing the stream function is the total mass contained in the chamber.

It should be mentioned that $H(\psi,t)$ is exactly the same quantity as $r(\psi,t)$. With these additions, the master equation becomes

$$x^2(\phi,\tau) = X_s^2(\tau) + \varepsilon \int_{\phi}^{\phi_s = 1 - X_s^2(\tau)} \left\{ \frac{\left[\frac{dX_s}{d\tau}\left(\tau' = \tau\left(\sqrt{1-\phi}\right)\right)\right]^2}{\left[\frac{dX_s}{d}(\tau)\right]^2 + \frac{1}{2(1-\varepsilon)} \int_{\phi_s = 1 - X_s^2(\tau)}^{\phi} \frac{\partial^2 x(\phi,\tau)}{\partial \tau^2} \frac{d\phi}{x(\phi,\tau)}} \right\}^{\frac{1}{\gamma}} d\phi \qquad (13)$$

This equation implies that a particle ϕ is, at a time τ, at a position, away from the shock by a distance equal to ε multiplied by an integral, the integrand of which consists of a numerator representing the isentropic condition and a denominator which is a constant fraction of the pressure. ε varies from 0 for $\gamma = 1$ to .25 for $\gamma = 1.667$. Recall that the entire integrand is $\frac{1}{\rho}$. The piston trajectory is $x(0,t)$; that is, the piston is always at the particle $\phi = 0$.

Since the right hand side of the equation involves a second derivative of the desired solution, a method of iteration must be used. Due to the difficulty in taking derivatives numerically, we should search for an analytic solution that may be placed back into the equation. From this analytic solution, a second approximation may be found, numerically if necessary, and compared to the first solution. If the comparison is good, then the solution may be assumed good. From streak and Kerr-Cell photographs, it seems that the luminous front in the plasma pinch always propagates toward the center in a relatively smooth manner. Hence, most interesting cases may be covered by assuming a parabolic shock trajectory:

$$R_s = L - \frac{ALt}{T_o} - \frac{BLt^2}{T_o^2}$$

or:

$$X_s = 1 - A\tau - B\tau^2$$

A is the initial nondimensional velocity of the shock and B is the shock's constant acceleration or deceleration toward the axis, depending upon whether B is positive or negative, respectively. Substituting this shock trajectory and equation (10), modified for the nondimensional variables and shock trajectory, into the nondimensional master equation, we find

$$x^2(\phi,\tau) = X_s^2(\tau) + \epsilon \int_{\phi}^{1 - X_s^2(\tau)} \left\{ \frac{A^2 + 4B\left(1 - \sqrt{1-\phi}\right)}{(A+2B\tau)^2 - \frac{1}{2(1-\epsilon)} \int_{\phi}^{1-X_s^2(\tau)} \frac{\partial^2 x(\phi,\tau)}{\partial \tau^2} \frac{d\phi}{x(\phi,\tau)}} \right\}^{\frac{1}{\gamma}} d\phi \quad (14)$$

As a first approximation, it may be assumed that the pressure does not change much between the shock and the piston. In effect, we assume

$$\frac{1}{2(1-\epsilon)} \int_{\phi}^{1 - X_s^2} \frac{\partial^2 x}{\partial \tau^2} \frac{d\phi}{x} << (A + 2B\tau)^2$$

This assumption is good when two conditions are met. First, we must not be too close to the axis. Otherwise, x will be small, making 1/x large. Secondly, the second derivative of x with respect to τ should be small and preferably negative. If $\frac{\partial^2 x}{\partial \tau^2} > 0$, then the denominator of the integrand of the large integral may approach zero. These conditions are met at least in the beginning stages of the piston's propagation. With this assumption, equation (12) becomes

$$x^2(\phi,\tau) = X_s^2(\tau) + \frac{\varepsilon}{(A+2B\tau)^{2/\gamma}} \int_{\phi}^{1-X_s^2} \left(A^2+4B-4B\sqrt{1-\phi}\right)^{1/\gamma} d\phi$$

which readily integrates to a first approximation for x

$$x_{first}^2(\phi,\tau) = X_s^2(\tau) + Q\,G(\tau)\,F(\tau) - Q\,G(\tau)\,H(\phi) \tag{15}$$

where

$$Q = \frac{2\,\varepsilon\,\gamma\,(A^2+4B)^{\frac{1}{\gamma}}}{P^2(\gamma+1)} \qquad \text{with} \quad P = \frac{4B}{A^2+4B}$$

$$G(\tau) = (A+2B\tau)^{-\frac{2}{\gamma}}$$

(16)

$$F(\tau) = PX_s(1-PX_s)^{\frac{\gamma+1}{\gamma}} + \frac{\gamma}{2\gamma+1}(1-PX_s)^{\frac{2\gamma+1}{\gamma}}$$

$$H(\phi) = P\sqrt{1-\phi}\left(1-P\sqrt{1-\phi}\right)^{\frac{\gamma+1}{\gamma}} + \frac{\gamma}{2\gamma+1}\left(1-P\sqrt{1-\phi}\right)^{\frac{2\gamma+1}{\gamma}}$$

In order to find the second approximation, we must substitute the second derivative of $x_{first}(\phi,\tau)$ with respect to τ into the subintegral of equation (14). Denoting all first derivatives with respect to τ as primes and second derivatives as double primes, the second derivation may be written as

$$\frac{1}{x}\frac{\partial^2 x}{\partial \tau^2} = \frac{x''}{x} = \frac{2(A+2B\tau)^2 - 4BX_s + Q(GF'' + 2G'F + G''F - G''H)}{2x^2} - \frac{[2X_s(-A-2B\tau) + QGF' + QG'F - QG'H]^2}{4x^4} \tag{17}$$

Letting $s = \frac{1}{\gamma}$ we find for the derivatives used above that

$$G' = -4sB(A + 2B\tau)^{-2s-1} \quad \text{and} \quad G'' = 8sB^2(2s+1)\ (A+2B\tau)^{-2s-2}$$

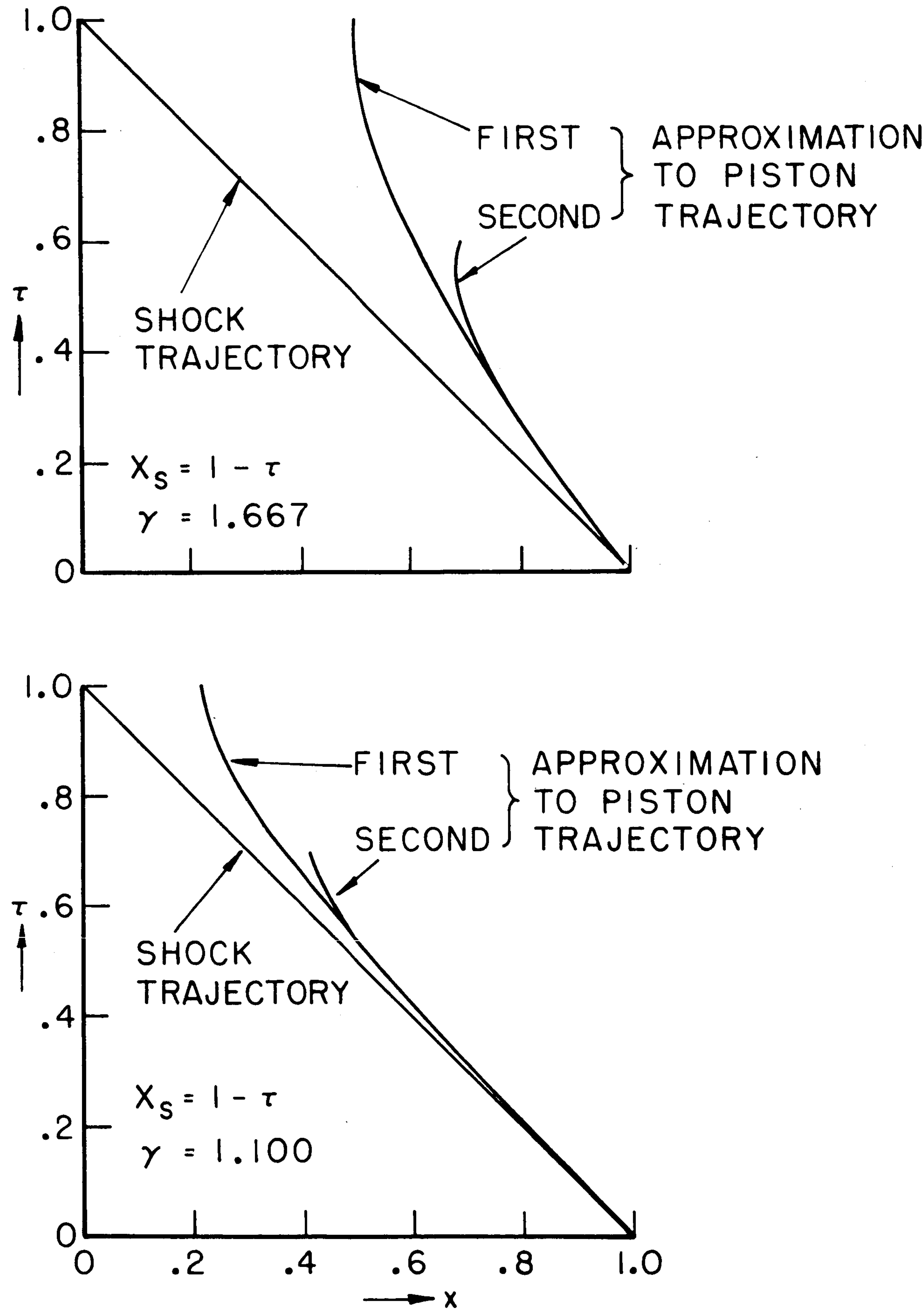
$$F' = P^2(s+1)X_s(1 - PX_s)^s(A + 2B\tau)$$

$$F'' = -P^2(s+1)(1 - PX_s)^s \left\{(A+2B\tau)^2 - \frac{PsX_s(A+2B\tau)^2}{1 - PX_s} - 2BX_s\right\}$$

From this point, we invoke a computer program to perform a numerical integration to determine a second approximation. This program is shown and discussed in Appendix A.

