VARIATIONS OF THE EARTH'S GRAVITATIONAL FIELD FROM CAMERA TRACKING OF SATELLITES[1]

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Abstract

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7,234 Baker-Nunn camera observations of five satellites were analyzed to determine simultaneously 44 tesseral harmonic coefficients of the gravitational field, 36 station coordinates and 511 orbital elements. Supplementary observational data incorporated in the solution included accelerations of 24-hour satellites and directions between tracking stations from simultaneous observations; observation equations were also written for the differences between geometrical and gravitational geoid heights at tracking stations. Several variations in relative weighting of different observational data and a priori variances of parameters were tested. The previous independent solution most closely approached was that by Anderle based on Doppler data, from which the rms discrepancy was $\pm$ 0.18 x 10^{-6} for 38 normalized harmonic coefficients, or $\pm$ 7 meters in total geoid height. An equatorial radius of 6 378 153 $\pm$ 8 m. was obtained.

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Introduction. The analyses described in this paper are a continuation of those reported three years ago [Kaula, 1963a, b]. They are an appreciable improvement over the previous analyses not only in using observations of a better variety of more recent orbits, but also in better methods of analysis and in using supplemental data. This investigation is one of four principal efforts in the determination of tesseral harmonics of the gravitational field. The complexity of such investigations makes it desirable that there be independent efforts which differ not only in the tracking data but also in the techniques of analysis applied.

Changes from previous solutions. The dynamical theory applied, formation of partial derivatives, use of observational and timing variances, formation of observational equations, and accumulation of normal equations are essentially the same as described by Kaula [1963a, b; see also Kaula, 1966a]. The most significant improvement is in the data, Baker-Nunn camera observations of the Smithsonian Astrophysical Observatory. The satellites used are somewhat better distributed in inclination, and, all being later than 1962 March 7, are appreciably less affected by drag than those used in the earlier analyses. The satellite data is summarized in Table 1. In determining the preliminary orbits, arcs were rejected for the final analysis not only if the number of observations were insufficient but also if excessive iterations were required to obtain a satisfactory fit. The greatest deficiency of camera tracking using solar illumination appears to be an inability to obtain a good distribution of observations of satellites which are low enough to be sensitive to the variations of the gravitational field (perigee below 1200 km) and are of inclination appreciably higher than the latitudes of the tracking stations (less than 37°). Thus the most sensitive satellite used in this study, 1961 o 1, is the poorest observed, while the best observed, 1961 $\alpha\ \delta$ 1, is so high as to be useless to determine gravitational harmonics above the 4th degree.

To enable solution for a maximum number of tesseral harmonics, the central term GM was held fixed at 3.986009×10^{14} m^3/sec^2, the mean of values determined from Ranger lunar probes [Sjogren & Trask, 1965], and the zonal harmonics J_2 through J_7 were held fixed at the values [Kozai, 1964; King-Hele et al, 1965a,b] given in Table 2. Perturbations due to these zonal harmonics, as well as luni-solar perturbations of more than 10^{-5} amplitude, were calculated in both preliminary and final orbit analyses. Arbitrary polynomials were limited to a t^2 term in the mean anomaly, making seven orbital constants for each arc.

To enable solution, in effect, for an indefinite number of orbital constants simultaneously with tesseral harmonic coefficients and corrections to station coordinates, the technique of partitioned normals was used; i.e., writing the normal equations as [Kaula, 1966a, sec. 5.3]:

$$\begin{Bmatrix} \underset{\sim}{N}_{11} & \underset{\sim}{N}_{12} \\ \underset{\sim}{N}_{21} & \underset{\sim}{N}_{22} \end{Bmatrix} \begin{Bmatrix} \underset{\sim}{z}_1 \\ \underset{\sim}{z}_2 \end{Bmatrix} = \begin{Bmatrix} \underset{\sim}{s}_1 \\ \underset{\sim}{s}_2 \end{Bmatrix} \tag{1}$$

where $\underset{\sim}{N}$ is the matrix of normal equation coefficients, $\underset{\sim}{z}$ is the vector of corrections of parameters, and $\underset{\sim}{s}$ is the vector of normal equation constants, then a solution of z_1 alone can be written as:

$$\underset{\sim}{z}_1 = \left[\underset{\sim}{N}_{11} - \underset{\sim}{N}_{12}\, \underset{\sim}{N}_{22}^{-1}\, \underset{\sim}{N}_{21}\right]^{-1} \left[\underset{\sim}{s}_1 - \underset{\sim}{N}_{12}\, \underset{\sim}{N}_{22}^{-1}\, \underset{\sim}{s}_2\right] \tag{2}$$

If $\underset{\sim}{z}_2$ comprises corrections to orbital constants, which are peculiar to each arc, then the non-zero elements in the matrix $\underset{\sim}{N}_{22}$ will be in a series of square blocks down the main diagonal, one block per arc. Hence the inversion $\underset{\sim}{N}_{22}^{-1}$ and the subtractions of $\underset{\sim}{N}_{12}\, \underset{\sim}{N}_{22}^{-1}\, \underset{\sim}{N}_{21}$ and $\underset{\sim}{N}_{12}\, \underset{\sim}{N}_{22}^{-1}\, \underset{\sim}{s}_2$ in equation (2) can be made separately for each arc one at a time. Therefore at any time there need be stored in the computer only those parts of the normal equations pertaining to the parameters common to all arcs -- the corrections $\underset{\sim}{z}_1$ to tesseral harmonic coefficients and stations coordinates -- plus the parts peculiar to the one arc being analysed. This technique is also used by Anderle [1966] and Guier & Newton [1965] in analyzing Transit Doppler tracking data; it is probably the principal difference of this paper in method from the iterative technique used by Izsak [1966] and Gaposhkin [1966] in analyzing the Baker-Nunn camera tracking data.

The principal inaccuracies in the calculations, aside from neglect of drag, are believed to be the absence of short period J_2^2 terms in the orbital theory of Brouwer [1959] and the failure to correct station positions to a common epoch for latitude variation [Veis, 1960, pp 97-98]. Both of these defects are on the order of $\pm$ 10 meters or less in effect.