B. RESULTS FROM MASTER EQUATION

Graphs showing the piston trajectories for a straight shock, an accelerating shock, and a decelerating shock are shown in Figs. 2, 3, and 4. Two values of γ are used: 1.1 and 5/3. The latter value is the value for Argon at standard temperatures and pressures. Though this will not be close to the real value of γ after the Argon has experienced ionization, this value



PISTON TRAJECTORIES FOR CONSTANT VELOCITY SHOCK

FIGURE 2

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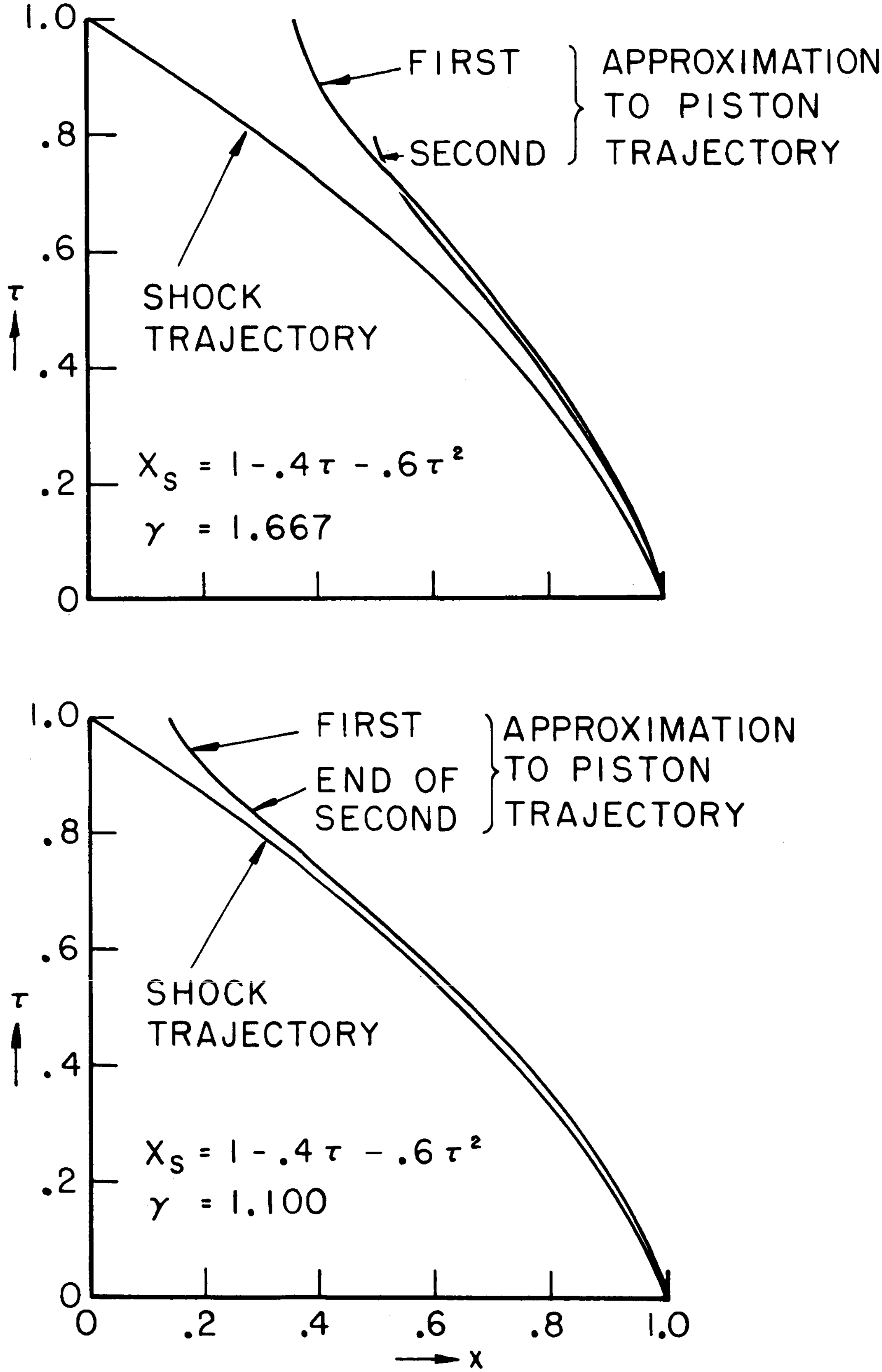

PISTON TRAJECTORIES FOR ACCELERATING SHOCK

FIGURE 3

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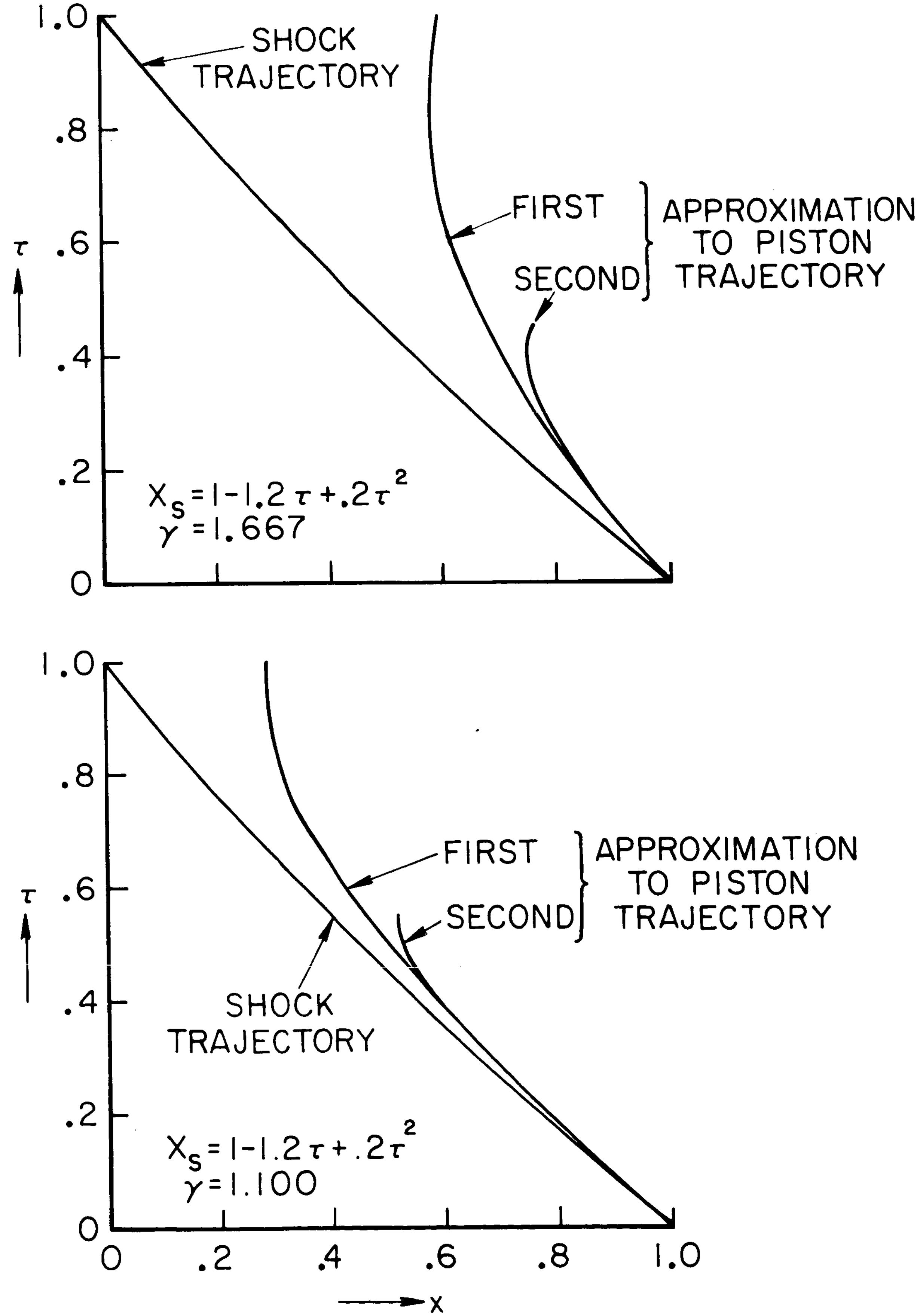


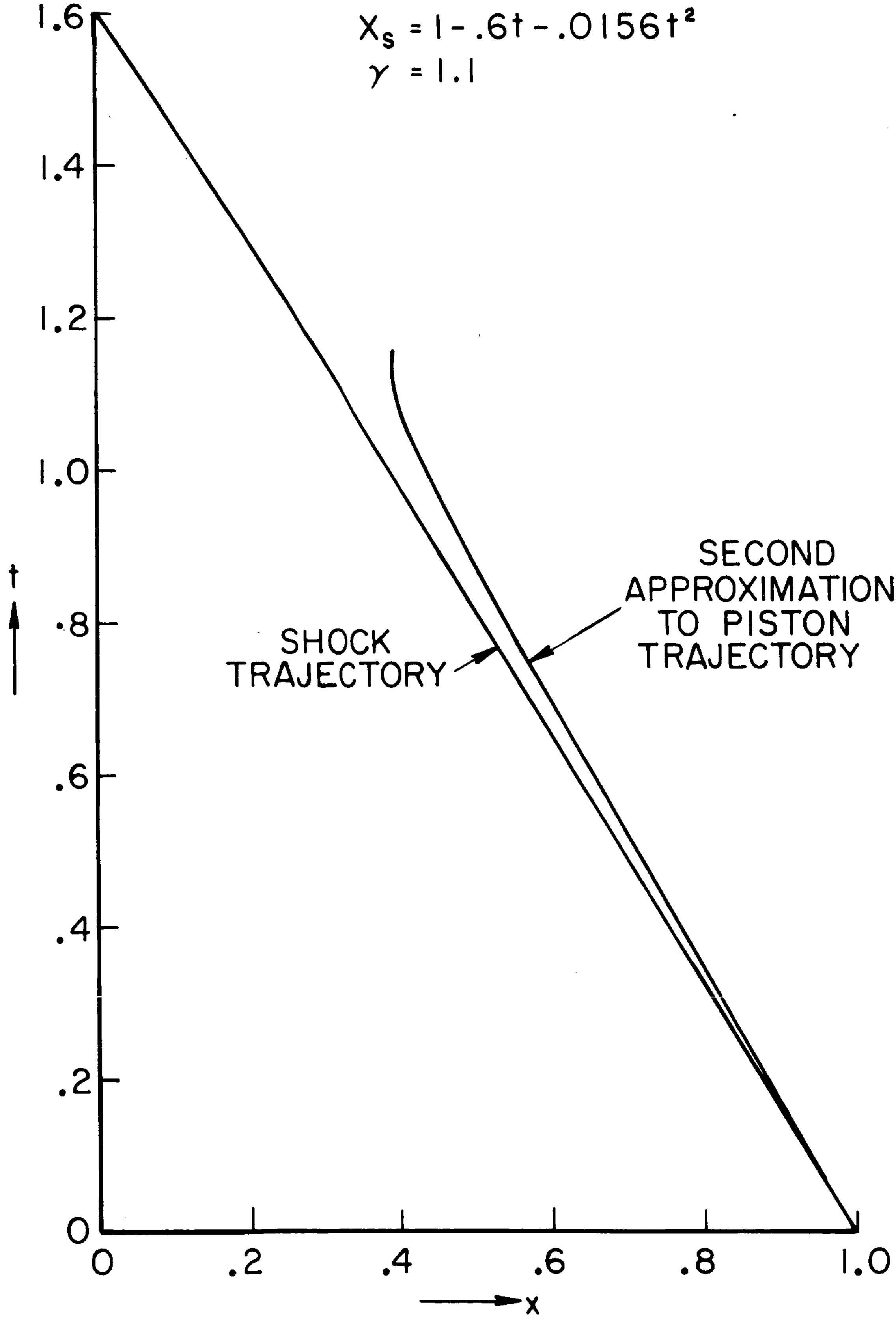
PISTON TRAJECTORIES FOR DECELERATING SHOCK

FIGURE 4

AP25-R4049-65

will show the worst qualities of the master equation. As $\varepsilon = \frac{\gamma - 1}{\gamma + 1}$ becomes larger, due to a larger γ, the first approximation becomes worse in that the entire integral is multiplied by a larger number. Further, the subintegral, which is neglected in the first approximation, is made more significant by the factor $\frac{1}{1-\varepsilon}$. In the pinch process, as observed in the laboratory, the Argon experiences first and second ionization, possibly even third; therefore, much of the energy available is absorbed in the ionization process. Hence, the real value of γ that may be expected would be closer to 1.1 than to 5/3.

Figures 2, 3, and 4 are based on a pinch time for the shock of 1.0 microseconds. This gives a convenient basis for comparison. There is no qualitative difference apparent if the time scale is expanded as may be seen from Fig. 5 which shows a pinch time of 1.6 microseconds. Figure 2 shows a constant velocity shock (B = 0) with the piston path computed for $\gamma = 1.1$ and 1.667. For $\gamma = 1.667$ it is seen that the first and second approximations for the piston trajectory are very close together until the piston reaches a radial position of 3/4. At this point, the second approximation diverges from the first approximation and ultimately turns back toward its initial position. This occurs as the denominator of the integrand of the master equation approaches zero, beyond which the second approximation is no longer calculated. Physically, a decrease in the denominator corresponds to a decrease in pressure at the piston. It is logical that the pressure decrease from the shock to the piston at a given time because in this region of the flow, there is, effectively, quasi-steady supersonic flow into a converging channel, which implies a decrease in velocity and a corresponding adverse pressure gradient. Since the conditions behind the shock are fixed, the pressure at the piston must be steadily decreasing as the gap between the shock and piston widens.



SECOND APPROXIMATION TO NEARLY LINEAR SHOCK

FIGURE 5

AD25-R4052-65

When the pressure at the piston face reaches zero, the model develops a singularity. Note that for $\gamma = 1.1$ the piston travels slightly more than half way to the axis of the cylinder before turning around.

Figure 3 shows the results for an accelerating shock. For $\gamma = 1.1$ there is no difference large enough to be seen between the first and second approximations until the second approximation reaches the zero pressure limit. This occurs at $x(o\ ,\tau) = .27$. Note that this radius is much smaller than the final radius of the piston for the constant velocity shock. This fact implies that the piston pushing an accelerating shock has control over the shock for a longer time than the piston pushing a constant velocity shock. For $\gamma = 1.667$, the accelerating shock has a piston path given by the second approximation that is closer to the center than the first approximation. This is due to the effect of the subintegral in the master equation. That is, in order to accelerate the flow, the pressure at the piston must be greater than the pressure at the shock. For the first approximation, this pressure difference is neglected. In the second approximation it is included. This effect is also present for the $\gamma = 1.1$ case; however, it is so small that it cannot be seen on the scale of Fig. 3.

Figure 4 shows the case of a decelerating shock. There is little new on this graph except that the piston turns back even sooner than it does for the constant velocity shock.

C. METHOD OF CHARACTERISTICS

It is of interest to determine what the real piston trajectory looks like after the time when the first and second approximations diverge. A third approximation was not attempted because of the inherent limitations on numerical calculations of the needed derivatives. Instead, it was decided to leave the master equation and to attempt to find a solution by the method of characteristics.