Hence, the parameters to be determined were selected as being of greater expected effect. Experience indicates that tracking stations as far apart as the Baker-Nunn cameras should, to this level of accuracy, be considered as moving separately. Hence 36 of the unknowns in $\underset{\sim}{z}_1$ are corrections to station coordinates. To select the tesseral harmonic coefficients to be determined in addition to the low degree terms up to degree and order ℓ,m of 4,4 and the small divisor terms for which m is approximately equal to the number of revolutions per day and ℓ is odd, a calculation of orbital perturbations was carried under the assumption that the normalized coefficients $\bar{C}_{\ell m}$, $\bar{S}_{\ell m}$ are $\pm$ 8 x $10^{-6}/\ell^2$ in magnitude, a rule-of-thumb which appears quite good up to about degree 15 [Kaula, 1966b]. The results of this calculation appear in Table 3. 22 coefficients of degrees 5 through 8 were thus selected.

The small divisor, or near resonant, harmonics [Anderle, 1965; Yionoulis, 1965] under the $\pm$ 8 x $10^{-6}/\ell^2$ assumption were significant for 1960 ι 2 (12th order), 1961 o 1 (14th order), and 1962 $\beta\mu$ 1 (13th order) but not for 1959 α 1 or 1961 α δ 1. The particular degrees selected for solution were those which happened to have the largest partial derivatives. The procedure for evaluating these partial derivatives is exactly the same as for the lower degree harmonics, with the important precaution that the rate for a perturbation of the mean anomaly through the perturbation of the semi-major axis is not assumed to be an integer multiple of the mean motion: i.e., for a disturbing function term

$$R_{\ell mpq} = \frac{\mu}{a}\left(\frac{a_e}{a}\right)^{\ell} F_{\ell mp}(I)\; G_{\ell pq}(e)\; S_{\ell mpq}(\omega, M, \Omega, \theta) \tag{3}$$

the indirect perturbation of the mean anomaly is:

$$\Delta_2 M_{\ell mpq} = \frac{3\mu a_e^{\ell} F_{\ell mp}\, G_{\ell pq}\, \bar{\bar{S}}_{\ell mpq}\, (\ell-2p+q)}{a^{\ell+3}\left\{(\ell-2p)\,\dot{\omega} + (\ell-2p+q)\,\dot{M} + m\,(\dot{\Omega}-\dot{\theta})\right\}^2} \tag{4}$$

where the overbar indicates integration of the sinusoidal function with respect to its argument. [Kaula, 1966a, Sec. 3.5].

To strengthen the solution, two types of supplemental data were included: the accelerations of 24-hour synchronous satellites and the mutual directions of tracking stations obtained from simultaneous satellite observations.

The acceleration in longitude of a 24-hour satellite appears in an observation equation of the form

$$\sum_{(\ell-m)\ \text{even}} Q_{\ell m}\left[\bar{C}_{\ell m}\sin m\lambda - \bar{S}_{\ell m}\cos m\lambda\right] = \ddot{\lambda}_o + \delta\ddot{\lambda}_o \tag{5}$$

where

$$Q_{\ell m} = \left[\frac{(\ell+m)!\ (2\ell+1)}{(\ell-m)!}\right]^{\frac{1}{2}} 3n^2 m\left(\frac{a_e}{a}\right)^{\ell} F_{\ell mp}(I)\ G_{\ell po}(e), \tag{6}$$

$$p = (\ell-m)/2$$

[Kaula, 1966a, Sec. 3.6]. The observed accelerations $\ddot{\lambda}_o$ (corrected for luni-solar perturbations) and their standard deviations $\sigma(\ddot{\lambda}_o)$ were taken from the work of Wagner [1966]. Five accelerations of 1963 - 31A at a variety of longitudes and one acceleration each of 1964 - 47A and 1965 - 28A were used, as summarized in Table 4.

The direction of one tracking station from another as obtained by simultaneous observations of satellites appears in an observation equation of the form

$$\begin{Bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{Bmatrix} \underset{\sim}{R}_{\ell u}\left[\underset{\sim}{u}_j + \Delta\underset{\sim}{u}_j - \left(\underset{\sim}{u}_i + \Delta\underset{\sim}{u}_i\right)\right] \Big/ \left|\underset{\sim}{u}_j - \underset{\sim}{u}_i\right| = \delta\underset{\sim}{\ell} \tag{7}$$

where $\underset{\sim}{R}_{\ell u}$ is the rotation matrix from coordinates referred to the earth's pole and Greenwich meridian to coordinates with the 1-axis along the line from station i to station j and the 2-axis along the major axis of the error ellipse of the observed direction:

$$\underset{\sim}{R}_{\ell u} = \underset{\sim}{R}_1(\rho)\ \underset{\sim}{R}_3(-\varphi)\ \underset{\sim}{R}_3(\lambda) \tag{8}$$

In equation (7), φ and λ constitute the observed direction of station j from station i in the form of latitude and longitude, and ρ is the angle between the normal to the meridian plane defined by λ and the major axis of the error ellipse.

The directions between 14 pairs of Baker-Nunn camera stations derived by Aardom et al [1965] from 615 pairs of quasi-simultaneous observations of satellites of about 3700 km altitude are given in the form of direction cosines $\underset{\sim}{c}$ with respect to polar-Greenwich axes of station j from station i. The standard deviations are given in the form of the semi-major and semi-minor axes a, b of the error ellipse and the angle θ between the major axis and the normal to the plane defined by the stations and the earth's center. To apply these observations in equations (6), (7) we have:

$$\left.\begin{aligned}
\varphi &= \sin^{-1} c_3 \\
\lambda &= \tan^{-1} c_2/c_1 \\
n &= \underset{\sim}{u}_2 \times \underset{\sim}{u}_1 \\
\underset{\sim}{m} &= \begin{cases} -\sin\varphi\cos\lambda \\ -\sin\varphi\sin\lambda \\ \cos\varphi \end{cases} \\
\underset{\sim}{k} &= \begin{cases} -\sin\lambda \\ \cos\lambda \end{cases} \\
\rho &= \tan^{-1}(\underset{\sim}{n}\cdot\underset{\sim}{m}/\underset{\sim}{n}\cdot\underset{\sim}{k}) + \theta - \pi/2
\end{aligned}\right\} \qquad (9)$$

The major semi-axis of the error ellipse was always within 18° of the station-center plane. The number of observation pairs used for each position line varied from 5 to 90; the standard ellipse major semi-axis, from $\pm$ 2.3 to $\pm$ 10.5 x 10^{-6}; the minor semi-axis, from $\pm$ 0.9 to $\pm$ 3.9 x 10^{-6}. The stations appearing in these 14 equations are noted in the last column of Table 7.