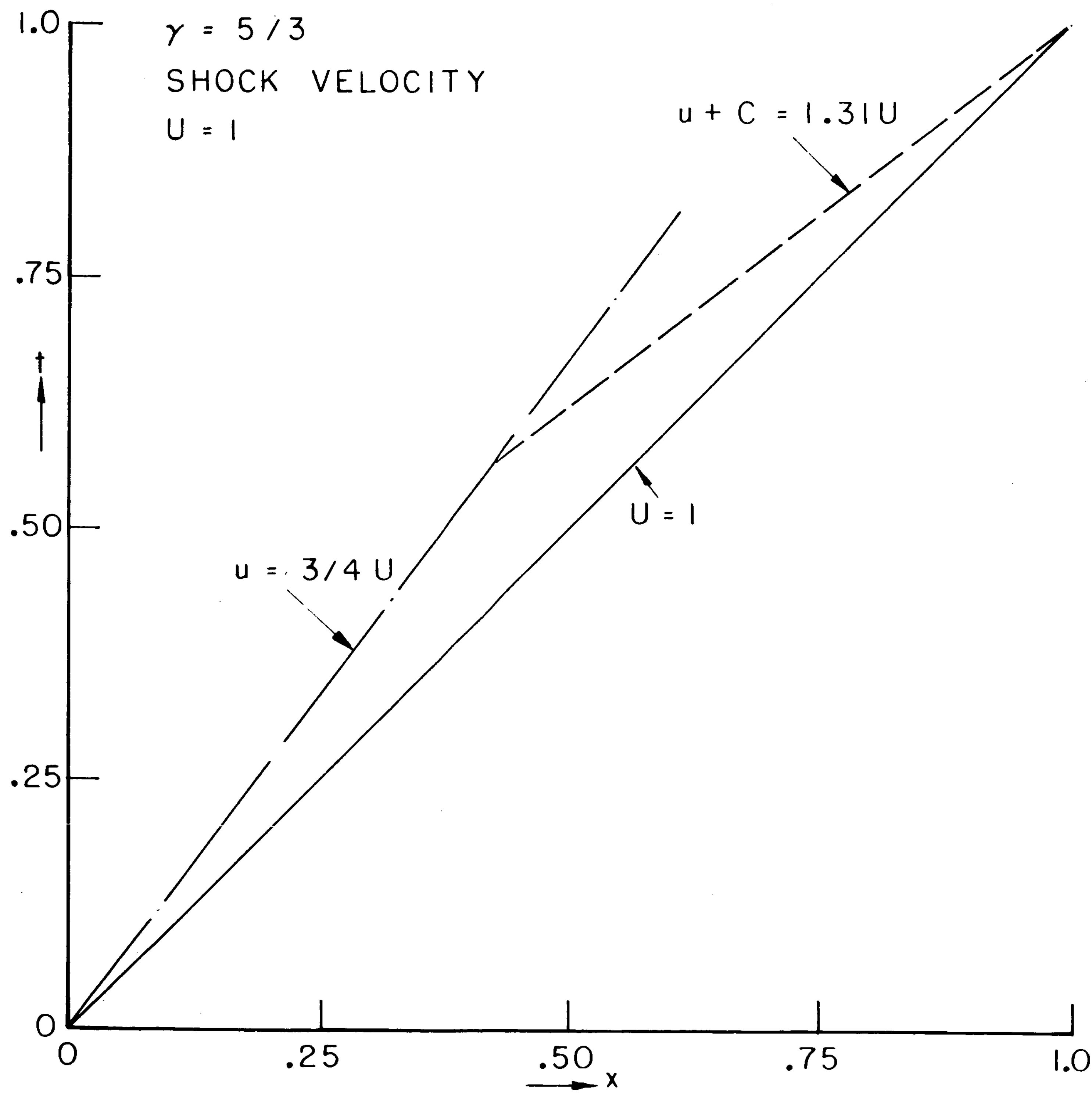
If we make the assumption of isentropic flow behind the shock, we may use the equations derived in other references.[8,9] To have isentropic flow behind the shock, we must limit ourselves to a linear shock in the r-t plane. This will be adequate to examine the nature of the divergence of the previously found first and second approximations.

It is first instructive to consider the problem of one-dimensional, unsteady, isentropic flow in rectangular coordinates with no area change. For this problem, it is known that the characteristic directions in the t-x plane are u + c and u - c where u is the local flow velocity and c is the local sound speed. If a piston is pushed down a tube in such a manner as to set up a strong shock propagating ahead of it, u and c will remain constant throughout the flow field as long as the piston velocity is constant. From the strong shock relations, u and c are calculated to be

$$u = \frac{2}{\gamma + 1} U \qquad\qquad c = \frac{\sqrt{2\gamma(\gamma - 1)}}{\gamma + 1} U$$

where U is the velocity of the shock. In this problem, the piston velocity is equal to the flow velocity. For $\gamma = 1.667$, Fig. 6 shows a piston traveling at a speed of u - 3/4 pushing a strong shock with speed U = 1. As seen from the figure, the last signal that may be sent from the piston and received by the shock before the shock reaches x = 1 must leave the piston before the piston is at x = ½. Thus the piston loses contact with the shock after the piston has propagated half way into the medium. The similar problem posed in cylindrical coordinates becomes slightly more difficult. The equations governing one-dimensional, unsteady, isentropic flow in cylindrical coordinates from the Eulerian viewpoint are written:

Mass conservation or continuity $$\frac{\partial(\rho u)}{\partial r} + \frac{\partial \rho}{\partial t} + \frac{\rho u}{r} = 0$$



PISTON AND SHOCK TRAJECTORY

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FIGURE 6

momentum or Newton's Law $$\frac{1}{\rho}\frac{\partial p}{\partial r} + u\frac{\partial u}{\partial r} + \frac{\partial u}{\partial t} = 0$$

isentropic relation $$\frac{P}{\rho^{\gamma}} = \text{constant}$$

perfect gas relation $$P = \rho R T$$

and sound speed $$c^2 = \frac{dP}{d\rho}$$

These equations may be put in a more useful form as shown by Courant and Friedrichs.[12]

Continuity $$\rho_t + u\rho_r + \rho u_r + \frac{\rho u}{r} = 0$$

Newton's Law $$c^2\rho_r + \rho u\, u_r + \rho u_t = 0 \qquad (19)$$

sound speed $$c^2 = \text{constant}\ \rho^{\gamma-1}$$

where the subscripts here denote partial differentiation with respect to the subscript. The shock conditions apply across the shock trajectory $R_s = L - At$:

$$u_1 = -\frac{2A}{\gamma+1} \qquad \rho_1 = \frac{\rho_\infty}{\varepsilon} \qquad M_1 = \frac{2}{\sqrt{2\gamma(\gamma-1)}}$$

$$p_1 = \frac{2\rho_\infty A^2}{\gamma+1} \qquad c_1 = A\sqrt{\frac{2\gamma\varepsilon}{\gamma+1}}$$

giving

$$c^2 = \frac{2A^2 \gamma \varepsilon^{\gamma}}{(\gamma + 1)\rho_\infty^{\gamma-1}} \rho^{\gamma-1}$$

By the normal recipe outlined in Appendix B, we find the characteristic equations to be

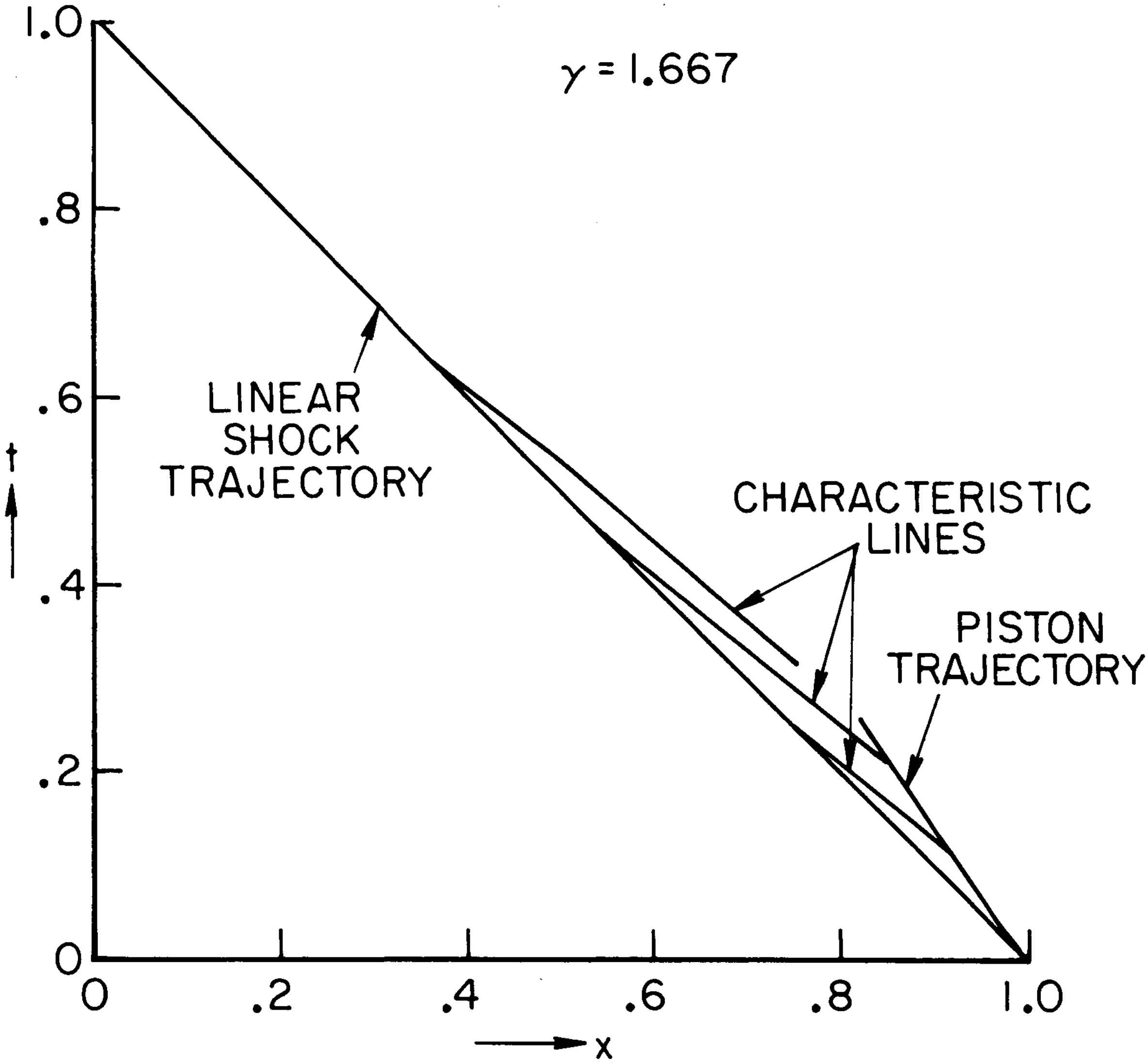
$$\text{I:} \quad dr = (u + c)\, dt \qquad\qquad \text{II:} \quad dr - (u - c)\, dt$$

$$\text{I:} \quad du + \frac{2}{\gamma-1}\, dc + \frac{uc}{r}\, dt = 0 \qquad \text{II:} \quad du - \frac{2}{\gamma-1}\, dc - \frac{uc}{r}\, dt = 0 \tag{20}$$

These equations are exactly the same as those for one-dimensional, isentropic, unsteady flow with constant area, except for the term $\frac{uc}{r}$ dt which adds the only complication to the problem. Once again we must resort to a computer program to find the solution. The program used may be found in Appendix C along with some comments on its structure.

D. RESULTS FROM CHARACTERISTICS

In Fig. 7 are shown the prescribed shock path, the piston trajectory as found from the method of characteristics, and three characteristic lines. The graph is drawn for $\gamma = 1.667$. If we consider the characteristics to run backwards from the shock to the piston, we note that they bend in such a manner as to intersect the piston trajectory much earlier than they would have if they had remained straight. Turning this around so that the characteristics run forward in time, we find that the piston loses control of the shock much sooner in the imploding cylindrical piston problem than it does in the constant-area piston-shock



PISTON TRAJECTORY AND CHARACTERISTIC LINES

AP25-R4050-65

FIGURE 7

problem. Specifically, the piston loses control of the shock by the time the piston is only about ¼ of the way toward the center of the cylinder. Recall that this is the same distance that the first and second approximations to the master equation for a linear shock coincided. When the results from the approximate method are compared to the results obtained by the method of characteristics, it is seen that the piston trajectories for the given shock compare exactly. The longest characteristic line shown in the figure intersects the shock only at $x = .35$, because the program did not produce any characteristic lines that extended further. These lines were not produced due to convergence problems for the portion of the cylinder close to the axis. Even the longest characteristic line is cut off before it reaches the shock; this is also due to a fault of the program. However, by estimating the paths of future characteristic lines, it appears as though the fate of the shock is determined by that portion of the piston's trajectory that is very close to the beginning. After this very early control of the shock by the piston, it matters not what path the piston takes. Any message that is sent out from the piston toward the shock after the piston has traveled more than ¼ of the way toward the center, will not be received by the shock before the shock reaches the center of the cylinder. This discussion has been for $\gamma = 1.667$. For γ any smaller, the piston would control the shock for a longer period of time, but would behave qualitatively the same.

E. INTERPRETATION

We now know that the diverging portion of the first and second approximations to the master equation solution for the straight shock does not have physical relevance. However, the portion of the two approximations that do coincide is the true solution for the piston driving the prescribed straight shock.

Any answer from this time on must not be considered accurate unless some further boundary condition is prescribed.

When this argument is extended to the accelerating and decelerating shock trajectories, it would seem to imply that where the first and second approximations are very close together, the solution is good for the given boundary condition. Referring to Fig. 2, 3, and 4 it is seen that for a given pinch time and for a given γ, an accelerating shock is controlled more by its piston than is a decelerating shock.

III. SUMMARY

The two major conclusions from this analysis are given some qualified endorsement by experimental observations. First, the small separation between the piston and shock for the low γ cases evident in Figs. 2-5 is not inconsistent with the experimental inability to distinguish a shock front from the intense luminosity of the current sheet on streak and Kerr-Cell photographs. The theory suggests that if such separation is to be observed, it will be most evident in cases of rapidly decelerating fronts. These cases have not been studied intensively in the past because of their low dynamic efficiencies for propulsion purposes.

Second, the prediction that the shock trajectory is determined by the very early portion of the piston path concurs with the observed insensitivity of the pinch processes to the nature of the inner portions of the electrodes. In many experiments, all but the outer inch of electrode has been replaced by glass, or removed entirely, with minimal effect on the development of the pinch pattern.[1]

Clearly, the definitive experiments must involve a positive identification of the shock fronts in relation to the current sheets, presumably by sensitive pressure gauges. Such experiments are currently in progress, and their results will soon be compared with this theory.

APPENDIX A

The following discussion concerns the numerical solution of the nondimensional master equation, equation (14), given a parabolic shock trajectory. If the shock has a constant velocity, B = 0, then P = 0, making Q = ∞ in equations (16). Therefore, the equations must be recast for the degenerate case of B = 0. Equation (14) becomes

$$x^2(\phi,\tau) = X_s^2(\tau) + \varepsilon \int_{\phi}^{1 - X_s^2} \left\{ \frac{A^2}{A^2 - \frac{1}{2(1-\varepsilon)} \int_{\phi}^{1 - X_s^2} \frac{\partial^2 x(\phi, \quad)}{\partial \tau^2} \frac{d\phi}{x(\phi, \tau)}} \right\}^{\frac{1}{\gamma}} d\phi \qquad (14')$$

For the first approximation, neglecting the subintegral, eq. (14') integrates to

$$x^2_{first} = X_s^2 + \varepsilon\,(1 - X_s^2) - \varepsilon\,\phi \qquad (15')$$

When x_{first} is substituted back into (14'), x_{second} is found to be

$$x^2_{second} = X_s^2 + \varepsilon \int_{\phi}^{1 - X_s^2} \left\{ \frac{2\,\varepsilon}{Q - \ln(F - \phi) - \frac{R}{(F - \phi)}} \right\}^{\frac{1}{\gamma}} d\phi \qquad (16')$$

where

$$Q = 1 + \varepsilon + \ln \frac{x_s^2}{\varepsilon} \quad ; \quad R = \frac{(1-\varepsilon)x_s^2}{\varepsilon} \quad \text{and } F = R + 1 \qquad (17')$$

The actual program used appears at the end of this discussion. The card numbers shown are found at the end of each card printout. Cards numbered from 90 to 930 are for the equations (14), (15), (16), (17), and (18). Cards numbered from 940 to 1440 are for equations (14'), (15'), (16') and (17'). The program was run for A = .999 and B = .001 and was also run for A = 1 and B = 0. The answers were the same to three significant figures. The general philosophy of the program is first to determine what time steps to use, and then when the time steps are known, to calculate all quantities that are dependent only upon time for the first $\Delta\tau$. Then, at the time under consideration, the quantities that depend on ϕ are calculated. Specifically, the value of the integrand for $\phi = \phi_s$ is first calculated (the subintegral is zero at this point). Then the values of ϕ for which we want the particle positions spelled out are determined. The program does this by taking ϕ_s, rounding it off to the next lowest .05, and then using $\Delta\phi$ in steps of .05, the steps for which the two numerical integrations are performed. The subintegral is found for the interval from ϕ_s to the next lower ϕ. With this value of the subintegral, the total integral may be found giving x for the rounded off ϕ and the time $\Delta\tau$. Using the next lower value of ϕ, the next portion of the subintegral is added to the value obtained above. Similarly, the next portion of the entire integral is added on to the part already found. This procedure is carried on until we reach the value $\phi = 0$, which is the piston. At this time, we proceed to the next time and repeat the entire process. Below are some comments on some cards that may not be obvious.

Card Number	Comments
85	If the shock is linear, B = 0, control is transferred to statement 200, card #940, to avoid dividing by zero as mentioned above.
130	TMAX is the pinch time of the shock.
140	R is the coefficient of the subintegral.
180	DELTAU is the time step used.
260	G1 = G', G2 = G", etc.
310,320	E and W are the parts of x"/x that depend only upon time.
330	XT is the part of x^2_{first} that depends only upon time.
340	PHIS = $\emptyset_s$ the value of $\emptyset$ at the shock at the time TAU.
370	HS is the value $H(\emptyset_s)$.
380	GR = x"/x at the shock.
390	GAR is the value of the integrand at the shock.
410	If the difference between $\emptyset_s$ and 0 is small, then the whole integral will be performed in one step.
420	HO is the value H(0).
440	PF is the value of x"/x at $\emptyset$ = 0.
460	PAM is the value of the integrand at $\emptyset$ = 0.
530	If the difference between $\emptyset_s$ and 0 is large, then the integral will be computed in steps.
540	First the integral is evaluated from $\emptyset_s$ to SIGMA.
650, 660	GR and GAR are replaced by PF and PAM respectively for the next part of the integration.
715	If the denominator of the integrand is zero or negative, control is switched to card #820.