In addition, we can write as an observation equation the fact that the geometrical geoid height derived from the position of a tracking station should differ from the gravitational geoid height calculated for the same point from the harmonic coefficients only by the contribution δN_{GR} of variations in the gravitational field of higher degree than those represented by the coefficients:

$$\{1\ 0\ 0\}\underset{\sim}{R}_{\ell m}\,\Delta\underset{\sim}{u} - a_e \sum_{\ell,m} \overline{P}_{\ell m}(\sin\varphi)\left[\overline{C}_{\ell m}\cos m\lambda + \overline{S}_{\ell m}\sin m\lambda\right] = \delta N_{GR} - N_{GE}$$

where $\underset{\sim}{R}_{\ell u}$ is defined by equation (8) (the rotation about the 1 axis being of no effect) using the position φ,λ of the station, $\overline{P}_{\ell m}$ is the normalized associated Legendre function, and N_{GE} is the geometrically calculated geoid height, obtained from station position $\underset{\sim}{u}$, the station height above sea level h, and the reference ellipsoid of semi-major axis 6378165 m. and flattening of 1/298.25, corresponding to the J_2 in Table 1. Also applied as a fixed correction are the contributions of the fixed zonal harmonics to the gravitational geoid height. Since the semi-major axis a_e is used to calculate the geometrical geoid height N_{GE} in equation (10), the mean radial shift of the tracking stations can be considered as a correction to the semi-major axis. The standard deviation of the "observation" δN_{GR} in equation (10) was estimated to be $\pm$ 20 meters as follows. The 49 coefficients fixed or being determined on the $\pm\ 8 \times 10^{-6}/\ell^2$ rule contribute a mean square of $(26\text{ m})^2$ to the geoid height, which was subtracted from the $(33\text{ m})^2$ mean square estimated from autocovariance analysis of gravimetry [Kaula, 1959, p. 2418].

In combining widely differing types of data, the relative weighting is necessarily somewhat arbitrary, particularly when the observational variances are derived in different ways. For the satellite observations, variances based on observational residuals of previous analyses were used: $(12.0'')^2$ direction and $(0.050^s)^2$ time [Kaula, 1963b, p. 5184]. For the 24-hour satellite accelerations and the directions between stations, the variances produced by the least-squares analyses of Wagner [1966] and Aardom et al [1965], respectively, were used.

Furthermore, when one type of data is represented by many more observations than another, as was the case for the close satellite data (14,468 equations) compared to the supplemental data (47 equations), then the neglect of covariances in the former will be much more significant, and the use of the correct variances in simple least squares will result in an overweighting of the more numerous relative to the less numerous.

For the foregoing considerations the computer program was modified so that when the normal equations for a particular satellite had been generated, they were saved on tape to be read off and multiplied by the weighting factor before being added to the combined normal equations. In this manner, additional solutions with different combinations of weights could be made with less than one minute computer time each. A further capability which was included for these short-time additional solutions was change in pre-assigned variances and starting values for the parameters.

Some of the data weighting and pre-assigned standard deviations of parameters tried are given in Table 5. The varied satellite weights and the supplemental equation weights in excess of 100 were calculated on the basis of making each satellite and each block of supplemental data of equal weight. However, since the satellite variances are probably too large and the supplemental variances probably too small, the smaller weights for the supplemental data are probably more realistic. In any case, over a quite wide range of weights the influence in the solution will appear for any data which differs significantly from the bulk of the data in its sensitivity to certain parameters.

As discussed by Kaula [1966b], solutions for a set of station coordinates from close satellite tracking are subject to systematic error in orientation. In the iterative solutions from camera data by Izsak [1966], Veis [1965], and Gaposhkin [1966], the overall orientation is essentially fixed by correcting orbital longitudes and station longitudes at alternate stages. In the solutions from Doppler data by Anderle [1966] and Guier & Newton [1965], one station is held fixed to establish a longitude reference. In the analyses described in this paper, several solutions (A through E, L and N in Table 5) were made in which all stations were left free to move, in the hope that adequate orientation would be obtained from the inertially referred directions constituted by the camera observations. The opposite alternative of fixing one station in one or more coordinates was also tried (solutions O, P, and Q). However, there is no reason to give preference to one station over another. Hence it seems better to treat all stations equally and to allow some influence to the camera directions by preassigning variances to all station positions (solutions F through K and M).

Missing from Table 5 are some obvious alternatives, such as: omitting or giving higher weight to the 24-hour satellite data; restraining the 5th-8th Degree gravitational coefficients completely; including or omitting mutual direction and geoid height equations separately; etc. Most of these alternatives were tested at an earlier stage, with a set of close satellite data differing in some respects from that used in the final analysis. In these tests the variations in the weighting of the 24-hour satellites had a considerable effect: their omission resulted in a wider scatter of results for the coefficients $\bar{C}_{33}$, $\bar{S}_{33}$ as well as some others, while weighting them heavily distorted $\bar{C}_{31}$, $\bar{S}_{31}$ from the values strongly indicated by the close satellite data. Variations in the weights of the geometrical data and restraining the higher gravitational coefficients appeared to have little effect on the solution for the low degree coefficients. Also tested was omission of each close satellite, one at a time, in a solution for the low degree gravitational coefficients. As anticipated, omission of 1961$\alpha\delta$1, the least sensitive satellite, had least effect, while omission of 1961o1 had greatest effect.

Results.

The principal test of the value of different solutions was intended to be the chi-square test: if the original estimates of weights, variances and covariances are good (and if the formulation of the problem is correct), then the corrected quadratic sum should be close to the degrees of freedom. In other words, the quantity

$$q = \left[\underset{\sim}{f}^T \underset{\sim}{W}^{-1} \underset{\sim}{f} - \underset{\sim}{z}^T \underset{\sim}{s} \right] \Big/ (n-p) \tag{11}$$

should be close to unity, where $\underset{\sim}{f}$ is the vector of observation equation constants; $\underset{\sim}{W}$ is the weighted covariance matrix; n is the number of observations; p is the number of parameters; and $\underset{\sim}{z}$ and $\underset{\sim}{s}$ are the solution and normal equation constant vectors, as in equation (1). The q's obtained varied from 1.18 (solution B) to 1.54 (solution E). However, much of this variation is due to the weights which are incorporated in the sums in the numerator, but not in the denominator, of equation (11). If the number of observations n is changed from $\sum_i n_i$ to $\sum_i \omega_i n_i$, where ω_i is the weight of data type i, then the q's vary from 1.01 (solution E) to 1.33 (solution F), with A, D, F thru K, and M thru Q all between 1.25 and 1.33. Of those which are distinctly lower, B, C, and L all fail to utilize the mutual direction and geoid height data. On the other hand, E over-utilizes this data: i.e., some of the geometrical geoid heights resulting from solution E agree with the gravitational geoid heights within a meter, which is not possible without distorting the lower degree gravitational coefficients by forcing them to absorb a lot of the higher degree contributions to the station geoid heights.

Hence the choice of preferred solution must be based on more selective indicators of the essential quality of sensitivity of data to parameters determined. The most obvious weakness is that of overall orientation: when all 36 station coordinates are free to shift, erratic results are obtained, as shown by Solutions A and N in Table 7. Some constraint must be applied, as it has been in all previous analyses of close satellite tracking. Such constraint necessarily amounts to allowing some influence to previous solutions. The station positions obtained by the iterative satellite orbit analysis of Izsak [1966] and Gaposhkin [1966] now seem superior to starting values based on terrestrial data, as used by Kaula [1963a,b]: certainly so for stations not connected to continental datums. The next choice is between expressing this influence by fixing one station (Solutions O, P, Q) or by assigning a priori variances to all station positions (Solutions F thru K and M). As previously discussed, the latter seems preferable on general grounds; the results in Tables 6 and 7 do not appear to markedly contradict this preference.

The two solutions which assigned a priori variances to gravitational coefficients, K and M, differed negligibly in their results from solutions J and G, respectively, the maximum changes being decreases in absolute magnitude of .09 to $.11 \times 10^{-6}$ in two or three 5th and 6th degree coefficients. Of the remaining solutions F thru J, F, I, and J are preferable to G and H because they incorporate the supplemental data, while H, I, and J are preferable to F and G because they give relatively greater weight to the sensitive lower satellites 1961ol and 1959α1 than to the insensitive high satellite 1961$\alpha\delta$1. The two preferred solutions I and J differ in the weight assigned to the supplemental equations, the effect of which shows most markedly in the sectorial harmonic coefficients $\bar{C}_{33}$, $\bar{C}_{44}$, and $\bar{S}_{44}$. For these three coefficients solution J is much closer than I to the independent results based on Doppler data of Anderle [1966] and Guier & Newton [1965]. Perhaps the differences are a reflection of the variances adopted for the direction data being too small relative to those for the close satellite data. We adopt solution J, but the preference is mild.

Eight solutions for gravitational coefficients through the 8th degree are given in Table 6, which suffices to demonstrate the more important effects of variations in weighting. The standard deviations $\sigma_{\ell m}$ resulting from the least squares calculation are also given for solution J; the one figure given pertains to both $\bar{C}_{\ell m}$ and $\bar{S}_{\ell m}$, since their standard deviations always agreed within $.01 \times 10^{-6}$. The highest correlations between different harmonics produced by the least squares occurred in the expected places; i.e., (1) between coefficients both appearing in the 24-hour satellite equations, for example -0.754 for $r(\bar{C}_{22}, \bar{C}_{33})$; -0.321 for $r(\bar{C}_{33}, \bar{S}_{33})$; -0.311 for $r(\bar{S}_{22}, \bar{C}_{33})$; and 0.240 for $r(\bar{S}_{33}, \bar{C}_{42})$; and (2) between coefficients of the same order m and degree ℓ differing by an even number, for example -0.534 for $r(\bar{C}_{41}, \bar{C}_{61})$; 0.692 for $r(\bar{C}_{41}, \bar{C}_{81})$; 0.480 for $r(\bar{C}_{42}, \bar{C}_{82})$; 0.446 for $r(\bar{C}_{44}, \bar{C}_{64})$; etc. All correlation coefficients not in these two categories were less than 0.18, most of them less than 0.08. Most correlations between gravitational coefficients and station coordinates were less than 0.05, the largest being 0.152 for $r(\bar{C}_{44}, u_{3,2})$ and -0.143 for $r(\bar{C}_{31}, u_{1,12})$.