Card Number	Comments
750	If SIGMA is equal to zero, then the piston position has been calculated for this time, in which case there is a special print out, TAU is increased by DELTAU and the entire integration is performed again. Otherwise, the value XFIRST, XSECOND, and Ø = SIGMA are printed out followed by the next step of the integration.
820	Since the pressure, according to the second approximation, is now zero or negative, card #715, only the first approximation is calculated and printed by card #920.
940-1440	The same procedure is followed except that it is simplified by having a linear shock and no subintegral. The equations for this portion are shown in the initial part of this Appendix.

```
C       GLEN A. ROWELL--APPROXIMATE SOLUTION TO THE MASTER EQUATION
   10 FORMAT (3F20.4)
   20 FORMAT (3H A=F7.4,4H  B=F7.4,8H  GAMMA=F6.3,10H  EPSILON=F6.4,
     1     7H  TMAX=F9.6//)
   30 FORMAT(53H1   FIRST AND SECOND APPROXIMATION TO THE PISTON PATH//)
   40 FORMAT (5H TAU=F7.4, 13H          XS=F6.4, 15H            PHIS=F6.4)
   50 FORMAT (3F20.6)
   60 FORMAT (2X,7HXFIRST=F8.6,13H     XSECOND=F8.6,10H       TAU=F7.4,
     1     6H  PHI=F4.2//)
   61 FORMAT (1H0///)
   65 FORMAT(4H XS=F7.4, 13H       XFIRST=F7.4, 10H       TAU=F8.4/)
   18 READ 10,A, B, GAMMA
      PRINT 30
      IF(B) 12, 200, 12
   12 S = 1./GAMMA
      P = 4.*B/(A*A+4.*B)
      EPSLON =(1.-S)/(1.+S)
      Q = 2.*EPSLON*GAMMA*(A*A+4.*B)**S/(P*P*(GAMMA+1.))
    8 TMAX = (-A+SQRTF(A*A+4.*B))/(2.*B)
      R = 1./(2.*(1.-EPSLON))
      PRINT 20,A, B, GAMMA, EPSLON, TMAX
      NMAX = TMAX*20.
      TAUMAX = NMAX
      DELTAU = TAUMAX*.0025
      TAU = DELTAU
    3 XS = 1.-A*TAU-B*TAU**2
      U = A+2.*B*TAU
      V = U*U
      Y = 1. - P*XS
      Z = Y**S
      G = U**(-2.*S)
      G1 = -4.*S*B*G/U
      G2 = 16.*S*B*B*(S+.5)*G/V
      F = P*XS*Z*Y+Z*Y*Y/(S+2.)
      F1 = P*P*(S+1.)*XS*Z*U
      F2 = -P*P*(S+1.)*Z*(V-P*S*XS*V/Y-2.*B*XS)
      E = 2.*V-4.*B*XS+Q*(G*F2 + 2.*G1*F1 +G2*F)
      W = Q*(G*F1+G1*F)-2.*XS*U
      XT = XS**2 + Q*G*F
      PHIS = 1.-XS**2
      PRINT 40, TAU, XS, PHIS
      RS = P*SQRTF(1.-PHIS)
      HS = RS*(1.-RS)**(S+1.)+(1.-RS)**(S+2.)/(S+2.)
      GR = (E-Q*G2*HS)/(2.*XS**2)-(W-Q*G1*HS)**2/(4.*XS**4)
      GAR = ((A*A+4.*B*(1.-XS))/V)**S
      L = 20.*PHIS
      IF(L-1) 70, 70, 80
   70 HO = P*(1.-P)**(S+1.)+(1.-P)**(S+2.)/(S+2.)
      XFIRST = SQRTF(XT-Q*G*HO)
      PF = (E - Q*G2*HO)/(2.*XFIRST**2) - (W -Q*G1*HO)**2/(4.*XFIRST**4)
      SUBINT = (PF + GR)/2.*PHIS
      PAM = (A*A/(V-SUBINT*R))**S
      BIGINT = (PAM + GAR)/2.*PHIS
      X = SQRTF(XS*XS + EPSLON*BIGINT)
      PHI = 0.0
      PRINT 60, XFIRST, X, TAU, PHI
      TAU = TAU + DELTAU
      GO TO 3
   80 C = L
      SIGMA = C*.05
      RSIGMA = P*SQRTF(1.-SIGMA)
```

```
      HSIGMA = RSIGMA*(1.-RSIGMA)**(S+1.)+(1.-RSIGMA)**(S+2.)/(S+2.)
      XFIRST = SQRTF(XT-Q*G*HSIGMA)
      PF = (E-Q*G2*HSIGMA)/(2.*XFIRST**2)-(W-Q*G1*HSIGMA)**2
     1     /(4.*XFIRST**4)
      SUBINT = (PF + GR)/2.*(PHIS - SIGMA)
      PAM = ((A*A + 4.*B*(1. - SQRTF(1. - SIGMA)))/(V - R*SUBINT))**S
      BIGINT = (PAM + GAR)*.5*(PHIS - SIGMA)
      X = SQRTF(XS*XS + EPSLON*BIGINT)
      PRINT 50, XFIRST, X, SIGMA
      SIGMA = SIGMA - .05
   85 GR = PF
      GAR = PAM
      RSIGMA = P*SQRTF(1.-SIGMA)
      HSIGMA = RSIGMA*(1.-RSIGMA)**(S+1.)+(1.-RSIGMA)**(S+2.)/(S+2.)
      XFIRST = SQRTF(XT-Q*G*HSIGMA)
      PF = (E-Q*G2*HSIGMA)/(2.*XFIRST**2)-(W-Q*G1*HSIGMA)**2
     1     /(4.*XFIRST**4)
      SUBINT = SUBINT + (PF + GR)/40.
      IF ( V - R*SUBINT ) 120,120, 87
   87 PAM = ((A*A + 4.*B*(1. - SQRTF(1. - SIGMA)))/(V - R*SUBINT))**S
      BIGINT = BIGINT + (PAM + GAR)*.025
      X = SQRTF(XS*XS + EPSLON*BIGINT)
      IF (SIGMA) 100, 90, 100
  100 PRINT 50, XFIRST, X, SIGMA
      SIGMA = SIGMA - .05
      GO TO 85
   90 PRINT 60, XFIRST, X, TAU, SIGMA
  110 TAU = TAU + DELTAU
      IF (TAU - TMAX)  3, 18, 18
  120 PRINT 61
      H  = P*(1.-P)**(S+1.)+(1.-P)**(S+2.)/(S+2.)
      XFIRST = SQRTF(XS*XS + Q*G*F - Q*G*H)
      PRINT 65, XS, XFIRST, TAU
  130 TAU = TAU + DELTAU
      G = (A + 2.*B*TAU)**(-2.*S)
      XS = 1. - A*TAU - B*TAU**2
      F = P*XS*(1. - P*XS)**(1.+S) + (1.-P*XS)**(2.+S)/(2.+S)
      H  = P*(1.-P)**(S+1.)+(1.-P)**(S+2.)/(S+2.)
      XFIRST = SQRTF(XS*XS + Q*G*F - Q*G*H)
      PRINT 65, XS, XFIRST, TAU
      IF (TAU-TMAX) 130, 18, 18
  200 S = 1./GAMMA
      EPSLON =(1.-S)/(1.+S)
      TMAX = 1./A
      PRINT 20,A, B, GAMMA, EPSLON, TMAX
      NMAX = 20.*TMAX
      TAUMAX = NMAX
      DELTAU = TAUMAX*.0025
      TAU = DELTAU
    4 XS = 1. - A*TAU
      XFIRST = SQRTF ((1. - EPSLON) * XS**2 + EPSLON )
      Q = 1. + EPSLON + LOGF(XS*XS/EPSLON)
      R = (1. - EPSLON)*XS*XS/EPSLON
      F = R + 1.
      PHIS = 1.-XS**2
      PRINT 40, TAU, XS, PHIS
      GR = (2.*EPSLON/(Q - LOGF(F-PHIS) - R/(F-PHIS)))**S
      L = 20.*PHIS
      IF(L-1)170,170,180
  170 RO = (2.*EPSLON/(Q-LOGF(F) - R/F))**S
      BIGINT = (RO+GR)*PHIS/2.
```

```
      X = SQRTF(XS*XS + EPSLON*BIGINT)
      PHI = 0.0
      PRINT 60, XFIRST, X, TAU, PHI
      TAU = TAU + DELTAU
      GO TO 4
  180 C = L
      PHI = C*.05
      PF = (2.*EPSLON/(Q - LOGF(F-PHI ) - R/(F-PHI )))**S
      BIGINT = (PF+GR)*(PHIS-PHI)/2.
      X = SQRTF(XS*XS + EPSLON*BIGINT)
      X1 = SQRTF((1.-EPSLON)*XS**2 + EPSLON*(1.-PHI))
      PRINT 50, X1, X, PHI
  185 GR = PF
      PHI = PHI - .05
      Z = Q - LOGF(F-PHI)-R/(F-PHI)
      IF(Z) 300, 300, 182
  182 PF = (2.*EPSLON/Z)**S
      BIGINT = BIGINT + (PF+GR)*.025
      X = SQRTF(XS*XS + EPSLON*BIGINT)
      IF(PHI) 190,190,1100
 1100 X1 = SQRTF((1.-EPSLON)*XS**2 + EPSLON*(1.-PHI))
      PRINT 50, X1, X, PHI
      GO TO 185
  190 PRINT 60, XFIRST, X, TAU, PHI
 1110 TAU = TAU + DELTAU
      IF (TAU - TMAX)  4, 18, 18
  300 X = SQRTF((1. - EPSLON)*XS**2 + EPSLON)
      PRINT 65, XS, X, TAU
      TAU = TAU + DELTAU
      XS = 1. - A*TAU
      IF (TAU-TMAX) 300, 18, 18
      END
```