The solutions for the 15th degree coefficients are not shown in Table 6 because they always came out the same:

$$\bar{C}_{15,12} = -0.043\ (\pm .002) \times 10^{-6}, \quad \bar{S}_{15,12} = -0.031\ (\pm .002) \times 10^{-6}$$

$$\bar{C}_{15,13} = -0.032\ (\pm .007) \times 10^{-6}, \quad \bar{S}_{15,13} = -0.065\ (\pm .007) \times 10^{-6}$$

$$\bar{C}_{15,14} = 0.010\ (\pm .003) \times 10^{-6}, \quad \bar{S}_{15,14} = -0.011\ (\pm .003) \times 10^{-6}$$

The geoid corresponding to solution J (plus Table 2) is shown in Figure 1. For 38 tesseral harmonic coefficients in common with the solution of Anderle [1966], the quadratic sum of differences in the coefficients was 1.29×10^{-12}, equivalent to $\pm$ 7.3 meters in geoid height, or an rms discrepancy of $\pm 0.18 \times 10^{-6}$ per coefficient. For other solutions, the comparable figures are: Guier & Newton [1965]: 38 coefficients, 1.91×10^{-12}, $\pm$ 8.8 m., $\pm 0.22 \times 10^{-6}$; Izsak [1966]: 32 coefficients, 1.94×10^{-12}, $\pm$ 8.9 m., $\pm 0.25 \times 10^{-6}$; and Gaposhkin [1966]: 40 coefficients, 1.00×10^{-12}, $\pm$ 6.4 m., $\pm 0.16 \times 10^{-6}$.

The results for station coordinate shifts are given in Table 7, together with the standard deviations for the preferred solution J. The ill-conditioning and orientation problems occuring when the stations are allowed to move freely are evident from the results for solutions A and N: for formal standard deviations for station coordinates generated by the least squares solutions were about $\pm$ 11 meters, but the rms difference between solutions A and N is $\pm$ 25 meters. Covariance between different stations also appears to be high; for example, the solution N $\Delta u_{1,12}$ has 16 correlation coefficients which are higher than 0.20. The fluctuations of station positions between different solutions in Table 7 is considerably more than that implied by the fluctuations of gravitational coefficients in Table 6. Multiplying the range of variation of a coefficient (e.g., 0.10×10^{-6} for $\overline{C}_{31}$) in Table 6 by the average partial derivative of satellite position with respect to the coefficient yields a range of about 6 meters in orbital position. From this, we would expect a range of about $\sqrt{12} \times 6$, or 20 meters, in station position, since a station coordinate appears in 1/12 as many equations. This is about equal to the absolute average discrepancy between coordinates for solutions O and J, which utilize the two alternative methods of fixing orientation. It is also about equal to the rms deviation of the coordinate shifts (not including $\Delta u_{1,12}$) of solution J, $\pm$ 22 m., from the iterated solution of Gaposhkin [1966]. For the one outstanding shift of + 127 meters for $u_{1,12}$, no special explanation has been found. Rotating to local coordinates, the shift of station 12 for solution J is - 99 m. in radius, + 41 m. in prime vertical, and + 83 m. in latitude. Solution I, which gives heavier weight to the mutual direction and geoid height equations, reduces the radial shift to - 35 m., but increases the northeastward shift by about 30 meters.

Geometrical geoid heights with respect to an ellipsoid of equatorial radius 6378165 meters and flattening 1/298.25 were calculated from the final positions for solution J. These geoid heights together with gravitational geoid heights obtained from Figure 1 are given in Table 8. If the mean value of a geometrical minus gravitational geoid height is taken as a correction to the semi-major axis, a value of 6378153 $\pm$ 8 m. is obtained. Using this radius with the GM of 3.986009×10^{14} m^3/sec^2 gives an equatorial gravity γ_e of 978.0284 cm sec^{-2}, which is in better accord with terrestrial solutions than previously obtained [Kaula, 1966b].

Conclusions. This investigation demonstrates that a good solution for the non-zonal harmonics of the gravitational field can be obtained from a relatively small amount of data. The agreement of the gravitational coefficients with other solutions using different data or methods of analysis is also quite satisfying, indicating that the amplitudes of persistent oscillations in the orbits are being determined within about ± 5 meters. The results for station coordinate shifts are not so satisfactory: the limitations on directions with respect to inertial space in which observations can be made for a given orbital arc of 18 days or so apparently results in poor separation of station coordinates from orbital parameters. Some constraint in orientation is needed for the entire system, as well as probably a considerably larger amount of data to gain an improvement over the accuracy of ± 20 meters obtained in this study.

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TABLE 1 Close Satellite Orbit Specifications

Satellite	1959α1	1960ι2	1961o1	1961$\alpha\delta$1	1962$\beta\mu$1
Name	VANGUARD 2	ECHO 1 ROCKET	TRANSIT 4A	MIDAS 4	ANNA 1B
Epoch	1963Jan27.0	1963Jan10.0	1962May21.0	1962Aug.18.0	1963Jan9.0
Semimajor Axis	1.301994	1.250052	1.146988	1.568120	1.177254
Eccentricity	0.16417	0.01139	0.00799	0.01209	0.00707
Inclination	0.57383	0.82437	1.16620	1.67302	0.87514
Argument of Perigee	3.13491	1.93573	1.18658	1.67305	0.94214
Longitude of Node	2.87158	0.79776	0.46898	6.27650	2.84671
Mean Anomaly	1.76589	5.92654	3.92748	0.67818	0.80524
Min. Acceleration*	-0.51×10^{-9}	-1.00×10^{-9}	$+0.02\times10^{-9}$	-0.25×10^{-9}	-0.44×10^{-9}
Max. Acceleration*	$+7.01\times10^{-9}$	$+1.35\times10^{-9}$	$+0.90\times10^{-9}$	$+0.41\times10^{-9}$	$+0.27\times10^{-9}$
Perigee motion/Day	+0.09238	+0.05200	-0.01210	-0.01708	+0.04364
Node motion/Day	-0.06141	-0.05415	-0.01438	+0.00367	-0.04119
Periods/Day	11.48	12.20	13.86	8.68	13.35
Max. A/m, cm^2/g	0.21	0.27	0.12	0.08	0.07
Min. A/m, cm^2/g	0.21	0.08	0.11	0.02	0.07
Perigee Height, km	560	1500	880	3500	1080
Starting Date	1963Jan.18	1963Jan.1	1962May12	1962Aug.3	1962Dec.31
Ending Date	1963Nov.20	1963Sep.28	1963Jul.24	1963Oct.27	1963Nov.2
Number of Arcs	13	15	15	15	15
Days/Arc	18	18	18	30	18
Min. Obs./Arc	42	67	32	61	61
Total Observations	790	1628	612	2882	1322
SAO Spec. Rep. Nos.	185	185	148,185	147,185	168

*Units for acceleration: dn/dt in radians/(806.8 secs.)^2 where n is mean motion

TABLE 2 Fixed Zonal Harmonics

ℓ	J_ℓ 10^{-6}	$\overline{C}_{\ell o}$ 10^{-6}
2	1082.70	-484.198
3	-2.55	0.965
4	-1.50	0.500
5	-0.15	0.045
6	0.50	-0.140
7	-0.37	0.090

TABLE 3 Amplitude of Perturbations of Close Satellite Orbits by 5th and Higher Degree Harmonics ℓm of Normalized Magnitude $\pm 8 \times 10^{-6}/\ell^2$ (excluding zonal and near-resonant harmonics).

Satellite	a	e	I	More than 20 Meters	10 to 20 Meters	5 to 10 Meters
1959α1	1.302	.16	32.9°	51,52,61,62, 63,71,81.	53,72,83, 101,111.	54,64,73,74,82,84,92, 93,102,104,122,141.
1960ι2	1.250	.01	47.2°	51,61.	52,63,64, 82,101.	53,54,62,65,71,72,81, 85.
1961o1	1.147	.01	66.8°	51,61,62,63, 65.	52,53,54, 55,64,66, 72,81, 84,101,121.	71,73,74,75,76,86,87, 91,92,102,103,111.
1961αδ1	1.568	.01	95.9°			61,62.
1962βμ1	1.77	.01	50.1°	51,52,61,63, 64.	53,62,65, 71,81,82, 101.	54,55,72,73,74,75,83, 85,86,92,102,111,121

TABLE 4 24-Hour Satellite Orbits

Satellite	1963-31A					1964-47A	1965-28A
Name	SYNCOM II					SYNCOM III	EARLY BIRD
Inclination	33°					0.1°	0.2°
Start Longitude	305.1°	244.7°	174.0°	118.0°	81.0°	179.2°	330.7°
End Longitude	302.4°	197.5°	161.5°	102.2°	52.0°	178.2°	330.7°
Observed Acceleration $\times 10^9$	-1.962	1.888	0.435	-2.203	0.849	1.476	-1.291
	± 28	± 74	± 44	± 44	± 54	± 62	± 9
Amplitude Factors of Partial Derivatives Q_{22}			$.7775 \times 10^{-3}$			$.9144 \times 10^{-3}$	
Q_{31}			$-.0155 \times 10^{-3}$			$-.0582 \times 10^{-3}$	
Q_{33}			$.1752 \times 10^{-3}$			$.2253 \times 10^{-3}$	
Q_{42}			$.0008 \times 10^{-3}$			$-.0182 \times 10^{-3}$	
Q_{44}			$.0344 \times 10^{-3}$			$.0482 \times 10^{-3}$	

Accelerations and Partial Derivatives in radians/(planetary time unit)^2, where planetary time unit = 806.8137 secs.

TABLE 5 Data Weights and A Priori Standard Deviations of Parameters

Solution	Data Weights			Parameter A Priori Standard Deviations	
	Close Satellites[1]	24-Hour Satellites	Mutual Directions & Geoid Heights[2]	Station Positions[3] m.	Gravity Coefficients $\bar{C}_{\ell m}$, $\bar{S}_{\ell m}$ 10^{-6}
A	1	1	1	∞	∞
B	1	1	0	∞	∞
C	Varied	1	0	∞	∞
D	Varied	21.2	Moderate	∞	∞
E	Varied	21.2	High	∞	∞
F	1	1	1	10.	∞
G	1	1	0	10.	∞
H	Varied	1	0	10.	∞
I	Varied	21.2	Moderate	10.	∞
J	Varied	1	1	10.	∞
K	Varied	1	1	10.	Deg 2-4:∞; 5-8:$8/\ell^2$
L	Varied	1	0	∞	Deg 2-4:∞; 5-8:$8/\ell^2$
M	1	1	0	10.	All $8/\ell^2$
N	Varied	1	1	∞	∞
O	Varied	1	1	a	∞
P	Varied	1	1	b	∞
Q	Varied	1	1	c	∞

NOTES: [1]: Varied satellite weighting: 1959α1, 2.05; 1960ι2, 1.00; 1961o1, 2.70; 1961$\alpha\delta$1, 0.55; 1962$\beta\mu$1, 1.20.

[2]: Moderate weighting: Directions 10.5, Heights 16.4; High weighting: Directions 110., Heights 270.

[3]: Station weighting a-c: all stations ∞, except a: Station 1 fixed in all coordinates; b: Station 1 fixed in longitude and radius; c: Station 1 fixed in longitude only.