APPENDIX B

The equations for continuity, Newton's law, and sound speed are

$$\rho_t + u\rho_r + \rho u_r + \frac{\rho u}{r} = 0$$

$$c^2 \rho_r + \rho u u_r + \rho u_t = 0$$

$$c^2 = \text{constant}\ \rho^{\gamma - 1}$$

The equation for the speed of sound may be differentiated to yield

$$\frac{2}{\gamma - 1}\frac{dc}{c} = \frac{d\rho}{\rho}$$

Substituting this into the two partial differential equations, we find

$$u_t + uu_r + \frac{2c}{\gamma - 1} c_r = 0$$

$$cu_r + \frac{2}{\gamma - 1} c_t + \frac{2u}{\gamma - 1} c_r + \frac{uc}{r} = 0$$

In order to find the characteristic directions for this pair of equations, we form an arbitrary linear combination of them by multiplying the first equation by λ and adding them.[12,13]

$$L = \lambda u_t + (\lambda u + c)u_r + \frac{2}{\gamma - 1} c_t + \left(\frac{2\lambda c}{\gamma - 1} + \frac{2u}{\gamma - 1}\right) c_r + \frac{uc}{r} = 0$$

We ask that this linear combination produce directional derivatives of u and c in the same directions. These directions, which depend upon r and t as well as the values of u and c at the point r,t, are the characteristic directions. For example, the derivative of u is taken in the directions

$$\frac{dt}{dr} = \frac{\lambda}{\lambda u + c}$$

If the derivative of c is taken in the same direction, it means that

$$\frac{\lambda}{\lambda u + c} = \frac{\frac{2}{\gamma - 1}}{\frac{2 \lambda c}{\gamma - 1} + \frac{2u}{\gamma - 1}} = \frac{dt}{dr}$$

Solving the first equality for λ, we find that

$$\lambda = \pm 1$$

Hence there are two characteristic directions

$$dr = (u + c)\ dt \qquad \text{and} \qquad dr = (u - c)\ dt$$

Substituting $\lambda = 1$ and $u + c = \frac{dr}{dt}$ into L, we find

$$L = \frac{\partial u}{\partial t} + \frac{\partial u}{\partial r}\frac{dr}{dt} + \frac{2}{\gamma-1}\frac{\partial c}{\partial t} + \frac{2}{\gamma-1}\frac{\partial c}{\partial r}\frac{dr}{dt} + \frac{uc}{r} = 0$$

or

$$L = \frac{du}{dt} + \frac{2}{\gamma - 1}\frac{dc}{dt} + \frac{uc}{r} = 0$$

This is the characteristic equation that corresponds to the characteristic direction given by $\lambda = 1$. Similarly, for $\lambda = -1$, we find the characteristic equation to be

$$du - \frac{2}{\gamma - 1}\,dc - \frac{uc}{r}\,dt = 0.$$

The general philosophy of this program is to calculate from the characteristic equations, equations (20), the characteristic directions and the values of the velocity and sound speed on a grid in the r-t plane. One set of grid lines is parallel to the shock trajectory at intervals of DT. The other set is perpendicular to the r axis and is spaced at intervals of DR. The piston position is calculated from the flow velocity of the particle that began at x = 1. Note that the flow velocities are negative since the flow direction is in the -r direction. The notation used is shown in the figure below

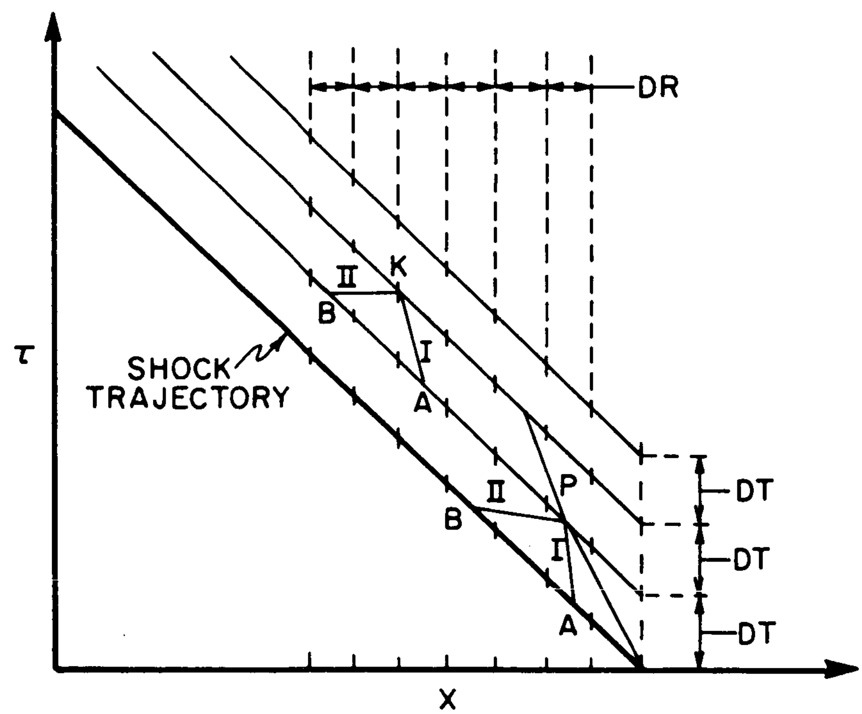


Figure 8

UI refers to the average flow velocity along the characteristic I. That is UI = ½(UA + UP), where UA refers to the flow velocity at the point A and UP at the point P. Similarly UII is the average flow velocity along a II characteristic. CI, CII, RI, and RII are the average sound speeds and positions. The quantities RA, UA, CA, TA, RP,..., RB,..., RK,... are the values of the position, flow velocity, sound speed, and time at the points A, P, B, and K. Initially, the piston position is found by assuming UP = U, the velocity behind the shock. Having this position, I and II characteristics are found assuming UI = U - UII and CI = C = CII where C is the speed of sound immediately following the shock. With the initial assumption that DELCI, ΔC along the I characteristic, is zero, DELUI, the change in flow velocity along the I characteristic may be found from the characteristic equations. With this information, DELUII may be found. Then DELCII may be calculated. With all this information, a new UP and CP may be found. Finally, a new average flow velocity of the piston may be found giving a new piston position. Meanwhile, the new values of UI are compared to the old values to determine whether the procedure should be done again. Further, the new values of RP and TP are also compared. When this has been done, the first grid point on the line parallel to the shock trajectory that is closer to the center of the cylinder is determined. For this point, RA, TA, RB, TB, UA, CA, UB, CB, RK, TK, UK, CK, UI, UII, CI, and CII are found and recorded. When this first line parallel to the shock is found, the conditions along the second line are found in much the same manner. However, the conditions at A and B are not so easily found because these points may fall between the previously recorded grid points. Therefore, a simple interpolation is performed.