TABLE 6 Coefficients of the Gravitational Field

Deg.	Ord.	Solution																
		A		F		H		I		N		O		Q		J		
ℓ	m	$\bar{C}_{\ell m}$	$\bar{S}_{\ell m}$	$\bar{C}_{\ell m}$	$\bar{S}_{\ell m}$	$\bar{C}_{\ell m}$	$\bar{S}_{\ell m}$	$\bar{C}_{\ell m}$	$\bar{S}_{\ell m}$	$\bar{C}_{\ell m}$	$\bar{S}_{\ell m}$	$\bar{C}_{\ell m}$	$\bar{S}_{\ell m}$	$\bar{C}_{\ell m}$	$\bar{S}_{\ell m}$	$\bar{C}_{\ell m}$	$\bar{S}_{\ell m}$	$\sigma_{\ell m}$
		10^{-6}		10^{-6}		10^{-6}		10^{-6}		10^{-6}		10^{-6}		10^{-6}			10^{-6}	
2	2	2.48	-1.38	2.46	-1.38	2.45	-1.40	2.49	-1.37	2.47	-1.39	2.43	-1.41	2.46	-1.43	2.45	-1.40	±.02
3	1	2.04	0.17	2.13	0.16	2.07	0.19	2.13	0.11	2.02	0.21	1.95	0.23	1.94	0.23	2.09	0.20	.05
3	2	0.60	-0.71	0.75	-0.79	0.78	-0.71	0.86	-0.76	0.67	-0.65	0.83	-0.81	0.65	-0.68	0.76	-0.73	.10
3	3	0.38	1.29	0.49	1.28	0.53	1.26	0.25	1.27	0.42	1.26	0.56	1.27	0.51	1.24	0.53	1.29	.09
4	1	-0.68	-0.55	-0.64	-0.49	-0.61	-0.48	-0.61	-0.53	-0.62	-0.51	-0.57	-0.44	-0.60	-0.54	-0.61	-0.48	.03
4	2	0.32	0.75	0.41	0.76	0.37	0.73	0.37	0.73	0.35	0.77	0.31	0.70	0.32	0.70	0.38	0.73	.05
4	3	0.91	0.08	0.82	0.14	0.87	0.09	0.92	0.06	0.92	0.06	0.88	0.06	0.91	0.08	0.87	0.09	.06
4	4	-0.39	-0.23	-0.42	0.14	-0.27	0.32	-0.01	-0.03	-0.35	-0.04	-0.09	0.08	-0.19	0.07	-0.27	0.27	.11
5	1	-0.10	-0.02	-0.07	0.01	-0.03	0.02	-0.01	0.03	-0.07	0.01	-0.03	0.00	-0.03	-0.00	-0.03	0.02	.04
5	2	0.71	-0.02	0.69	-0.23	0.74	-0.18	0.70	-0.28	0.77	-0.05	0.80	-0.13	0.80	-0.04	0.74	-0.21	.09
5	3	-0.68	0.40	-0.52	0.27	-0.46	0.27	-0.65	0.37	-0.71	0.35	-0.60	0.11	-0.58	0.26	-0.52	0.34	.11
6	1	-0.14	0.19	-0.19	0.15	-0.19	0.10	-0.18	0.16	-0.15	0.16	-0.16	0.13	-0.16	0.17	-0.19	0.11	.04
6	2	0.07	-0.32	0.10	-0.39	0.05	-0.39	0.09	-0.34	0.09	-0.33	0.06	-0.37	0.10	-0.36	0.06	-0.38	.06
6	3	0.13	0.36	-0.03	0.43	0.09	0.36	0.01	0.34	0.15	0.28	0.11	0.32	0.12	0.33	0.07	0.35	.08
6	4	0.21	-0.86	0.07	-0.48	0.17	-0.39	0.30	-0.47	0.23	-0.79	0.23	-0.54	0.27	-0.71	0.18	-0.42	.10
6	5	-0.03	-0.51	-0.01	-0.39	-0.09	-0.40	-0.27	-0.30	-0.18	-0.51	-0.19	-0.43	-0.18	-0.54	-0.11	-0.38	.11
7	1	0.25	0.14	0.25	0.10	0.25	0.12	0.31	0.14	0.25	0.16	0.22	0.14	0.23	0.16	0.26	0.13	.05
8	1	-0.09	-0.04	-0.02	0.01	-0.04	0.05	-0.05	0.02	-0.08	0.01	-0.00	0.08	-0.05	-0.01	-0.04	0.05	.05
8	2	-0.11	-0.08	0.15	-0.39	0.12	-0.04	0.02	-0.08	-0.09	-0.07	0.03	-0.07	-0.07	-0.09	0.10	-0.05	±.08

TABLE 7 Station Positions

No.	Name (number of observations)		Starting Coordinates m	A m	F m	H m	I m	N m	O m	Q m	J m	σ m	In Direction eq. with Sta. No.
1.	Organ	u_1	-1 535 753	-92	-53	-38	-27	-108	0	+19	-41	±6	7,9,10,12
	Pass	u_2	-5 167 000	+119	+16	+20	+11	+98	0	+64	+14	5	
	(926)	u_3	3 401 047	+208	+19	+47	+21	+181	0	+144	+27	5	
2.	Olifants-		5 056 133	-38	+10	+19	+18	+1	+24	-17	+19	6	8,9,10
	fontein		2 716 489	-73	-24	-40	-32	-105	-68	-50	-36	7	
	(664)		-2 775 832	-66	+4	-3	+5	-45	-14	-36	-1	7	
3.	Woomera		-3 983 738	+61	+10	-11	-9	-38	+18	+15	+6	7	
	(719)		3 743 127	-119	-39	-44	-40	-85	-61	-121	-43	6	
			-3 275 615	-50	+3	+8	+9	-27	+8	-16	+10	6	
4.	San		5 105 610	-108	-15	-28	-6	-94	-48	-85	-15	5	8,9,10
	Fernando		-0 555 226	-55	-22	-31	+3	-64	-36	-13	-23	7	
	(790)		3 769 693	+210	+31	+59	+26	+193	+73	+175	+40	5	
5.	Tokyo		-3 946 697	+143	+18	+23	+29	+140	+65	+89	+22	7	6
	(339)		3 366 293	-44	-1	-1	-19	-30	-17	-70	-4	7	
			3 698 858	+214	+16	+13	+22	+174	+56	+152	+14	7	
6.	Naini		1 018 206	+34	+17	+29	+7	+52	+26	-2	+19	7	5,8
	Tal		5 471 103	-111	-4	-3	-1	-97	+25	-69	-4	5	
	(678)		3 109 620	+245	+32	+21	+44	+217	+88	+193	+36	5	
7.	Are-		1 942 768	+3	+6	+9	+5	-29	-15	+36	+1	5	1,9,10,11
	quipa		-5 804 089	+64	-1	+3	-5	+41	-5	+40	+3	5	
	(518)		-1 796 968	+67	+8	+4	+33	+55	+43	+94	+11	7	
8.	Shiraz		3 376 887	-30	+8	+18	-10	-15	-1	-47	+6	6	2,4,6
	(564)		4 403 994	-120	-21	-39	-10	-105	-45	-80	-22	6	
			3 136 264	+235	+27	+36	+33	+207	+80	+186	+33	5	
9.	Curacao		2 251 822	+6	+10	-6	+9	-26	-15	+35	+5	5	1,2,4,7,10, 11
	(484)		-5 816 923	+87	+8	-22	+11	+63	+6	+60	+6	5	
			1 327 171	+168	+3	-3	+11	152	+17	+159	+10	5	
10.	Jupiter		0 976 281	-82	-10	-57	-3	-64	-31	+3	-13	5	1,2,4,7,9
	(567)		-5 601 390	+91	+2	+4	-1	+71	-1	+60	+1	5	
			2 880 247	+225	+33	+32	+35	+198	+34	+190	+39	5	
11.	Villa		2 280 572	+7	+1	-23	+8	-23	-22	+29	+6	5	7,9
	Dolores		-4 914 580	+131	+22	+20	+6	+89	+33	+98	+16	6	
	(552)		-3 355 464	-46	+1	+4	+26	-27	+4	-11	+8	6	
12.	Maui	u_1	-5 466 063	+245	+101	+138	+84	+236	+212	+261	+127	6	1
	(623)	u_2	-2 404 286	+36	+4	+28	-35	+60	+3	+11	+12	6	
		u_3	2 242 180	+251	+40	+38	+65	+203	+61	+194	+43	±6	