L is defined in the program and printed out so that we know what leg of the program is being computed. The quantity A is read into the program from the data card, determines the shock pinch time as 1/A. G represents Gamma.

```
 10 FORMAT(51H1 STRAIGHT SHOCK BY CHARACTERISTICS, GLEN A. ROWELL///)
 15 FORMAT(5F12.6)
 17 FORMAT(6HGAMMA=F6.3, 11H   DELTA R=F5.3, 11H   DELTA T=F6.4,
  1          5H   A=F5.2,12H  1 PART IN F6.0//)
 20 FORMAT (20H                   O=F13.6, 14H       LEG NO.I2)
 30 FORMAT (2F20.6)
 40 FORMAT (9HORPISTON=F7.4, 12H    TPISTON=F7.4 , 16H    PISTON MACH=
  1          F 7.4 /)
 50 FORMAT(3H R(I2,2H)=F6.4,4H  T(I2,2H)=F7.4,4H  M(I2,2H)=F7.4)
 60 FORMAT (2F9.4,12H      UI+CI=F7.4, 2F9.4,14H       UII-CII=F7.4)
 70 FORMAT (4H RA=F6.4,6H   TA=F7.4,10H       RB=F6.4,6H   TB=F7.4///)
 80 FORMAT(1H1)
  5 READ  15, G, DR, DT, A, PCENT
    PRINT  10
    PRINT 17,  G, DR, DT, A, PCENT
    EPSLON = (G-1.)/(G + 1.)
    U = -2./(G+1.)
    C = SQRTF(-U*G*EPSLON)
    N = 1./DR
    DIMENSION R1( 100), U1( 200), T1( 100), C1( 200)
    R1(1) = 1. - DR
    T1(1) = DR
    DO 100  I = 2,N
    T1(I)=T1(I-1) + DR
100 R1(I)=R1(I-1) -DR
    V = U
    UI = U
    UII = U
    CI  = C
    CII = C
    DELCI = 0.
    DELCII = 0.
    TP = DT/(1. + V)
    RP = 1. + V*TP
120 DELTI  =  DT/(1. + UI   +  CI )
    RA = RP - (UI   +  CI )* DELTI
    RI  = (RA + RP) * .5
130 DELUI = -2. * DELCI/(G-1.) - UI*CI*DELTI/RI
    UP = U + DELUI
    UIA = (U +UP)*.5
    UII = UIA
    DELUII = DELUI
    DELTII =  DT/(1. + UII -  CII)
    RB = RP - (UII -  CII)* DELTII
    RII = (RB + RP) * .5
    DELCII =  (DELUII - UII * CII *DELTII/RII)*(G-1.)/2.
    CP = C + DELCII
    CII = (CP + C)*.5
    CI =  (CII+CI)*.5
    O = ABSF((UIA - UI)/UIA*PCENT)
    L = 1
    PRINT 20, O, L
    DELCI =(DELCII+DELCI)*.5
    UI = (UIA + UI)*.5
    IF(O-1.)   150, 120, 120
150 V = (UP + U )*.5
    TG = DT/(1. + V)
    RG = 1. + V*TG
    P = ABSF((TG - TP)/TG * PCENT)
    Q = ABSF((RG - RP)/RG * PCENT)
    PRINT 30, P, Q
```

```
      RP = RG
      TP = TG
      IF ( P+Q-1.)   160, 160, 120
  160 TPISTN = TP*A
      GMPSTN =-UP/CP
      W = UI/A
      X = CI/A
      Y = UII/A
      Z = CII/A
      E = DELTI*A
      F = DELTII*A
      B = W + X
      D = Y - Z
      TA = TPISTN - E
      TB = TPISTN - F
      PRINT 40, RP, TPISTN, GMPSTN
      PRINT 60, W, X, B, Y, Z, D
      PRINT 70, RA, TA, RB, TB
      K = (1. - RP)/DR +1.
      R =R1(K)
      T1(K)=T1(K) + DT
      T =T1(K)
  140 DELTI  =  DT/(1. + UI  +  CI )
      RA = R  - (UI  +  CI )* DELTI
      RI  = (RA + R ) * .5
  170 DELUI = -2. * DELCI/(G-1.) - UI*CI*DELTI/RI
      UK = U + DELUI
      UII = (U + UK)*.5
      DELTII =  DT/(1. + UII -  CII)
      RB = R  - (UII -  CII)* DELTII
      RII = (RB + R ) * .5
      DELUII = DELUI
      DELCII =  (DELUII - UII * CII *DELTII/RII)*(G-1.)/2.
      DELCI =(DELCII+DELCI)*.5
      CK = C + DELCII
      CII = (CK + C ) *.5
      CI = (CI+CII)*.5
      UIA = (U +UK)*.5
      O = ABSF((UIA - UI)/UIA*PCENT)
      L = 2
      PRINT 20, O, L
      UI  =(UIA+UI)*.5
      IF (O - 100.)  175, 190, 190
  175 IF(O-1.)   180, 140, 140
  180 U1(K)= UK
      C1(K)= CK
      GMACH =-UK/CK
      TR = T*A
      W = UI/A
      X = CI/A
      Y = UII/A
      Z = CII/A
      E = DELTI*A
      F = DELTII*A
      B = W + X
      D = Y - Z
      TA = TR - E
      TB = TR - F
      PRINT 50, K, R, K, TR, K, GMACH
      PRINT 60, W, X, B, Y, Z, D
      PRINT 70, RA, TA, RB, TB
```

```
      K = K+1
      R =R1(K)
      T1(K)=T1(K) + DT
      T =T1(K)
      IF (T-1.) 140, 190, 190
  190 TPAM = TP
      RPAM = RP
      UPAM = UP
      CPAM = CP
      PRINT 80
      KMAX = K - 2
      V = UP
      TP = TP + DT/(1.+V)
      IF (1. - TP) 5, 5, 195
  195 RP= RP+V* DT/(1.+V)
      UI = UP
      UII = UP
      CI = CP
      CII = CP
      DELCI = 0.
  200 DELTI  =  DT/(1. + UI   +  CI  )
      RA = RP - (UI   +  CI )* DELTI
      RI  = (RA + RP) * .5
      IA = (1.-RA)/DR
      CA = (RA -R1(IA))*(C1(IA)-C1(IA + 1))/DR +C1(IA)
      UA = (RA -R1(IA))*(U1(IA)-U1(IA + 1))/DR +U1(IA)
      UI = (UA + UP)*.5
      CI = (CA+CP)*.5
      DELUI = -2. * DELCI/(G-1.) - UI*CI*DELTI/RI
      UP = UA + DELUI
      DELTII =  DT/(1. + UII -  CII)
      RB = RP - (UII -  CII)* DELTII
      RII = (RB + RP) * .5
      IB = (1.-RB)/DR
      IF (KMAX-IB) 5, 205, 205
  205 CB = (RB -R1(IB))*(C1(IB)-C1(IB + 1))/DR +C1(IB)
      UB = (RB -R1(IB))*(U1(IB)-U1(IB + 1))/DR +U1(IB)
      UII = (UB + UP)*.5
      CII= (CB+CP)*.5
      DELUII= UP - UB
      DELCII =  (DELUII - UII * CII *DELTII/RII)*(G-1.)/2.
      CP = CB + DELCII
      CII = (CP + CB)*.5
      DELCI =(CP-CA+DELCI)*.5
      CI =((CP + CA) * .5+CI)*.5
      UIA = (UA+UP)*.5
      O = ABSF((UIA - UI)/UIA*PCENT)
      L = 3
      PRINT 20, O, L
      UI  =(UIA+UI)*.5
      IF(O-1.)   210,  200, 200
  210 V = (UPAM+UP)*.5
      TG = DT/(1. + V) + TPAM
      RG = RPAM + V*DT/(1.+V)
      P = ABSF((TG - TP)/TG * PCENT)
      Q = ABSF((RG - RP)/RG * PCENT)
      PRINT 30, P, Q
      RP = RG
      TP = TG
      IF ( P+Q-1.)   220, 220, 200
  220 TPISTN = TP*A
```

```
      GMPSTN =-UP/CP
      W = UI/A
      X = CI/A
      Y = UII/A
      Z = CII/A
      E = DELTI*A
      F = DELTII*A
      B = W + X
      D = Y - Z
      TA = TPISTN - E
      TB = TPISTN - F
      PRINT 40, RP, TPISTN, GMPSTN
      PRINT 60, W, X, B, Y, Z, D
      PRINT 70, RA, TA, RB, TB
      K = (1. - RP)/DR +1.
      KKEEP = K
      R =R1(K)
      T1(K)=T1(K) + DT
      T =T1(K)
      UK = UP
      CK = CP
  230 DELTI  =  DT/(1. + UI   +  CI )
      RA = R   - (UI   +   CI )* DELTI
      RI   = (RA + R ) * .5
      IA = (1.-RA)/DR
      CA = (RA -R1(IA))*(C1(IA)-C1(IA + 1))/DR +C1(IA)
      UA = (RA -R1(IA))*(U1(IA)-U1(IA + 1))/DR +U1(IA)
      UI = (UA + UK)*.5
      CI = (CA+CK)*.5
      DELUI = -2. * DELCI/(G-1.) - UI*CI*DELTI/RI
      UK = UA + DELUI
      DELTII =  DT/(1. + UII -  CII)
      RB = R   - (UII -  CII)* DELTII
      RII = (RB + R ) * .5
      IB = (1.-RB)/DR
      IF(IB-KMAX) 233, 233, 250
  233 CB = (RB -R1(IB))*(C1(IB)-C1(IB + 1))/DR +C1(IB)
      UB = (RB -R1(IB))*(U1(IB)-U1(IB + 1))/DR +U1(IB)
      UII = (UB + UK)*.5
      CII= (CB+CK)*.5
      DELUII = UK - UB
      DELCII =  (DELUII - UII * CII *DELTII/RII)*(G-1.)/2.
      CK = CB + DELCII
      DELCI =(CK-CA+DELCI)*.5
      CI =((CK + CA)*.5+CI)*.5
      UIA = (UA + UK)*.5
      O = ABSF((UIA - UI)/UIA*PCENT)
      L = 4
      PRINT 20, O, L
      UI  =(UIA+UI)*.5
      IF (O - 100.)  235, 250, 250
  235 IF(O-1.)   240, 230, 230
  240 J = N+K
      U1(J)= UK
      GMACH =-UK/CK
      TR = T*A
      W = UI/A
      X = CI/A
      Y = UII/A
      Z = CII/A
      E = DELTI*A
```

```
      F = DELTII*A
      B = W + X
      D = Y - Z
      TA = TR - E
      TB = TR - F
      PRINT 50, K, R, K, TR, K, GMACH
      PRINT 60, W, X, B, Y, Z, D
      PRINT 70, RA, TA, RB, TB
      C1(J)= CK
      K = K+1
      R = R1(K)
      T1(K)=T1(K) + DT
      T =T1(K)
      IF (T-1.)  230, 250, 250
  250 K = K - 1
      KMAX = K
      DO 260  I = KKEEP, K
      II = I + N
      U1(I)=U1( II )
  260 C1(I)=C1( II )
      GO TO 190
      END
```

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