TABLE 8 Comparison of Geoid Heights (Solution J)

Referred to an ellipsoid a_e = 6 378 165 m., f = 1/298.25

Station Number	Latitude Deg.	Longitude East Deg.	Elevation above MSL m	Geoid Height Geometrical m	Geoid Height Gravitational m
1	32.4	253.4	1651	-32	-23
2	-26.0	28.2	1544	21	27
3	-31.1	136.8	162	-28	1
4	36.5	353.8	24	60	52
5	35.7	139.5	58	22	20
6	29.4	79.5	1927	-59	-52
7	-16.5	288.5	2451	25	4
8	29.6	52.5	1596	-27	-13
9	12.1	291.2	7	-43	-20
10	27.0	279.9	15	-44	-27
11	-31.9	294.9	598	25	10
12	20.7	203.7	3035	-101	-17

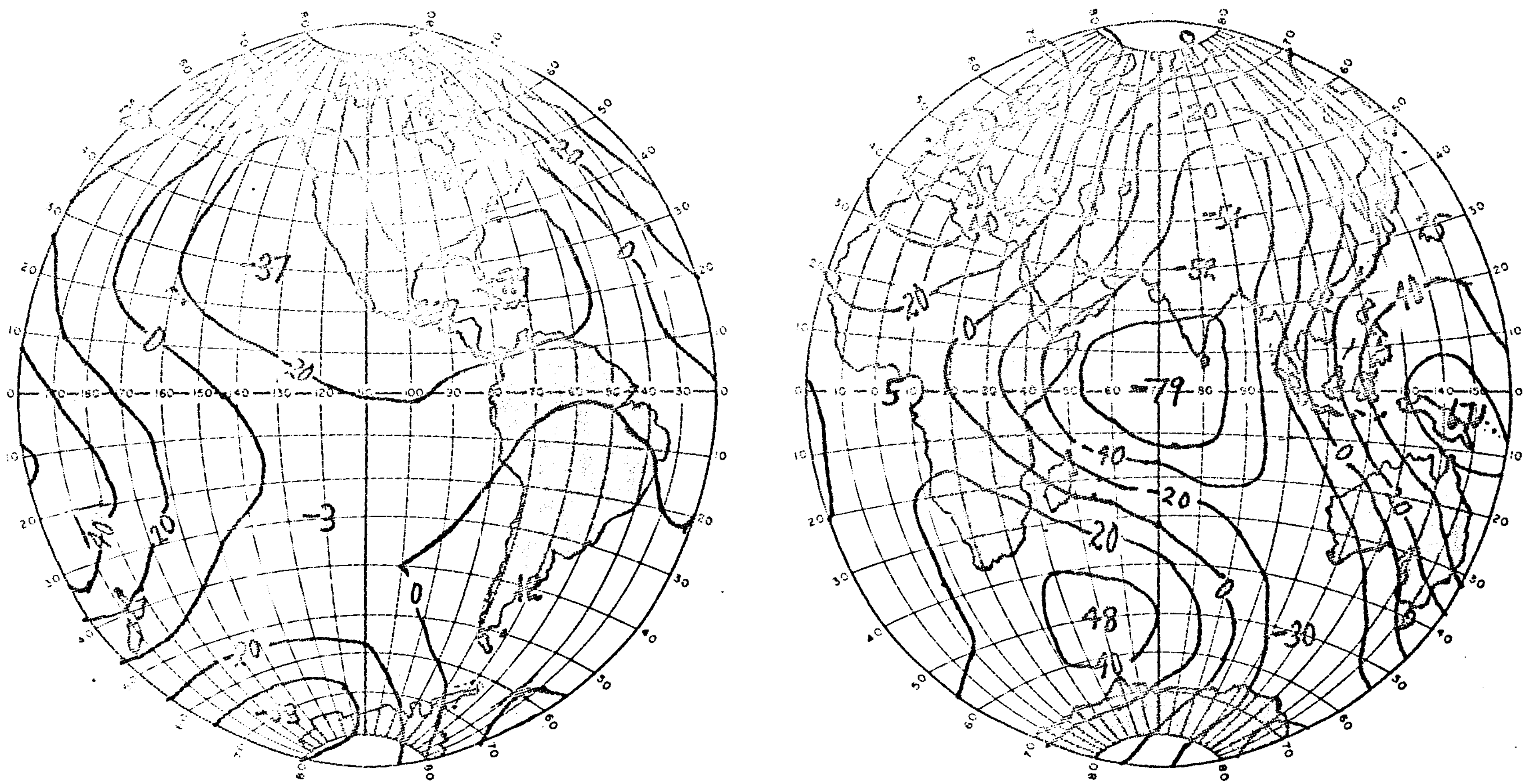

Fig. 1 Geoid Heights in Meters referred to an ellipsoid of flattening 1/298.25.
Based on Solution J, Table 6.